# Chapter 5: $L^p$ spaces

Let $(\Omega, \Sigma, \mu)$ be a measure space, i.e.

(i) $\Omega$ is a set

(ii) $\Sigma$ is a collection of subsets of $\Omega$ s.t.

- $\phi, \Omega \in \Sigma$
- if $A \in \Sigma$, then $A^c = \Omega \setminus A \in \Sigma$
- If $\{A_n\}_{n=1}^{\infty}$ s.t $A_n \in \Sigma$, $\forall n \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \Sigma$.

Then $\Sigma$ is called a sigma-algebra and the sets in $\Sigma$ are measurable sets.

(iii) $\mu : \Sigma \longrightarrow [0, \infty]$ is a positive measure

- $\mu(\phi) = 0$
- If $\{A_n\}_{n=1}^{\infty}$, $A_n \in \Sigma$ are disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

In practice, we will often write $(\Omega, \mu)$ and the information of $\Sigma$ is implicitly contained in $\mu$.

Examples:

① $\Omega = \mathbb{R}^d$, $\Sigma$ = Borel sets

= smallest sigma-algebra containing all open / closed sets.

$\exists!$ measure $\mu : \Sigma \to [0, \infty]$ s.t.

$$\begin{cases} \mu([0,1]^d) = 1 \\ \mu(x + A) = \mu(A), \quad \forall A \in \Sigma, \forall x \in \mathbb{R}^d \end{cases}$$

② $\Omega = \mathbb{R}^d$, $\Sigma$ = Lebesgue sets

= smallest sigma-algebra containing all open / closed sets and all negligible sets (sets of zero measure).

$\exists!$ measure $\mu : \Sigma \to [0, \infty]$ s.t.

$$\begin{cases} \mu([0,1]^d) = 1 \\ \mu(x + A) = \mu(A), \quad \forall A \in \Sigma, \forall x \in \mathbb{R}^d \end{cases}$$

The measure $\mu$ is called the Lebesgue measure.

Remarks:

- The Borel measurable sets are not <u>complete</u> in the sense that $\exists A, B \subset \mathbb{R}^d$ s.t.

$$A \in \Sigma, \ \mu(A) = 0$$

but $B \notin \Sigma$, $B \subset A$.

- The Lebesgue measurable sets are complete; if $A \in \Sigma$, $\mu(A) = 0$, then

$\forall B \subset A$, we have $B \in \Sigma$, $\mu(B) = 0$.

(By definition, a set $A$ is negligible if $A \subset \tilde{A}$, $\tilde{A}$ is Borel measurable and $\mu(\tilde{A}) = 0 \iff \forall \varepsilon > 0, \exists A_\varepsilon$ Borel measurable s.t. $A \subset A_\varepsilon$ & $\mu(A_\varepsilon) < \varepsilon$.)

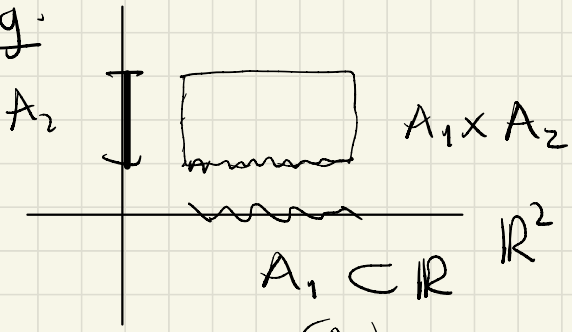
- The Borel measurable sets have the projection property, i.e. if $A \subset \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ is Borel measurable, then:

$$B = \{ x \in \mathbb{R}^m : \exists y \in \mathbb{R}^n : (x, y) \in A \}$$

is Borel measurable in $\mathbb{R}^m$.

However, the Lebesgue measurable sets don't have the projection property.

Eg:




$\mu(A_1) = 0 \rightsquigarrow A_1$ is Lebesgue measurable

Then $\forall A_2 \subset \mathbb{R}$, then $A_1 \times A_2$ is negligible in $\mathbb{R}^2 \rightsquigarrow$ Lebesgue measurable.

But $A_2$ is <u>not</u> necessarily Lebesgue measurable in $\mathbb{R}$.

This will be a problem if we want to apply the product measure & Fubini theorem.

<u>Example:</u> $\Omega = \mathbb{N}$, $\Sigma =$ all subsets,

$\mu =$ counting measure

$\mu(A) = |A| =$ # elements of $A$.

Let $(\Omega, \mu)$ be a measure space.

Def: A function $f: \Omega \to \mathbb{C}$ is measurable if $f^{-1}$(measurable sets) are measurable.

Here $\mathbb{C} \simeq \mathbb{R}^2$ with Lebesgue measurable sets.

Def: Let $f: \Omega \to [0, \infty]$ be a measurable function. Then we can define

$$\int_\Omega f d\mu = \int_\Omega f(x)\, d\mu(x) \in [0, \infty]$$

If $\int_\Omega f d\mu < \infty$, then $f$ is called integrable

In general, if $f: \Omega \to \mathbb{C}$ is measurable and $|f|$ is integrable, then $\int_\Omega f d\mu \in \mathbb{C}$ is well-defined. Actually,

$$f = f_1 - f_2 + i f_3 - i f_4 \text{ where}$$

$f_1, f_2, f_3, f_4 \geq 0$, measurable functions

$$\int f d\mu = \int f_1 d\mu - \int f_2 d\mu + i \int f_3 d\mu - i \int f_4 d\mu$$

Basic properties Let $(\Omega, \mathcal{A})$ be a measure space.

Thm (Monotone convergence). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions s.t.

$$0 \le f_1(x) \le f_2(x) \le \dots \quad \text{and} \quad \limsup_{n\to\infty} \int_\Omega f_n \, d\mu < \infty.$$

Then $\exists$ an integrable function $f$ s.t.

$$\begin{cases} \lim_{n\to\infty} f_n(x) = f(x) < \infty & \text{a.e. } x \in \Omega \\ \int f_n \, d\mu \to \int f \, d\mu & \text{as } n \to \infty \end{cases}$$

(Here a property holds for a.e. $x \in \Omega$ $\Leftrightarrow$ this holds except a set of 0 measure.)

Thm: (Dominated convergence) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions s.t.

$$\begin{cases} f_n(x) \to f(x) & \text{a.e. } x \in \Omega \\ |f_n(x)| \le F(x) & \text{a.e. } x \in \Omega \end{cases}$$

and $F$ is integrable. Then:

$$\int f_n \, d\mu \to \int f \, d\mu.$$

Actually: $\int |f_n - f| \, d\mu \to 0$ as $n \to \infty$.

**Remark:** $\forall\, 0<p<\infty$, if

$$\begin{cases} f_n(x) \to f(x) & \text{a.e. } x\in\Omega \\ |f_n(x)| \le F(x) & \text{a.e. } x\in\Omega \end{cases}$$

and $\int F(x)^p \, d\mu(x) < \infty$. Then:

$$\int |f_n - f|^p \, d\mu \to 0 \quad \text{as } n\to\infty.$$

In particular:

$$\int |f_n|^p \, d\mu \to \int |f|^p \, d\mu.$$

**Proof:** Consider $|f_n - f|^p(x) \to 0$ a.e. $x\in\Omega$ and

$$|f_n - f|^p \le C_p \left(|f_n|^p + |f|^p\right) \le 2C_p F^p, \quad \int F^p < \infty$$

$$\rightsquigarrow \quad \int |f_n - f|^p \to 0 \quad \text{as } n\to\infty.$$

**Thm** (Fatou lemma) Let $\{f_n\}$ be a sequence of measurable functions s.t.

$$f_n \ge 0, \qquad f_n(x) \to f(x) \quad \text{a.e. } x\in\Omega$$

Then:

$$\liminf_{n\to\infty} \int f_n \, d\mu \ge \int f \, d\mu.$$

**Remark:** If $f_n \geq 0$, $f_n(x) \to f(x)$ a.e. $x \in \Omega$.

Then
$$\int f_n \, d\mu \to \int f \, d\mu < \infty \iff \int |f_n - f| \, d\mu \to 0.$$

**Proof:**

Consider $g_n(x) = f_n(x) - f(x) - |f_n(x) - f(x)| \to 0$ a.e.

$$|g_n(x)| \leq \big||f_n(x)| - |f_n - f(x)|\big| + |f(x)|$$
$$\leq 2|f(x)|$$

By Dominated convergence, $\int g_n \to 0$

$$\Rightarrow \underbrace{\int f_n - \int f}_{\to 0 \text{ by assumption}} - \int |f_n - f| \to 0$$

$$\Rightarrow \int |f_n - f| \to 0.$$

**Thm** ( Brezis–Lieb refinement of Fatou's Lemma)

Let $\{f_n\}_{n=1}^{\infty}$ be measurable functions s.t.

$$\begin{cases} f_n(x) \to f(x) \quad \text{a.e. } x \in \Omega \\ \limsup_{n \to \infty} \int |f_n|^p \, d\mu < \infty \end{cases}$$

Then:

$$\int_\Omega \left| |f_n(x)|^p - |f(x)|^p - |f_n(x) - f(x)|^p \right| d\mu \to 0$$

Consequently, if $\int_\Omega |f_n|^p d\mu \to \int |f|^p d\mu$

then:

$$\int_\Omega |f_n - f|^p \, d\mu \to 0.$$

This holds for all $0 < p < \infty$. (Exercise)

Def: Let $(\Omega, \mu)$ be a measure space. We say that it is sigma-finite if $\exists$ measurable sets $\{A_n\}_{n=1}^\infty$ s.t. $\mu(A_n) < \infty$ $\forall n$ and $\Omega = \bigcup_{n=1} A_n$.

Theorem (Fubini) Let $(\Omega_1, \mu_1)$, $(\Omega_2, \mu_2)$ be two measure spaces which are sigma-finite. Then we can define the product measure space $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$. Assume that $f \geq 0$ be a measurable function on $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$.

Then:

$$\int_{\Omega_1 \times \Omega_2} f(x,y)\, d(\mu_1 \times \mu_2)(x,y)$$

$$= \int_{\Omega_1} \left( \int_{\Omega_2} f(x,y)\, d\mu_2(y) \right) d\mu_1(x)$$

$$= \int_{\Omega_2} \left( \int_{\Omega_1} f(x,y)\, d\mu_1(x) \right) d\mu_2(y)$$

(with the possibility that all $= \infty$)

Moreover, if $f : \Omega_1 \times \Omega_2 \to \mathbb{C}$ is integrable, then the same equality occurs.

Remark: The reverse direction also holds by Tonelli theorem: if $f : \Omega_1 \times \Omega_2 \to \mathbb{C}$ s.t.

$$\int_{\Omega_2} |f(x,y)|\, d\mu_2(y) < \infty \quad \text{a.e. } x \in \Omega_1$$

and

$$\int_{\Omega_1} \left( \int_{\Omega_2} |f(x,y)|\, d\mu_2(y) \right) d\mu_1(x) < \infty$$

Then: $f$ is integrable in $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$.

**Def ($L^p$ spaces)** Let $(\Omega, \mu)$ be a measure space. Let $1 \le p < \infty$. Define

$$L^p(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ measurable}, \int |f|^p d\mu < \infty \}$$

and

$$\|f\|_{L^p} = \left( \int_\Omega |f|^p \, d\mu \right)^{1/p}.$$

When $p = \infty$, then

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ measurable}, \exists C : |f(x)| \le C \text{ a.e. } x \in \Omega \}$$

$$\|f\|_{L^\infty} = \inf \{ C : |f(x)| \le C \text{ a.e. } x \in \Omega \}$$

**Remark:** Here we identify $f$ and $g$ if $f = g$ a.e. Thus $L^p$ spaces are the spaces of the equivalent classes of functions.

**Theorem:** (Fischer - Riesz) For any measure space $(\Omega, \mu)$ and any $1 \le p \le \infty$, then $L^p(\Omega)$ is a Banach space.

**Thm (Hölder inequality)** If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ s.t. $p, p' \in [1, \infty]$, and $\frac{1}{p} + \frac{1}{p'} = 1$,

then $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg| \, d\mu \leq \|f\|_{L^p} \|g\|_{L^q}.$$

**Remind:** The proof is based on Young's inequality

$\forall a, b \geq 0$, $\quad ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$.

$\Leftrightarrow p, p' > 1$

**Thm** (Minskowski inequality) For all $1 \leq p \leq \infty$, $\forall f, g \in L^p(\Omega)$, then:

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

(The proof is based on Hölder's inequality)

**Proof that $L^p(\Omega)$ is complete**

- $p = \infty$, then the proof is similar to the case of $C(K)$ space.

- $1 \le p < \infty$. Take a sequence $\{f_n\}_{n=1}^{\infty} \subset L^p(\Omega)$ s.t.

$$\sum_{n=1}^{\infty} \|f_n\|_{L^p} < \infty.$$

Then we prove that the series $\sum_{n=1}^{\infty} f_n$ converges namely $g_n = \sum_{m=1}^{n} f_m$ converges when $n \to \infty$.

**p=1** Define $G_n = \sum_{m=1}^{n} |f_m|$. Then

$$0 \le G_1 \le G_2 \le \dots$$

and

$$\int G_n \, d\mu = \sum_{m=1}^{n} \int |f_m| \le \sum_{m=1}^{\infty} \int |f_m| < \infty$$

By Monotone convergence, $\exists G \in L^1(\Omega)$ s.t.

$$\begin{cases} G_n \uparrow G \text{ a.e.} \\ \int G_n \longrightarrow \int G. \end{cases}$$

Thus: $|g_n(x)| \le G(x)$ $\forall n = 1, 2, \dots$ and a.e. $x$

Moreover, for any $x \in \Omega$ s.t. $G(x) < \infty$ then:

$$\sum_{n=1}^{\infty} |f_n(x)| = G(x) < \infty$$

Using the completeness of $\mathbb{C}$ and $\{f_n(x)\}_{n=1}^{\infty} \subset \mathbb{C}$,

then $\sum_{n=1}^{\infty} f_n(x)$ converges in $\mathbb{C}$

i.e. $g_n(x) = \sum_{m=1}^{n} f_m(x) \to g(x)$ in $\mathbb{C}$.

In summary,

$$\begin{cases} g_n(x) \to g(x) & \text{a.e. } x \in \Omega \\ |g_n(x)| \le G(x), & G \in L^1(\Omega) \end{cases}$$

By Dominated convergence,

$$\int |g_n - g| \to 0 \quad \text{as } n \to \infty$$

$\hookrightarrow$ $g_n \to g$ in $L^1(\Omega)$.

$\boxed{1<p<\infty}$ Similar (exercise) □

Remark From the proof of the completeness, we can also obtain the following reverse statement of the Dominated convergence.

Dominated c.v.

$$\begin{cases} f_n(x) \to f(x) \text{ a.e. } x \in \Omega \\ |f_n| \le F \in L^p(\Omega) \end{cases}$$

$$\Rightarrow f_n \to f \text{ in } L^p(\Omega).$$

Reverse statement: If $f_n \to f$ in $L^p(\Omega)$, then $\exists$ a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ s.t.

$$\begin{cases} f_{n_k}(x) \to f(x) \quad \text{a.e. } x \in \Omega \\ |f_{n_k}| \le F \in L^p(\Omega) \end{cases}$$

Proof of the reverse statement: Since $f_n \to f$ in $L^p \Rightarrow \{f_n\}$ is a Cauchy sequence $\Rightarrow$ $\exists$ a subsequence $\{f_{n_k}\}$ s.t.

$$\|f_{n_k} - f_{n_{k+1}}\|_{L^p} \le 2^{-k}.$$

Then:

$$\sum_k \|f_{n_k} - f_{n_{k+1}}\|_{L^p} < \infty.$$

Thus

$$f_{n_k} = f_{n_1} + (f_{n_2} - f_{n_1}) + \dots + (f_{n_k} - f_{n_{k-1}}) \to f \text{ in } L^p(\Omega).$$

**Theorem:** Let $(\Omega, \mu)$ be a measure space. Let $1<p<\infty$. Then $L^p(\Omega)$ is uniformly convex. Consequently, $L^p(\Omega)$ is reflexive.

**Proof:**

**Step 1:** $2 \leq p < \infty$.

**Lemma (Clarkson's inequality)** If $2 \leq p < \infty$,

$$\left\| \frac{f+g}{2} \right\|_{L^p}^p + \left\| \frac{f-g}{2} \right\|_{L^p}^p \leq \frac{\|f\|_{L^p}^p + \|g\|_{L^p}^p}{2}$$

$$\Leftrightarrow \int \left| \frac{f+g}{2} \right|^p + \int \left| \frac{f-g}{2} \right|^p \leq \int \frac{|f|^p + |g|^p}{2}.$$

**Proof:** We claim that $\forall a, b \in \mathbb{C}$, then:

$$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \leq \frac{|a|^p + |b|^p}{2}.$$

Actually, if $p=2$, then we have the equality.

For $p \geq 2$, then we have:

$$\alpha^p + \beta^p \leq \left(\alpha^2 + \beta^2\right)^{\frac{p}{2}}, \quad \forall \alpha, \beta \geq 0.$$

This follows from:

$$\frac{\alpha^p+\beta^p}{(\alpha^2+\beta^2)^{p/2}} = \underbrace{\left(\frac{\alpha^2}{\alpha^2+\beta^2}\right)^{p/2}}_{\leq 1} + \underbrace{\left(\frac{\beta^2}{\alpha^2+\beta^2}\right)^{p/2}}_{\leq 1} \overbrace{\geq 1}^{p/2}$$

$$\leq \frac{\alpha^2}{\alpha^2+\beta^2} + \frac{\beta^2}{\alpha^2+\beta^2} = 1.$$

Using that for $\alpha = \left|\frac{a+b}{2}\right|$, $\beta = \left|\frac{a-b}{2}\right|$, we have:

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p = \alpha^p+\beta^p \leq (\alpha^2+\beta^2)^{p/2}$$

$$= \left(\frac{|a|^2+|b|^2}{2}\right)^{p/2} \leq \frac{|a|^p+|b|^p}{2}. \quad \square$$

Consequently, if $\|f\|_{L^p} \leq 1$, $\|g\|_{L^p} \leq 1$, $\|f-g\|_{L^p} \geq \varepsilon$

then by Clarkson's inequality

$$\left\|\frac{f+g}{2}\right\|_{L^p}^p + \underbrace{\left\|\frac{f-g}{2}\right\|_{L^p}^p}_{\geq \frac{\varepsilon}{2}} \leq \underbrace{\frac{\|f\|_{L^p}^p + \|g\|_{L^p}^p}{2}}_{\leq 1}$$

$$\Rightarrow \left\|\frac{f+g}{2}\right\|_{L^p}^p \leq 1 - \frac{\varepsilon}{2} \Rightarrow L^p(\Omega) \text{ uniformly convex}$$

Consequently, if $2 \le p < \infty$, then $L^p(\Omega)$ is reflexive.

**Step 2** $1 < p < 2$.

Lemma: (Clarkson's second inequality) If $1 < p < 2$

$$\left\| \frac{f+g}{2} \right\|_{L^p}^{p'} + \left\| \frac{f-g}{2} \right\|_{L^p}^{p'} \le \left( \frac{\|f\|_{L^p}^p + \|g\|_{L^p}^p}{2} \right)^{\frac{p'}{p}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The proof of Clarkson's second inequality is more complicated than the first one.

Another way to get the uniform convexity of $L^p(\Omega)$ is to use

Lemma (Hanner's inequality) If $1 < p < 2$, then

$$\left( \|f+g\|_{L^p} + \|f-g\|_{L^p} \right)^p + \left| \|f+g\|_{L^p} - \|f-g\|_{L^p} \right|^p \le 2^p \left( \|f\|_{L^p}^p + \|g\|_{L^p}^p \right).$$

## Proof of Hanner inequality

Denote $F = f+g$, $G = f-g \rightsquigarrow f = \frac{F+G}{2}$, $g = \frac{F-G}{2}$

Then we need to prove that

$$\left(\|F\|_{L^p} + \|G\|_{L^p}\right)^p + \left|\|F\|_{L^p} - \|G\|_{L^p}\right|^p \leq \|F+G\|_{L^p}^p + \|F-G\|_{L^p}^p$$

**Lemma:** For any $a, b \in \mathbb{C}$, and $1 < p < 2$, we have

$$|a+b|^p + |a-b|^p \geq \left||a|+|b|\right|^p + \left||a|-|b|\right|^p = \sup_{0<r\leq 1} \left(\alpha(r) A + \beta(r) B\right)$$

Here
$$\begin{cases} \alpha(r) = (1+r)^{p-1} + (1-r)^{p-1} \\ \beta(r) = \left((1+r)^{p-1} - (1-r)^{p-1}\right) r^{1-p} \\ A = \max\left(|a|^p, |b|^p\right), \quad B = \min\left(|a|^p, |b|^p\right). \end{cases}$$

**Proof of the lemma:** Consider the first inequality.

WDLG, we can assume $a \geq 0$, $b = b_0 e^{i\theta}$ with $b_0 \geq 0$ and $\theta \in [0, 2\pi)$. We have: ($e^{i\theta} = \cos\theta + i\sin\theta$)

$$|a+b|^p + |a-b|^p = \left(|a+b_0 e^{i\theta}|^2\right)^{p/2} + \left(|a-b_0 e^{i\theta}|^2\right)^{p/2}$$

$$= \left(a^2+b_0^2 + 2ab_0\cos\theta\right)^{p/2} + \left(a^2+b_0^2 - 2ab_0\cos\theta\right)^{p/2}$$

since

$$\begin{aligned}|a+be^{i\theta}|^2 &= |(a+b_0\cos\theta) + ib_0\sin\theta|^2 \\ &= (a+b_0\cos\theta)^2 + (b_0\sin\theta)^2 \\ &= a^2 + b_0^2\left[(\cos\theta)^2 + (\sin\theta)^2\right] + 2ab_0\cos\theta \\ &= a^2+b_0^2 + 2ab_0\cos\theta\end{aligned}$$

Consider $X \geq 0$, $Y \in \mathbb{R}$, $|Y| \leq Y_0 \leq X$, then

$$|X+Y|^{p/2} + |X-Y|^{p/2} \geq |X+Y_0|^{p/2} + |X-Y_0|^{p/2}$$

This follows from the concavity $t \mapsto t^{p/2}$ since $p/2 \in (0,1)$. Applying that to

$$X = a^2+b_0^2, \quad Y = 2ab_0\cos\theta, \quad Y_0 = 2ab_0$$

We find:

$$\begin{aligned}|a+b|^p + |a-b|^p &\geq \left|a^2+b_0^2+2ab_0\right|^{p/2} + \left|a^2+b_0^2-2ab_0\right|^{p/2} \\ &= |a+b_0|^p + |a-b_0|^p \\ &= \left||a|+|b|\right|^p + \left||a|-|b|\right|^p \text{ as desired.}\end{aligned}$$

Consider the second inequality. WLOG, assume $a \geq b \geq 0$ and $a > 0$ and we prove that

$$|a+b|^p + |a-b|^p = \sup_{0<r\leq 1} \left( \alpha(r)|a|^p + \beta(r)|b|^p \right)$$

with $\begin{cases} \alpha(r) = (1+r)^{p-1} + (1-r)^{p-1} \\ \beta(r) = \left((1+r)^{p-1} - (1-r)^{p-1}\right) r^{1-p} \end{cases}$

Define

$$H(r) = \alpha(r)|a|^p + \beta(r)|b|^p, \quad 0 < r \leq 1.$$

$$H'(r) = \alpha'(r)|a|^p + \beta'(r)|b|^p$$

where $\alpha'(r) = (p-1)\left[(1+r)^{p-2} - (1-r)^{p-2}\right]$

$$\beta'(r) = (p-1)\left[(1+r)^{p-2} + (1-r)^{p-2}\right] r^{1-p} + (1-p)\left[(1+r)^{p-1} - (1-r)^{p-1}\right] r^{-p}$$

$$= (p-1)\left[(1+r)^{p-2} r + (1-r)^{p-2} r - \underbrace{(1+r)^{p-1}}_{(1+r)^{p-2}(1+r)} + \underbrace{(1-r)^{p-1}}_{(1-r)^{p-2}(1-r)}\right] r^{-p}$$

$$= (p-1)\left[-(1+r)^{p-2} + (1-r)^{p-2}\right] r^{-p}.$$

Thus $H'(r) = \underbrace{(p-1)}_{>0}\left[\underbrace{(1+r)^{p-2} - (1-r)^{p-2}}_{<0 \text{ as } p<2}\right](a^p - r^{-p}b^p)$

$\rightsquigarrow$ sign of $H'(r)$ is the same with $r^{-p}b^p - a^p$

and $H'(r) = 0 \iff r = \frac{b}{a}$




Conclusion: $\sup_{0<r\leq 1} H(r) = H\left(\frac{b}{a}\right) = \alpha\left(\frac{b}{a}\right)a^p + \beta\left(\frac{b}{a}\right)b^p$

$$= \left[\left(1+\frac{b}{a}\right)^{p-1} + \left(1-\frac{b}{a}\right)^{p-1}\right]a^p + \left[\left(1+\frac{b}{a}\right)^{p-1} - \left(1-\frac{b}{a}\right)^{p-1}\right]\left(\frac{b}{a}\right)^{1-p}b^p$$

$$= (a+b)^{p-1}a + (a-b)^{p-1}a + (a+b)^{p-1}b - (a-b)^{p-1}b$$

$$= (a+b)^p + (a-b)^p \quad \text{as desired.} \qquad \square$$

# Conclusion of the proof of Hanner inequality

$$\left(\|F\|_{L^p}+\|G\|_{L^p}\right)^p+\left|\|F\|_{L^p}-\|G\|_{L^p}\right|^p \le \|F+G\|_{L^p}^p+\|F-G\|_{L^p}^p$$

$$RHS = \int |F+G|^p+|F-G|^p \, d\mu$$

By the Lemma, $\forall x\in\Omega$, $F(x), G(x)\in\mathbb{C}$, we have:

$$|F(x)+G(x)|^p+|F(x)-G(x)|^p \ge \sup_{r\in(0,1)} \left(\alpha(r)\max(F,G)^p+\beta(r)\min(F,G)^p\right)$$

$$\Rightarrow \int |F+G|^p+|F-G|^p \ge \int \sup_{r\in(0,1]}(\cdot) \ge \sup_{r\in(0,1]} \int \left(\alpha(r)\max(F,G)^p+\beta(r)\min(F,G)^p\right)$$

Observation: when $r\in(0,1]$ and $1<p<2$, then

$$\alpha(r)=(1+r)^{p-1}+(1-r)^{p-1} \ge \beta(r)=\left[(1+r)^{p-1}-(1-r)^{p-1}\right]r^{1-p}$$

(why?)

## Rearrangement inequality:

If $a_1 \geq a_2$ and $b_1 \geq b_2$, then

$$a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1.$$

Proof: $a_1 b_1 + a_2 b_2 - a_1 b_2 - a_2 b_1 = (a_1 - a_2)(b_1 - b_2) \geq 0$

## Application:

$$\alpha(r) \max(F,G)^p + \beta(r) \min(F,G)^p$$

$$\geq \max\left(\alpha(r) F^p + \beta(r) G^p,\ \alpha(r) G^p + \beta(r) F^p\right)$$

Thus

$$\|F+G\|_{L^p}^p + \|F-G\|_{L^p}^p = \int |F+G|^p + |F-G|^p$$

$$\geq \sup_{r \in (0,1)} \int \max(\dots)$$

$$= \sup_{r \in (0,1)} \max\left[\int \alpha(r) F^p + \beta(r) G^p,\ \int \alpha(r) G^p + \beta(r) F^p\right]$$

$$= \sup_{r \in (0,1)} \max\Big(\alpha(r)\|F\|_{L^p}^p + \beta(r)\|G\|_{L^p}^p,\ \alpha(r)\|G\|_{L^p}^p + \beta(r)\|F\|_{L^p}^p\Big)$$

$$= \sup_{r \in (0,1)} \Big( \alpha(r) \max(\|F\|_{L^p}, \|G\|_{L^p})^p + \beta(r) \min(\|F\|_{L^p}, \|G\|_{L^p})^p \Big)$$

$$= \left(\|F\|_{L^p} + \|G\|_{L^p}\right)^p + \left|\|F\|_{L^p} - \|G\|_{L^p}\right|^p$$

by the lemma. This completes the proof of Hanner's inequality.

## Conclusion of the uniform convexity of $L^p$ for $1<p<2$

If $\|f\|_{L^p} \le 1$, $\|g\|_{L^p} \le 1$ and $\|f-g\|_{L^p} \ge \varepsilon$, then by Hanner's inequality

$$\left(\|f+g\|_{L^p} + \|f-g\|_{L^p}\right)^p + \left|\|f+g\|_{L^p} - \|f-g\|_{L^p}\right|^p$$

$$\le 2^p\left(\|f\|_{L^p}^p + \|g\|_{L^p}^p\right) \le 2^p \cdot 2.$$

we obtain $\|f+g\|_{L^p} \le 2-\delta$ with $\delta = \delta_\varepsilon > 0$

Assume by contradiction $\|f+g\|_{L^p} \ge 2-\delta$ for $\delta > 0$ small. Then, using $p>1$ ($t^p$ convex)

$$\left(\|f+g\|_{L^p} + \|f-g\|_{L^p}\right)^p + \left|\|f+g\|_{L^p} - \|f-g\|_{L^p}\right|^p$$

$$= |X+Y|^p + |X-Y|^p \ge |2-\delta+\varepsilon|^p + |2-\delta-\varepsilon|^p$$

$$\ge 2 \cdot 2^p \quad \text{if} \quad \delta < \text{const.}\,\varepsilon.$$

We can also use an exercise:

The space $X$ is uniformly convex if $\|f_n\| \to 1$, $\|g_n\| \to 1$ & $\|f_n + g_n\| \to 2 \Rightarrow \|f_n - g_n\| \to 0$.

Here by Hanner inequality

$$\left(\|f_n+g_n\|_{L^p} + \|f_n-g_n\|_{L^p}\right)^p + \left|\|f_n+g_n\|_{L^p} - \|f_n-g_n\|_{L^p}\right|^p \leq 2^p\left(\|f_n\|_{L^p}^p + \|g_n\|_{L^p}^p\right)$$

$\|f_n+g_n\| \to 2$, $\|f_n\| \to 1$, $\|g_n\| \to 1$

$$\Rightarrow \|f_n - g_n\| \to 0.$$

This completes the proof of the uniform convexity of $L^p(\Omega)$, for $1<p<\infty$. □

Remark: The spaces $L^1(\Omega)$ & $L^\infty(\Omega)$ are never uniformly convex.

If $(\Omega, \mu)$ is "nice", then $(L^1)^* = L^\infty$, but $(L^\infty)^* \supsetneq L^1$ so both $L^1$ & $L^\infty$ not reflexive.

## Theorem (Riesz representation theorem)

Let $(\Omega, \mu)$ be a measure space. Let $1 < p < \infty$.

Then: $\left(L^p(\Omega)\right)^* = L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$

**Proof:** Take $T: L^{p'} \to \left(L^p\right)^*$

$$u \mapsto T_u \in \left(L^p\right)^*$$

$$T_u(g) = \int_\Omega ug, \quad \forall g \in L^p.$$

By Hölder inequality

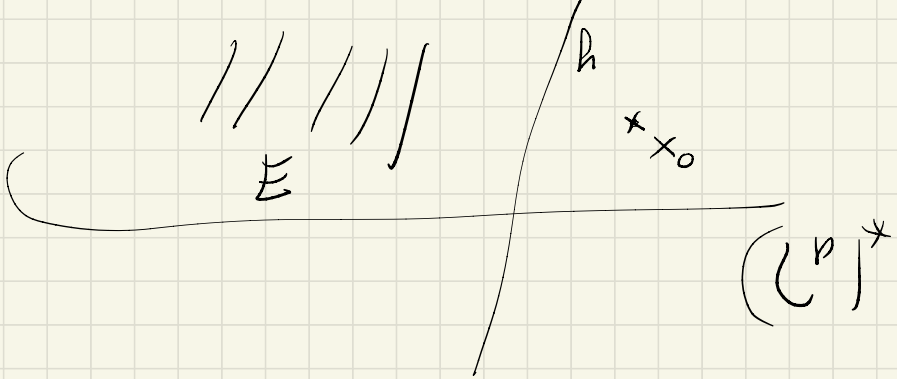
$$|T_u(g)| = \left|\int_\Omega ug\right| \leq \|u\|_{L^p} \|g\|_{L^{p'}}$$

$$\Rightarrow \|T_u\| \leq \|u\|_{L^p}$$

Actually, $\|T_u\| = \|u\|_{L^p}$ (why?)

Thus: $E = T\left(L^{p'}\right)$ is a closed subspace of $\left(L^p\right)^*$. We prove that $E = \left(L^p\right)^*$.

Assume by contradiction that $E \subsetneq \left(L^p\right)^*$.




By Hahn-Banach theorem, $\exists\, 0 \neq h \in (L^p)^{**}$ s.t. $h(e) = 0$, $\forall e \in E$.

Since $L^p$ is reflexive, $(L^p)^{**} = L^p$. Thus $\exists\, v \in L^p$ s.t. $h(\xi) = \xi(v)$ $\forall \xi \in (L^p)^*$.

In particular: $\forall e \in E \Rightarrow 0 = T_u$, $u \in L^{p'}$

$$0 = h(e) = e(v) = T_u(v) = \int uv$$

Thus: $\int uv = 0$, $\forall u \in L^{p'}$

$\Rightarrow v = 0$ (why) $\Rightarrow h = v = 0$

but it contradicts to the choice $h \neq 0$.

Thus we conclude that $E = T(L^{p'}) = (L^p)^*$.

Consequence, if $1 < p < \infty$, then $f_n \to f$ weakly in $L^p(\Omega)$ if and only if

$$\int_\Omega f_n \varphi \longrightarrow \int_\Omega f \varphi \quad , \forall \varphi \in L^{p'}(\Omega)$$

$$\left( \frac{1}{p} + \frac{1}{p'} = 1 \right)$$

Also, since $L^p(\Omega)$ is reflexive, then $\forall \{f_n\}$ bounded in $L^p(\Omega)$, $\exists$ a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ in $L^p(\Omega)$ by Banach - Alaoglu theorem.

Moreover, we know that if $f_n \to f$ weakly in $L^p$, then $\{f_n\}$ is bounded in $L^p$ by Uniform boundedness principle.

# Approximation theory for $L^p(\mathbb{R}^d)$

Consider $\mathbb{R}^d$ with the Lebesgue measure.

**Thm:** If $1 \leq p < \infty$, then $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

Consequently: • $L^p(\mathbb{R}^d)$ is separable

• $\left(L^1(\mathbb{R}^d)\right)^* = L^\infty(\mathbb{R}^d)$.

( but $\left(L^\infty(\mathbb{R}^d)\right)^* \supsetneq L^1(\mathbb{R}^d)$ )

**Proof:** $\forall f \in L^p(\Omega)$, we can approximate $f$ by step functions of the form

$$\sum_{i \in I \text{ finite}} \lambda_i \, \mathbb{1}_{\Omega_i}(x)$$

& bounded

where $\lambda_i \in \mathbb{C}$, $\Omega_i \subset \mathbb{R}^d$ measurable and

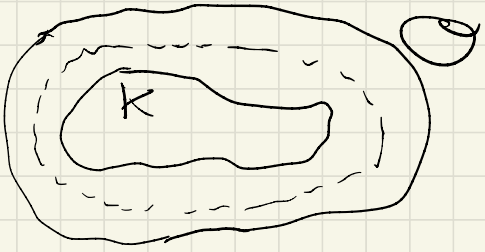
$$\mathbb{1}_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

characteristic / indicator function

Consider $\Omega$ measurable and bounded. Then we have:

- (Inner regularity) $\forall \varepsilon > 0$, $\exists K$ compact set s.t. $K \subset \Omega$, $|\Omega \setminus K| \leq \varepsilon$.

- (Outer regularity) $\forall \varepsilon > 0$, $\exists O$ open set s.t. $\Omega \subset O$, $|O \setminus \Omega| \leq \varepsilon$.




<u>Urysohn's lemma:</u> Let $O$ be an open set in $\mathbb{R}^d$ and $K$ be a compact set s.t. $K \subset O$. Then $\exists \varphi \in C_c^\infty(\mathbb{R}^d)$ such that

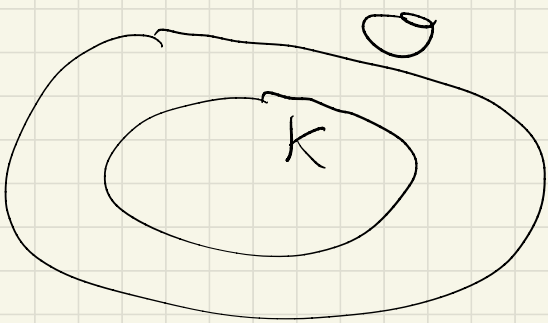
- $0 \leq \varphi \leq 1$
- $\varphi(x) = 1$ if $x \in K$
- $\varphi(x) = 0$ if $x \in \mathbb{R}^d \setminus O$.

This means that we can approximate $\mathbb{1}_\Omega$ by $\varphi$, i.e.

$$\int_{\mathbb{R}^d} |\mathbb{1}_\Omega - \varphi|^p \leq |O \setminus K|$$
$$= |O \setminus \Omega| + |\Omega \setminus K| \leq 2\varepsilon$$

Proof: It is easy to prove Urysohn Lemma if we only require that $\varphi \in C_c(\mathbb{R}^d)$.

In fact:




we can choose $\varphi(x) = \dfrac{\operatorname{dist}(x,K)}{\operatorname{dist}(x,K) + \operatorname{dist}(x,O^c)}$

However, it is more difficult to find a smooth function $\varphi$. For that we need the technique "convolution".

## Definition (convolution)

If $f, g : \mathbb{R}^d \to \mathbb{C}$, then $f * g : \mathbb{R}^d \to \mathbb{C}$

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

**Remark:**

•) $f * g = g * f$

Since $(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$

$$\overset{z = x-y}{=} \int_{\mathbb{R}^d} f(z)\, g(x-z)\, dz$$

$$= (g * f)(x).$$

•) $(f * g) * h = f * (g * h)$ by Fubini

•) $\exists$ no regular function $f$ s.t.

$$f * g = g$$

But actually we have:

$$\delta_0 * g = g$$

with $\delta_0$ the delta function

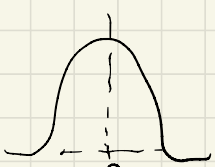
which is defined formally as:

$$\begin{cases} \delta_0(x) = \infty & \text{if } x = 0 \\ \delta_0(x) = 0 & \text{if } x \neq 0 \\ \int_{\mathbb{R}^d} \delta_0(x)\,dx = 1 \end{cases}$$

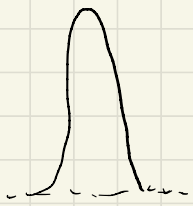
This "function" $\delta_0$ can be defined properly as a distribution, i.e. $\delta_0 \in (C_c^\infty(\mathbb{R}^d))'$

Mathematically, we can approximate $\delta_0$ by a sequence $\{\rho_n\}_{n=1}^\infty$ where $\rho_n(x) = n^d \rho_1(nx)$

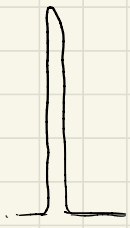
$$\int \rho_n = \int \rho_1 = 1$$



$\rho_1$



$\rho_2$



$\rho_n$

In this way we expect that

$$\rho_n * g \to g \quad \text{as } n \to \infty.$$

**Theorem:** Given $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$.

(a) Then $f * g \in L^p(\mathbb{R}^d)$ and

$$\|f * g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}.$$ (Young's inequality)

(b) If $\int_{\mathbb{R}^d} f = 1$ and $f_n(x) = n^d f(nx)$, then:

$$f_n * g \longrightarrow g \quad \text{strongly in } L^p(\mathbb{R}^d).$$

**Proof:** (a) By Hölder inequality if $1 < p < \infty$

$$|(f*g)(x)| = \left| \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \right|$$

$$\le \left( \int_{\mathbb{R}^d} |f(x-y)|\, dy \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^d} |f(x-y)|\, |g(y)|^p\, dy \right)^{\frac{1}{p}}$$

$$= \|f\|_{L^1}^{\frac{1}{p'}} \left( \int_{\mathbb{R}^d} |f(x-y)|\, |g(y)|^p\, dy \right)^{\frac{1}{p}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$

$$\Rightarrow \quad |(f*g)(x)|^p \le \|f\|_{L^1}^{\frac{p}{p'}} \int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|^p\,dy$$

$$\Rightarrow \int_{\mathbb{R}^d} |(f*g)(x)|^p dx \le \|f\|_{L^1}^{\frac{p}{p'}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|^p dy\,dx$$

$$= \|f\|_{L^1}^{\frac{p}{p'}} \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^d} |f(x-y)|\,dx\right)}_{\|f\|_{L^1}} |g(y)|^p\,dy$$

$$= \|f\|_{L^1}^{\frac{p}{p'}+1} \|g\|_{L^p}^p$$

$$= \left(\|f\|_{L^1}\,\|g\|_{L^p}\right)^p \quad \text{since}$$

$$\frac{p}{p'}+1 = p\underbrace{\left(\frac{1}{p'}+\frac{1}{p}\right)}_{=1} = p$$

$$\Rightarrow \|f*g\|_{L^p} \le \|f\|_{L^1}\,\|g\|_{L^p}.$$

(b) Take $f_n(x) = n^d f(nx)$ for $\int_{\mathbb{R}^d} f = 1$, $f \in L^1$

then we prove

$$f_n * g \to g \text{ in } L^p(\mathbb{R}^d), \ \forall g \in L^p(\mathbb{R}^d)$$

Step 1: Assume $f, g \in C_c(\mathbb{R}^d)$. Then:

$$(f_n * g)(x) - g(x) = \int f_n(y) g(x-y)\,dy - \int f_n(y) g(x)\,dy$$

$$= \int_{\mathbb{R}^d} n^d f(ny) \big(g(x-y) - g(x)\big)\,dy$$

Assume $\operatorname{supp} f \subset B(0,R)$ i.e. $f(x) = 0$ if $|x| > R$

$\Rightarrow \operatorname{supp} f(n\cdot) \subset B(0, \frac{R}{n})$

$$\Rightarrow |(f_n * g)(x) - g(x)| \le \int_{|y| \le \frac{R}{n}} n^d |f(ny)| \underbrace{|g(x-y) - g(x)|}\,dy$$

$$\le \sup_{|z| \le \frac{R}{n}} |g(x-z) - g(x)|$$

$$\le \sup_{|z| \le \frac{R}{n}} |g(x-z) - g(x)| \int_{|y| \le \frac{R}{n}} n^d |f(ny)|\,dy$$

$$\Rightarrow \quad |(f_n * g)(x) - g(x)| \leq \|f\|_{L^1} \sup_{|z| \leq \frac{R}{n}} |g(x-z) - g(x)|$$

Assume $\operatorname{supp} g \subset B(0, R_1)$

$$\Rightarrow \quad \sup_{|z| \leq \frac{R}{n}} |g(x-z) - g(x)| = 0 \text{ if } |x| > R_1 + \frac{R}{n}$$

{ and bounded by $2\|g\|_{L^\infty}$

Thus:

$$|(f_n * g)(x) - g(x)| \longrightarrow 0 \quad \text{for all } x \in \mathbb{R}^d$$

$$|(f_n * g)(x) - g(x)| \leq 2\|g\|_{L^\infty} \mathbb{1}_{B(0, R_1+1)}(x)$$

for all $n$ s.t. $R_1 + \frac{R}{n} < R_1 + 1$

$$\Rightarrow \quad f_n * g - g \longrightarrow 0 \text{ strongly in } L^p(\mathbb{R}^d)$$

by Dominated convergence.

Step 2: Let $f \in C_c(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Then we can find a sequence $g_m \in C_c(\mathbb{R}^d)$ s.t. $g_m \to g$ in $L^p(\mathbb{R}^d)$. Then:

$$\|f_n * g - g\|_{L^p} \le \|f_n * (g - g_m)\|_{L^p} + \|f_n * g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p}$$

$$\le \underbrace{\|f_n\|_{L^1}}_{=\|f\|_{L^1}} \|g - g_m\|_{L^p} + \|f_n * g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p}$$

$$\le (\|f\|_{L^1} + 1) \|g_m - g\|_{L^p} + \|f_n * g_m - g_m\|_{L^p}$$

$$\Rightarrow \limsup_{n \to \infty} \|f_n * g - g\|_{L^p} \le (\|f\|_{L^1} + 1) \|g_m - g\|_{L^p}$$

since $\lim_{n \to \infty} \|f_n * g_m - g_m\|_{L^p} = 0$ by Step 1

Take $m \to \infty \Rightarrow \|f_n * g - g\|_{L^p} \to 0$.

Step 3. Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$.
$\exists F_m \in C_c(\mathbb{R}^d)$ s.t. $F_m \to f$ in $L^1(\mathbb{R}^d)$.
and $\int F_m = 1$

Define $F_{m,n}(x) = n^d F_m(nx)$. Then:
by the triangle & Young inequalities

$$\|f_n * g - g\|_{L^p} \le \|(f_n - F_{m,n}) * g\|_{L^p} + \|F_{m,n} * g - g\|_{L^p}$$

$$\le \underbrace{\|f_n - F_{m,n}\|_{L^1}}_{\int n^d |f(nx) - F_m(nx)|\,dx = \|f - F_m\|_{L^1}} \|g\|_{L^p} + \|F_{m,n} * g - g\|_{L^p}$$

By Step 2:

$$\lim_{n \to \infty} \|F_{m,n} * g - g\|_{L^p} = 0$$

$$\Rightarrow \limsup \|f_n * g - g\|_{L^p} \le \|f - F_m\|_{L^1} \|g\|_{L^p}.$$

Take $m \to \infty$ to get $f_n * g - g \to 0$ in $L^p$.

**Remark:** given $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $\exists\ \tilde{F}_m$ s.t.

$\tilde{F}_m \in C_c(\mathbb{R}^d)$, $\tilde{F}_m \to f$ in $L^1(\mathbb{R}^d)$.

Then:

$$\int \tilde{F}_m \longrightarrow \int f = 1$$

We can define

$$F_m = \frac{1}{\left(\int \tilde{F}_m\right)} \tilde{F}_m \qquad \text{for } m \text{ large}$$

$$\Rightarrow F_m \in C_c(\mathbb{R}^d) \text{ and } F_m \to f \text{ in } L^1(\mathbb{R}^d)$$

$$\int F_m = 1.$$

**Theorem:** If $f \in C_c^\infty(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ for $1 \le p < \infty$. Then $f * g \in C^\infty(\mathbb{R}^d)$ and

$$D^\alpha(f * g) = (D^\alpha f) * g, \quad \forall \alpha.$$

Here for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ we denote

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$$

Proof: First we prove that $f * g$ is continuous.

Take $\{y_n\} \subset \mathbb{R}^d$, $y_n \to y$ in $\mathbb{R}^d$. Then

$$(f*g)(y_n) = \int_{\mathbb{R}^d} f(y_n - x)\, g(x)\, dx$$

$$\longrightarrow \int_{\mathbb{R}^d} f(y - x)\, g(x)\, dx$$

by Dominated convergence. In fact:

$$f(y_n - x)\, g(x) \longrightarrow f(y - x)\, g(x)$$

and $|f(y_n - x)\, g(x)| \le \|f\|_{L^\infty} \mathbb{1}_{B(0,R)}(x)\, |g(x)|$ $\in L^1(\mathbb{R}^d)$

where $R$ is chosen s.t.

$$f(y_n - x) = 0 \quad \text{if } |x| \ge R$$

which is doable since $y_n \to y$ & $f$ is compactly supported.

Similarly: $e_i = (0, \dots, \underset{\text{ith}}{1}, \dots) \in \mathbb{R}^d$

$$\partial_{x_i}(f*g)(x) = \lim_{h\to 0} \int \frac{f(x+he_i-y) - f(x-y)}{h} g(y)\,dy$$

Dominated Convergence

$$= \int \lim_{h\to 0} \frac{f(x+he_i-y) - f(x-y)}{h} g(y)\,dy$$

$$= \int (\partial_{x_i} f)(x-y)\, g(y)\,dy$$

$$= (\partial_{x_i} f) * g$$

And $\partial_{x_i} f * g \in C(\mathbb{R})$ by the previous step

Since $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$. The same argument gives

$$D^\alpha(f*g) = (D^\alpha f) * g \in C(\mathbb{R})$$

$\forall \alpha$ as conclusion.

**Theorem:** Let $\Omega \subset \mathbb{R}^d$ be an open set, with Lebesgue measure. Then: $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $1 \leq p < \infty$.

Consequently: $L^p(\Omega)$ is separable with $1 \leq p < \infty$.

**Proof:** **Step 1:** Consider $\Omega = \mathbb{R}^d$. Take $g \in L^p(\mathbb{R})$ and $g$ is compactly supported.

Take $f \in C_c^\infty(\mathbb{R}^d)$ and $\int f = 1$. Define

$$f_n(x) = n^d f(nx) \rightsquigarrow f_n \in C_c^\infty, \int f_n = 1.$$

We know that

$$f_n * g \longrightarrow g \quad \text{strongly in } L^p(\mathbb{R}^d),$$

and also $f_n * g \in C^\infty$. Moreover, $f_n * g$ is also compactly supported. Indeed:

$$(f_n * g)(x) = \int_{\mathbb{R}^d} f_n(x-y)\, g(y)\, dy$$

If $\operatorname{supp} f_n \subset B(0, R_n)$ and $\operatorname{supp} g \subset B(0, R_0)$

$\Rightarrow (f_n * g)(x) = 0$ if $|x| > R_0 + R_n$.

Step 2. $\Omega = \mathbb{R}^d$ and $g \in L^p(\mathbb{R}^d)$.

Then we can approximate $g$ by $L^p_c(\mathbb{R}^d)$

e.g. $g\,\mathbb{1}_{(|x|\le R)} \longrightarrow g$ in $L^p(\mathbb{R}^d)$

$R \to \infty$ (Dominated C.V.)

From step 1, any $g\,\mathbb{1}_{(|x|\le R)}$ can be approximated by $C_c^\infty(\mathbb{R}^d)$ functions.

More precisely, $\forall \varepsilon > 0$, $\exists\, R = R_\varepsilon$ s.t.

$$\| g\,\mathbb{1}_{(|x|\le R)} - g \|_{L^p} \le \varepsilon$$

and $\exists\, \varphi \in C_c^\infty(\mathbb{R}^d)$

$$\| g\,\mathbb{1}_{(|x|\le R)} - \varphi \|_{L^p} \le \varepsilon$$

$$\Rightarrow \quad \| g - \varphi \|_{L^p} \le 2\varepsilon \quad \text{by triangle ineq.}$$

Step 3: $\Omega$ open bounded subset of $\mathbb{R}^d$.

Take $g \in L^p(\Omega)$. Then $\forall \varepsilon > 0$, $\exists\, g_\varepsilon \in L^p(\Omega)$ and $\operatorname{supp} g_\varepsilon \subset K$ compact $\subset \Omega$ and

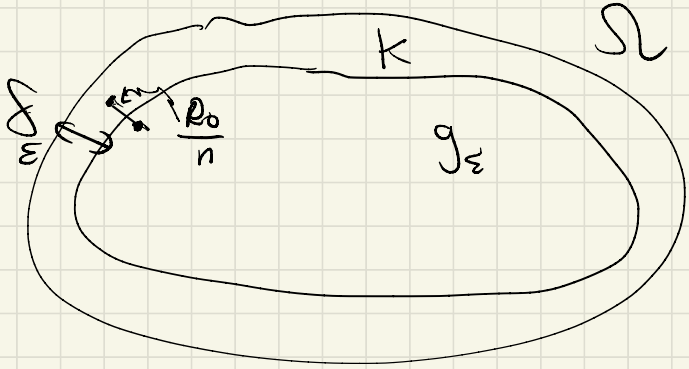
$$\| g_\varepsilon - g \|_{L^p} \le \varepsilon.$$

For example, we can take

$$g_\varepsilon = g \, \mathbb{1}_{K(\varepsilon)}$$

where $K = \{x \in \Omega : \operatorname{dist}(x, \Omega^c) \leq \delta_\varepsilon\}$.

Define $\tilde{g}_\varepsilon(x) = \begin{cases} g_\varepsilon(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}$




$\tilde{g}_\varepsilon(x) = 0$ on $\Omega^c$

Take $\rho \in C_c^\infty(\mathbb{R}^d)$ and $\int \rho = 1$. Define

$$\rho_n(x) = n^d \rho(nx), \quad \forall x \in \mathbb{R}^d.$$

Then

$$\rho_n * \tilde{g}_\varepsilon \longrightarrow \tilde{g}_\varepsilon \quad \text{as } n \to \infty.$$

Here $\operatorname{supp} \tilde{g}_\varepsilon \subset K = \{x \in \Omega : \operatorname{dist}(x, \Omega^c) \leq \delta_\varepsilon\}$

and $\operatorname{supp} \rho_n \subset \{|x| \leq R_0/n\}$ where

$\operatorname{supp} \rho \subset \{|x| \leq R_0\}$

Then: $(f_n * \tilde{g}_\varepsilon)(x) = \int_{\mathbb{R}^d} f_n(x-y)\,\tilde{g}_\varepsilon(y)\,dy$

supported $K + \{|z| \le R_0/n\}$

$= \{y \in \Omega : \operatorname{dist}(y, \Omega^c) \le \delta_\varepsilon\} + \{|z| \le R_0/n\}$

$\subset \{x \in \Omega : \operatorname{dist}(x, \Omega^c) \le \delta_\varepsilon - R_0/n\}.$

by the triangle inequality. Thus by taking $n$ large enough, we know that.

$$f_n * \tilde{g}_\varepsilon \in C_c^\infty(\Omega)$$

and $\|f_n * \tilde{g}_\varepsilon - \tilde{g}_\varepsilon\|_{L^p(\mathbb{R}^d)} \le \varepsilon.$

Thus:

$$\|f_n * g_\varepsilon - g\|_{L^p(\Omega)} \le \|f_n * g_\varepsilon - \tilde{g}_\varepsilon\|_{L^p(\Omega)} + \|\tilde{g}_\varepsilon - g\|_{L^p(\Omega)}$$

$$\le \|f_n * g_\varepsilon - \tilde{g}_\varepsilon\|_{L^p(\mathbb{R}^d)} + \|g_\varepsilon - g\|_{L^p(\Omega)}$$

$$\in 2\varepsilon.$$

Step 4: Take $\Omega$ open set in $\mathbb{R}^d$ and $g \in L^p(\Omega)$. $\forall R > 0$, take $\Omega_R = \Omega \cap B(0,R)$. Then $\Omega_R$ open and bounded. Then

$$g \mathbb{1}_{\Omega_R} \to g \quad \text{strongly in } L^p(\Omega)$$

by Dominated c.v.

Then since $g \mathbb{1}_{\Omega_R} \in L^p(\Omega_R)$, this can be approximated by functions in $C_c^\infty(\Omega_R) \subset C_c^\infty(\Omega)$. Then we get the conclusion by the triangle inequality, i.e. $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$. □

Remark: $\forall g \in L^p(\Omega)$, we can approximate it by $C_c(\Omega)$. $\forall g \in C_c(\Omega)$, $\operatorname{supp} g \subset K$ compact in $\Omega$.

Weierstrass theorem: $g \in C(\underset{\text{compact}}{K})$ can be approximated in sup-norm by polynomials. We can choose the coefficients of polynomials to be elements of $\mathbb{Q}$ as countable.

Thus we conclude that $L^p(\Omega)$ is separable.

**Thm:** (Fundamental theorem of calculus)
Let $\Omega$ be an open set in $\mathbb{R}^d$. Let $f \in L^1_{loc}(\Omega)$ i.e. $f \mathbb{1}_K \in L^1(\Omega)$ for any $K_{compact} \subset \Omega$.
If

$$\int_\Omega f\varphi = 0 \quad , \forall \varphi \in C_c^\infty(\Omega)$$

then $f = 0$.

**Proof:** Take $g \in C_c^\infty(\Omega)$. Then:

$$\int_\Omega fg\varphi = 0 \quad , \forall \varphi \in C_c^\infty(\Omega)$$

Since $fg \in L^1(\Omega)$ and compactly supported.
Take $h \in C_c^\infty(\mathbb{R}^d)$ and $\int h = 1$. Define

$$h_n(x) = n^d \, h(nx) \in C_c^\infty, \quad \int h_n = 1.$$

Then: $h_n * (fg) \longrightarrow fg$ in $L^1(\mathbb{R}^d)$
Here $h_n * (fg) = \int_{\mathbb{R}^d} (fg)(y) \, h_n(x-y) \, dy$

$$= \int_\Omega (fg)(y)\ \underbrace{h_n(x-y)}_{\varphi}\ dy$$

This function is compactly supported in $\Omega$

$$\underbrace{\operatorname{supp}(fg)}_{\subset K \subset \Omega_{\text{open}}} + \underbrace{\operatorname{supp} h_n}_{\subset \{|x| \le \frac{R_0}{n}\}}$$
compact

For any $x$ in the supp, then

$$K \ni y \mapsto h_n(x-y) \in C_c^\infty(\Omega)$$

Thus:

$$\int_\Omega (fg)(y)\ h_n(x-y)\, dy$$

$$= \int_\Omega (fg)(y)\ \underbrace{\varphi_{n,x}(y)}_{\in C_c^\infty(\Omega)}\, dy \quad = 0$$

Conclusion:

$$\underbrace{(fg) * h_n}_{=0} \longrightarrow fg \Rightarrow fg = 0$$

□

Since $fg = 0$ $\forall g$ compactly supp in $\Omega \Rightarrow f = 0$.

## Fourier transform:

**Def:** Given $f : \mathbb{R}^d \to \mathbb{C}$, define

$$(\mathcal{F}f)(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i k\cdot x}\, dx$$

[ Here $k\cdot x = \sum_{i=1}^{d} k_i \cdot x_i$ with $k=(k_i)$, $x=(x_i)$ ]

**Basic properties.**

- If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^\infty(\mathbb{R}^d)$ and
$$\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}.$$
- If $f, g \in L^1(\mathbb{R}^d)$, then $f * g \in L^1$ by Young inequality. Then: by Fubini theorem
$$\widehat{f*g}(k) = \hat{f}(k)\,\hat{g}(k)$$

This identity should hold as soon as $\hat{f}$, $\hat{g}$, $\widehat{f*g}$ are well-defined.

- $\mathcal{F}\left(e^{-\pi|x|^2}\right) = e^{-\pi|k|^2}$ exercise

more generally: $\mathcal{F}\left(e^{-\pi\lambda^2|x|^2}\right) = \lambda^{-d} e^{-\pi\frac{|k|^2}{\lambda^2}}$, $\forall \lambda > 0$.

**Thm:** (a) If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R})$, then:

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2} \quad \text{(Plancherel identity)}$$

(b) $\mathcal{F}: L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ can be extended to be an isometry on $L^2(\mathbb{R}^d)$, i.e.

$$\mathcal{F}: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}.$$

(c) $\mathcal{F}$ is a unitary transformation, i.e.

$\mathcal{F}^{-1}$ is well-defined. In fact $\forall f \in L^1 \cap L^2$,

$$(\mathcal{F}^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(k)\, e^{2\pi i k\cdot x}\, dk.$$

**Proof:** (a) Consider:

$$\int_{\mathbb{R}^d} |\hat{f}(k)|^2\, e^{-\pi\varepsilon|k|^2}\, dk, \quad \varepsilon > 0,$$

$$A_\varepsilon := \int_{\mathbb{R}^d} |\hat{f}(k)|^2 \, e^{-\pi\varepsilon|k|^2} \, dk$$

$$= \int_{\mathbb{R}^d} \overline{\hat{f}(k)} \, \hat{f}(k) \, e^{-\pi\varepsilon|k|^2} \, dk$$

$$= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x) \, e^{-2\pi i k \cdot x} \, dx \right) \overline{\left( \int_{\mathbb{R}^d} f(y) \, e^{-2\pi i k \cdot y} \, dy \right)} \, e^{-\pi\varepsilon|k|^2} \, dk$$

$$= \int_{\mathbb{R}^{3d}} \overline{f(x)} \, f(y) \, e^{2\pi i k \cdot (x-y)} \, e^{-\pi\varepsilon|k|^2} \, dk \, dx \, dy$$

(by Fubini theorem)

$$= \int_{\mathbb{R}^{2d}} \overline{f(x)} \, f(y) \left( \int_{\mathbb{R}^d} e^{-2\pi i k \cdot (y-x)} \, e^{-\pi\varepsilon|k|^2} \, dk \right) dx \, dy$$

$e^{-\pi\varepsilon|k|^2}$ $(y-x)$

$$= \int_{\mathbb{R}^{2d}} \overline{f(x)} \, f(y) \, \varepsilon^{-\frac{d}{2}} \, e^{-\pi \frac{(x-y)^2}{\varepsilon}} \, dx \, dy$$

$$= \int_{\mathbb{R}^d} \overline{f(x)} \, (G_\varepsilon * f)(x) \, dx$$

where $G_\varepsilon(x) = \varepsilon^{-d/2} \, e^{-\pi \frac{x^2}{\varepsilon}}$

$$(G_\varepsilon * f)(x) = \int_{\mathbb{R}^d} \varepsilon^{-d/2} \, e^{-\pi \frac{(x-y)^2}{\varepsilon}} f(y) \, dy$$

The function $G_\varepsilon(x) = (\sqrt{\varepsilon})^{-d} \, G_1\left(\frac{x}{\sqrt{\varepsilon}}\right)$ has

$$\int G_\varepsilon = \int G_1 = \int_{\mathbb{R}^d} e^{-\pi x^2} = 1 .$$

Then when $\varepsilon \to 0$ (or $\varepsilon = \frac{1}{n}$ and $n \to \infty$)

$$G_\varepsilon * f \longrightarrow f \text{ strongly in } L^1 \cap L^2 .$$

Conclusion:

$$A_\varepsilon = \int_{\mathbb{R}^d} \overline{f(x)} \, (G_\varepsilon * f)(x) \, dx$$

$$\xrightarrow[\varepsilon \to 0]{} \int_{\mathbb{R}^d} \overline{f(x)} \, f(x) \, dx = \int_{\mathbb{R}^d} |f(x)|^2 \, dx$$

Namely:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |\hat{f}(k)|^2 e^{-\pi \varepsilon |k|^2} dk = \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Moreover, $|\hat{f}(k)|^2 e^{-\pi \varepsilon |k|^2} \uparrow |\hat{f}(k)|^2$ as $\varepsilon \downarrow 0$

$\Rightarrow$ By monotone c.v., $|\hat{f}(k)|^2 \in L^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} |\hat{f}(k)|^2 = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} |\hat{f}(k)|^2 e^{-\pi \varepsilon |k|^2} dk$$

$$= \int_{\mathbb{R}^d} |f(x)|^2 dx.$$

(b) The operator $\mathcal{F}: L^1 \cap L^2 \to L^2$ is well-defined and $\|\mathcal{F} f\|_{L^2} = \|f\|_{L^2}$, $\forall f \in L^1 \cap L^2$.

Then $\forall f \in L^2$, since $L^1 \cap L^2$ is dense in $L^2$, $\exists \{f_n\} \subset L^1 \cap L^2$ s.t. $f_n \to f$ strongly in $L^2$. Then the sequence $\{f_n\}$ is Cauchy

sequence in $L^2$. By (a)

$$\|\mathcal{F}\varphi_n - \mathcal{F}\varphi_m\|_{L^2} = \|\mathcal{F}(\varphi_n - \varphi_m)\|_{L^2} = \|\varphi_n - \varphi_m\|_{L^2}$$

$\rightsquigarrow$ $\{\mathcal{F}\varphi_n\}$ is a Cauchy sequence in $L^2$

$\rightsquigarrow$ $\exists \lim_{n\to\infty} \mathcal{F}\varphi_n =: \mathcal{F}f$ by definition.

We can check that the definition of $\mathcal{F}f$ is independent of the choice of $\{\varphi_n\} \subset L^1 \cap L^2$.

In fact, if $\varphi_n \to f$, $g_n \to f$, and $\varphi_n, g_n \subset L^1 \cap L^2$.

Then: $\|\mathcal{F}\varphi_n - \mathcal{F}g_n\|_{L^2} = \|\mathcal{F}(\varphi_n - g_n)\|_{L^2}$

$$= \|\varphi_n - g_n\|_{L^2} \le \|\varphi_n - f\|_{L^2} + \|g_n - f\|_{L^2}$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

(c) We prove that $\mathcal{F}^{-1}$ exists.

Take $f \in L^2(\mathbb{R}^d)$, we have:

Exercise

(*) $\int_{\mathbb{R}^d} \hat{G}_\varepsilon(x-y) f(y)\, dy = \int_{\mathbb{R}^d} G_\varepsilon(k) \hat{f}(k) e^{2\pi i k\cdot x}\, dk$

where $G_\varepsilon(x) = e^{-\pi\varepsilon|x|^2}$, $\varepsilon > 0$

$$\text{LHS}(*) = \int_{\mathbb{R}^d} \varepsilon^{-d/2} e^{-\pi \frac{(x-y)^2}{\varepsilon}} f(y)\, dy$$

$\xrightarrow{\varepsilon \to 0}$ $f(x)$ strongly in $L^2(\mathbb{R}^d)$

$$\text{RHS}(*) = \int_{\mathbb{R}^d} \hat{f}(k)\, e^{-\pi\varepsilon|k|^2} e^{2\pi i k\cdot x}\, dk.$$

$\xrightarrow{\varepsilon \to 0}$ $\int_{\mathbb{R}^d} \hat{f}(k)\, e^{2\pi i k\cdot x}\, dk$

if $\hat{f} \in L^1$, by Dominated c.v.

Thus: if $f \in L^1 \cap L^2$, $\hat{f} \in L^1$, then

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(k)\, e^{2\pi i k\cdot x}\, dk.$$

The assumption $\hat{f} \in L^1$ can be removed

by an approximation.

Actually, if $f \in C_c^\infty(\mathbb{R}^d)$, then $\hat{f} \in L^1 \cap L^\infty$

(indeed $\hat{f}$ is bounded and decay faster than any polynomial) $\square$

Remark: We know that if $f \in L^1(\mathbb{R}^d)$ or $f \in L^2(\mathbb{R}^d)$ then we can define $\hat{f}$. If $f \in L^p(\mathbb{R}^d)$ with $1 < p < 2$, then we can write

$$f = g + h, \quad g \in L^1, \ h \in L^2$$

$$\rightsquigarrow \quad \hat{f} = \hat{g} + \hat{h}.$$

This is trivial! The non-trivial thing is that $\mathcal{F}: L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded operator if $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem: (Hausdorff - Young inequality)
If $1 \le p \le 2$, and $f \in L^p(\mathbb{R}^d)$, then

$$\|\hat{f}\|_{L^q} \le \|f\|_{L^p} \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

**Remark:** If $p=1$, then

$$\hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k\cdot x} \, dx$$

$$\Rightarrow |\hat{f}(k)| \leq \int_{\mathbb{R}^d} |f(x) e^{-2\pi i k\cdot x}| \, dx$$

$$= \int_{\mathbb{R}^d} |f(x)| \, dx = \|f\|_{L^1}$$

$$\Rightarrow \|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$$

If $p=2$, then

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2} \quad \text{by Plancherel identity.}$$

Thus: $\mathcal{F}: L^1 \to L^\infty$ and $L^2 \to L^2$. The Hausdorff–Young inequality tells us that

$\mathcal{F}: L^p \to L^q$ and

$$\|F\|_{\mathcal{L}(L^p, L^q)} \leq \max\left( \|F\|^{\frac{1}{2}}_{\mathcal{L}(L^1, L^\infty)}, \|F\|^{\frac{1}{2}}_{\mathcal{L}(L^2, L^2)} \right)$$

This is a particular case of a family of interpolation inequalities. For example:

**Theorem:** (Riesz-Thorin interpolation ineq.)
Let $(\Omega, \mu)$ be a measure space which is $\sigma$-finite. Let $T$ be a linear operator

$$T: L^{p_1} + L^{p_2} \longrightarrow L^{q_1} + L^{q_2}.$$

If

$$\|T\|_{\mathcal{L}(L^{p_1}, L^{q_1})} \leq 1$$

$$\|T\|_{\mathcal{L}(L^{p_2}, L^{q_2})} \leq 1$$

Then

$$\|T\|_{\mathcal{L}(L^{p}, L^{q})} \leq 1$$

where $1 \leq p_1 < p_2 \leq \infty$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\theta \in (0,1)$

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

In our case: $\begin{cases} p_1 = 1, q_1 = \infty \\ p_2 = 2, q_2 = 2 \end{cases} \Rightarrow \frac{1}{p} + \frac{1}{q} = 1.$

The proof of the Riesz–Thorin interpolation inequality is difficult (it use "complex interpolation").

## A direct proof of the Hausdorff–Young ineq.

**Step 1:** (Layer cake representation)

If $f \in L^p$, then:

$$\int |f|^p = \int_0^\infty p\,\alpha^{p-1}\,\lambda_f(\alpha)\,d\alpha$$

where $\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\})$, $\mu$ Lebesgue measure

**Proof:** Using

$$\lambda_f(\alpha) = \int_{\mathbb{R}^d} \chi_{\{x:\,|f(x)|>\alpha\}}\,dx$$

$$\Rightarrow \int_0^\infty p\,\alpha^{p-1}\,\lambda_f(x)\,d\alpha$$

$$= \int_0^\infty p\alpha^{p-1}\left(\int_{\mathbb{R}^d} \chi_{\{x:\,|f(x)|>\alpha\}}\,dx\right)d\alpha$$

Fubini

$$= \int_{\mathbb{R}^d}\left(\int_0^\infty p\alpha^{p-1}\,\chi_{\{x:\,|f(x)|>\alpha\}}\,d\alpha\right)dx$$

$$= \int_{\mathbb{R}^d} \left( \int_0^{|f(x)|} p\alpha^{p-1} d\alpha \right) dx$$

$$= \int_{\mathbb{R}^d} |f(x)|^p dx .$$

Step 2: given $a>0$ and decompose

$$f = f_a^+ + f_a^-$$

$$= f \mathbb{1}(|f| \geq a) + f \mathbb{1}(|f|<a)$$

Then:

$$\lambda_f(\alpha) = \{x : |f(x)| > \alpha\}$$

$$\leq \mu\left(\{x : |f_a^+(x)| + |f_a^-(x)| > \alpha\}\right)$$

$$\leq \mu\left(\{x : |f_a^+(x)| > \tfrac{\alpha}{2}\}\right) + \mu\left(\{x : |f_a^-(x)| > \tfrac{\alpha}{2}\}\right)$$

$$= \lambda_{f_a^+}\left(\tfrac{\alpha}{2}\right) + \lambda_{f_a^-}\left(\tfrac{\alpha}{2}\right) .$$

Note that if $1<p<2$

$$\int |f_a^+| = \int |f| \mathbb{1}(|f|>a) \leq \int \frac{|f|^p}{a^{p-1}}$$

and

$$\lambda_{f_a^-}(\alpha) \begin{cases} = 0 & \text{if } \alpha > a \\ \leq \int \frac{|f_a^-|^2}{\alpha^2} = \int \frac{|f|^2 \mathbb{1}(f<a)}{\alpha^2} & \\ \leq \frac{a^{2-p} \int |f|^p}{\alpha^2} & \text{if } \alpha \leq a \end{cases}$$

Here we used the estimate:

$$\int |g|^p \geq \int |g|^p \mathbb{1}(|g|>a)$$

$$\geq \int a^p \mathbb{1}(|g|>a) = a^p \lambda_g(a).$$

Step 3: Take $f \in L^p(\mathbb{R}^d)$ with $\int |f|^p = 1$.

Then:

$$\|Ff\|_{L^q}^q = \int_0^\infty q\beta^{q-1} \lambda_{Ff}(\beta)\, d\beta$$

$$Ff = Ff_a^+ + Ff_a^-$$

$$= \int_0^\infty q\beta^{q-1} \lambda_{Ff_a^+ + Ff_a^-}(\beta)\, d\beta$$

$$\leq \int_0^\infty q\beta^{q-1}\left[\lambda_{Tf_a^+}\left(\frac{\beta}{2}\right) + \lambda_{Tf_a^-}\left(\frac{\beta}{2}\right)\right] d\beta$$

Note that

$$\|Tf_a^+\|_{L^\infty} \leq \|f_a^+\|_{L^1} = \int |f|\, \mathbb{1}(|f|>a)$$

$$\leq \int \frac{|f|^p}{a^{p-1}} = a^{1-p}$$

$$\Rightarrow \lambda_{Tf_a^+}\left(\frac{\beta}{2}\right) = 0 \quad \text{if} \quad a^{1-p} = \frac{\beta}{4}.$$

Moreover,

$$\lambda_{Tf_a^-}\left(\frac{\beta}{2}\right) \leq \frac{\int |Tf_a^-|^2}{\left(\frac{\beta}{2}\right)^2} = \frac{\int |f_a^-|^2}{\left(\frac{\beta}{2}\right)^2}$$

$$= \frac{4}{\beta^2} \int_0^\infty 2\alpha\, \lambda_{f_a^-}(\alpha)\, d\alpha$$

$$\leq \frac{8}{\beta^2} \int_0^a \alpha\, \lambda_{f_a}(\alpha)\, d\alpha$$

Here we used $\lambda_{f_a^-}(\alpha) \leq \lambda_{f_a}(\alpha)\, \mathbb{1}(\alpha \leq a)$.

In summary, with the choice $a^{1-p} = \frac{\beta}{4}$,

$$\|Fg\|_{L^{q,q}}^q \le C_p \int_0^\infty \frac{\beta^{q-1}}{\beta^2} \left( \int_0^a \alpha \, \lambda_g(\alpha) \, d\alpha \right) d\beta$$

$$= C_p \iint_{\alpha^{1-p} \ge a^{1-p} = \frac{\beta}{4}} \beta^{p-3} \alpha \, \lambda_g(\alpha) \, d\alpha \, d\beta$$

$$= C_p \int_0^\infty \left( \int_0^{4\alpha^{1-p}} \beta^{q-3} \, d\beta \right) \alpha \, \lambda_g(\alpha) \, d\alpha$$

$q > 2$ since $\frac{1}{p} + \frac{1}{q} = 2$ & $p < 2$

$$= \tilde{C}_p \int_0^\infty p \underbrace{\left(\alpha^{1-p}\right)^{q-2} \alpha}_{\alpha^{p-1}} \lambda_g(\alpha) \, d\alpha$$

$$= \tilde{C}_p \int |g|^p = \tilde{C}_p$$

Here we used: $(1-p)(q-2) + 1 = p - 1$

$\Leftrightarrow (1-p)q - 2 + 2p + 1 = p - 1$

$\Leftrightarrow p = (p-1)q \Leftrightarrow q = \frac{p}{p-1} \Leftrightarrow \frac{1}{p} + \frac{1}{q} = 1$

Thus we proved that $\forall f \in L^p$, $\int |f|^p = 1$ then:

$$\|\mathcal{F} f\|_{L^q}^q \leq C_p$$

$$\Rightarrow \|\mathcal{F} f\|_{L^q} \leq C_p \|f\|_{L^p}, \forall f \in L^p. \quad (*)$$

Step 4: We can replace $C_p$ in $(*)$ by 1 by "tensor trick". Take $f \in C_c(\mathbb{R}^d)$. Define:

$$f^{\otimes N} \in L^p\left((\mathbb{R}^d)^N\right)$$

$$f^{\otimes N}(x_1, \dots, x_N) = f(x_1) \cdots f(x_N) \qquad x_i \in \mathbb{R}^d$$

$$\Rightarrow \|f^{\otimes N}\|_{L^p(\mathbb{R}^{dN})} = \|f\|_{L^p}^N \quad \text{by Fubini.}$$

Moreover:

$$\mathcal{F}(f^{\otimes N})(k) = \int f^{\otimes N}(x_1, \dots, x_N)\, e^{-2\pi i (k_1 x_1 + \dots + k_N x_N)}\, dx_1 \dots dx_N$$

$$k = (k_1, k_2, \dots, k_N) \in \mathbb{R}^{dN}$$

$$k_i \in \mathbb{R}^d$$

$$= \int f(x_1)\cdots f(x_N)\, e^{-2\pi i k_1 x_1} \cdots e^{-2\pi i k_N x_N}\, dx_1 \cdots dx_N$$

Fubini

$$= \left(\int f(x_1) e^{-2\pi i k_1 x_1} dx_1\right) \cdots \left(\int f(x_N) e^{-2\pi i k_N x_N} dx_N\right)$$

$$= \hat{f}(k_1) \cdots \hat{f}(k_N) = (\hat{f})^{\otimes N}(k)$$

$$(k_1 \ldots k_N)$$

$$\Rightarrow \|\mathcal{F}(f^{\otimes N})\|_{L^q(\mathbb{R}^{dN})} = \|\hat{f}^{\otimes N}\|_{L^q(\mathbb{R}^{dN})} = \|\hat{f}\|^N_{L^q(\mathbb{R}^d)}$$

By Step 3, independent of $N$

$$\|\mathcal{F}(f^{\otimes N})\|_{L^q(\mathbb{R}^{dN})} \le C_p^N \|f^{\otimes N}\|_{L^p(\mathbb{R}^{dN})}$$

$$\Rightarrow \|\hat{f}\|^N_{L^q} \le C_p \|f\|^N_{L^p}, \quad \forall N \ge 1$$

$$\Rightarrow \|\hat{f}\|_{L^q} \le C_p^{1/N} \|f\|_{L^p}, \quad \forall N \ge 1$$

$N \to \infty$

$$\Rightarrow \|\hat{f}\|_{L^q} \le \|f\|_{L^p}.$$

This holds $\forall f \in C_c(\mathbb{R}^d)$ or $L^1 \cap L^2$. But

by a density argument, we obtain

$$\|\hat{f}\|_{L^q} \leq \|f\|_{L^p} \quad , \forall f \in L^p(\mathbb{R}^d).$$

---

**Theorem:** $\left(L^1(\mathbb{R}^d)\right)^* = L^\infty(\mathbb{R}^d)$.

More precisely, if $T \in \left(L^1(\mathbb{R}^d)\right)^*$, then $\exists!\ u \in L^\infty(\mathbb{R}^d)$ such that

$$T(f) = \int_{\mathbb{R}^d} uf \quad , \forall f \in L^1(\mathbb{R}^d).$$

**Proof:** Clearly, $\forall u \in L^\infty$, then

$$f \longmapsto \int_{\mathbb{R}^d} uf \quad \text{is linear \& continuous}$$

since $\left|\int_{\mathbb{R}^d} uf\right| \leq \|u\|_{L^\infty} \|f\|_{L^1}$.

The non-trivial direction is to construct $u \in L^\infty(\mathbb{R}^d)$ from $T \in \left(L^1(\mathbb{R}^d)\right)^*$.

We need to use that $\mathbb{R}^d$ is $\sigma$-finite,

namely $\exists \{\Omega_n\}_{n=1}^{\infty}$ bounded, disjoint set s.t.

$$\mathbb{R}^d = \bigcup_{n=1}^{\infty} \Omega_n \quad (\text{e.g } \Omega_n = \text{cube})$$

For every $n \geq 1$, consider the mapping

$$f \in L^2(\Omega_n) \longrightarrow T(\mathbb{1}_{\Omega_n} f)$$

$$\mathbb{1}_{\Omega_n} f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$$

linear & continuous from $L^2(\Omega_n) \to \mathbb{C}$.

By Riesz representation theorem for $L^2(\Omega_n)$, $\exists v_n \in L^2(\Omega_n)$ s.t.

$$(*) \quad T(\mathbb{1}_{\Omega_n} f) = \int_{\Omega_n} v_n f \quad , \forall f \in L^2(\Omega_n).$$

Define: $u = \sum_{n=1}^{\infty} \mathbb{1}_{\Omega_n} v_n$.

Why $u \in L^\infty(\mathbb{R}^d)$? We prove that

$$\|v_n\|_{L^\infty(\Omega_n)} = \|u\|_{L^\infty(\Omega_n)} \leq \|T\|_{(L^1)^*} \quad , \forall n \geq 1.$$

Using (*) with $f = \overline{v_n}$, we have:

$$\int_{\Omega_n} |v_n|^2 = \int_{\Omega_n} v_n \overline{v_n} = T\left(\mathbb{1}_{\Omega_n} \overline{v_n}\right)$$

$$\leq \|T\|_{(L^1)^*} \left\| \mathbb{1}_{\Omega_n} \overline{v_n} \right\|_{L^1(\mathbb{R}^d)}$$

$$= \|T\|_{(L^1)^*} \int_{\Omega_n} |v_n|$$

This can be improved by introduce the set

$$A = \left\{ x \in \Omega_n : |v_n(x)| > \|T\|_{(L^1)^*} \right\}$$

and use (*) with $f = \mathbb{1}_A \overline{v_n}$. Then:

$$\int_A |v_n|^2 \leq \|T\|_{(L^1)^*} \int_A |v_n|$$

$$\Rightarrow \int_A \underbrace{\left( |v_n|^2 - \|T\|_{(L^1)^*} |v_n| \right)}_{> 0 \text{ on } A} \leq 0$$

$$\Rightarrow \mu(A) = 0 \Rightarrow |v_n(x)| \leq \|T\|_{(L^1)^*} \text{ a.e.}$$

Thus we conclude that

$$\|v_n\|_{L^\infty} \leq \|T\|_{(L^p)^*}, \quad \forall n \geq 1$$

$$\Rightarrow \|u\|_{L^\infty} \leq \|T\|_{(L^p)^*}.$$

So far, we know that $\forall f \in L^2(\mathbb{R}^d)$

$$T(\mathbb{1}_{\Omega_n} f) = \int_{\Omega_n} v_n f = \int_{\mathbb{R}^d} u(\mathbb{1}_{\Omega_n} f)$$

$$\Rightarrow T(\mathbb{1}_{\Omega_n} f) = \int_{\mathbb{R}^d} u(\mathbb{1}_{\Omega_n} f), \quad \forall f \in L^1(\mathbb{R}^d)$$

by a density argument. Thus:

$$T(f) = T\left(\sum_n \mathbb{1}_{\Omega_n} f\right) = \sum_n \int_{\mathbb{R}^d} u(\mathbb{1}_{\Omega_n} f)$$

$$= \int_{\mathbb{R}^d} u\underbrace{\left(\sum_n \mathbb{1}_{\Omega_n} f\right)}_{=1} = \int_{\mathbb{R}^d} uf$$

by Dominated convergence, $\forall f \in L^1(\mathbb{R}^d)$.

Finally, the choice of $u$ is unique since

if $T(\varphi) = \int_{\mathbb{R}^d} u\varphi = \int_{\mathbb{R}^d} \tilde{u}\varphi$, $\forall \varphi \in \mathcal{L}^1$

$$\Rightarrow \int_{\mathbb{R}^d} (u - \tilde{u})\varphi = 0, \ \forall \varphi \in \mathcal{L}^1$$

$$\Rightarrow u - \tilde{u} = 0$$

by the fundamental lemma of calculus.

Remark: We prove $\|u\|_{\mathcal{L}^\infty} \leq \|T\|_{(\mathcal{L}^1)^*}$.

But $|T(\varphi)| = \left| \int_{\mathbb{R}^d} u\varphi \right| \leq \|u\|_{\mathcal{L}^\infty} \|\varphi\|_{\mathcal{L}^1}$

$$\Rightarrow \|T\|_{(\mathcal{L}^1)^*} \leq \|u\|_{\mathcal{L}^\infty}.$$

Thus: $\|T\|_{(\mathcal{L}^1)^*} = \|u\|_{\mathcal{L}^\infty}$.

i.e. $(\mathcal{L}^1)^* = \mathcal{L}^\infty$ by an isometry.

Remark: $(\mathcal{L}^1(\Omega))^* = \mathcal{L}^\infty(\Omega)$, $\forall$ $\Omega$ open set in $\mathbb{R}^d$. □

**Theorem.** $\left(L^\infty(\mathbb{R}^d)\right)^* \supsetneq L^1(\mathbb{R}^d)$

and $L^\infty(\mathbb{R}^d)$ is not separable.

The same holds for $L^\infty(\Omega)$ with $\Omega^{\text{open}} \subset \mathbb{R}^d$.

**Proof.** **Step 1:** $L^1(\mathbb{R}^d) \subset \left(L^\infty(\mathbb{R}^d)\right)^*$.

Indeed, $\forall\, u \in L^1(\mathbb{R}^d)$, define $T_u \in \left(L^\infty(\mathbb{R}^d)\right)^*$ by

$$T_u(g) = \int_{\mathbb{R}^d} ug \quad , \forall\, g \in L^\infty(\mathbb{R}^d).$$

$$\rightsquigarrow \quad \|u\|_{L^1} = \|T_u\|_{(L^\infty)^*}.$$

**Step 2:** We prove that $L^1(\mathbb{R}^d) \subsetneq \left(L^\infty(\mathbb{R}^d)\right)^*$.

Define $T \in \left(L^\infty(\mathbb{R}^d)\right)^*$ as follows:

$$T: C_c(\mathbb{R}^d) \to \mathbb{C}$$

$$g \longmapsto T(g) = g(0)$$

$\Rightarrow$ $T$ is linear & continuous, i.e.

$$|T(g)| \leq \|g\|_{L^\infty}.$$

Since $C_c(\mathbb{R}^d)$ is a subspace of $L^\infty(\mathbb{R}^d)$, we can extend $T$ to be a function from $L^\infty(\mathbb{R}^d) \to \mathbb{C}$ which is linear & continuous, by the Hahn-Banach theorem.

Thus $\exists\, T \in (L^\infty(\mathbb{R}^d))^*$ s.t.

$$T(f) = f(0), \quad \forall f \in C_c(\mathbb{R}^d).$$

Now we prove that $\nexists\, u \in L^1(\mathbb{R}^d)$ s.t.

$$T(f) = \int uf, \quad \forall f \in L^\infty(\mathbb{R}^d).$$

By contradiction, assume $\exists\, u \in L^1(\mathbb{R}^d)$ s.t.

$$\int_{\mathbb{R}^d} uf = T(f) = f(0), \quad \forall f \in C_c(\mathbb{R}^d).$$

This implies that

$$\int_{\mathbb{R}^d} u\chi f = (\chi f)(0) = 0, \quad \forall f \in C_c(\mathbb{R}^d)$$

$$\forall \chi \in C_c(\mathbb{R}^d),\ \chi(0) = 0$$

By the fundamental lemma of Calculus,

$$\int_{\mathbb{R}^d} (u\chi) f = 0, \ \forall f \in C_c(\mathbb{R}^d) \Rightarrow u\chi = 0 \text{ a.e.}$$

From $u\chi = 0$ a.e. $\forall \chi \in C_c(\mathbb{R}^d)$ s.t. $\chi(0) = 0$

$\Rightarrow u = 0$ a.e. $\Rightarrow$ by the choice of $u$

$$0 = \int uf = f(0), \quad \forall f \in C_c(\mathbb{R}^d)$$

We can choose $f \in C_c(\mathbb{R}^d)$ s.t. $f(0) \neq 0$.

$\rightsquigarrow$ contradiction.

Step 3. We prove that $L^\infty(\mathbb{R}^d)$ is not separable.

Claim: $\exists$ uncountable subsets $\{\Omega_i\}_{i\in I}$ of $\mathbb{R}^d$ s.t. $\begin{cases} \Omega_i \text{ is measurable } \forall i \in I \\ \| \mathbb{1}_{\Omega_i} - \mathbb{1}_{\Omega_j} \|_{L^\infty} = 1 \text{ if } i \neq j. \end{cases}$

In fact, we can take $\Omega_i = B_{\mathbb{R}^d}(x_i, r_i)$, the balls in $\mathbb{R}^d$.

Now, $\forall i \in I$, define the set

$$\mathcal{O}_i = \{ f \in L^\infty(\mathbb{R}^d) : \| f - \mathbb{1}_{\Omega_i} \|_{L^\infty} < \frac{1}{3} \}$$
$$= B(\mathbb{1}_{\Omega_i}, \tfrac{1}{3}) \subset L^\infty(\mathbb{R}^d).$$

Then clearly

$$\begin{cases} \mathcal{O}_i \text{ is open, } \forall i \in I \\ \mathcal{O}_i \cap \mathcal{O}_j = \emptyset, \ \forall i \neq j \end{cases}$$

In fact, if $\mathcal{O}_i \cap \mathcal{O}_j \ni f$ for $i \neq j$

$\Rightarrow \| f - \mathbb{1}_{\Omega_i} \| < 1/3$ and $\| f - \mathbb{1}_{\Omega_j} \| < 1/3$

$\Rightarrow \| \mathbb{1}_{\Omega_i} - \mathbb{1}_{\Omega_j} \| < 2/3$ by the triangle ineq

$\nu$ contradiction to the choice of $\Omega_i$.

We conclude that $L^\infty(\mathbb{R}^d)$ is not separable.

Assume by contradiction that $L^\infty(\mathbb{R}^d)$ is separable, i.e. $\exists$ countable set $\{f_n\}_{n=1}^\infty$ s.t. it is dense in $L^\infty(\mathbb{R}^d)$.

Then $\forall i \in I$, since $\mathcal{O}_i$ is open in $L^\infty(\mathbb{R}^d)$ and $\{f_n\}_{n=1}^\infty$ is dense in $L^\infty(\mathbb{R}^d) \Rightarrow \exists m = m_i$ s.t. $f_{m_i} \in \mathcal{O}_i$. Since $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ $\forall i \neq j \Rightarrow m_i \neq m_j$, $\forall i \neq j$.

Thus $\{\mathcal{O}_i\}_{i\in I} \hookrightarrow \{f_{m_i}\}_{i\in I}$ where $\{f_{m_i}\}_{i\in I} \subset \{f_n\}_{n=1}^\infty$ is countable.

$\Rightarrow \{\mathcal{O}_i\}_{i\in I}$ is countable, but it is a contradiction.

Thus $L^\infty(\mathbb{R}^d)$ is <u>not</u> separable. $\square$

In summary, if $\Omega$ open set in $\mathbb{R}^d$, then:

| $L^p(\Omega)$ | reflexive | dual space | separable |
|---|---|---|---|
| $1 < p < \infty$ | Yes | $L^{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$ | Yes |
| $p = 1$ | No | $L^\infty$ | Yes |
| $p = \infty$ | No | bigger $L^1$ | No |

Note that $\ell^p(\mathbb{N})$ can be interpretted as $L^p(\Omega,\mu)$ where $\Omega = \mathbb{N}$ and $\mu$ the counting measure, i.e.

$$\mu(A) = |A| = \#\text{ elements of } A.$$

This holds for all $1 \leq p \leq \infty$. As consequence, we have the following properties:

- $\ell^p(\mathbb{N})$ is a Banach space $\forall\, 1 \leq p \leq \infty$.
- If $1 < p < \infty$, then $\ell^p(\mathbb{N})$ is uniformly convex and hence it is reflexive. In fact,

$$(\ell^p(\mathbb{N}))^* = \ell^{p'}(\mathbb{N}), \quad \frac{1}{p}+\frac{1}{p'} = 1.$$

- $(\ell^1(\mathbb{N}))^* = \ell^\infty(\mathbb{N})$
- $(\ell^\infty(\mathbb{N})) \supsetneq \ell^1(\mathbb{N})$ & $\ell^\infty(\mathbb{N})$ not separable

One remarkable difference from $\ell^p$ to $L^p$ is:

**Theorem** (Schur) In $\ell^1(\mathbb{N})$, the weak c.v. implies the strong c.v. Namely, if

$$f_n \to f \text{ weakly in } \ell^1(\mathbb{N}),$$

then $f_n \to f$ strongly in $\ell^1(\mathbb{N})$.

Proof: WLOG, assume $f_n \to 0$ and prove that $f_n \to 0$ in $\ell^1(\mathbb{N})$. By contradiction, assume $f_n \not\to 0$ in $\ell^1(\mathbb{N})$. Up to a subsequence, we can assume that $\|f_n\|_{\ell^1} \geq \varepsilon$, $\forall n = 1, 2, \dots$

Let us write $f_n = \left(f_n(1), f_n(2), \dots\right) \in \ell^1(\mathbb{N})$

$f_n(1) \to \mathbb{C}$, $f_n(2) \to \mathbb{C}$

The assumption $f_n \to 0 \Rightarrow \forall k \geq 1$,

$$\lim_{n \to \infty} f_n(k) \to 0.$$

(This is because the mapping $\forall k \geq 1$

$f = (f(1), f(2) \dots) \longmapsto f(k)$ is linear & continuous)

We construct $x = (x(1), x(2), \dots) \in \ell^\infty(\mathbb{N}) = (\ell^1)^*$ and a subsequence $\{f_{n_k}\}_{k\geq 1} \subset \{f_n\}$ s.t.

$$\langle x, f_{n_k} \rangle = \sum_{\ell=1}^{\infty} x(\ell) f_{n_k}(\ell) \geq \frac{\varepsilon}{3}, \quad \forall k \geq 1.$$

which is a contradiction to $f_{n_k} \to 0$ in $\ell^1(\mathbb{N})$.

We will take $|x(\ell)| = 1, \forall \ell \geq 1$.

Step 1: Since $f_1 = (f_1(\ell))_{\ell \geq 1} \in \ell^1(\mathbb{N})$

$$\Rightarrow \sum_{\ell \geq 1} |f_1(\ell)| < \infty$$

$$\Rightarrow \exists n_1 \geq 0 \text{ s.t. } \sum_{\ell > n_1} |f_1(\ell)| < \frac{\varepsilon}{3}.$$

Define $x(\ell) = \operatorname{sign}(f_1(\ell)), \forall \ell \leq n_1$

$$\text{s.t. } \begin{cases} x(\ell) f_1(\ell) = |f_1(\ell)|. \\ |x(\ell)| = 1. \end{cases}$$

Thus:

$$\langle x, f_1 \rangle = \sum_{\ell=1}^{\infty} x(\ell) f_1(\ell) = \sum_{\ell \leq n_1} + \sum_{\ell > n_1}$$

$$\geq \sum_{\ell \leq n_1} |f_1(\ell)| - \sum_{\ell > n_1} |f_1(\ell)|$$

$$= \sum_{\ell=1}^{\infty} |f_1(\ell)| - 2 \sum_{\ell > n_1} |f_1(\ell)| \geq \varepsilon - \frac{2}{3}\varepsilon = \frac{\varepsilon}{3}$$

Thus we already define $x(\ell)$ for $\ell \le n_1$.

Consider $\{f_n\}_{n=1}^{\infty}$. Since $f_n \to 0 \Rightarrow \forall \ell$,

$$f_n(\ell) \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \sum_{\ell \le n_1} x(\ell) f_n(\ell) \to 0 \text{ as } n \to \infty$$

$\Rightarrow$ Up to a subsequence of $\{f_n\}$, we can assume

$$\sum_{\ell \le n_1} |f_n(\ell)| < \frac{\varepsilon}{6}, \quad \forall n \ge 1.$$

Then since $f_2 \in \ell^1(\mathbb{N})$

$$\Rightarrow \sum_{\ell \ge 1}^{\infty} |f_2(\ell)| < \infty$$

$$\Rightarrow \exists n_2 > n_1 \text{ s.t. } \sum_{\ell \ge n_2} |f_2(\ell)| < \frac{\varepsilon}{6}.$$

Take $\begin{cases} x(\ell) = \operatorname{sign} f_2(\ell) & \text{if } n_1 < \ell \le n_2 \\ \text{i.e. } |x(\ell)| = 1 \text{ s.t. } x(\ell) f_2(\ell) = |f_2(\ell)| \end{cases}$

Then:

$$\langle x, f_2 \rangle = \sum_{\ell \geq 1}^{\infty} x(\ell) f_2(\ell)$$

$$= \sum_{1 \leq \ell \leq n_1} + \sum_{n_1 < \ell \leq n_2} + \sum_{\ell > n_2}$$

$$\geq -\sum_{1 \leq \ell \leq n_1} |f_2(\ell)| + \sum_{n_1 < \ell \leq n_2} |f_2(\ell)| - \sum_{\ell > n_2} |f_2(\ell)|$$

$$= \sum_{\ell=1}^{\infty} |f_2(\ell)| - 2 \sum_{1 \leq \ell \leq n_1} |f_2(\ell)| - 2 \sum_{\ell > n_2} |f_2(\ell)|$$

$$\geq \varepsilon - 2 \cdot \frac{\varepsilon}{6} - 2 \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

By the same argument, we can define $x(\ell)$ with $n_2 < \ell \leq n_3$ where $n_3$ depends on $f_3$ and so on.

This gives $\{x(\ell)\}_{\ell \geq 1} \in \ell^\infty(\mathbb{N})$ since $|x(\ell)| = 1$, $\forall \ell \geq 1$ and

$\langle x, f_n \rangle \geq \frac{\varepsilon}{3}$, $\forall n \geq 1$ $\rightarrow$ contradiction to $f_n \to 0$.