Exercise sheet 4

Exercise 1.

1) Let A be a commutative ring and let $S \subset A$ and $T \subset A$ be multiplicative subsets of A. We assume that $T \subset S$. Show that there is a canonical morphism of (commutative) rings:

$$T^{-1}A \to S^{-1}A$$

Let $S' \subset T^{-1}A$ be the image of S through the above morphism. Show that the above morphism induces an isomorphism of (commutative) rings:

$$(S')^{-1}(T^{-1}A) \cong S^{-1}A$$

2) Let $\phi : A \to B$ be a morphism of commutative rings. Let $\mathcal{Q}$ be a prime ideal of B and let $\mathcal{P} \in Spec(A)$ be the inverse image of $\mathcal{Q}$ through ϕ. Sow that the obvious composition $A \to B \to B_{\mathcal{Q}}$ induces a canonical morphism of (commutative) rings:

$$A_{\mathcal{P}} \to B_{\mathcal{Q}}$$

3) Let A be a commutative ring and let $S \subset A$ be a multiplicative subset. Let $\mathcal{Q}$ be a prime ideal of $S^{-1}A$ and let $\mathcal{P} \in Spec(A)$ be the inverse image of $\mathcal{Q}$ through $A \to S^{-1}A$. Show that the ring homomorphism $A \to S^{-1}A$ induces a isomorphism of (commutative) rings:

$$A_{\mathcal{P}} \cong (S^{-1}A)_{\mathcal{Q}}$$

Exercise 2.

Let $\phi : A \to B$ be a morphism of (commutative) rings. If A and B are both local rings, one says that the morphism ϕ is a local morphism of local rings if the inverse image of the maximal ideal in B is the maximal ideal in A.

1) Let $\phi : A \to B$ be a morphism of commutative rings. Let $\mathcal{Q}$ be a prime ideal of B and let $\mathcal{P} \in Spec(A)$ be the inverse image of $\mathcal{Q}$ through ϕ. Sow that canonical morphism of (commutative) rings:

$$A_{\mathcal{P}} \to B_{\mathcal{Q}}$$

obtained in question 2) of the previous exercise is local.

2) Give an example of morphism of (commutative) rings $\phi : A \to B$ with A and B both local rings but which is not a local morphism.

Exercise 3.

Let A be a commutative ring.

Let $f \in A$ and $g \in A$ be elements such that one has the equality of open subsets of $Spec(A)$:

$$D(f) = D(g)$$

Show that there is a canonical isomorphism of A-algebras:

$$A_f \cong A_g$$

Exercise 4.

Let A be a commutative ring.

Show that the canonical morphism of rings

$$A \to \Pi_{\mathcal{M}} \, A_{\mathcal{M}}$$

is injective, where $\mathcal{M}$ runs in the set of maximal ideals of A.

[Hint: let $x \in A$ be in the kernel. To show $x = 0$, use the annihilator ideal $ann(x) := \{a \in A | a.x = 0\}$....]