Algebraic Number Theory

Exercises Tutorium 10

Dr. Tom Bachmann Winter Semester 2021–22

Exercise 1. Let $d \not\equiv 1(4)$ be squarefree. Using the "$efg = n$ formula", classify primes which ramify in $\mathbb{Q}(\sqrt{d})$.

Exercise 2. Let
$$\Phi_{p^r}(X) = \frac{X^{p^r} - 1}{X^{p^{r-1}} - 1} \in \mathbb{Z}[X].$$
We shall give another proof that Φ_{p^r} is irreducible, as follows.

(1) Let $K = \mathbb{Q}(\zeta_{p^r})$, where ζ_{p^r} is a primitive p^r-th root of unity. Show that
$$[K : \mathbb{Q}] \leq \deg \Phi_{p^r} =: e,$$
with equality if and only if Φ_{p^r} is irreducible.

(2) Let $z_1, \ldots, z_e$ be the roots of Φ_{p^r}. Show that
$$\prod_i (z_i - 1) = \pm p.$$

(3) Show that $(z_j - 1)\mathcal{O}_K$ is independent of j.

(4) Deduce that $p\mathcal{O}_k = P^e$, for some ideal P. Conclude.