Exercise sheet 2

Exercise 1.

Give an example of a continuous map $f : Y \to X$ such that the direct image functor

$$f_* : Pre - Shv(Y) \to Pre - Shv(X)$$

doesn't commute to the sheafification functor.

[Hint: you might take Y very small]

Exercise 2.

Let $f : Y \to X$ be a continuous map which is local homeomorphism. Let $\Gamma_f \in Shv(X)$ be the corresponding sheaf of sets on X of local sections of f.

Show that for any $x \in X$, the stalk $(\Gamma_f)_x$ is naturally in bijection with $f^{-1}(x)$.

Deduce the construction of an homeomorphism (over X) of the form $Y \cong X_{\Gamma_f}$.

Exercise 3.

Let X be a topological space. Let S be a set, considered a topological space for the discrete topology (every subset is open).

Show that the sheaf sets $C^0(-, S) : U \mapsto C^0(U, S)$ ($C^0(U, S)$ is the set of continuous maps from U to S) is canonically isomorphic to the sheafification $\underline{S}$ of the constant presheaf $U \mapsto S$. [Hint: you may prove that the associated "étalé" space X_S associated to the constant presheaf S is $S \times X$]

Exercise 4.

Let $f : Y \to X$ be a continuous map. Let

$$f^{-1} : Shv(X) \to Shv(Y)$$

be the inverse image functor of sheaves of sets, left adjoint to the direct image functor

$$f_* : Shv(Y) \to Shv(X)$$

Describe f^{-1} in the following two cases:

1) the space Y is reduced to a single point $Y = \{y\}$.

2) f is the inclusion $j : U \subset X$ of an open subset of X.