

Homework Sheet 10

(Released 19.12.2025 – Discussed 9.01.2026)

E10.1 Let $\{u_n\}_{n=1}^\infty$ be a sequence in $H^1(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^d)$. Prove that, up to a subsequence, the following hold:

(a) For every $R > 0$,

$$\mathbf{1}_{B(0,R)} u_n \rightarrow \mathbf{1}_{B(0,R)} u \text{ in } L^2(\mathbb{R}^d) \text{ as } n \rightarrow \infty$$

(b) $u_n(x) \rightarrow u(x)$ for almost every $x \in \mathbb{R}^d$.

E10.2 Let $A : D(A) \rightarrow \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that every isolated point of $\sigma(A)$ is an eigenvalue of A .

E10.3 Let $A : D(A) \rightarrow \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that $\mu_1 \in \sigma_{\text{ess}}(A)$ and that there exists an orthonormal family $\{u_n\}_{n=1}^\infty$ in $D(A)$ such that $\|(A - \mu_1)u_n\| \rightarrow 0$ as $n \rightarrow \infty$. Fix $k \in \mathbb{N}$, and define $M_m = \text{span}\{u_{m+1}, \dots, u_{m+k}\}$ for each $m \in \mathbb{N}$. Prove that

$$\lim_{m \rightarrow \infty} \sup_{\substack{u \in M_m \\ \|u\|=1}} \langle u, Au \rangle = \mu_1$$

E10.4 Let $A : D(A) \rightarrow \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and assume that $\mu_1 = \min \sigma(A) < \inf \sigma_{\text{ess}}(A)$, with corresponding normalized eigenvector u_1 . We decompose $\mathcal{H} = W_1 \oplus W_1^\perp$, where $W_1 = \text{span}\{u_1\}$. Prove that

$$\mu_n(A_1) = \mu_{n+1}(A),$$

where $A_1 = A|_{W_1^\perp}$ and $\mu_n(\cdot)$ denotes the n th min-max value of the corresponding operator.

E10.5 Let $V \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ be a nonnegative function such that $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. Let $A = -\Delta + V$ be the self-adjoint operator on $L^2(\mathbb{R}^d)$ defined via Friedrichs extension. Prove that

$$\lim_{n \rightarrow \infty} \mu_n(A) = \infty.$$

Homework Sheet 9

(Released 12.12.2025 – Discussed 19.12.2025)

E9.1 Let $V \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ with $p \geq 2$. Prove that the operator $A = -\Delta + V$ on $L^2(\mathbb{R}^3)$ is a self-adjoint with domain $D(A) = H^2(\mathbb{R}^3)$. Hint: Use **E8.3**.

E9.2 Let $V \in L^p(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ with $p > 1$. Prove that the operator $A = -\Delta + V$ on $L^2(\mathbb{R}^2)$ is bounded from below when defined on the domain $D(A) = C_c^\infty(\mathbb{R}^2)$.

E9.3 Let $\{v_n\}_{n=1}^\infty$ be a normalized sequence in a Hilbert space $\mathcal{H}$ such that $v_n \rightharpoonup 0$. Prove that for every $\varepsilon > 0$, there exists a subsequence $\{v_{n_k}\}_{k=1}^\infty$ such that

$$\sum_{k=1}^{\infty} |v_{n_k}\rangle\langle v_{n_k}| \leq 1 + \varepsilon.$$

E9.4 Let $d \geq 3$, and assume that $V \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let $\{u_n\}_{n=1}^N$ be a sequence in $H^1(\mathbb{R}^d)$ such that $\{\sqrt{-\Delta}u_n\}_{n=1}^N$ is an orthonormal family in $L^2(\mathbb{R}^d)$.

(a) Prove that $\{\nabla u_n\}_{n=1}^N$ is an orthonormal family in $L^2(\mathbb{R}^d)^d$.

(b) Show that

$$N = \sum_{n=1}^N \int_{\mathbb{R}^d} |\nabla u_n(x)|^2 dx \geq C_d \int_{\mathbb{R}^d} \rho_N^{d/(d-2)}(x) dx$$

for some constant $C_d > 0$ independent of N and $\{u_n\}_{n=1}^N$, where

$$\rho_N(x) = \sum_{n=1}^N |u_n(x)|^2.$$

(c) Assume that $-\Delta + V$ on $L^2(\mathbb{R}^d)$ has N negative eigenvalues λ_n , with corresponding eigenvectors w_n . Prove that there exists $\{w_n\}_{n=1}^N \subset H^1(\mathbb{R}^d)$ such that $\{\sqrt{-\Delta}w_n\}_{n=1}^N$ is an orthonormal family in $L^2(\mathbb{R}^d)$ and

$$\langle w_n, (-\Delta + V)w_n \rangle_{L^2(\mathbb{R}^d)} \leq 0$$

for all $1 \leq n \leq N$. Hint: Choose $\{w_n\}_{n=1}^N$ inside $\text{span}\{v_n\}_{n=1}^N$ so that $\{\sqrt{-\Delta}w_n\}_{n=1}^N$ becomes an orthonormal basis of $\text{span}\{\sqrt{-\Delta}v_n\}_{n=1}^N$.

(d) Conclude that if $-\Delta + V$ has N negative eigenvalues, then

$$N \leq C_d \int_{\mathbb{R}^d} |V(x)|^{d/2} dx$$

for some constant $C_d > 0$ independent of V and N .

Homework Sheet 8

(Released 05.12.2025 – Discussed 12.12.2025)

E8.1 Let $A : D(A) \rightarrow \mathcal{H}$ be a symmetric operator on a Hilbert space $\mathcal{H}$ satisfying $A \geq 1$, and let

$$Q(A) = \{x \in \mathcal{H} : \exists (\psi_n)_n \text{ a } \|\cdot\|_Q\text{-Cauchy sequence in } D(A) \text{ such that } x_n \rightarrow x \text{ in } \mathcal{H}\}$$

be the quadratic form domain of A , where the form norm $\|x\|_Q^2 = \langle x, Ax \rangle$. Define

$$D(A_F) = \{x \in Q(A) : \exists z \in \mathcal{H}, \langle x, y \rangle_Q = \langle z, y \rangle \ \forall y \in Q(A)\},$$

and set $A_F x = z$. Prove that $A_F : D(A_F) \rightarrow \mathcal{H}$ is a symmetric extension of A .

E8.2 Let $A : D(A) \rightarrow \mathcal{H}$ be a symmetric operator on a Hilbert space $\mathcal{H}$ satisfying $A \geq 1$. Prove that if B is a self-adjoint extension of A such that

$$\inf_{x \in D(A), \|x\|=1} \langle x, Ax \rangle = \inf_{x \in D(B), \|x\|=1} \langle x, Bx \rangle,$$

then $Q(A) = Q(B)$.

E8.3 Given $m, d \in \mathbb{N}$ such that $d < 2m$, prove that there exists a constant $C > 0$ such that

$$\|u\|_{H^m(\mathbb{R}^d)} \geq C \|u\|_{L^p(\mathbb{R}^d)}$$

for all $u \in H^m(\mathbb{R}^d)$ and every $p \in [2, \infty]$.

Homework Sheet 7

(Released 28.11.2025 – Discussed 05.12.2025)

E7.1 Let $\mathfrak{A} = \text{Mat}(2 \times 2, \mathbb{C})$, and let ω be a state on $\mathfrak{A}$. Describe the GNS representation associated with ω . More precisely, find a Hilbert space $\mathcal{H}$, a representation $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$, and a normalized vector $\Omega \in \mathcal{H}$ such that they realize ω as

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$$

for all $A \in \mathfrak{A}$, and, moreover, $\overline{\pi(\mathfrak{A})\Omega} = \mathcal{H}$. Hint: You can assume that ω is given in terms of a diagonal density matrix (why?). You might have to distinguish cases.

E7.2 For each $x \in \mathbb{Z}$, let $\mathcal{H}_x = \mathbb{C}^2$, and for each finite subset $\Lambda \subset \mathbb{Z}$, define the Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$, from which we can consider the local C^* -algebra $\mathfrak{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$. Finally, let $\mathfrak{A} = \overline{\bigcup_{\Lambda \in \mathbb{Z}} \mathfrak{A}_\Lambda}$ as explained in the lecture, where $\Lambda \Subset \mathbb{Z}$ means that Λ is a finite subset of $\mathbb{Z}$. For $\Lambda \subset \tilde{\Lambda} \Subset \mathbb{Z}$, there is a natural embedding

$$\mathfrak{A}_\Lambda \ni A \mapsto A \otimes \mathbf{1}_{\tilde{\Lambda} \setminus \Lambda} \in \mathfrak{A}_{\tilde{\Lambda}},$$

where $\mathbf{1}_\Sigma$ denotes the identity operator on $\mathcal{H}_\Sigma$, with $\Sigma \Subset \mathbb{Z}$.

(a) For $\Lambda \Subset \mathbb{Z}$, define $\psi_{+, \Lambda} = \bigotimes_{x \in \Lambda} (1, 0)$ and $\psi_{-, \Lambda} = \bigotimes_{x \in \Lambda} (0, 1)$, and let $\omega_{\pm, \Lambda}$ be the corresponding vector state on the local algebra $\mathfrak{A}_\Lambda$, i.e., $\omega_{\pm, \Lambda}(A) = \langle \psi_{\pm, \Lambda}, A \psi_{\pm, \Lambda} \rangle$ for all $A \in \mathfrak{A}_\Lambda$. Prove that the local states are compatible, in the sense that for $\Lambda \subset \tilde{\Lambda} \Subset \mathbb{Z}$ and $A \in \mathfrak{A}_\Lambda$, we have $\omega_{\pm, \tilde{\Lambda}}(A \otimes \mathbf{1}_{\tilde{\Lambda} \setminus \Lambda}) = \omega_{\pm, \Lambda}(A)$, and conclude that we can extend $\omega_{\pm, \Lambda}$ to a state $\omega_\pm$ on $\mathfrak{A}$.

(b) Let $(\mathcal{H}^\pm, \pi^\pm, \Omega^\pm)$ be the GNS representations of $\omega_\pm$. For each $n \in \mathbb{N}$, consider the finite interval $\Lambda_n = [-n, n] \cap \mathbb{Z}$ and the local magnetization operator

$$M_n = \frac{1}{2n+1} \sum_{x=-n}^n \sigma_x,$$

where

$$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove that

$$\lim_{n \rightarrow \infty} \langle \psi, \pi^\pm(M_n)\varphi \rangle = \pm \langle \psi, \varphi \rangle$$

for all $\varphi, \psi \in \mathcal{H}^\pm$. Hint: It suffices to show that for $\varphi, \psi \in \bigcup_{\Lambda \Subset \mathbb{Z}} \pi^\pm(\mathfrak{A}_\Lambda)\Omega^\pm$.

(c) Conclude that $\pi^\pm$ are not unitary equivalent. Hint: Look at the spectrum of $\pm \mathbf{1}$.

Homework Sheet 6

(Released 21.11.2025 – Discussed 28.11.2025)

E6.1 Let $A : D(A) \rightarrow \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that if $D(A) = \mathcal{H}$, then A is bounded.

E6.2 Let $A : D(A) \rightarrow \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$.

(a) For each $n \in \mathbb{N}$, define

$$A_n = \frac{nA}{n + |A|}.$$

Prove that A_n is bounded for all $n \in \mathbb{N}$, and that $\|A_n x - Ax\| \rightarrow 0$ for all $x \in D(A)$.

(b) Prove that $\|A(A \pm in)^{-1}\| \leq 1$ and $\|(A \pm in)^{-1}\| \leq n^{-1}$ for all $n \in \mathbb{N}$

E6.3 Let $A : D(A) \rightarrow \mathcal{H}$ be a symmetric operator on a Hilbert space $\mathcal{H}$. Prove that A is essentially self-adjoint if and only if $\overline{\text{ran}(A \pm i)} = \mathcal{H}$.

E6.4 Let $A : D(A) \rightarrow \mathcal{H}$ be a symmetric operator on a Hilbert space $\mathcal{H}$ satisfying $A \geq 1$. Prove that

$$\langle x, y \rangle_Q = \langle x, Ay \rangle$$

is an inner product on $D(A)$.

Homework Sheet 5

(Released 14.11.2025 – Discussed 21.11.2025)

E5.1 Let $A : D(A) \rightarrow \mathcal{H}$ be a (densely defined) operator on a Hilbert space $\mathcal{H}$. Prove that

$$\mathcal{H} = \overline{\text{ran}(A)} \oplus \ker(A^*).$$

E5.2 Let $A : D(A) \rightarrow \mathcal{H}$ be a (densely defined) operator on a Hilbert space $\mathcal{H}$. Prove that the spectrum $\sigma(A) \subset \mathbb{C}$ is closed.

E5.3 Let $A : D(A) \rightarrow \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. In the lecture, we have proved that $S = (A + i)^{-1}$ is a normal bounded operator, and hence it is unitarily equivalent to M_f for some complex-valued function $f \in L^\infty(\Omega, \mu)$.

- (a) Prove that $f \neq 0$ almost everywhere.
- (b) Prove that $a = (f^{-1} - i)$ is real-valued almost everywhere.
- (c) Prove that $U[D(A)] = D(M_a)$, where $U : L^2 \rightarrow \mathcal{H}$ is the unitary transformation such that $USU^{-1} = M_f$.

E5.4 Let $\mathfrak{A} = \text{Mat}(2 \times 2, \mathbb{C})$. Argue that all states ρ on $\mathfrak{A}$ are of the form

$$\rho = a\mathbf{1} + \vec{b} \cdot \vec{\sigma}$$

for some $a \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^3$, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Here, $\vec{b} \cdot \vec{\sigma} = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3$. For which values of $a \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^3$ does this yield a pure state?

E5.5 Let $\mathfrak{B} \subseteq \mathfrak{A}$ be a C^* -subalgebra of the C^* -algebra $\mathfrak{A}$ that contains the unit of $\mathfrak{A}$, and let ω be a state on $\mathfrak{A}$

- (a) Prove that $\omega|_{\mathfrak{B}}$ is a state.
- (b) Suppose $\mathfrak{A} = \text{Mat}(N \times N, \mathbb{C}) \otimes \text{Mat}(M \times M, \mathbb{C}) \simeq \text{Mat}(MN \times MN, \mathbb{C})$ and $\mathfrak{B} = \{A \otimes \mathbf{1}_M : A \in \text{Mat}(N \times N, \mathbb{C})\}$. If $\omega(a) = \text{Tr}(\rho a)$ for some $\rho \in \text{Mat}(MN \times MN, \mathbb{C})$, find $\tilde{\rho} \in \text{Mat}(N \times N, \mathbb{C})$ such that

$$\omega|_{\mathfrak{B}}(A \otimes \mathbf{1}_n) = \text{Tr}(\tilde{\rho}A).$$

Homework Sheet 4

(Released 7.11.2025 – Discussed 14.11.2025)

E4.1 Let A be a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Prove that

$$(fg)(A) = f(A)g(A)$$

for all real-valued functions $f, g \in C(\sigma(A))$.

E4.2 In the lecture, we proved that for any $u \in \mathcal{H}$, there exists a Borel measure μ_u on $\mathbb{R}$ such that

$$\langle u, f(A)u \rangle = \int_{\sigma(A)} f(\lambda) d\mu_u(\lambda)$$

for every real-valued function $f \in C(\sigma(A))$.

(a) Prove that this allows us to extend $f(A)u$ to all complex-valued functions $f \in L^2(\sigma(A), \mu_u)$, and in this case

$$\|f(A)u\| = \|f\|_{L^2}.$$

(b) Prove that there exists a unitary operator $U : \mathcal{H}_u \rightarrow L^2(\sigma(A), \mu_u)$ such that

$$UAU^* = M_a,$$

where $a(\lambda) = \lambda$ and $\mathcal{H}_u = \overline{\text{span}\{A^n u : n = 0, 1, \dots\}}$

E4.3 Let $\mathfrak{A}$ be a non-unital C^* -algebra. Define its unitization $\mathfrak{A}^+$ as the algebra whose underlying vector space is $\mathfrak{A} \oplus \mathbb{C}$, equipped with multiplication and involution

$$(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu), \quad (a, \lambda)^* = (a^*, \bar{\lambda}),$$

and with the norm

$$\|(a, \lambda)\| = \sup_{b \in \mathfrak{A}, \|b\| \leq 1} \|ab + \lambda b\|.$$

(a) Prove that $\mathfrak{A}^+$ is a C^* -algebra with unit $(0, 1)$.

(b) Prove that $\mathfrak{A}$ is isometrically and $*$ -isomorphic to $\{(a, 0) : a \in \mathfrak{A}\}$.

Homework Sheet 3

(Released 31.10.2025 – Discussed 7.11.2025)

E3.1 Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint bounded operator on a Hilbert space $\mathcal{H}$. Prove that if $u_1 \in \mathcal{H}$ is an minimizer of the variational problem

$$\mu_1 = \inf\{\langle u, Au \rangle : u \in \mathcal{H}, \|u\| = 1\},$$

then $Au_1 = \mu_1 u_1$.

E3.2 Let A be a compact self-adjoint operator on a separable ∞ -dimensional Hilbert space $\mathcal{H}$. Prove that its lowest eigenvalues $\mu_1 \leq \mu_2 \leq \dots$ (counted with multiplicity) satisfy the *min-max principle*

$$\mu_n = \min_{\substack{M \subset \mathcal{H} \\ \dim M = n}} \max_{\substack{u \in M \\ \|u\|=1}} \langle u, Au \rangle, \quad \forall n = 1, 2, \dots$$

E3.3 Let (Ω, μ) be a σ -finite measure space, and let $a : \Omega \rightarrow \mathbb{C}$ be a measurable function. Define the multiplication operator M_a on $L^2(\Omega, \mu)$ by

$$(M_a f)(x) = f(x)a(x), \quad D(M_a) = \{f \in L^2(\Omega, \mu) : af \in L^2(\Omega, \mu)\}.$$

Prove the following statements:

- (a) $D(M_a)$ is dense in $L^2(\Omega, \mu)$.
- (b) The spectrum of M_a is equal to the essential range of a .
- (c) M_a is bounded if and only if a is bounded. Moreover, in this case $\|M_a\| = \|a\|_{L^\infty}$.
- (d) M_a is self-adjoint if and only if a is real-valued almost everywhere.
- (e) λ is an eigenvalue of M_a if and only if $\mu(a^{-1}(\{\lambda\})) > 0$.

E3.4 Let $\mathfrak{A}$ be a unital C^* -algebra. For each $a \in \mathfrak{A}$, define its spectral radius by

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\},$$

where $\sigma(a) = \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } \mathfrak{A}\}$. Prove the following statements:

- (a) $r(a) = \lim_n \|a^n\|^{1/n}$.
- (b) If $a = a^*$, then $\|a\| = r(a)$.
- (c) If $\|\cdot\|_1$ is a another norm on $\mathfrak{A}$ that also makes it a C^* -algebra, then

$$\|a\| = \|a\|_1, \quad \forall a \in \mathfrak{A}.$$

- (d) If $\mathfrak{B}$ is another unital C^* -algebra and $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a linear map satisfying

$$\Phi(ab) = \Phi(a)\Phi(b), \quad \Phi(a^*) = \Phi(a)^*, \quad \text{and} \quad \Phi(\mathbf{1}) = \mathbf{1}, \quad \forall a, b \in \mathfrak{A},$$

(with $\mathbf{1}$ the identity element), then Φ is continuous and $\|\Phi\|_{\text{op}} \leq 1$.

Homework Sheet 2

(Released 24.10.2025 – Discussed 28.10.2025)

E2.1 Let $A : D(A) \rightarrow \mathcal{H}$ be a densely defined operator on a Hilbert space $\mathcal{H}$. Prove that

$$D(A^*) = \{x \in \mathcal{H} : y \mapsto \langle x, Ay \rangle \text{ is bounded in } (D(A), \|\cdot\|_{\mathcal{H}})\},$$

where the domain of A^* is given by

$$D(A^*) = \{x \in \mathcal{H} : \exists z \in \mathcal{H}, \langle z, y \rangle = \langle x, Ay \rangle \forall y \in D(A)\}.$$

E2.2 Let $A : D(A) \rightarrow \mathcal{H}$ be a densely defined operator on a Hilbert space $\mathcal{H}$ such that

$$\sup_{\substack{x \in D(A) \\ x \neq 0}} \frac{\|Ax\|}{\|x\|} < \infty.$$

Prove that A can be uniquely extended to a bounded linear operator.

E2.3 Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Let X be a subspace of $\mathcal{H}$. Prove that if A leaves X invariant, namely $A(X) \subset X$, then A also leaves $X^\perp$ invariant.

E2.4 Let $\{x_n\}_{n=1}^\infty$ be a sequence in a Hilbert space $\mathcal{H}$. Prove that:

- (i) If $x_n \rightarrow x$, then $x_n \rightharpoonup x$.
- (ii) If $x_n \rightharpoonup x$, then $\liminf_n \|x_n\| \geq \|x\|$.
- (iii) If $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$.
- (iv) If $\{x_n\}_{n=1}^\infty$ is an orthonormal family, then $x_n \rightharpoonup 0$.

E2.5 Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Prove that the following are equivalent:

- (i) $\overline{A(B(0,1))}$ is compact in $\mathcal{H}$, where $B(0,1)$ is the open unit ball in $\mathcal{H}$.
- (ii) A maps weakly convergent sequences to strongly convergent sequences; that is, for all sequences $\{x_n\}$, if $x_n \rightharpoonup x$, then $Ax_n \rightarrow Ax$.

Homework Sheet 1

(Released 17.10.2025 – Discussed 21.10.2025)

E1.1 Let $\mathcal{H}$ be a (complex) vector space and let $\langle \cdot, \cdot \rangle$ be an inner product. Define the norm $\|x\| = \sqrt{\langle x, x \rangle}$.

(i) Prove the Cauchy–Schwarz inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\|, \quad \forall x, y \in \mathcal{H}.$$

(ii) Prove the triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathcal{H}.$$

(iii) Give an example where $(\mathcal{H}, \|\cdot\|)$ is complete, and an example where $(\mathcal{H}, \|\cdot\|)$ is not complete. Justify your examples.

E1.2 Let $\mathcal{H}$ be a Hilbert space and $W \subset \mathcal{H}$ be a subspace. Let $\overline{W}$ be the closure and $W^\perp$ the orthogonal complement of W .

(i) Prove that $W^\perp = (\overline{W})^\perp$.

(ii) Prove that $W^\perp$ is closed.

E1.3 Prove Heisenberg’s uncertainty principle

$$\left(\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \right) \left(\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 dx \right) \geq \frac{d^2}{4}, \quad \forall f \in C_c^2(\mathbb{R}^d).$$

E1.4 Prove that

$$\inf_{\|f\|_{L^2}=1} \left\{ \frac{9}{4 \int_{\mathbb{R}^3} |x|^2 |f(x)|^2 dx} - \int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|} dx \right\} = -\infty$$

where the infimum is taken over all $f \in C_c^2(\mathbb{R}^3)$ such that $\|f\|_{L^2(\mathbb{R}^3)} = 1$.

This implies that Heisenberg’s uncertainty principle is insufficient to explain the stability of the hydrogen atom.