Algebraic Number Theory

Exercises Tutorium 5

Dr. Tom Bachmann — Winter Semester 2021–22

Let A be a commutative ring. Recall the definition of a fractional ideal, and a locally free module of rank 1.

Exercise 1. Let A be a field. Show that any locally free module of rank 1 is isomorphic to A, and any fractional ideal is principal.

Exercise 2. Let M be a locally free module of rank 1. Show that M is finitely generated. Deduce that if A is a domain, every locally free module of rank 1 is isomorphic to a fractional ideal.

Exercise 3. Let A be noetherian. Show that the set of isomorphism classes of locally free modules of rank 1 forms a group, with operation induced by tensor product. This is the *Picard group* $Pic(A)$.

[*Hint*: you may wish to prove first that if M is a finitely generated module, N is any module and S is a multiplicative subset, then $S^{-1}\mathrm{Hom}(M, N) \simeq \mathrm{Hom}(S^{-1}M, S^{-1}N)$.]

Exercise 4. Let A be a Dedekind domain. Show that $Pic(A)$ is isomorphic to the ideal class group of A.

[You may use that if I, J are ideals, then $IJ \simeq I \otimes J$ (sheet 5).]

Exercise 5. *Extra.* Let A be a commutative ring, M, N arbitrary A-modules such that $M \otimes_A N \simeq A$. Show that M is locally free of rank 1, as follows:

(1) Show that tensoring with M is an autoequivalence of the category of A-modules.
(2) Deduce that M is projective.
(3) Under the isomorphism $A \simeq M \otimes N$, let 1 correspond to $\sum_{i=1}^r m_i \otimes n_i$. Using the map

$$A^r \otimes N^r \to M \otimes N \simeq A, (a_1, \ldots, a_r) \otimes (n_1, \ldots, n_r) \mapsto \sum_{i,j} a_i m_i \otimes n_j,$$

show that M s finitely generated.
(4) Show that a finitely generated projective module is locally free.