Mathematisches Institut LMU
Phan Thành Nam & Dong Hao Ou Yang

Homework Sheet 12

(Released 14.7.2025 – Discussed 18.7.2025)

E12.1 Let $d \in \mathbb{R}$. Recall that the Hardy–Littlewood maximal function

$$M_f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)| \mathrm{d}y$$

satisfies the weak type estimate

$$|\{x : M_f(x) > \lambda\}| \leq C_d \lambda^{-1} \|f\|_{L^1(\mathbb{R}^d)}, \quad \forall\, 0 < \lambda < \infty.$$

Prove that we have the strong type estimate

$$\|M_f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|f\|_{L^p(\mathbb{R}^d)}, \quad \forall\, 1 < p < \infty.$$

Hint: You can use the layer-cake representation $\|f\|_{L^p}^p = p \int_0^\infty |\{x : |f(x)| > t\}| t^{p-1} \mathrm{d}t$.

E12.2 Let f be a normalized vector in the Hilbert space $\mathcal{H}$. For every $\lambda \in \mathbb{R}$ define the Bogoliubov transformation U_λ on the bosonic Fock space $\mathcal{F}(\mathcal{H})$:

$$U_\lambda = \exp\left(\frac{\lambda}{2}(a^*(f)^2 - a(f)^2)\right)$$

(a) Prove that

$$U_\lambda^* a(f) U_\lambda = \cosh(\lambda) a(f) + \sinh(\lambda) a^*(f).$$

(b) Let $A = a^*(f)a(f) + a(f)a^*(f)$. Prove the operator inequality

$$U_\lambda^* A U_\lambda \leq e^{2|\lambda|} A.$$

E12.3 Let u_0 be a normalized vector on a Hilbert space $\mathcal{H}$. Recall from the direct sum $\mathcal{H} = \mathrm{span} u_0 \oplus \mathcal{H}_+$ with $\mathcal{H}_+ = \{u_0\}^\perp$, we have the isometry of bosonic Fock spaces

$$\mathcal{F}(\mathcal{H}) \approx \mathcal{F}_0 \otimes \mathcal{F}_+, \quad \mathcal{F}_0 = \mathcal{F}(\mathrm{span}\{u_0\}), \quad \mathcal{F}_+ = \mathcal{F}(\mathcal{H}_+).$$

Consider the unitary map $U_N : \mathcal{H}_s^N \to \mathcal{F}_+^{\leq N} = \bigoplus_{n=0}^N (\mathcal{H}_+)_s^n$ defined by

$$U_N^*(\oplus_{n=0}^N \varphi_n) = \sum_{n=0}^N u_0^{\otimes(N-n)} \otimes_s \varphi_n = \sum_{n=0}^N \frac{(a^*(u_0))^{N-n}}{\sqrt{(N-n)!}} \varphi_n.$$

Let $f \perp u_0$ and let $\mathcal{N}$ is the number operator on Fock space. Prove that

$$U_N (a(u_0)^* a(f)) U_N^* = \sqrt{N - \mathcal{N}}\, a(f).$$

E12.4 Consider the torus $\Lambda = [0,1]^d$ and $0 \leq \hat{w} \in \ell^1(\Lambda^*)$. Prove that

$$E^{\mathrm{Bog}} = \sum_{0 \neq p \in \Lambda^*} \left(\sqrt{|p|^4 + 2|p|^2 \hat{w}(p)} - |p|^2 - \hat{w}(p) \right) \in (-\infty, 0].$$

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Mathematical Quantum Mechanics
Summer 2025

Homework Sheet 11

(Released 7.7.2025 – Discussed 11.7.2025)

E11.1 In this exercise we discuss Bach's proof of Lieb's variational principle. Let T be a self-adjoint non-negative operator on $L^2(\mathbb{R}^d)$ and let $W : \mathbb{R}^d \to \mathbb{R}$ be a non-negative even function. For $0 \leq \gamma \leq 1$ trace class on $L^2(\mathbb{R}^d)$, consider the Hartree–Fock functional

$$\mathcal{E}^{\mathrm{HF}}(\gamma) = \mathrm{Tr}(T\gamma) + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y)[\rho_\gamma(x)\rho_\gamma(y) - |\gamma(x,y)|^2]\mathrm{d}x\mathrm{d}y.$$

(a) Assume that $\gamma = \sum_i \lambda_i |u_i\rangle\langle u_i|$ with $\{u_i\}$ orthonormal and $0 \leq \lambda_i \leq 1$. For every given pair $i \neq j$, we consider two trial states

$$\gamma_1 = \sum_{k\neq i,j} \lambda_k |u_k\rangle\langle u_k| + (\lambda_i - \delta)|u_i\rangle\langle u_i| + (\lambda_j + \delta)|u_j\rangle\langle u_j|,$$

$$\gamma_2 = \sum_{k\neq i,j} \lambda_k |u_k\rangle\langle u_k| + (\lambda_i + \delta)|u_i\rangle\langle u_i| + (\lambda_j - \delta)|u_j\rangle\langle u_j|,$$

where $0 \leq \delta \leq \min\{\lambda_i, \lambda_j, 1-\lambda_i, 1-\lambda_j\}$. Prove that

$$\frac{\mathcal{E}^{\mathrm{HF}}(\gamma_1) + \mathcal{E}^{\mathrm{HF}}(\gamma_2)}{2} \leq \mathcal{E}^{\mathrm{HF}}(\gamma).$$

(b) Prove that if γ is a finite-rank operator and $\mathrm{Tr}\,\gamma = N$, we can find a rank-N projection $\tilde{\gamma}$ such that $\mathcal{E}^{\mathrm{HF}}(\tilde{\gamma}) \leq \mathcal{E}^{\mathrm{HF}}(\gamma)$.

(c) Deduce that $E^{\mathrm{HF}}(N) = \inf\{\mathcal{E}^{\mathrm{HF}}(\gamma) : 0 \leq \gamma \leq 1, \mathrm{Tr}\,\gamma = N\}$ for all $N \in \mathbb{N}$. Note that originally, $E^{\mathrm{HF}}(N)$ is defined by taking the infimum over all rank-N projections.

E11.2 (a) Prove that for every trace class operator $0 \leq \gamma \leq 1$ on $\mathcal{H} = L^2(\mathbb{R}^d)$, we have

$$\mathrm{Tr}(-\Delta\gamma) \geq C\int_{\mathbb{R}^d} \rho_\gamma(x)^{1+\frac{2}{d}}\mathrm{d}x$$

with a constant $C_d > 0$ independent of γ.

Hint: You may use E4.5 if γ is a projection. How about the general case?

(b) Let $V_- \in L^{1+d/2}(\mathbb{R}^d)$ and consider the non-interacting Hamiltonian

$$H_N = \sum_{i=1}^N (h^2(-\Delta_i) + V(x_i)), \quad h = N^{-1/d},$$

on the fermionic space $\mathcal{H}_a^N$. Prove that $H_N \geq -CN$ for all $N \in \mathbb{N}$. Here the constant C may depend on V, but it is independent of N.

(c) Prove that if a normalized state $\Psi_N \in \mathcal{H}_a^N$ satisfies that $\langle\Psi, H_N\Psi\rangle \leq O(N)$, then

$$\int_{\mathbb{R}^d} \rho_{\Psi_N}(x)^{1+2/d}\mathrm{d}x \leq O(N^{1+2/d}).$$

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Homework Sheet 10

(Released 2.7.2025 – Discussed 4.7.2025)

In this homework we consider the Hartree–Fock theory for a trapped Fermi gas. Let $V : \mathbb{R}^d \to \mathbb{R}$ with $V(x) \to \infty$ as $|x| \to \infty$. Let $w : \mathbb{R}^d \to [0, \infty)$ be an even function. Assume that

$$w(x) + |V_-(x)| \leq \varepsilon(-\Delta) + C_\varepsilon \quad \text{on } L^2(\mathbb{R}^d), \quad \forall \varepsilon > 0.$$

Consider the Hartree–Fock functional

$$\mathcal{E}^{\mathrm{HF}}(\gamma) = \mathrm{Tr}((-\Delta + V)\gamma) + \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\rho_\gamma(x)\rho_\gamma(y) - |\gamma(x,y)|^2) w(x-y) \mathrm{d}x \mathrm{d}y$$

where $0 \leq \gamma \leq 1$, $\rho_\gamma(x) = \gamma(x,x)$.

E9.1 Prove that for all $0 \leq \gamma \leq 1$ and a.e. $x, y \in \mathbb{R}^d$ we have

$$|\gamma(x,y)|^2 \leq \rho_\gamma(x)\rho_\gamma(y).$$

Deduce that $\mathcal{E}^{\mathrm{HF}}(\gamma) \geq \frac{1}{2} \mathrm{Tr}((-\Delta + V_+)\gamma) - C \,\mathrm{Tr}\, \gamma$.

E9.2 Let $\{\gamma_n\}_{n \geq 1}$ be a sequence of operators on $L^2(\mathbb{R}^d)$ such that $\mathrm{Tr}((-\Delta + V_+)\gamma_n) \leq C$. Prove that up to a subsequence, we have $\gamma_N \to \gamma$ in trace class. Moreover, if γ_n is a rank-N projection for all n, then γ is also a rank-N projection.

E9.3 Prove that for every $N \in \mathbb{N}$, the Hartree–Fock variational problem

$$E^{\mathrm{HF}}(N) = \inf\{\mathcal{E}^{\mathrm{HF}}(\gamma) : 0 \leq \gamma = \gamma^2 \leq 1, \mathrm{Tr}\, \gamma = N\}$$

has a minimizer.

E9.4 Prove that the Hartree–Fock minimizer in the previous problem can be written as $\gamma = \sum_{i=1}^N |u_i\rangle\langle u_i|$ where the orthonormal functions u_i are the eigenfunctions of the N-lowest eigenvalues of the mean-field operator

$$H_\gamma^{\mathrm{HF}} = -\Delta + V + \rho_\gamma * w - X_\gamma$$

where X_γ is the operator with kernel $X_\gamma(x,y) = \gamma(x,y)w(x-y)$.

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Homework Sheet 9

(Released 20.6.2025 – Discussed 27.6.2025)

In this homework we consider the Hartree theory for a trapped Bose gas. Let $h = -\Delta + V$ on $L^2(\mathbb{R}^d)$ with $V \geq 0$ and $V(x) \to \infty$ as $|x| \to \infty$. Let $w : \mathbb{R}^d \to \mathbb{R}$ be an even function such that

$$|w(x)| \leq \varepsilon(-\Delta) + C_\varepsilon \quad \text{on } L^2(\mathbb{R}^d), \quad \forall \varepsilon > 0.$$

Consider the Hartree functional

$$\mathcal{E}^{\mathrm{H}}(u) = \langle u, hu\rangle_{L^2(\mathbb{R}^d)} + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(x)|^2|u(x)|^2 w(x-y)\mathrm{d}x\mathrm{d}y.$$

E9.1 Prove that for all $u \in H^1(\mathbb{R}^d)$ and $\|u\|_{L^2} = 1$ we have

$$\mathcal{E}^{\mathrm{H}}(u) \geq \frac{1}{2}\langle u, hu\rangle_{L^2(\mathbb{R}^d)} - C.$$

E9.2 Prove that the following variational problem has (at least) a minimizer u_0:

$$e^{\mathrm{H}} = \inf\{\mathcal{E}^{\mathrm{H}}(u) : u \in H^1(\mathbb{R}^d), \|u\|_{L^2} = 1\}.$$

Hint: You can show that for every minimizing sequence $\{u_n\}$, up to a subsequence as $n \to \infty$, then $u_n \to u_0$ weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^2(\mathbb{R}^d)$.

E9.3 Assume that e^{H} has a unique minimizer u_0 (up to a phase). For every $\varepsilon > 0$, consider the perturbed variational problem

$$e^{\mathrm{H}}_\varepsilon = \inf\{\mathcal{E}^{\mathrm{H}}(u) + \varepsilon|\langle u_0, u\rangle|^2 : u \in H^1(\mathbb{R}^d), \|u\|_{L^2} = 1\}.$$

Prove that

$$\lim_{\varepsilon\to 0^+} \frac{e^{\mathrm{H}}_\varepsilon - e^{\mathrm{H}}}{\varepsilon} = 1.$$

Hint: You can show that if u_ε is a minimizer for $e^{\mathrm{H}}_\varepsilon$, then $u_\varepsilon \to u_0$ in $L^2(\mathbb{R}^d)$ when $\varepsilon \to 0$.

E9.4 Assume that w is sufficiently regular and the Fourier transform $\hat{w}(k) > 0$ for almost every k (for example, w is a Gaussian function). Prove that e^{H} has a unique minimizer.
Hint: You may use the diamagnetic inequality $|\nabla u(x)| \geq |\nabla|u|(x)|$.

Note: If $\hat{w} \geq 0$, then we can still show that e^{H} has a unique minimizer. But it requires a more complicated argument, using the fact that any minimizer u_0 of e^{H} can be chosen up a phase such that $u_0 > 0$. The latter can be deduced from the Hartree equation

$$(-\Delta + V(x) + w * |u_0|^2(x) - \mu)u_0(x) = 0, \quad \mu \in \mathbb{R}.$$

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Homework Sheet 8

(Released 13.6.2025 – Discussed 20.6.2025)

E8.1 Let Γ be a state on the fermionic space $\mathcal{H}_a^N$. Prove that the following statements are equivalent:

(a) $\gamma_\Gamma^{(1)}$ is a projection, namely $(\gamma_\Gamma^{(1)})^2 = \gamma_\Gamma^{(1)}$.

(b) Γ is a Slater determinant, namely $\Gamma = |u_1 \wedge u_2 \wedge ... \wedge u_N\rangle\langle u_1 \wedge u_2 \wedge ... \wedge u_N|$, with $\{u_j\}_{j=1}^N$ an orthonormal family in $\mathcal{H}$.

Hint: We have proved that (b) implies (a) in E6.2.

E8.2 Let Γ be a state on the bosonic space $\mathcal{H}_s^N$. Prove that the following statements are equivalent:

(a) $\gamma_\Gamma^{(1)} = N|u\rangle\langle u|$ for a normalized vector $u \in \mathcal{H}$.

(b) $\gamma_\Gamma^{(2)} = N(N-1)|u^{\otimes 2}\rangle\langle u^{\otimes 2}|$ for a normalized vector $u \in \mathcal{H}$.

(c) $\Gamma = |u^{\otimes N}\rangle\langle u^{\otimes N}|$ for a normalized vector $u \in \mathcal{H}$.

Hint: We have proved that (c) implies (a) and (b) in E6.2 and E7.2.

E8.3 Consider the bosonic Fock space $\mathcal{F}(\mathcal{H})$ with $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Define the operator $U : \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2) \to \mathcal{F}(\mathcal{H})$ by

$$U(\Phi_N \otimes \Psi_M) = \sqrt{\frac{(N+M)!}{N!M!}} P_{N+M}(\Phi_N \otimes \Psi_M)$$

for all $\Phi_N \in (\mathcal{H}_1)_s^N$, $\Psi_M \in (\mathcal{H}_2)_s^M$, and all $M, N \geq 0$. Here P_{N+M} is the bosonic symmetrization operator with $N+M$ variables, which is defined in E1.2.

(a) Prove that U is a unitary transformation.

(b) Prove that $a^*(f)U(\Phi_N \otimes \Psi_M) = U((a^*(f)\Phi_N) \otimes \Psi_M)$ for all $f \in \mathcal{H}_1$.

E8.4 Let A be a trace class operator on $X = X_1 \otimes X_2$. Define the partial trace $A_1 = \mathrm{Tr}_{X_2}(A)$ as an operator on X_1 satisfying that for all bounded operators B on X_1:

$$\mathrm{Tr}_{X_1}[A_1 B] = \mathrm{Tr}_X[A(B \otimes \mathbb{1}_{X_2})].$$

(a) Prove that if $A \geq 0$, then $A_1 \geq 0$.

(b) Consider the case $X_1 = X_2 = L^2(\mathbb{R}^d)$ and $X = L^2(\mathbb{R}^{2d})$. Prove that if A has the integral kernel $A(x_1, x_2; y_1, y_2)$ with $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$, $y = (y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d$, then A_1 has the integral kernel

$$A_1(x_1; y_1) = \int_{\mathbb{R}^d} A(x_1, z; y_1, z)\mathrm{d}z.$$

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Homework Sheet 7

(Released 7.6.2025 – Discussed 13.6.2025)

E7.1 Let Γ be a state on Fock space $\mathcal{F}(\mathcal{H})$ such that $\mathrm{Tr}[\mathcal{N}^2\Gamma] < \infty$.

(a) Prove that the two-body density matrix $\gamma_\Gamma^{(2)}$ is a non-negative trace class operator on $\mathcal{H}^2 = \mathcal{H} \otimes \mathcal{H}$ and

$$\mathrm{Tr}_{\mathcal{H}^2}[\gamma_\Gamma^{(2)}] = \mathrm{Tr}_{\mathcal{F}(\mathcal{H})}[\mathcal{N}(\mathcal{N}-1)\Gamma].$$

(b) Prove that $\mathrm{Ex}\gamma_\Gamma^{(2)} = \pm\gamma_\Gamma^{(2)}$, where $+$ is for bosons and $-$ is for fermions. Recall that Ex is the exchange operator on $\mathcal{H}^2$, which satisfies $\mathrm{Ex}(f \otimes g) = g \otimes f$.

E7.2 Consider the pure state $\Gamma = |\psi\rangle\langle\psi|$ with $\psi \in \mathcal{H}_{a/s}^N$.

(a) Prove that if $\psi = u^{\otimes N} \in \mathcal{H}_a^N$ with a normalized function $u \in \mathcal{H}$, then

$$\gamma_\Gamma^{(2)} = N(N-1)|u^{\otimes 2}\rangle\langle u^{\otimes 2}|.$$

(b) Prove that if $\psi = u_1 \wedge u_2 \wedge ... \wedge u_N \in \mathcal{H}_a^N$ with $\{u_j\}_{j=1}^N \subset \mathcal{H}$ orthonormal, then

$$\gamma_\Gamma^{(2)} = \gamma_\Gamma^{(1)} \otimes \gamma_\Gamma^{(1)} - \mathrm{Ex}\gamma_\Gamma^{(1)} \otimes \gamma_\Gamma^{(1)}.$$

E7.3 Let A be a self-adjoint bounded operator on $\mathcal{H}_s^N$ (in particular $\mathrm{Ex}_{ij}A = A$ for all $i \neq j$). Prove that if $\langle u^{\otimes N}, Au^{\otimes N}\rangle = 0$ for all $u \in \mathcal{H}$, then $A = 0$. Hint: $u \mapsto u + tv$.

E7.4 Prove that on the bosonic Fock space $\mathcal{F}(\mathcal{H})$, we have

$$a(v)^2 a^*(v)^2 = a^*(v)^2 a(v)^2 + 4\|v\|^2 a^*(v)a(v) + 2\|v\|^4, \quad \forall v \in \mathcal{H}.$$

We used this to compare the Wick and anti-Wick quantizations in the proof of quantum de Finetti theorem.

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Homework Sheet 6

(Released 1.6.2025 – Discussed 6.6.2025)

E6.1 Let $\mathcal{H} = L^2(\mathbb{R}^d)$ and let Ψ be a normalized function in $\mathcal{H}^N_{a/s}$. Prove that the one-body density matrix $\gamma^{(1)}$ of $\Gamma = |\Psi\rangle\langle\Psi|$ is a trace class operator on $L^2(\mathbb{R}^d)$ with kernel

$$\gamma^{(1)}(x;y) = N \int_{\mathbb{R}^{d(N-1)}} \overline{\Psi(x, x_2, ..., x_N)}\Psi(y, x_2, ..., x_N)\mathrm{d}x_2...\mathrm{d}x_N.$$

Hint: You need to show that for all $f, g \in L^2(\mathbb{R}^d)$,

$$\langle f, \gamma^{(1)} g\rangle = N \int \overline{f(x)\Psi(x, x_2, ..., x_N)}g(y)\Psi(y, x_2, ..., x_N)\mathrm{d}x\mathrm{d}y\mathrm{d}x_2...\mathrm{d}x_N.$$

E6.2 (a) Compute the one-body density matrix of the Hartree state $u^{\otimes N} \in \mathcal{H}^N_s$, with u a normalized vector in the Hilbert space $\mathcal{H}$.

(b) Compute the one-body density matrix of the Slater determinant $u_1 \wedge u_2 \wedge ... \wedge u_N \in \mathcal{H}^N_a$, with $\{u_j\}_{j=1}^N$ an orthonormal family in $\mathcal{H}$.

E6.3 Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with the min-max values $-\infty < \mu_1 \leq \mu_2 \leq ...$ Prove that

$$\begin{aligned}\sum_{i=1}^N \mu_i &= \inf\left\{\sum_{i=1}^N \langle u_i, Au_i\rangle \,|\, \{u_i\}_{i=1}^N \text{ orthonormal in } \mathcal{H}\right\} \\ &= \inf\left\{\mathrm{Tr}(A\gamma) \,|\, 0 \leq \gamma \leq 1 \text{ on } \mathcal{H}, \mathrm{Tr}\,\gamma = N\right\}.\end{aligned}$$

Hint: You may use induction similarly to the proof of the min-max principle. We have used this result to deduce the ground state energy of non-interacting fermionic systems.

E6.4 Let A be a non-negative trace class operator on a Hilbert space $\mathcal{H}$.

(a) Prove that on the fermionic Fock space $\mathcal{F}(\mathcal{H})$ we have

$$\mathrm{d}\Gamma(A) \leq \mathrm{Tr}\, A.$$

(b) Prove that $\mathrm{d}\Gamma(A)$ is an unbounded operator on the bosonic Fock space if $A \neq 0$.

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Homework Sheet 5

(Released 23.5.2025 – Discussed 30.5.2025)

In this homework, we will give two different descriptions of the Fock space $\mathcal{F}_+(\mathbb{C})$.

E5.1 Consider the bosonic Fock space $\mathcal{F}_+(\mathcal{H})$ with a one-dimensional space $\mathcal{H} = \mathrm{Span}(f) \simeq \mathbb{C}$, where $\|f\| = 1$.

(a) Prove that the vectors $|n\rangle = (0, ..., f^{\otimes n}, 0, ...)$, with $n = 0, 1, 2, ...$, form an orthonormal basis for $\mathcal{F}_+(\mathcal{H})$. Here $|0\rangle = (1, 0, 0, ...)$ is the vacuum. Deduce that $\mathcal{F}_+(\mathcal{H})$ can be naturally identified with $\ell^2(\mathbb{N})$.

(b) Define the creation and annihilation operators $a^* = a^*(f)$, $a = a(f)$ by

$$a^*|n\rangle = \sqrt{n+1}|n+1\rangle \quad \text{for all } n \geq 0, \qquad a|n\rangle = \sqrt{n}|n-1\rangle \text{ for all } n \geq 1.$$

Prove that they satisfy the canonical commutation relation $[a(f), a^*(f)] = 1$.

E5.2 Define the operators a and a^* on $L^2(\mathbb{R})$ by

$$a = \frac{1}{\sqrt{2}}\Big(x + \frac{d}{dx}\Big), \quad a^* = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big).$$

Define the functions $\{f_n\}_{n\geq 0} \subset L^2(\mathbb{R})$ by

$$f_0(x) = \pi^{-1/4} e^{-|x|^2/2}, \quad f_{n+1} = \frac{a^* f_n}{\sqrt{n+1}}, \quad \forall n \geq 0.$$

(a) Prove the canonical commutation relation $[a, a^*] = 1$.

(b) Prove that $af_0 = 0$ and $af_{n+1} = \sqrt{n+1} f_n$ for all $n \geq 0$.

(c) Prove that $\{f_n\}_{n\geq 0}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Note: Problem E5.2 allows us to think of $L^2(\mathbb{R})$ as the Fock space $\mathcal{F}_+(\mathbb{C})$. In the proof you may use the fact that $\mathrm{Span}\{p(x)e^{-x^2/2} \,|\, p(x) \text{ a polynomial}\}$ is dense in $L^2(\mathbb{R})$.

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Homework Sheet 4

(Released 16.5.2025 – Discussed 23.5.2025)

E4.1 Use Chemin–Xu method to prove the Sobolev inequality

$$\|u\|_{L^p(\mathbb{R}^2)} \leq C\|u\|_{H^1(\mathbb{R}^2)}$$

for all $2 \leq p < \infty$. Here $C = C_p$ is independent of u.

E4.2 Let $d \geq 1$ and $2 < p < 2^*$ (with $2^* = 2d/(d-2)$ if $d \geq 3$ and $2^* = \infty$ if $d = 1, 2$). Use the sub-critical Sobolev inequality

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C\|u\|_{H^1(\mathbb{R}^d)}$$

and a scaling argument to prove the interpolation inequality

$$\|u\|_{L^p(\mathbb{R}^d)} \leq \widetilde{C}\|\nabla u\|_{L^2}^{\theta}\|u\|_{L^2}^{1-\theta}$$

where $\theta \in (0, 1)$ determined by $2d/p = \theta(d-2) + (1-\theta)d$.

E4.3 Let X be a Hilbert space and let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in X.

(a) Prove that if $\sum_{n\geq 1} |u_n\rangle\langle u_n| \leq C$, then $u_n \rightharpoonup 0$ weakly in X.

(b) Prove that if $u_n \rightharpoonup 0$ weakly in X, then up to a subsequence, $\sum_n |u_n\rangle\langle u_n| \leq C$.

We used this property to prove Sobolev's compact embedding theorem.

E4.4 Let $U \subset \mathbb{R}^d$ be a bounded set. Let A be a self-adjoint operator on $L^2(U)$ with quadratic form

$$\langle f, Af\rangle = \int_U |\nabla f(x)|^2 dx.$$

Prove that A has compact resolvent.

Hint: You may use the min-max principle and Sobolev's compact embedding.

E4.5 Use Chemin–Xu–Rumin method to prove the following extension of the Lieb–Thirring inequality: For all $d \geq 1$ and $d/2 > \alpha \geq 0$, if $\{|2\pi k|^{\alpha}\hat{u}_n(k)\}_{n=1}^N$ is an orthonormal family in $L^2(\mathbb{R}^d)$, then

$$\sum_{n=1}^N \int_{\mathbb{R}^d} |\nabla u_n|^2 \geq C \int_{\mathbb{R}^d} \rho_N^{1+\frac{2}{d-2\alpha}}$$

where $\rho_N(x) = \sum_{n=1}^N |u_n(x)|^2$ and $C = C_{d,\alpha} > 0$ is independent of N.

The case $\alpha = 0$ corresponds to the standard Lieb–Thirring inequality, while the case $\alpha = 1$ is called Lieb's inequality (which is equivalent to the Cwikel–Lieb–Rozenblum (CLR) inequality). In all cases, this is an extension of the bound in E4.2 for $p = 2(1 + \frac{2}{d-2\alpha})$.

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Homework Sheet 3

(Released 11.5.2025 – Discussed 16.5.2025)

E3.1 Let X and Y be two Hilbert spaces with orthonormal bases $\{x_i\}_{i\geq 1}$ and $\{y_j\}_{j\geq 1}$. Prove that $\{x_i \otimes y_j\}_{i,j\geq 1}$ is an orthonormal basis for the tensor product space $X \otimes Y$.

E3.2 Let (X,μ) and (Y,ν) be two measure spaces with σ-finite measures. Prove that the tensor product space $L^2(X,\mu)\otimes L^2(Y,\nu)$ can be identified with $L^2(X\times Y,\mu\times\nu)$.

E3.3 Let (X,μ) be a measure space and let $\mathcal{H}$ be a Hilbert space. Show that the tensor product space $L^2(X,\mu)\otimes\mathcal{H}$ can be identified with the Hilbert space $L^2(X,\mu;\mathcal{H})$ – the space of $\mathcal{H}$-valued L^2-functions on X – via the map

$$(f\otimes u)(x) = f(x)u.$$

E3.4 Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{u_n\}_{n\geq 1}$.

(a) Prove that $\{P_+(u_{i_1}\otimes ...\otimes u_{i_N})\}_{i_1,...,i_N\geq 1}$ is an orthogonal basis for the bosonic space $\mathcal{H}_s^N$.

(b) Prove that $\{P_-(u_{i_1}\otimes ...\otimes u_{i_N})\}_{1\leq i_1<\cdots<i_N}$ is an orthogonal basis for the fermionic space $\mathcal{H}_a^N$.

E3.5 Let h be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with the first two min-max values $\mu_1<\mu_2$. Let $N\geq 2$. Let $H_N = h_1+...+h_N$ be the non-interacting Hamiltonian on the tensor product space $\mathcal{H}^N$, and let H_N^b be its restriction on the bosonic space $\mathcal{H}_s^N$.

(a) Prove that the first and the second min-max values of H_N are the same as those of H_N^b.

(b) What about the third min-max values of H_N and H_N^b?

Homework Sheet 2

(Released 2.5.2025 – Discussed 9.5.2025)

E2.1 Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that H is bounded from below and $\mu_\infty = \inf \sigma_{\rm ess}(H)$ is finite.

(a) Prove that for every $N \in \mathbb{N}$, there exists an orthonormal family $\{u_n\}_{n=1}^N$ such that

$$\max_{1\leq n\leq N} \langle u_n, Hu_n\rangle \leq \mu_\infty.$$

(b) Prove that there exists an orthonormal family $\{u_n\}_{n=1}^\infty$ such that $\langle u_n, Hu_n\rangle \to \mu_\infty$.

E2.2 Let $A \geq 0$ be a self-adjoint compact operator on a Hilbert space $\mathcal{H}$. Prove that the following statements are equivalent:

(a) A is trace class;

(b) $\sum_{n\geq 1}\langle u_n, Au_n\rangle$ is finite for some orthonormal basis $\{u_n\}_{n\geq 1}$ for $\mathcal{H}$.

Prove also that in this case, $\operatorname{Tr} A = \sum_{n\geq 1}\langle u_n, Au_n\rangle$ for every orthonormal basis $\{u_n\}_{n\geq 1}$.

Note: If $A \geq 0$ is self-adjoint and $\sum_{n\geq 1}\langle u_n, Au_n\rangle < \infty$ for some orthonormal basis $\{u_n\}_{n\geq 1}$, then we can still deduce that A is trace class. But the proof is more difficult.

E2.3 Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ which is bounded from below, namely $\mathcal{H} \geq -C$. Let Γ be a (mixed) state, namely $\Gamma \geq 0$ and $\operatorname{Tr}\Gamma = 1$. Prove that

$$\operatorname{Tr}[\Gamma^{1/2} H \Gamma^{1/2}] = \operatorname{Tr}[(H+C)^{1/2}\Gamma(H+C)^{1/2}] - C$$

where the both sides make sense as a number in $[-C, \infty]$. We denote this by $\operatorname{Tr}[H\Gamma]$.

E2.4 Let Γ be a (mixed) state on a Hilbert space $\mathcal{H}$. Let $S(\Gamma) = -\operatorname{Tr}[\Gamma \log \Gamma]$.

(a) Prove that $S(\Gamma) = 0$ if and only if Γ is a pure state (i.e. a rank-1 operator).

(b) Prove that if Γ is a rank-N operator (namely $\dim \Gamma\mathcal{H} = N$), then

$$0 \leq \Gamma(\Gamma) \leq N \log N.$$

Deduce that if $S(\Gamma) = \infty$, then Γ is an infinite-rank operator.

(c) Give an example where Γ is infinite-rank but $S(\Gamma) < \infty$.

E2.5 Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ such that $\operatorname{Tr}(e^{-H/T}) = \infty$ for some temperature $T > 0$. Prove that

$$F_T := \inf\{\operatorname{Tr}[H\Gamma] + T\operatorname{Tr}[\Gamma\log\Gamma] : \Gamma \geq 0, \operatorname{Tr}\Gamma = 1\} = -\infty.$$

Hint: If $\mathcal{H}$ has compact resolvent, you may construct a sequence of trial states Γ using eigenfunctions of $\mathcal{H}$. If H has essential spectrum, then E2.1 is helpful.

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Mathematical Quantum Mechanics
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Homework Sheet 1

(Released 25.4.2025 – Discussed 2.5.2025)

E1.1 Let $f \in L^2(\mathbb{R})$. Prove that if both f and $\hat{f}$ are compactly supported, then $f = 0$. This is an illustration for the uncertainty principle.

E1.2 Let Ω be a measurable subset of $\mathbb{R}^d$. Let $N \geq 2$. Consider the two operators $P_\pm$ on $\mathcal{H}^N = L^2(\Omega^N)$ defined by

$$(P_+\Psi)(x_1,...,x_N) = (N!)^{-1} \sum_{\sigma\in S_N} \Psi(x_{\sigma(1)},...,x_{\sigma(N)}), \quad x_j \in \Omega$$
$$(P_-\Psi)(x_1,...,x_N) = (N!)^{-1} \sum_{\sigma\in S_N} (-1)^\sigma \Psi(x_{\sigma(1)},...,x_{\sigma(N)}).$$

Here S_N is the permutation group of $\{1,2,...,N\}$ and $(-1)^\sigma$ is the sign of $\sigma \in S_N$.

(a) Prove that P_+ and P_- are two orthogonal projections on $\mathcal{H}^N$ (i.e. $P = P^*$, $P^2 = P$).

(b) Prove that the subspaces $\mathcal{H}^N_{\rm s} = P_+\mathcal{H}^N$ and $\mathcal{H}^N_{\rm a} = P_-\mathcal{H}^N$ satisfy $\mathcal{H}^N_{\rm s} \cap \mathcal{H}^N_{\rm a} = \{0\}$.

The symmetric space $\mathcal{H}^N_{\rm s}$ corresponds to bosons, while the antisymmetric space $\mathcal{H}^N_{\rm a}$ corresponds to fermions.

E1.3 Let Ω be a measurable subset of $\mathbb{R}^d$. Let $u_1, u_2, ..., u_N$ be orthonormal functions in $L^2(\Omega)$. Consider the *Slater determinant*

$$(u_1 \wedge \cdots \wedge u_N)(x_1,\dots,x_N) = (N!)^{-1/2} \det \begin{pmatrix} u_1(x_1) & \cdots & u_N(x_1) \\ u_1(x_2) & \cdots & u_N(x_2) \\ & \vdots & \\ u_1(x_N) & \cdots & u_N(x_N) \end{pmatrix}.$$

Prove that $u_1 \wedge \cdots \wedge u_N$ is a normalized vector in the antisymmetric space $\mathcal{H}^N_{\rm a}$.

E1.4 Let Ω be a measurable subset of $\mathbb{R}^d$. Let $w : \mathbb{R}^d \to \mathbb{R}$ be a regular even function.

(a) Consider the Hamiltonian $H_N = \sum_{j=1}(-\Delta_{x_j}) + \sum_{j<k} w(x_j - x_k)$ and a function $u \in H^1(\Omega)$, $\|u\|_{L^2(\Omega)} = 1$. Compute the expectation

$$\mathcal{E}(u) = N^{-1}\langle u^{\otimes N}, H_N u^{\otimes N}\rangle, \quad u^{\otimes N}(x_1,...,x_N) = u(x_1)...u(x_N).$$

(b) Prove that if u_0 is a minimizer for $\mathcal{E}(u)$ under the constraint $\|u\|_{L^2(\Omega)} = 1$, then

$$-\Delta u_0 + (N-1)(w * |u_0|)u_0 = \mu u_0$$

in the distributional sense $\mathcal{D}'(\Omega)$, with a parameter $\mu \in \mathbb{R}$.