Exercise sheet 3

Exercise 1.

Let X be a topological space. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on X.

1) Prove the following equivalence:

(for any $U \in Op(X)$, the map $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective) $\Leftrightarrow$ (for any $x \in X$, the induced map on the stalk $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective)

In that case one says that ϕ is monomorphism.

2) Let $\mathcal{F}$ be a sheaf of sets on X. Show that the canonical morphism of sheaves

$$\mathcal{F} \to \Pi_{x \in X} \ x_*(\mathcal{F}_x)$$

is a monomorphism in the sense of 1). [Hint: show that on each stalk it induces an injective map, but beware that the stalk of a product is not the product of the stalks in general!]

Exercise 2.

Let A be a commutative ring.

1) Let $I \subset A$ be an ideal. Show that the radical $\sqrt{I}$ of I defined as the set

$$\sqrt{I} = \{x \in A | \ \exists n \geq 1 \ / \ x^n \in I\}$$

is an ideal of A containing I.

2) Show that for any ideal I one has the equality (of closed subsets of $Spec(A)$):

$$V(I) = V(\sqrt{I})$$

3) Show that for any ideal I one has the equality in A:

$$\sqrt{I} = \bigcap_{\mathcal{P} \in V(I)} \mathcal{P}$$

4) One says that an ideal I of A is a radical ideal if $I = \sqrt{I}$. Exibit a bijection between the set of closed subsets of $Spec(A)$ and the set of radical ideals of A.

Exercise 3.

Let A be a commutative ring.

1) If A is an integral domain we let $\eta_A \in Spec(A)$ be the ideal 0 (which is a prime ideal). Show that the closure of $\{\eta_A\}$ is $Spec(A)$.

2) For $\mathcal{P} \in Spec(A)$ describe the closure of $\{\mathcal{P}\}$ in $Spec(A)$.

3) Give an example of A not being an integral domain, where there is a prime ideal $\eta \in Spec(A)$ such that the closure of $\{\eta\}$ is $Spec(A)$.

Exercise 4.

Let A be an integral domain with fraction field K.

1) Show that for any prime ideal $\mathcal{P} \in Spec(A)$ the canonical morphism of localised rings

$$A_{\mathcal{P}} \to K$$

is injective.

2) Let $x \in K$ be an element. Show that $\mathcal{I}_x = \{\alpha \in A | \alpha.x \in A\}$ is an ideal of A and that for $\mathcal{P}$ a prime ideal, if $x \in A_{\mathcal{P}} \subset K$ then $\mathcal{I}_x \not\subset \mathcal{P}$.

3) Show the equalities in K:

$$A = \bigcap_{\mathcal{P} \in Spec(A)} A_{\mathcal{P}} = \bigcap_{\mathcal{M} \in Spec_{max}(A)} A_{\mathcal{M}}$$