Some multilinear algebra

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See also: Lang *Algebra*, van der Waerden *Algebra I*.

1 Multilinear maps

1.1 General multilinear maps

We work with vector spaces over a fixed field K.

Let $V_1, \dots, V_k, W$ be vector spaces. A map

$$V_1 \times \ldots \times V_k \xrightarrow{\mu} W$$

is called *multilinear* if it is linear in each variable, that is, if all maps $\mu(v_1, \ldots, v_{i-1}, \cdot, v_{i+1}, \ldots, v_k) : V_i \to W$ are linear. The multilinear maps form a vector space $\mathrm{Mult}(V_1, \ldots, V_k; W)$.

Multilinear maps in k variables are also called *k-linear.* The 1-linear maps are the linear maps, the 2-linear maps are the bilinear maps.

Multilinear maps are determined by their values on bases, and these values are independent of each other. More precisely, if $(e^{(i)}_{j_i} \,|\, j_i \in J_i)$ are bases of the V_i, then a multilinear map μ is determined by the values $\mu(e^{(1)}_{j_1}, \ldots, e^{(k)}_{j_k}) \in W$ for $(j_1, \ldots, j_k) \in J_1 \times \ldots \times J_k$, and these values can be arbitrary, i.e. for any vectors $w_{j_1 \ldots j_k} \in W$ there is a unique multilinear map μ with $\mu(e^{(1)}_{j_1}, \ldots, e^{(k)}_{j_k}) = w_{j_1 \ldots j_k}$. Indeed, for the values on general vectors $v_i = \sum_{j_i \in J_i} a_{ij_i} e^{(i)}_{j_i} \in V_i$, we obtain the representation

$$\mu(v_1, \ldots, v_k) = \sum_{j_1, \ldots, j_k} \Big(\prod_{i=1}^{k} a_{ij_i} \Big) \cdot \mu(e^{(1)}_{j_1}, \ldots, e^{(k)}_{j_k}). \tag{1.1}$$

In particular, if the dimensions of the vector spaces are finite, then

$$\dim \mathrm{Mult}(V_1, \ldots, V_k; W) = \Big(\prod_j \dim V_j \Big) \cdot \dim W.$$

Examples: Products in algebras. Composition $\mathrm{Hom}(U, V) \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W)$. Scalar produtcs, symplectic forms, volume forms, determinant. Natural pairing $V \times V^* \to K$.

1.2 The sign of a permutation

Let S_k denote the *symmetric group* of k symbols, realized as the group of bijective self-maps of the set $\{1, \ldots, k\}$. The *sign* of a permutation $\pi \in S_k$ is defined as

$$\mathrm{sgn}(\pi) = \prod_{1 \le i < j \le k} \frac{\pi(i) - \pi(j)}{i - j} \in \{\pm 1\}.$$

It counts the parity of the number of *inversions* of π, i.e. of pairs (i, j) such that $i < j$ and $\pi(i) > \pi(j)$. The sign is positive if and only if the number of inversions is even, and such permutations are called *even.* Transpositions are odd, and a permutation is even if and only if it can be written as the product of an even number of transpositions. The sign is multiplicative,

$$sgn(\pi\pi') = sgn(\pi)\,\mathrm{sgn}(\pi'),$$

i.e. the sign map

$$S_k \xrightarrow{\mathrm{sgn}} \{\pm 1\}$$

is a *homomorphism* of groups. Its kernel A_n is called the *alternating group.*

Briefly, the sign map is *characterized* as the unique homomorphism $S_k \to \{\pm 1\}$ which maps transpositions to -1. The remarkable fact is that such a homomorphism exists at all. A consequence is that, when representing a permutation as a product of transpositions, the *parity* of the number of factors is well-defined.

Remark. For $n \ge 5$ the alternating group A_n is non-abelian and *simple.*

1.3 Symmetric multilinear maps

We now consider multilinear maps whose variables take their values in the same vector space V. One calls such maps also multilinear maps *on* V (instead of on V^n). We abbreviate

$$\mathrm{Mult}_k(V;W) := \mathrm{Mult}(\underbrace{V,\dots,V}_{k};W)$$

and denote by $\mathrm{Mult}_k(V) := \mathrm{Mult}_k(V;K)$ the space of *k-linear forms* on V. Then $\mathrm{Mult}_1(V) = V^*$ and, by convention, $\mathrm{Mult}_0(V) = K$.

A k-linear map

$$V^k = \underbrace{V\times\ldots\times V}_{k} \xrightarrow{\mu} W$$

is called *symmetric* if

$$\mu(v_{\sigma(1)},\dots,v_{\sigma(k)}) = \mu(v_1,\dots,v_k) \tag{1.2}$$

for all permutations $\sigma\in S_k$ and all $v_i\in V$.

Remark. There is a natural action

$$S_k \curvearrowright \mathrm{Mult}_k(V;W) \tag{1.3}$$

of permutations on multilinear forms given by

$$(\sigma\mu)(v_1,\dots,v_k) = \mu(v_{\sigma(1)},\dots,v_{\sigma(k)}).$$

Indeed,

$$(\tau(\sigma\mu))(v_1,\dots) = (\sigma\mu)(v_{\tau(1)},\dots) = \mu(v_{\tau(\sigma(1))},\dots) = ((\tau\sigma)\mu)(v_1,\dots)$$

for $\sigma,\tau\in S_k$. Thus condition (1.2) can be rewritten as

$$\sigma\mu = \mu,$$

i.e. μ is symmetric if and only if it is a fixed point for the natural S_k-action (1.3).

We describe the *data* necessary to determine a symmetric multilinear map.

If $(e_i\,|\,i\in I)$ is a basis of V, then $\mu\in\mathrm{Mult}_k(V)$ is symmetric if and only if the values on basis vectors satisfy $\mu(e_{i_{\sigma(1)}},\dots,e_{i_{\sigma(k)}}) = \mu(e_{i_1},\dots,e_{i_k})$ for all $i_1,\dots,i_k\in I$ and $\sigma\in S_k$, compare the representation (1.1) of the values on general vectors. Hence, if I is equipped with a total ordering "$\prec$", then μ is deteremined by the values $\mu(e_{i_1},\dots,e_{i_k})$ for $i_1\preceq\ldots\preceq i_k$, and these values can be arbitrary.

If dimensions are finite, we conclude that

$$\dim\mathrm{Mult}_k^{sym}(V;W) = \binom{\dim V+k-1}{k}\cdot\dim W.$$

Further discussion: Polynomials and symmetric multilinear forms.

1.4 Alternating multilinear maps

We now consider a modified symmetry condition for multilinear maps, namely which is "twisted" by the signum homomorphism on permutations:

Definition (Antisymmetric). A map $\mu \in \mathrm{Mult}_k(V;W)$ is called *anti-* or *skew-symmetric* if

$$\mu(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma) \cdot \mu(v_1, \ldots, v_k) \tag{1.4}$$

for all $v_i \in V$ and permutations $\sigma \in S_k$.

In terms of the natural action $S_k \curvearrowright \mathrm{Mult}_k(V;W)$ we can rewrite (1.4) as

$$\sigma\mu = \mathrm{sgn}(\sigma) \cdot \mu, \tag{1.5}$$

for all $\sigma \in S_k$.

A closely related family of conditions will turn out to be more natural to work with:

Lemma 1.6. The following three conditions on a k-linear map $\mu \in \mathrm{Mult}_k(V;W)$ are equivalent:

(i) $\mu(v_1, \ldots, v_k) = 0$ whenever $v_i = v_{i+1}$ for some $1 \leq i < k$.

(ii) $\mu(v_1, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $1 \leq i < j \leq k$.

(iii) $\mu(v_1, \ldots, v_k) = 0$ whenever the v_i are linearly dependent.

They imply that μ is antisymmetric.

Proof. Obviously (iii)⇒(ii)⇒(i).

(i)⇒antisymmetric and (ii): Suppose that (i) holds. The computation

$$\beta(u,v) + \beta(v,u) = \beta(u+v, u+v) - \beta(u,u) - \beta(v,v)$$

for bilinear maps β shows that then (1.4) holds in the general k-linear case for the transpositions of pairs $(i, i{+}1)$ of adjacent numbers. Since these transpositions generate the group S_k, it follows that (1.4) holds for all permutations $\sigma \in S_k$,[1] that is, μ is antisymmetric. The antisymmetry together with (i) implies (ii).

(ii)⇒(iii): Suppose that the v_i are linearly dependent. In view of the antisymmetry (implied already by (i), as we just saw), we may assume that v_k is a linear combination of the other v_i, that is, $v_k = \sum_{i<k} a_i v_i$. Then $\mu(v_1, \ldots, v_k) = \sum_{i<k} a_i \mu(v_1, \ldots, v_{k-1}, v_i) = 0$ because of (ii). □

Definition (Alternating). A map $\mu \in \mathrm{Mult}_k(V;W)$ is called *alternating* if it satisfies (one of) the equivalent conditions (i-iii) of the lemma.

According to the lemma, *alternating multilinear maps are antisymmetric.*

If $\mathrm{char}\, K \neq 2$, then also the converse holds:[2]

[1] The permutations $\sigma \in S_k$, for which (1.4) holds, form a subgroup of S_k.

[2] The field K has *characteristic* $\neq 2$, if $2 := 1 + 1 \neq 0$ in K. In this case, 2 has a multiplicative inverse in K, i.e. one can *divide by 2* in K. On the other hand, if the field K has characteristic 2, i.e. if $2 = 0$ in K, then $-1 = 1$ and hence $-a = a$ for all $a \in K$, i.e. there are *no signs* in K.

Lemma. If $\operatorname{char} K \neq 2$, then antisymmetric multilinear maps are alternating.

Proof. It suffices to treat the bilinear case. An antisymmetric bilinear map β satisfies $\beta(v,v) = -\beta(v,v)$ for all $v \in V$, and hence $2\beta(v,v) = 0$. Dividing by 2 yields that β is alternating. $\square$

Moreover, in characteristic $\neq 2$ antisymmetry and symmetry are "transverse" conditions; the only multilinear maps, which are both symmetric and skew-symmetric, are the null-maps.

Remark. If $\operatorname{char} K \neq 2$, then bilinear forms can be uniquely decomposed as sums of symmetric and alternating ones, since

$$\beta(u,v) = \underbrace{\frac{\beta(u,v)+\beta(v,u)}{2}}_{\text{symmetric}} + \underbrace{\frac{\beta(u,v)+\beta(v,u)}{2}}_{\text{anti-symmetric}}.$$

If $\operatorname{char} K = 2$, then the relations between the conditions are different. Since there are no signs, skew-symmetry is the same as symmetry, whereas alternation is more restrictive if $k \geq 2$.

Since antisymmetry coincides with either alternation or symmetry, depending on the characteristic, alternation is the more interesting condition to consider. We denote by

$$\operatorname{Alt}_k(V;W) \subset \operatorname{Mult}_k(V;W)$$

the K-vectorspace of alternating multilinear maps, and we write $\operatorname{Alt}_k(V) := \operatorname{Alt}_k(V;K)$.

We now describe the *data* determining an alternating multilinear map.

Lemma 1.7. If $(e_i \,|\, i \in I)$ is a basis of V, then $\alpha \in \operatorname{Mult}_k(V;W)$ is alternating if and only if

(i) $\alpha(e_{i_1}, \dots, e_{i_k}) = 0$ if some of the e_{i_j} agree, and

(ii) $\alpha(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}) = \operatorname{sgn}(\sigma) \cdot \alpha(e_{i_1}, \dots, e_{i_k})$ for all $\sigma \in S_k$ if the e_{i_j} are pairwise different.

Proof. The conditions are obviously necessary.

To see that they are also sufficient, we first treat the case $k = 2$ of a bilinear form β. For a vector $v = \sum_i v_i e_i$, assumptions (i+ii) imply

$$\beta(v,v) = \sum_i v_i^2 \underbrace{\beta(e_i,e_i)}_{=0} + \sum_{i<j} v_i v_j \underbrace{(\beta(e_i,e_j) + \beta(e_j,e_i))}_{=0} = 0,$$

where we assume that I is equipped with a total ordering "$<$". Thus, β is alternating.

In the general k-linear case, it follows that the bilinear forms $\alpha(e_{i_1}, \dots, e_{i_{j-1}}, \cdot, \cdot, e_{i_{j+2}}, \dots, e_{i_k})$ for $1 \leq j < k$ are alternating, and consequently all bilinear forms $\alpha(v_1, \dots, v_{j-1}, \cdot, \cdot, v_{j+2}, \dots, v_k)$ since they are linear combinations of the former. Hence condition (i) of Lemma 1.6 is satisfied and α is alternating. $\square$

Corollary. *If I is equipped with a total ordering "$<$", then a map $\alpha \in \operatorname{Alt}_k(V;W)$ is determined by the values $\alpha(e_{i_1}, \dots, e_{i_k})$ for $i_1 < \dots < i_k$, and these values can be arbitrary.*

Thus, if dimensions are finite, then

$$\dim \operatorname{Alt}_k(V;W) = \binom{\dim V}{k} \cdot \dim W.$$

In particular, $\dim \operatorname{Alt}_k(V) = 0$ if $k > \dim V$ and $\dim \operatorname{Alt}_k(V;W) = \dim W$ if $k = \dim V$.

Corollary. *If* $\dim V = k$, *then* $\dim \operatorname{Alt}_k(V) = 1$, *i.e. there exists, up to scalar multiple, a unique non-zero alternating* k*-linear form.*

Moreover, if $0 \neq \alpha \in \operatorname{Alt}_k(V)$ *and* $(e_1, \ldots, e_k)$ *is a basis of* V, *then* $\alpha(e_1, \ldots, e_k) \neq 0$.

1.4.1 The determinant of a matrix

To represent general values of alternating multilinear maps in terms of the values on a basis, the key computation is the following. For vectors $v_j = \sum_{i \in I} a_{ji} e_i$, one obtains:

$$\begin{aligned}
\alpha(v_1, \ldots, v_k) &= \sum_{i_1, \ldots, i_k \in I} \alpha(a_{1i_1} e_{i_1}, \ldots, a_{ki_k} e_{i_k}) = \sum_{i_1 < \ldots < i_k} \sum_{\sigma \in S_k} \underbrace{\alpha(a_{1i_{\sigma(1)}} e_{i_{\sigma(1)}}, \ldots, a_{ki_{\sigma(k)}} e_{i_{\sigma(k)}})}_{(\prod_{j=1}^k a_{ji_{\sigma(j)}}) \cdot \alpha(e_{i_{\sigma(1)}}, \ldots, e_{i_{\sigma(k)}})} \\
&= \sum_{i_1 < \ldots < i_k} \Big(\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \prod_{j=1}^k a_{ji_{\sigma(j)}} \Big) \cdot \alpha(e_{i_1}, \ldots, e_{i_k})
\end{aligned}$$

The coefficients appearing in this formula are all derived from the same building block:

Definition (Determinant of a matrix). The *determinant* of a matrix $(a_{ij}) \in K^{k \times k}$ is defined as the quantity

$$\det(a_{ij}) := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^k a_{i\sigma(i)} \in K. \tag{1.8}$$

Note that

$$\det(a_{ij})_{i,j} = \sum_{\sigma \in S_k} \underbrace{\operatorname{sgn}(\sigma)}_{=\operatorname{sgn}(\sigma^{-1})} \cdot \prod_{i=1}^k a_{\sigma^{-1}(i)i} = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^k a_{\sigma(i)i} = \det(a_{ji})_{i,j}, \tag{1.9}$$

i.e. the determinant of a matrix equals the determinant of its *transpose*.

Matrices may be regarded as tuples of (column or row) vectors, and accordingly the determinant as a function in several vector variables. As such, the determinant of a $k \times k$-matrix is the up to scalar multiple unique alternating k-linear form on K^k. Indeed, the last corollary, the subsequent computation and (1.9) imply the following *characterization*:

Theorem 1.10. *The function* $\det : K^{k \times k} \to K$ *given by* (1.8) *is*

(i) alternating k*-linear in the columns,*

(i') alternating k*-linear in the rows,*

(ii) normalized by $\det E = 1$, *where* $E \in K^{k \times k}$ *denotes the identity matrix.*

It is uniquely determined by properties (i) and (ii), and also by (i') and (ii).

Formula (1.8) for the determinant is called the *Leibniz formula*.

Closely related to it is the *Laplace expansion* of the determinant. We denote by M_{rs} the matrix obtained from (a_{ij}) by cancelling the r-th row and the s-th column. Due to the Leibniz formula and the antisymmetry of the determinant in columns and rows, we have that $\det M_{rs} = (-1)^{r+s} \cdot \det M'_{rs}$, where M'_{rs} denotes the matrix obtained from (a_{ij}) by replacing all entries in the r-th row and the s-th column by 0, except a_{rs} which is replaced by 1. The linearity of the determinant in the r-th row yields the expansion

$$\det(a_{ij}) = \sum_{s=1}^{k} a_{rs} \det M'_{rs} = \sum_{s=1}^{k} (-1)^{r+s} a_{rs} \det M_{rs},$$

and similarly the linearity in the s-th column the expansion

$$\det(a_{ij}) = \sum_{r=1}^{k} (-1)^{r+s} a_{rs} \det M_{rs}.$$

1.4.2 The determinant of an endomorphism

A linear map of vector spaces $L : U \to V$ induces natural linear *pull-back* maps

$$\mathrm{Mult}_k(V) \xrightarrow{L^*} \mathrm{Mult}_k(U)$$

of multilinear forms,

$$(L^*\mu)(u_1, \ldots, u_k) = \mu(Lu_1, \ldots, Lu_k).$$

Alternating forms pull back to alternating ones, $L^*(\mathrm{Alt}_k(V)) \subset \mathrm{Alt}_k(U)$.

Suppose now that $\dim V = k$ and $L \in \mathrm{End}\, V$ is an *endomorphism*. Then $\mathrm{Alt}_k(V)$ is a one-dim vector space and the induced endomorphism L^* of $\mathrm{Alt}_k(V)$ must be the multiplication by a scalar. This factor is given by the determinant of a matrix for L. Indeed, if (e_i) is a basis of V and (a_{ij}) the matrix of L relative to this basis, then $Le_j = \sum_i a_{ij} e_i$ and, according to our computation above and also invoking (1.9), we have for $\alpha \in \mathrm{Alt}_k(V)$ that

$$\alpha(Le_1, \ldots, Le_k) = \det(a_{ij}) \cdot \alpha(e_1, \ldots, e_k), \tag{1.11}$$

that is,

$$L^*\alpha = \det(a_{ij}) \cdot \alpha.$$

In particular, $\det(a_{ij})$ is *independent* of the chosen basis and it makes sense to define:

Definition (Determinant of an endomorphism). The *determinant* of an *endomorphism* of a finite-dim vector space is defined as the determinant of its matrix relative to a basis.

The induced endomorphism L^* of $\mathrm{Alt}_k(V)$ can then be written as

$$L^*|_{\mathrm{Alt}_k(V)} = \det L \cdot \mathrm{id}_{\mathrm{Alt}_k(V)}, \tag{1.12}$$

which may be taken as an alternative direct definition of the determinant of an endomorphism.

For endomorphisms $A, B \in \mathrm{End}(V)$ it holds by the contravariance of pull-back that

$$(AB)^* = B^* A^*.$$

With (1.12) this immediately implies the *multiplication law* for determinants

$$\det(AB) = \det A \cdot \det B. \tag{1.13}$$

It amounts to the fact that the natural map

$$\mathrm{Aut}(V) \xrightarrow{\det} K^* \tag{1.14}$$

from the group of linear automorphisms of V to the multiplicative group of the field K given by the determinant is a *group homomorphism.* In particular, for $V = K^k$ this means that the map $\mathrm{GL}(k, K) \xrightarrow{\det} K^*$ given by the determinant of matrices is a group homomorphism.

1.4.3 Orientation

We now do geometry and work over the field $K = \mathbb{R}$. Let V be a n-dim vector space.

The most intuitive description of orientations is in terms of *bases.*

Suppose first that $n \geq 1$. Let $\mathcal{B}(V)$ denote the space of ordered bases $e = (e_1, \ldots, e_n)$ of V. It is a dense open subset of V^n.

Theorem 1.15. *$\mathcal{B}(V)$ has two path components. More precisely:*

Two ordered bases e and e' lie in the same path component of $\mathcal{B}(V)$ if and only if for one and hence every $0 \neq \alpha \in \mathrm{Alt}_n(V)$ the values $\alpha(e_1, \ldots, e_n)$ and $\alpha(e'_1, \ldots, e'_n)$ have the same sign.

Proof. A form $0 \neq \alpha \in \mathrm{Alt}_n(V)$ yields a surjective continuous map $\mathcal{B}(V) \to \mathbb{R}^*$ by evaluating it on bases, i.e. sending $(e_1, \ldots, e_n) \mapsto \alpha(e_1, \ldots, e_n)$. Hence, $\mathcal{B}(V)$ is not path connected.

To see that there are at most two path components, we note that any ordered basis e can be continuously deformed (a continuous deformation of bases being a continuous path in $\mathcal{B}(V)$) by shearings and stretchings (as in the proof of Lemma 1.18) to a permutation of any other ordered basis e'. Moreover, by rotations we can continuously deform e' to $(\ldots, e'_{i-1}, -e'_{i+1}, e'_i, e'_{i+2}, \ldots)$ for any $1 \leq i < n$. Hence, e can be deformed to one of the two bases $(\pm e'_1, e'_2 \ldots, e'_n)$. □

Definition (Orientation). An *orientation* of V is a path component of $\mathcal{B}(V)$.

In dimension $n = 0$, one defines an orientation of a trivial vector space $V = \{0\}$ as a choice of sign $\pm$. In particular, in this case there is the *natural* orientation $+$.

Thus, a finite-dim real vector space has *two* orientations.

If an orientation has been chosen, one calls the ordered bases belonging to this equivalence class *positively* oriented, the others *negatively* oriented, and the vector space *oriented.* An ordered basis determines an orientation, namely the component of $\mathcal{B}(V)$ containing it.

The *standard orientation* of $\mathbb{R}^n$ is determined by its standard basis $(e_1, \ldots, e_n)$.

One can also describe orientations in terms of top degree *alternating multilinear forms.* As a consequence of the second part of the theorem, a *volume form*, that is, a form $0 \neq \alpha \in \mathrm{Alt}_n(V)$ determines an orientation, namely by defining a basis e as positively oriented if

$$\alpha(e_1, \dots, e_n) > 0,$$

and all positive multiples of α yield the same orientation.

One can therefore alternatively define an orientation as a ray component of $\mathrm{Alt}_n(V) \setminus \{0\}$.

There is a natural simply transitive right action

$$\mathcal{B}(V) \curvearrowleft \mathrm{GL}(n, \mathbb{R}) \tag{1.16}$$

given by

$$e \cdot A = e' \qquad \text{with } e'_j = \sum_i a_{ij} e_i$$

for $A = (a_{ij})$. The group $\mathrm{GL}(n, \mathbb{R})$ decomposes as the disjoint union of the open subgroup $\mathrm{GL}_+(n, \mathbb{R})$ of matrices with positive determinant and its open coset $\mathrm{GL}_-(n, \mathbb{R})$ of matrices with negative determinant. (Recall that $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ is a homomorphism, cf (1.14).)

The above theorem implies a corresponding result for $\mathrm{GL}(n, \mathbb{R})$:

Corollary. *(i)* $\mathrm{GL}(n, \mathbb{R})$ *has two path components, namely* $\mathrm{GL}_\pm(n, \mathbb{R})$.

(ii) The path components of $\mathcal{B}(V)$ *are the* $\mathrm{GL}_+(n, \mathbb{R})$*-orbits for the action* (1.16).

Proof. The simple transitivity of the $\mathrm{GL}(n, \mathbb{R})$-action on $\mathcal{B}(V)$ yields that the orbit maps $o_A : \mathrm{GL}(n, \mathbb{R}) \to \mathcal{B}(V), A \mapsto e \cdot A$ are homeomorphisms. Thus, $\mathrm{GL}(n, \mathbb{R})$ is homeomorphic to $\mathcal{B}(V)$.

Since $\alpha(e \cdot A) = \det A \cdot \alpha(e)$ by (1.11), the subgroup $\mathrm{GL}_+(n, \mathbb{R})$ preserves the path components of $\mathcal{B}(V)$ and is therefore homeomorphic to both of them and in particular path connected. □

A linear isomorphism $L : V \to V'$ of n-dim vector spaces induces a homeomorphism $\mathcal{B}(V) \to \mathcal{B}(V')$ of spaces of ordered bases and a linear isomorphism $\mathrm{Alt}_n(V') \to \mathrm{Alt}_n(V)$ of lines of top-degree alternating multilinear forms. If V and V' are oriented, then L is called *orientation preserving* if it maps oriented bases to oriented bases, and orientation *reversing* otherwise.

A linear automorphism $L : V \to V$ preserves orientation if and only if $\det L > 0$, cf (1.12).

Complex vector spaces have *natural* orientations as real vector spaces. Indeed, as in the proof of the theorem one sees that, if W is a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} W = n$, then the space $\mathcal{B}_{\mathbb{C}}(W)$ of ordered complex bases is path connected. The image of the natural continuous embedding

$$\mathcal{B}_{\mathbb{C}}(W) \longrightarrow \mathcal{B}_{\mathbb{R}}(W), \quad (e_1, \dots, e_n) \mapsto (e_1, ie_1, \dots, e_n, ie_n)$$

is contained in a path component of $\mathcal{B}_{\mathbb{R}}(W)$ which one defines to be the natural orientation. Note that the group $GL(n, \mathbb{C})$ acts simply transitively on $\mathcal{B}_{\mathbb{C}}(W)$ and is hence *path connected.*

Remark. $\mathrm{GL}(n, \mathbb{R})$ continuously retracts to $\mathrm{O}(n)$, and $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{U}(n)$.

1.4.4 Determinant and volume

We keep working over the field $K = \mathbb{R}$.

Let $(V, \langle\cdot,\cdot\rangle)$ be a n-dim euclidean vector space. The scalar product induces a measurement not only of *lengths*, but also of *k-dim volumes* in any dimension $1 \le k \le n$. More precisely, it induces a natural k-dim Lebesgue measure on every k-dim linear or affine subspace.

For a k-tuple $(v_1, \ldots, v_k)$ of vectors in V, we denote by

$$\mathrm{vol}_k(v_1, \ldots, v_k)$$

the (non-oriented) k-dim volume of the parallelepiped $P(v_1, \ldots, v_k)$ spanned by the vectors v_i. Our aim is to give a concrete formula for the volume in terms of the scalar product.

The 1-dim volume is just the *length*, $\mathrm{vol}_1(v) = \|v\|$, equivalently, $\mathrm{vol}_1(v)^2 = \langle v, v\rangle$.

The 2-dim volume is the *area*. We compute the area of the parallelogram $P(u, v)$ as in elementary geometry as "base times height" and obtain using the Pythagorean theorem:

$$\mathrm{vol}_2(u,v)^2 = \|u\|^2 \cdot \Big(\|v\|^2 - \langle v, \frac{u}{\|u\|}\rangle^2\Big) = \|u\|^2 \cdot \|v\|^2 - \langle u, v\rangle^2 = \det\begin{pmatrix} \langle u,u\rangle & \langle u,v\rangle \\ \langle v,u\rangle & \langle v,v\rangle \end{pmatrix} \tag{1.17}$$

This computation can be generalized to arbitrary dimension by induction.

We take another approach and establish a close link between the volume function and alternating multilinear forms. It suffices to consider the top-dimensional case. Let (e_i) denote an ONB of V. One has the following *characterization* of the volume:

Lemma 1.18. The function vol_n is the *unique* function $V^n \to [0, \infty)$ with the properties:

(i) *symmetric*, i.e. $\mathrm{vol}_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \mathrm{vol}_n(v_1, \ldots, v_n)$ for $\sigma \in S_n$ and $v_i \in V$.

(ii) invariant under *shearing* (Cavalieri principle), i.e. it holds that

$$\mathrm{vol}_n(v_1 + w, v_2, \ldots, v_n) = \mathrm{vol}_n(v_1, v_2, \ldots, v_n)$$

for $v_i \in V$ and any vector w in the span of $v_2, \ldots, v_n$.

(iii) positively homogeneous under *stretching*, i.e.

$$\mathrm{vol}_n(a_1 v_1, \ldots, a_n v_n) = |a_1 \cdot \ldots \cdot a_n| \cdot \mathrm{vol}_n(v_1, \ldots, v_n)$$

for $v_i \in V$ and $a_i \in \mathbb{R}$.

(iv) *normalized* by $\mathrm{vol}_n(e_1, \ldots, e_n) = 1$.

Sketch of proof: The properties (i-iv) are clearly satisfied by vol_n. On the other hand, they determine vol_n, because any n-tuple (v_i) can be transformed by shearings and permutations in finitely many steps to a tuple $(a_i e_i)$ of multiples of the reference ONB vectors. □

These properties bear strong similarities to the properties of top-degree alternating multilinear forms and the determinant. In fact, for every form $0 \ne \alpha \in \mathrm{Alt}_n(V)$, its absolute value $|\alpha|$

satisfies properties (i-iii) of the lemma. Consequently, due to uniqueness of vol_n,

$$\mathrm{vol}_n(v_1,\dots,v_n) = \left|\frac{\alpha(v_1,\dots,v_n)}{\alpha(e_1,\dots,e_n)}\right|. \tag{1.19}$$

Note that, as a consequence, there exists a unique up to sign form $\omega \in \mathrm{Alt}_n(V)$ whose absolute value equals the volume,

$$\mathrm{vol}_n = |\omega|.$$

If V is in addition *oriented*, then there is a *unique* such form ω such that $\omega(e_1,\dots,e_n) = 1$ for every positively oriented ONB (e_i) of V, cf Theorem 1.15. It is called the *volume form* of the oriented euclidean vector space V.

Since endomorphisms act on top-degree alternating multilinear forms by multiplication with their determinant, see (1.12), we conclude from (1.19) that they act on volume by multiplication with the absolute value of the determinant,

$$L^* \mathrm{vol}_n = |\det L| \cdot \mathrm{vol}_n,$$

that is,

$$\mathrm{vol}_n(Lv_1,\dots,Lv_n) = |\det L| \cdot \mathrm{vol}_n(v_1,\dots,v_n)$$

for $L \in \mathrm{End}(V)$ and $v_i \in V$.

This provides a *geometric interpretation* for the *determinant* of an endomorphism, namely that the *volume distortion factor* of an endomorphism is given by the absolute value of its determinant. The determinant itself can be seen as a more refined *oriented* volume distortion which takes into account additionally whether the endomorphism preserves or changes the orientation of the vector space. The values of a volume form can be interpreted as *signed* or *oriented volumes* of parallelepipeds.

Returning to our task of generalizing (1.17) to arbitrary dimension, we consider the multilinear form

$$(u_1,\dots,u_n,v_1,\dots,v_n) \mapsto \det(\langle u_i, v_j\rangle) \tag{1.20}$$

on V. It is alternating in the u_i's as well as in the v_j's. Hence, by the uniqueness of alternating n-linear forms up to scalar multiple, it holds that

$$\det(\langle u_i, v_j\rangle) = \frac{\alpha(u_1,\dots,u_n)}{\alpha(e_1,\dots,e_n)} \cdot \det(\langle e_i, v_j\rangle) = \frac{\alpha(u_1,\dots,u_n)}{\alpha(e_1,\dots,e_n)} \cdot \frac{\alpha(v_1,\dots,v_n)}{\alpha(e_1,\dots,e_n)} \cdot \underbrace{\det(\langle e_i, e_j\rangle)}_{=1}.$$

In particular, with (1.19) we obtain the generalization

$$\mathrm{vol}_n(v_1,\dots,v_n)^2 = \det(\langle v_i, v_j\rangle) \tag{1.21}$$

of (1.17). The expression on the right-hand side is called the *Gram determinant.* Of course, this formula carries over to all intermediate dimensions $1 \le k \le n$.

2 Tensors

2.1 The tensor product of vector spaces

Can one make sense of *multiplying vectors* belonging to possibly different vector spaces with each other? This is certainly possibly. Given the freedom of constructions in mathematics, it is a matter of giving a suitable definition…

A *product*, in algebraic contexts, usually is a *bilinear map*, that is, a *distributive law* holds. Therefore, if U and V are K-vector spaces, by a product of vectors in U with vectors in V we just mean some bilinear map

$$U \times V \overset{\beta}{\to} W, \quad (u,v) \mapsto u \cdot v \tag{2.1}$$

with values in another vector space W.

Given such a product β, we can compose it with a linear map $l : W \to W'$ to obtain another product $\beta' = l \circ \beta$. If l has nontrivial kernel, then the new product β' is more "degenerate" than β in that there are additional (linear) relations between its values. We are looking for a *universal* product $\beta_{univ} : U \times V \to W_{univ}$ from which all other products can be derived by composing with a suitable linear map. It is thus natural to require that:

(i) W_{univ} is "no larger than necessary", i.e. W is the linear span of the values of the product.

(ii) W_{univ} is "as large as possible", i.e. there "no unnecessary relations" between the values.

One observes that, if (e_i) and (f_j) are bases of U and V, respectively, then the product of two general vectors $u = \sum_i u_i e_i$ and $v = \sum_j v_j f_j$ can be expressed due to bilinearity as a linear combination of the products of basis vectors,

$$u \cdot v = \sum_{i,j} u_i v_j \, e_i \cdot f_j . \tag{2.2}$$

If the product is universal, then (i) the products $e_i \cdot f_j$ should span and (ii) they should be linearly independent. This suggests to construct a universal product by choosing W_{univ} as a vector space with basis the set of symbols $e_i \cdot f_j$ and to then define the product map by (2.2). This is a possible approach, cf lemma 2.6 below, but we will give a "basis free" construction.

We first formulate the properties which we expect from a universal product, to be called a *tensor product*, namely that any product can be derived from it in a unique way:

Definition 2.3 (Tensor product). A *tensor product* of two vector spaces U and V is a vector space $U \otimes V$ together with a bilinear map

$$U \times V \xrightarrow{\otimes} U \otimes V$$

satisfying the following *universal property:* For every bilinear map β as in (2.1) there exists a unique linear map

$$U \otimes V \xrightarrow{\lambda} W$$

such that $\beta = \lambda \circ \otimes$.

In other words, the natural linear map

$$\mathrm{Hom}(U \otimes V, W) \longrightarrow \mathrm{Bil}(U, V; W), \quad \lambda \mapsto \lambda \circ \otimes \tag{2.4}$$

is an isomorphism. A tensor product thus serves as a device which *converts bilinear maps into linear ones.*

Often, one refers to the vector space $U \otimes V$ itself as the tensor product of U and V.

Theorem 2.5. *A tensor product exists and is unique up to natural isomorphism.*

We can therefore speak of *the* tensor product.

Proof. Uniqueness follows from the universal property (by a typical kind of argument referred to in category theory as "abstract nonsense"): Given two tensor products $U \times V \xrightarrow{\otimes} U \otimes V$ and $U \times V \xrightarrow{\widetilde{\otimes}} U \widetilde{\otimes} V$, there exist unique linear maps $U \otimes V \xrightarrow{\lambda} U \widetilde{\otimes} V$ and $U \widetilde{\otimes} V \xrightarrow{\widetilde{\lambda}} U \otimes V$ such that $\widetilde{\otimes} = \lambda \circ \otimes$ and $\otimes = \widetilde{\lambda} \circ \widetilde{\otimes}$. It follows that $\otimes = (\widetilde{\lambda} \circ \lambda) \circ \otimes$ and $\widetilde{\otimes} = (\lambda \circ \widetilde{\lambda}) \circ \widetilde{\otimes}$. The uniqueness part of the universal property then implies that $\widetilde{\lambda} \circ \lambda = \mathrm{id}_{U \otimes V}$ and $\lambda \circ \widetilde{\lambda} = \mathrm{id}_{U \widetilde{\otimes} V}$. Hence, between any two tensor products there is a natural isomorphism.

Existence. We start by forming a vector space with basis $U \times V$. Namely, let E be the vector space consisting, as a set, of all symbols

$$\sum_i a_i \, (u_i, v_i)$$

with $u_i \in U$, $v_i \in V$ and $a_i \in K$, the vector space operations (addition and scalar multiplication) defined in the obvious way. We denote by

$$U \times V \xrightarrow{\iota} E$$

the natural inclusion. It is only a map of sets and *not* bilinear. Accordingly, every bilinear map β as in (2.1) can be viewed as a map defined on the basis $\iota(U \times V)$ of E and there is a unique extension to a linear map $\hat{\lambda} : E \to W$ such that $\beta = \hat{\lambda} \circ \iota$. However, not for all linear maps λ the composition $\hat{\lambda} \circ \iota$ is bilinear, because ι is not bilinear.

In order to pass from ι to a bilinear map, we *impose relations* on the values by dividing out corresponding elements of E: Let $R \subset E$ be the linear subspace generated by the elements

$$(u_1, v) + (u_2, v) - (u_1 + u_2, v), \quad (u, v_1) + (u, v_2) - (u, v_1 + v_2),$$

$$(au, v) - a(u, v), \quad (u, av) - a(u, v)$$

for $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $a \in K$, and consider the quotient vector space

$$E/R =: U \otimes V.$$

The map ι descends to the map

$$U \times V \xrightarrow{\otimes} \underbrace{E/R}_{=: U \otimes V}, \quad (u, v) \mapsto (u, v) + R =: u \otimes v.$$

which is *bilinear* by construction. For instance:

$$\begin{aligned}(u_1+u_2)\otimes v &= (u_1+u_2,v)+R = (u_1,v)+(u_2,v)-\underbrace{\big((u_1,v)+(u_2,v)-(u_1+u_2,v)\big)}_{\in R}+R \\ &= \big((u_1,v)+R\big)+\big((u_2,v)+R\big) = u_1\otimes v+u_2\otimes v\end{aligned}$$

We must verify that the map $\otimes$ satisfies the desired universal property. Given a bilinear map β as in (2.1), let $\hat{\lambda}: E \to W$ with $\beta = \hat{\lambda}\circ\iota$ be the linear extension as above. Note that the bilinearity of β is equivalent to $\hat{\lambda}(R)=0$ since, for instance,

$$\hat{\lambda}((u_1,v)+(u_2,v)-(u_1+u_2,v)) = \beta(u_1,v)+\beta(u_2,v)-\beta(u_1+u_2,v).$$

Hence, $\hat{\lambda}$ descends to a linear map $U\otimes V \xrightarrow{\lambda} W$ with

$$\lambda(u\otimes v) = \hat{\lambda}\big((u,v)\big) = \beta(u,v).$$

The map λ is unique because already its lift $\hat{\lambda}$ is the unique extension of β. ☐

A more concrete idea of the tensor product is provided by the following fact.

Lemma 2.6. If $(e_i\,|\,i\in I)$ and $(f_j\,|\,j\in J)$ are bases of U and V, then $(e_i\otimes f_j\,|\,i\in I, j\in J)$ is a basis of $U\otimes V$. In particular, in the case of finite dimensions, it holds that

$$\dim(U\otimes V) = \dim U\cdot\dim V.$$

Proof. From the construction of the tensor product we know that the elements $u\otimes v$ generate $U\otimes V$. Consequently, also the elements $e_i\otimes f_j$ generate.

In order to show that they are linearly independent, we separate them by linear forms. Namely, we use the linear forms

$$U\otimes V \xrightarrow{\lambda_{kl}} K, \quad u\otimes v \mapsto e_k^*(u)f_l^*(v).$$

Their existence follows from the universal property of the tensor product; they are induced by the bilinear forms

$$U\times V \xrightarrow{\beta_{kl}} K, \quad (u,v)\mapsto e_k^*(u)f_l^*(v).$$

For a finite linear relation

$$\sum_{i,j} c_{ij}\, e_i\otimes f_j = 0$$

it follows by applying these linear forms that

$$c_{kl} = \lambda_{kl}\Big(\sum_{i,j} c_{ij}\, e_i\otimes f_j\Big) = 0.$$

Thus, the elements $e_i\otimes f_j$ form a basis. ☐

With respect to the bases, the tensor product is given by:

$$\Big(\sum_i a_i e_i\Big)\otimes\Big(\sum_j b_j f_j\Big) = \sum_{i,j} a_i b_j\, e_i\otimes f_j$$

When changing the bases of the factors, $\tilde{e}_k = \sum_i g_{ki} e_i$ and $\tilde{f}_l = \sum_j h_{lj} f_j$, the induced *change of basis* for the tensor product is given by:

$$\tilde{e}_k \otimes \tilde{f}_l = \sum_{i,j} g_{ki} h_{lj}\, e_i \otimes f_j$$

Remark 2.7. It follows that if $\{e_i : i \in I\}$ is a basis of U, then every element in $U \otimes V$ can be written as

$$\sum_i e_i \otimes v_i$$

with *unique* vectors $v_i \in V$.

Remark. (i) There is a natural isomorphism switching factors

$$U \otimes V \cong V \otimes U \tag{2.8}$$

which identifies the elements $u \otimes v$ with the elements $v \otimes u$. It is induced by the bilinear map $U \times V \to V \otimes U$ sending $(u, v) \mapsto v \otimes u$.

(ii) There are natural linear maps

$$U^* \otimes V \longrightarrow \mathrm{Hom}(U, V); \quad u^* \otimes v \mapsto u^*(\cdot)v \tag{2.9}$$

relating spaces of homomorphisms to tensor products. If $\dim U < \infty$, then these are *isomorphisms*, i.e. the space of homomorphisms can then be represented as a tensor product,

$$\mathrm{Hom}(U, V) \cong U^* \otimes V.$$

Indeed, if $(e_i \,|\, i \in I)$ is a basis of U, then $(e_i^* \,|\, i \in I)$ is a basis of U^* (due to finite dimensionality). Elements of $U^* \otimes V$ then have unique representations as sums $\sum_i e_i^* \otimes v_i$ and, correspondingly, elements of $\mathrm{Hom}(U, V)$ as sums $\sum_i e_i^*(\cdot) v_i$.

If also $\dim V < \infty$ and $\{f_j\}$ is a basis of V, then elements of $U^* \otimes V$ have unique representations of the form

$$\sum_{i,j} a_{ji}\, e_i^* \otimes f_j$$

and they correspond to the homomorphisms given with respect to the chosen bases by the matrices $(a_{ji})_{j,i}$.

In particular, if $V = U$ and $\dim U < \infty$, then we have the natural isomorphism

$$\mathrm{End}(U) \cong U^* \otimes U, \tag{2.10}$$

and id_U corresponds to the element $\sum_i e_i^* \otimes e_i$.

(ii') The homomorphism (2.9) is always injective, as one sees by restricting to finite dimensional subspaces of U.

By analogy with the twofold tensor product, the *multiple tensor product* of an arbitrary finite number of vector spaces $U_1, \ldots, U_n$ is a multilinear map

$$U_1 \times \ldots \times U_n \xrightarrow{\otimes} U_1 \otimes \ldots \otimes U_n$$

with the universal property that every multilinear map $U_1 \times \ldots \times U_n \to W$ is the composition of $\otimes$ with a unique linear map $U_1 \otimes \ldots \otimes U_n \to W$. Existence and uniqueness of the multiple tensor product are proven in the same way.

Hence, as in the case of two factors, cf (2.4), the natural linear map

$$\mathrm{Hom}(U_1 \otimes \ldots \otimes U_n, W) \xrightarrow{\circ\otimes} \mathrm{Mult}(U_1, \ldots, U_n; W) \tag{2.11}$$

given by precomposition with the tensor product is an isomorphism, and the tensor product can be viewed as a tool for *converting multilinear maps into linear ones.*

If $(e_{j_i} \,|\, j_i \in J_i)$ are bases of the U_i, then again $(e_{j_1} \otimes \ldots \otimes e_{j_n} \,|\, (j_1, \ldots, j_n) \in J_1 \times \ldots \times J_n)$ is a basis of $U_1 \otimes \ldots \otimes U_n$, cf. Lemma 2.6.

When building up multiple tensor products in several steps, the question of associativity arises, i.e. of the independence of the choice of partial steps.

Lemma 2.12 (Associativity). There are natural isomorphisms

$$(U_1 \otimes \ldots \otimes U_n) \otimes (U_{n+1} \otimes \ldots \otimes U_{n+m}) \longrightarrow U_1 \otimes \ldots \otimes U_{n+m} \tag{2.13}$$

mapping elements $(u_1 \otimes \ldots \otimes u_n) \otimes (u_{n+1} \otimes \ldots \otimes u_{n+m})$ to elements $u_1 \otimes \ldots \otimes u_{n+m}$.

Proof. The natural multilinear map

$$U_1 \times \ldots \times U_{n+m} \longrightarrow (U_1 \otimes \ldots \otimes U_n) \otimes (U_{n+1} \otimes \ldots \otimes U_{n+m})$$

induces a linear map

$$U_1 \otimes \ldots \otimes U_{n+m} \longrightarrow (U_1 \otimes \ldots \otimes U_n) \otimes (U_{n+1} \otimes \ldots \otimes U_{n+m})$$

mapping elements $u_1 \otimes \ldots \otimes u_{n+m}$ to elements $(u_1 \otimes \ldots \otimes u_n) \otimes (u_{n+1} \otimes \ldots \otimes u_{n+m})$. That it is an isomorphism, is seen by choosing bases. □

Remark 2.14. (i) *Permutations of factors.* Generalizing (2.8), for permutations $\sigma \in S_n$ there are the natural isomorphisms

$$U_1 \otimes \ldots \otimes U_n \cong U_{\sigma(1)} \otimes \ldots \otimes U_{\sigma(n)}$$

mapping elements $u_1 \otimes \ldots \otimes u_n$ to elements $u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(n)}$.

(ii) *Functoriality.* Linear maps $L_i : U_i \to V_i$ induce a linear map

$$L_1 \otimes \cdots \otimes L_n : U_1 \otimes \ldots \otimes U_n \longrightarrow V_1 \otimes \ldots \otimes V_n \tag{2.15}$$

mapping elements $u_1 \otimes \ldots \otimes u_n$ to elements $L_1(u_1) \otimes \ldots \otimes L_n(u_n)$. Indeed, it is induced by the multilinear map $U_1 \times \ldots \times U_n \longrightarrow V_1 \otimes \ldots \otimes V_n$ sending $(u_1, \ldots, u_n) \mapsto L_1(u_1) \otimes \ldots \otimes L_n(u_n)$

(iii) *Multilinear maps.* Generalizing (2.9), we have natural injective linear maps

$$U_1^* \otimes \ldots \otimes U_n^* \otimes V \longrightarrow \mathrm{Mult}(U_1, \ldots, U_n; V); u_1^* \otimes \ldots \otimes u_n^* \otimes v \mapsto \big((u_1, \ldots, u_n) \mapsto u_1^*(u_1) \cdot \ldots \cdot u_n^*(u_n) v\big)$$

which are isomorphisms if $\dim U_i < \infty$.

2.2 The tensor algebra of a vector space

Covariant tensors. Now we multiply vectors in a fixed vector space U with each other. For $m \in \mathbb{N}_0$, we call the m-fold tensor product

$$U^m = \underbrace{U \times \ldots \times U}_{m} \xrightarrow{\otimes} \underbrace{U \otimes \ldots \otimes U}_{m} =: \otimes^m U =: T_m U \tag{2.16}$$

of U with itself the m-th *tensor power* of U. Then $T_1 U = U$. By convention, $T_0 U := K$. The m-th tensor power of U can be regarded as the *universal m-multilinear map* from $U \times \ldots \times U$, universal in the sense that all others are obtained from it by postcomposition with a linear map. As in the case of the general tensor product, one often refers to the K-vector space $T^m U$ itself as the m-th tensor power of U.

We combine the tensor powers of the various degrees by forming the graded K-vector space

$$T_* U := \bigoplus_{m=0}^{\infty} T_m U$$

Due to the associativity of the tensor product, cf (2.13), there are natural bilinear maps

$$T_m U \times T_n U \xrightarrow{\otimes} T_{m+n} U$$

which, by bilinear extension, yield a *product*

$$T_* U \times T_* U \xrightarrow{\otimes} T_* U.$$

Equipped with this product, $T_* U$ becomes a *graded associative K-algebra with unity*, the *(covariant) tensor algebra* of U. Covariant, because the functor $U \mapsto T_* U$ from vector spaces to algebras is covariant, i.e. a linear map $L : U \to V$ induces a homomorphism of graded algebras

$$T_* U \xrightarrow{T_* L} T_* V$$

in the same direction. Indeed, cf (2.15), there are natural maps $L^{\otimes m} : \otimes^m U \to \otimes^m V$ sending $u_1 \otimes \ldots \otimes u_n \mapsto L_1(u_1) \otimes \ldots \otimes L_n(u_n)$. We obtain $T_* L$ by putting them together, $T_* L|_{\otimes^m U} = L^{\otimes m}$.

The covariant tensor algebra can also be characterized by a *universal property.* The algebra $T_* U$ is the "largest" associative K-algebra with unity "generated by U" in the sense that every linear map $L : U \to A$ to an associative K-algebra with unity uniquely extends to an algebra homomorphism

$$T_* U \to A.$$

It maps elements $u_1 \otimes \ldots \otimes u_n$ to $L(u_1) \cdot \ldots \cdot L(u_n)$ and can be obtained as the composition of the algebra homomorphism $L_* : T_* U \to T_* A$ induced by L with the natural "retraction" algebra endomorphism $T_* A \to A$ sending $a_1 \otimes \ldots \otimes a_n$ to $a_1 \cdot \ldots \cdot a_n$.

A basis $(e_i \,|\, i \in I)$ of U induces a (vector space) basis $(e_{i_1} \otimes \ldots \otimes e_{i_m} \,|\, i_1, \ldots, i_m \in I)$ of $T_m U$

Contravariant tensors. Multiplying covectors, that is, linear forms leads to the *contravariant* tensor algebra of U. It is defined as

$$T^* U := T_* U^*.$$

Contravariant, because a linear homomorphism $U \to V$ induces a linear homomorphism $V^* \to U^*$ of dual spaces and hence an algebra homomorphism $T^*V \to T^*U$ in the reverse direction.

Mixed tensors. Both types of tensors can be combined to *mixed* tensors with vector and covector components. One defines the tensor spaces

$$T_r^s U := T_r U \otimes T^s U$$

for $r, s \in \mathbb{N}_0$ with the convention $T_0^0 U = K$, and the *tensor algebra* of U as

$$T(U) := T_*^* U := \bigoplus_{r,s=0}^{\infty} T_r^s U.$$

Again, there is a natural product $\otimes$ on $T(U)$ satisfying

$$\begin{aligned}&(u_1 \otimes \ldots \otimes u_{r_1} \otimes u_1^* \otimes \ldots \otimes u_{s_1}^*) \otimes (v_1 \otimes \ldots \otimes v_{r_2} \otimes v_1^* \otimes \ldots \otimes v_{s_2}^*)\\ &= u_1 \otimes \ldots \otimes u_{r_1} \otimes v_1 \otimes \ldots \otimes v_{r_2} \otimes u_1^* \otimes \ldots \otimes u_{s_1}^* \otimes v_1^* \otimes \ldots \otimes v_{s_2}^*\end{aligned}$$

which makes $T(U)$ into a *bigraded associative algebra with unity.* There are natural inclusions $T_*U \subset T(U)$ and $T^*U \subset T(U)$ such that $T_r U = T_r^0 U$ and $T^s U = T_0^s U$. The elements of $T(U)$ are called *tensors*, and the elements of $T_r^s U$ are called *homogeneous* tensors of *type* (r, s). Among them, the tensors $u_1 \otimes \ldots \otimes u_r \otimes u_1^* \otimes \ldots \otimes u_s^*$ are called *decomposable* or *simple* or *monomials.* We note that not all homogeneous tensors are decomposable (by dimension reasons).

In low degrees, there are natural identifications of tensors with other linear algebra objects, for instance: Tensors of type $(1, 0)$ are vectors. Tensors of type $(0, 1)$ are covectors, i.e. linear forms. Tensors of type $(0, 2)$ are naturally identified with bilinear forms, compare (2.23) below, and tensors of type $(1, 1)$ with endomorphisms cf (2.10).

The natural linear inclusion

$$U \hookrightarrow U^{**}, \quad u \mapsto (u^* \mapsto u^*(u))$$

induces natural linear inclusions

$$T_r^s U \hookrightarrow T_s^r(U^*) \quad \text{and} \quad T(U) \hookrightarrow T(U^*) \tag{2.17}$$

of mixed tensor spaces and tensor algebras. If $\dim U < \infty$, these inclusions are *isomorphisms.*

Contractions. There are natural contraction maps between tensor spaces obtained by pairing vector with covector factors. The simplest and most basic such map is the linear form

$$T_1^1 U = U \otimes U^* \longrightarrow K, \quad u \otimes u^* \mapsto u^*(u) \tag{2.18}$$

induced by the natural non-degenerate bilinear pairing[3]

$$U \times U^* \longrightarrow \mathbb{R}, \quad (u, u^*) \mapsto u^*(u). \tag{2.19}$$

[3] A *pairing* of two vector spaces U and V is a bilinear map $\beta : U \times V \to K$. It induces linear maps $U \to V^*, u \mapsto \beta(u, \cdot)$ and $V \to U^*, v \mapsto \beta(\cdot, v)$, and can be recovered from either of them. The pairing β is called *non-degenerate* if for every $0 \neq u \in U$ exists $v \in V$ such that $\beta(u, v) \neq 0$ and for every $0 \neq v \in V$ exists $u \in U$ such that $\beta(u, v) \neq 0$. This is equivalent to the injectivity of both induced linear maps. If $\dim(U), \dim(V) < \infty$, the non-degeneracy of the pairing is equivalent to both linear maps being isomorphisms (implying $\dim(U) = \dim(V)$).

More generally, one can pair the i-th vector factor of a homogeneous tensor with the j-th covector factor and thus obtains the *contraction* homomorphisms

$$T_r^s U \xrightarrow{C_i^j} T_{r-1}^{s-1} U$$

for $1 \le i \le r$ and $1 \le j \le s$ satisfying

$$u_1 \otimes \ldots \otimes u_r \otimes u_1^* \otimes \ldots \otimes u_s^* \mapsto u_j^*(u_i) \cdot u_1 \otimes \ldots \widehat{u_i} \ldots \otimes u_r \otimes u_1^* \otimes \ldots \widehat{u_j^*} \ldots \otimes u_s^*,$$

the "hats" on two of the factors indicating that these factors are omitted. These homomorphisms are induced by the multilinear maps

$$(u_1, \ldots, u_r, u_1^*, \ldots, u_s^*) \mapsto u_j^*(u_i) \cdot u_1 \otimes \ldots \widehat{u_i} \ldots \otimes u_r \otimes u_1^* \otimes \ldots \widehat{u_j^*} \ldots \otimes u_s^*.$$

By composing partial contractions one obtains (various) total contractions

$$T_r^r U \longrightarrow K,$$

for instance,

$$C_1^1 \circ \ldots \circ C_r^r : u_1 \otimes \ldots \otimes u_r \otimes u_1^* \otimes \ldots \otimes u_r^* \mapsto \prod_i u_i^*(u_i). \tag{2.20}$$

If $\dim U < \infty$, then the contraction (2.18) is nothing but the *trace*

$$T_1^1 U = U \otimes U^* \overset{(2.10)}{\cong} \mathrm{End}(U) \xrightarrow{\mathrm{tr}} K.$$

Indeed, if (e_i) is a basis of U, then the endomorphism $A = \sum_{i,j} a_{ij}\, e_i \otimes e_j^*$ with matrix (a_{ij}) relative to this basis is mapped to $\sum_{i,j} a_{ij}\, e_j^*(e_i) = \sum_i a_{ii} = \mathrm{tr}\, A$. (This also shows that the expression $\sum_i a_{ii}$ is independent of the basis (e_i).)

Pairings and identifications. Generalizing (2.19), a natural non-degenerate[4] bilinear pairing

$$T_r^s U \times T_s^r U \longrightarrow K \tag{2.21}$$

is obtained by composing the tensor product $T_r^s U \times T_s^r U \overset{\otimes}{\to} T_{r+s}^{r+s} U$ with a total contraction. It induces natural linear inclusions

$$T_s^r U \hookrightarrow (T_r^s U)^*$$

If $\dim U < \infty$, then these are isomorphisms, and together with the (now) isomorphisms (2.17), we obtain the natural isomorphisms

$$(T_r^s U)^* \cong T_s^r U \cong T_r^s (U^*). \tag{2.22}$$

[4]The non-degeneracy can be verified by induction over the bigrade (r, s). The induction step follows from the observation: *If $\beta : U \times U' \to K$ and $\gamma : V \times V' \to K$ are non-degenerate pairings, then the induced pairing $\beta \otimes \gamma : (U \otimes V) \times (U' \otimes V') \to K$ is non-degenerate.* To verify this, note that a non-zero element in $\tau \in U \otimes V$ can be expressed as a finite sum $\sum_i u_i \otimes v_i$ with linearly independent $u_i \in U$ and non-zero $v_i \in V$, cf. Remark 2.7. In view of the inclusion $U \hookrightarrow U'^*$ induced by β, the linear forms $\beta(u_i, \cdot)$ on U' are linearly independent. Hence, there exists $u' \in U'$ so that $\beta(u_1, u') \neq 0$ and $\beta(u_i, u') = 0$ for $i \geq 2$. Furthermore, there exists $v' \in V'$ so that $\gamma(v_1, v') \neq 0$. Then $(\beta \otimes \gamma)(\tau, u' \otimes v') = \sum_i \beta(u_i, u')\gamma(v_i, v') \neq 0$.

Multilinear forms. Returning to the viewpoint of the tensor product as a device of converting multilinear maps into linear maps, we observe now that the isomorphism (2.22) allows us in the case $\dim U < \infty$ to identify multilinear forms with *contravariant tensors*,

$$\mathrm{Mult}_r(U) \cong (T_r U)^* \cong T^r U = T_r U^*, \tag{2.23}$$

a monomial $u_1^* \otimes \ldots \otimes u_r^* \in T_r U^*$ corresponding to the multilinear form $(u_1, \ldots, u_r) \mapsto \prod_i u_i^*(u_i)$, compare (2.20), i.e.

$$(u_1^* \otimes \ldots \otimes u_r^*)(u_1, \ldots, u_r) = \prod_i u_i^*(u_i). \tag{2.24}$$

If (e_i) is a basis of U, then $(e_{i_1}^* \otimes \ldots \otimes e_{i_r}^*)$ is a basis of $\mathrm{Mult}_r(U)$ under this identification. An r-linear form μ on U can be written with respect to this basis as

$$\mu = \sum_{i_1,\ldots,i_r} \underbrace{\mu(e_{i_1}, \ldots, e_{i_r})}_{\in K} e_{i_1}^* \otimes \ldots \otimes e_{i_r}^*.$$

For instance, a *bilinear form* $\beta \in \mathrm{Bil}(U) = \mathrm{Mult}_2(U)$, e.g. a *scalar product*, is identified with a type $(0,2)$ tensor and can be written as

$$\beta = \sum_{i,j} \underbrace{\beta(e_i, e_j)}_{\in K} e_i^* \otimes e_j^*. \tag{2.25}$$

Multiplying multilinear forms. The identification (2.23) of multilinear forms on U with contravariant tensors embeds the forms into the tensor algebra and thus gives rise to natural *product* maps

$$\mathrm{Mult}_k(U) \times \mathrm{Mult}_l(U) \xrightarrow{\otimes} \mathrm{Mult}_{k+l}(U) \tag{2.26}$$

such that

$$(\mu \otimes \nu)(u_1, \ldots, u_{k+l}) = \mu(u_1, \ldots, u_k) \cdot \nu(u_{k+1}, \ldots, u_{k+l})$$

for $\mu \in \mathrm{Mult}_k(U)$ and $\nu \in \mathrm{Mult}_l(U)$. The last formula is satisfied if μ, ν are monomials in view of (2.24), and by bilinear extension for arbitrary μ, ν.

Insertion. For a vector $u \in U$, there are natural linear maps

$$\mathrm{Mult}_k(U) \xrightarrow{i_u} \mathrm{Mult}_{k-1}(U)$$

for $k \geq 1$ given by inserting u for the first variable,

$$i_u \mu = \mu(u, \ldots).$$

For products $\mu \otimes \nu$ of multilinear forms $\mu \in \mathrm{Mult}_{k\geq 1}(U)$ and $\nu \in \mathrm{Mult}_{l \geq 0}(U)$ it holds that

$$i_u(\mu \otimes \nu) = (i_u \mu) \otimes \nu.$$

2.3 The exterior algebra of a vector space

Now we turn from general multilinear maps to *alternating* multilinear maps.

For a vector space V, we look, by analogy with the k-th tensor power (2.16), for the universal *alternating* k-fold product of vectors in V, or put differently, for the universal *alternating* k-multilinear map from V^k.

This amounts to *imposing additional relations* on the product by passing to a quotient of the tensor power. Indeed, a multilinear map arising as the postcomposition $V^k \xrightarrow{\otimes} T_kV \xrightarrow{\lambda} W$ of the tensor power with a linear map λ from T_kV is alternating if and only if λ annihilates the linear subspace $I_kV \subset T_kV$ spanned by the elements of the form

$$\ldots \otimes v \otimes v \otimes \ldots .$$

Note that $I_0V = 0$ and $I_1V = 0$.

We therefore define the k-th *exterior power* of V as the alternating k-multilinear map

$$\underbrace{V \times \ldots \times V}_{k} \xrightarrow{\wedge} T_kV/I_kV =: \Lambda_kV \tag{2.27}$$

obtained from postcomposing the k-th tensor power $V^k \to T_kV$ with the quotient map $T_kV \to T_kV/I_kV$. It is the *universal alternating k-multilinear map* from V^k in the sense that all others are obtained from it by postcomposition with a linear map. We have $\Lambda_0V = K$ and $\Lambda_1V = V$.

The image under $\wedge$ of a k-tuple $(v_1, \ldots, v_k)$ is denoted $v_1 \wedge \ldots \wedge v_k$. The product $\wedge$ is called the *exterior* or *wedge product*.[5] We thus have, besides the multilinearity of this product, the alternation relations $\ldots \wedge v \wedge v \wedge \ldots = 0$ and, more generally,

$$\ldots \wedge v \wedge \ldots \wedge v \wedge \ldots = 0,$$

compare Lemma 1.6. The wedge product is in particular *antisymmetric*,

$$v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(k)} = \mathrm{sgn}(\sigma) \cdot v_1 \wedge \ldots \wedge v_k \tag{2.28}$$

for $\sigma \in S_k$ and $v_i \in V$.

Terminology. An element of an exterior power is called a *multivector.* More specifically, an element of Λ_kV is called a *k-vector.* A k-vector of the form $v_1 \wedge \ldots \wedge v_k$ is called *decomposable* or a *simple k-vector* or a *k-blade.* A 0-vector is a scalar. A 1-vector is a vector, and it is always simple. However, for $k \geq 2$ not all k-vectors are simple (by dimension reasons).

To give a more concrete idea of the exterior powers, we again describe *bases*, cf Lemma 2.6:

Lemma. If $(e_i | i \in I)$ is a basis of V and I is equipped with a total ordering "<", then $(e_{i_1} \wedge \ldots \wedge e_{i_k} | i_1 < \ldots < i_k)$ is a basis of Λ_kV.

[5] In german: *Dachprodukt.*

Proof. Since the monomials $e_{i_1} \wedge \ldots \wedge e_{i_k}$ are the images of the basis elements $e_{i_1} \otimes \ldots \otimes e_{i_k}$ under the natural quotient projection $T_k V \to \Lambda_k V$, they clearly generate $\Lambda_k V$, and in view of the antisymmetry (2.28), already those for $i_1 < \ldots < i_k$ generate.

To see their linear independence, we note that for any $j_1 < \ldots < j_k$ in I there exists an alternating multilinear form $\alpha \in \mathrm{Alt}_k(V)$ such that $\alpha(e_{i_1}, \ldots, e_{i_k}) \neq 0$ if the i_l are a permutation of the j_l , and $= 0$ otherwise, compare section 1.4, in particular Lemma 1.7. By the universal property of the exterior power, α translates into a linear form on $\Lambda_k V$ which takes a nonzero value on $e_{j_1} \wedge \ldots \wedge e_{j_k}$ and vanishes on the other elements of the generating set. □

Again we combine the exterior powers of all degrees by forming the graded vector space

$$\Lambda_* V := \bigoplus_{k=0}^{\infty} \Lambda_k V.$$

It is the quotient vector space

$$\Lambda_* V \cong T_* V / I_* V$$

of the covariant tensor algebra $T_* V = \oplus_{k=0}^{\infty} T_k V$ by the graded linear subspace

$$I_* V := \bigoplus_{k=0}^{\infty} I_k V.$$

The latter is in fact the *two-sided ideal* in $T_* V$ generated by the elements $v \otimes v$ for $v \in V$. Therefore $\Lambda_* V$ inherits from $T_* V$ a natural structure as a graded associative *K-algebra*. Its product, the graded *wedge product*

$$\Lambda_* V \times \Lambda_* V \xrightarrow{\wedge} \Lambda_* V$$

induced by the tensor product, is the bilinear extension of the collection of wedge product maps

$$\Lambda_k V \times \Lambda_l V \xrightarrow{\wedge} \Lambda_{k+l} V.$$

Due to the associativity of the tensor product, cf Lemma 2.12, it holds that

$$(v_1 \wedge \ldots \wedge v_k) \wedge (v_{k+1} \wedge \ldots \wedge v_{k+l}) = v_1 \wedge \ldots \wedge v_{k+l}.$$

Furthermore, in view of (2.28), the wedge product is *graded commutative*, i.e. it satisfies the commutation law

$$b \wedge a = (-1)^{\deg a \cdot \deg b} a \wedge b \tag{2.29}$$

for homogeneous elements a and b. Thus, the exterior algebra $\Lambda_* V$ is an *alternating*[6] *$\mathbb{Z}$-graded associative K-algebra with unity.* The $\mathbb{Z}$-grading $\Lambda_* V = \oplus_{k \in \mathbb{Z}} \Lambda_k V$ (putting $\Lambda_k V = 0$ for $k < 0$) coarsens to a $\mathbb{Z}_2$-grading $\Lambda_* V = \Lambda_{even} V \oplus \Lambda_{odd} V$ by collecting the components of even and odd degrees, respectively. It is all what is needed to formulate the anticommutation law (2.29). The exterior algebra is characterized by the *universal property* that it is the "largest" alternating

[6] A $\mathbb{Z}$-graded algebra is called *alternating* if its product is graded commutative.

$\mathbb{Z}_2$-graded associative K-algebra[7] with unity "generated by V" in the sense that any linear map $L : V \to A$ into the odd part of an alternating $\mathbb{Z}_2$-graded associative K-algebra with unity uniquely extends to a homomorphism $\Lambda_* V \to A$ of graded algebras with unity.

If $\dim V < \infty$, then $\dim \Lambda_k V = \binom{\dim V}{k}$ and $\dim \Lambda_* V = 2^{\dim V}$.

Functoriality. The exterior power and exterior algebra functors $V \mapsto \Lambda_k V$ and $V \mapsto \Lambda_* V$ are *covariant*, that is, a linear map $L : U \to V$ induces linear maps

$$\Lambda_k L : \Lambda_k U \to \Lambda_k V, \quad u_1 \wedge \ldots \wedge u_k \mapsto L u_1 \wedge \ldots \wedge L u_k$$

which combine by linear extension to a homomorphism of graded algebras $\Lambda_* L : \Lambda_* U \to \Lambda_* V$.

Determinant revisited. If $\dim V = k$ and $L \in \mathrm{End}(V)$, then for a basis (e_i) of V and the matrix (a_{ij}) of L relative to this basis, $L e_j = \sum_i a_{ij} e_i$, one obtains

$$L e_1 \wedge \ldots \wedge L e_k = \underbrace{\det(a_{ij})}_{= \det L} \cdot e_1 \wedge \ldots \wedge e_k$$

and hence

$$\Lambda_k L = \det L \cdot \mathrm{id}_{\Lambda_k V}, \tag{2.30}$$

which is dual to the earlier observation (1.12). It also immediately yields the multiplication law (1.13) for determinants, since for $A, B \in \mathrm{End}\, V$ one has $\Lambda_k(AB) = (\Lambda_k A)(\Lambda_k B)$.

Pairings. As for tensor spaces there are natural bilinear pairings between the exterior powers of a vector space and its dual space. The natural pairing

$$T_k V \times T_k V^* \to K, \quad (v_1 \otimes \ldots \otimes v_k, v_1^* \otimes \ldots \otimes v_k^*) \mapsto \prod_i v_i^*(v_i)$$

itself, compare (2.21), does not descend to $\Lambda_k V \times \Lambda_k V^*$, however its antisymmetrization

$$(v_1 \otimes \ldots \otimes v_k, v_1^* \otimes \ldots \otimes v_k^*) \mapsto \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \prod_i v_i^*(v_{\sigma(i)}) = \det(v_i^*(v_j))$$

does descend, because it vanishes on $I_k V \times T_k V^*$ and $T_k V \times I_k V^*$, due to the fact that the determinant of a matrix is alternating in columns and rows. We thus obtain the natural non-degenerate[8] pairing

$$\Lambda_k V \times \Lambda_k V^* \to K, \quad (v_1 \wedge \ldots \wedge v_k, v_1^* \wedge \ldots \wedge v_k^*) \mapsto \det(v_i^*(v_j)). \tag{2.31}$$

[7] A $\mathbb{Z}_2$-graded algebra is sometimes also called a *superalgebra.* If the commutation law (2.29) holds, then a superalgebra it is called *commutative.* Thus, the exterior algebra is a commutative associative superalgebra over K with unity.

[8] This follows from the more general assertion: *If $\beta : V \times V' \to K$ is a non-degenerate pairing, then the induced pairing $\Lambda_k \beta : \Lambda_k V \times \Lambda_k V' \to K$ given by $(\Lambda_k \beta)(v_1 \wedge \ldots \wedge v_k, v_1' \wedge \ldots \wedge v_k') = \det(\beta(v_i, v_j'))$ is non-degenerate.* To see this, note that for $0 \neq a \in \Lambda_k V$ there exist linearly independent $v_1, \ldots, v_l \in V$, so that $a = v_1 \wedge \ldots \wedge v_k + b$ where b is a linear combination of monomials $v_{i_1} \wedge \ldots \wedge v_{i_k}$ with $i_k > k$. In view of the inclusion $V \hookrightarrow V'^*$ induced by β, the linear forms $\beta(v_i, \cdot)$ on V' are linearly independent. Hence, there exist $v_1', \ldots, v_l' \in V'$ so that $\beta(v_i, v_j') = \delta_{ij}$. Then $(\Lambda_k \beta)(a, v_1' \wedge \ldots \wedge v_k') = (\Lambda_k \beta)(v_1 \wedge \ldots \wedge v_k, v_1' \wedge \ldots \wedge v_k') = 1 \neq 0$.

Alternating multilinear forms. Due to the universal property of exterior powers, we have the identification $\mathrm{Alt}_k(V) \cong (\Lambda_k V)^*$ of the spaces of alternating multilinear forms on V. If $\dim V < \infty$, then the non-degenerate pairing (2.31) induces natural isomorphisms

$$\mathrm{Alt}_k(V) \cong (\Lambda_k V)^* \cong \Lambda_k V^* =: \Lambda^k V. \tag{2.32}$$

We put $\Lambda^* V := \Lambda_* V^*$.

One can thus identify alternating multilinear forms on V with elements in exterior powers of V^*, monomials $v_1^* \wedge \ldots \wedge v_k^* \in \Lambda^k V$ corresponding to forms $(v_1, \ldots, v_k) \mapsto \det(v_i^*(v_j))$, i.e.

$$(v_1^* \wedge \ldots \wedge v_k^*)(v_1, \ldots, v_k) = \det(v_i^*(v_j)). \tag{2.33}$$

If (e_i) is a basis of V, then $(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \,|\, i_1 < \cdots < i_k)$ is a basis of $\mathrm{Alt}_k(V)$ under this identification. A form $\alpha \in \mathrm{Alt}_k(V)$ can be written with respect to this basis as

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha(e_{i_1}, \ldots, e_{i_k})\, e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*.$$

Multiplying alternating multilinear forms. From the identification of spaces of alternating multilinear forms with exterior powers of the dual space, natural wedge product maps

$$\mathrm{Alt}_k(V) \times \mathrm{Alt}_l(V) \xrightarrow{\wedge} \mathrm{Alt}_{k+l}(V) \tag{2.34}$$

arise. If $\mathrm{char}\, K = 0$, they work as follows. For forms $\alpha \in \mathrm{Alt}_k(V)$ and $\beta \in \mathrm{Alt}_l(V)$, one has

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma) \cdot \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}). \tag{2.35}$$

Indeed, since both sides are alternating multilinear in the v_i and bilinear in α and β, it suffices to verify this formula in the case when the vectors v_i are linearly independent, i.e. constitute part of a basis, and the forms are monomials, $\alpha = v_1^* \wedge \ldots \wedge v_k^*$ and $\beta = v_{k+1}^* \wedge \ldots \wedge v_{k+l}^*$, such that the covector factors v_i^* are dual to the basis vectors v_i, i.e. $v_i^*(v_j) = \delta_{ij}$. However, in this case the formula it is easily confirmed, because the left-hand side equals 1 and in the right-hand sum exactly the $k!l!$ permutations $\sigma \in S_k \times S_l \subset S_{k+l}$ preserving the subset $\{1, \ldots, k\}$ contribute, each of them a summand 1.

The wedge product (2.34) on alternating multilinear forms is a *skew-symmetrization* of the tensor product (2.26) on general multilinear forms. Namely, there are natural linear projections

$$\mathrm{Mult}_k(V) \xrightarrow{\mathrm{alt}} \mathrm{Alt}_k(V)$$

onto the subspaces $\mathrm{Alt}_k(V) \subset \mathrm{Mult}_k(V)$ given by antisymmetrization (still assuming $\mathrm{char}\, K = 0$),

$$(\mathrm{alt}\,\mu)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \mu(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for $\mu \in \mathrm{Mult}_k V$ and $v_i \in V$, that is, in terms of the natural action $S_k \curvearrowright \mathrm{Mult}_k(V)$,

$$\mathrm{alt}\,\mu = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \sigma\mu.$$

We may then rewrite (2.35) as

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \operatorname{alt}(\alpha \otimes \beta).$$

For multiple products, the last formula generalizes to

$$\alpha_1 \wedge \dots \wedge \alpha_r = \frac{(k_1 + \dots + k_r)!}{k_1! \cdots k_r!} \operatorname{alt}(\alpha_1 \otimes \dots \otimes \alpha_r),$$

equivalently, (2.35) to

$$(\alpha_1 \wedge \dots \wedge \alpha_r)(v_1, \dots, v_k) = \frac{1}{k_1! \cdots k_r!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \alpha_1(v_{\sigma(1)}, \dots) \cdot \ldots \cdot \alpha_r(\dots, v_{\sigma(k)})$$

where $k = k_1 + \dots + k_r$.

Interior product. As in the case of general multilinear maps, for a vector $v \in V$, there are natural linear maps

$$\operatorname{Alt}_k(V) \xrightarrow{i_v} \operatorname{Alt}_{k-1}(V)$$

for $k \geq 1$ given by inserting v for the first variable,

$$i_v \alpha = \alpha(v, \dots),$$

and called *interior multiplication* or *contraction* with v. Invoking the identifications (2.32), we combine these maps by linear extension to a linear map $i_v : \Lambda^* V \to \Lambda^* V$, where we make the convention $i_v|_{\Lambda^0 V} = 0$. It satisfies

$$i_v^2 = 0$$

and is an *antiderivation* of degree -1 of the graded algebra $\Lambda^* V$, i.e. it lowers degrees by 1 and one has the product rule

$$i_v(\alpha \wedge \beta) = (i_v \alpha) \wedge \beta + (-1)^k \cdot \alpha \wedge (i_v \beta) \tag{2.36}$$

for $\alpha \in \Lambda^k V$ and $\beta \in \Lambda^* V$. Indeed, since both sides are bilinear in α and β, it suffices to verify this formula for monomials $\alpha = v_1^* \wedge \dots \wedge v_k^*$ and $\beta = v_{k+1}^* \wedge \dots \wedge v_{k+l}^*$ with $v_i^* \in \Lambda^1 V = V^*$. From (2.33) and the Laplace expansion of determinants, we obtain that

$$i_v \alpha = i_v(v_1^* \wedge \ldots \wedge v_k^*) = \sum_{i=1}^{k} (-1)^{i+1} \cdot v_i^*(v) \cdot (v_1^* \wedge \ldots \widehat{v_i^*} \ldots \wedge v_k^*)$$

where "$\widehat{v_i^*}$" indicates that the factor v_i^* is omitted. There are analogous expressions for $i_v \beta$ and $i_v(\alpha \wedge \beta)$, and it follows that the left-hand side of (2.36) equals

$$\underbrace{\sum_{i=1}^{k} (-1)^{i+1} \cdot v_i^*(v) \cdot (v_1^* \wedge \ldots \widehat{v_i^*} \ldots \wedge v_{k+l}^*)}_{=(i_v \alpha) \wedge \beta} + \underbrace{\sum_{i=1}^{l} (-1)^{k+i+1} \cdot v_{k+i}^*(v) \cdot (v_1^* \wedge \ldots \widehat{v_{k+i}^*} \ldots \wedge v_{k+l}^*)}_{=(-1)^k \cdot \alpha \wedge (i_v \beta)},$$

as claimed.

Suppose that V is a n-dim vector space equipped with a volume form $0 \neq \omega \in \Lambda^n V$. Then ω gives rise to a linear isomorphism

$$V \xrightarrow{\cong} \Lambda^{n-1} V, \quad v \mapsto i_v \omega$$

and, more generally, linear isomorphisms

$$\Lambda_k V \xrightarrow{\cong} \Lambda^{n-k} V, \quad v_1 \wedge \ldots \wedge v_k \mapsto \underbrace{\omega(v_1, \ldots, v_k, \ldots)}_{i_{v_k} \ldots i_{v_1} \omega}$$

for $0 \leq k \leq n$, induced by the alternating multilinear maps $(v_1, \ldots, v_k) \mapsto \omega(v_1, \ldots, v_k, \ldots)$. If (e_i) is an ordered basis so that $\omega = e_1^* \wedge \ldots \wedge e_n^*$, then $e_1 \wedge \ldots \wedge e_k \mapsto e_{k+1}^* \wedge \ldots \wedge e_n^*$.

Geometric notions. We now work over the field $K = \mathbb{R}$. Let V be a n-dim vector space.

Scalar products revisited. As we already pointed out in section 2.2, being bilinear forms, scalar products are type $(0,2)$ tensors. If $\langle \cdot, \cdot \rangle$ is a scalar product on V and (e_i) is an ONB with respect to it, then in view of $\langle e_i, e_j \rangle = \delta_{ij}$ we can write

$$\langle \cdot, \cdot \rangle = \sum_{i=1}^n e_i \otimes e_i,$$

compare (2.25).

Orientation revisited. The orientation determined by an ordered basis (e_i) of V is induced by the volume form $e_1^* \wedge \ldots \wedge e_n^* \in \Lambda^n V$, where (e_i^*) denotes the dual basis of V^*, cf section 1.4.3.

Orientations can also be described in terms of top degree *multivectors*: We have $\dim \Lambda_n V = 1$. Every oriented base e gives rise to an n-vector $0 \neq e_1 \wedge \cdots \wedge e_n \in \Lambda_n V$. For bases e, e' with $e \cdot A = e'$, the induced n-vectors are related by

$$e'_1 \wedge \cdots \wedge e'_n = \det A \cdot e_1 \wedge \cdots \wedge e_n,$$

compare (2.30). An orientation of V thus corresponds to a *ray component* of $\Lambda_n V \setminus \{0\}$.

Volume revisited. We now can make sense of the fact that the k-dim volume, compare our discussion in section 1.4.4, can be regarded as a *norm* on k-vectors. A scalar product $\langle \cdot, \cdot \rangle_V$ on V induces scalar products $\langle \cdot, \cdot \rangle_{\Lambda_k V}$ on the exterior powers $\Lambda_k V$ satisfying

$$\langle u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \wedge v_k \rangle_{\Lambda_k V} = \det(\langle u_i, v_j \rangle_V)_{i,j=1,\ldots,k}.$$

They are the symmetric bilinear forms on $\Lambda_k V$ induced, via the universal property of exterior powers, by the $2k$-linear forms $(u_1, \ldots, u_k, v_1, \ldots, v_k) \mapsto \det(\langle u_i, v_j \rangle_V)$ on V which are alternating in the u_i's as well as in the v_j's, compare (1.20). That the $\langle \cdot, \cdot \rangle_{\Lambda_k V}$ are positive definite, can be seen using bases. Namely, if (e_i) is an ONB for $\langle \cdot, \cdot \rangle_V$, then $(e_{i_1} \wedge \ldots \wedge e_{i_k} \,|\, i_1 < \ldots < i_k)$ is an ONB for $\langle \cdot, \cdot \rangle_{\Lambda_k V}$. We observe that, on decomposable multivectors $v_1 \wedge \ldots \wedge v_k$, the quadratic forms associated to the $\langle \cdot, \cdot \rangle_{\Lambda_k V}$ are given by Gram determinants and equal the squares of the k-dim volume functionals, cf (1.21). The associated norms $\| \cdot \|_{\Lambda_k V}$ thus satisfy

$$\mathrm{vol}_k(v_1, \ldots, v_k) = \| v_1 \wedge \ldots \wedge v_k \|_{\Lambda_k V}.$$

If the euclidean vector space $(V, \langle\cdot,\cdot\rangle_V)$ is in addition equipped with an *orientation*, then there is a unique volume form $0 \neq \omega \in \Lambda^n V$ with the property that

$$\omega(e_1, \dots, e_n) = 1$$

for every positively oriented ONB (e_i) of V, compare sections 1.4.3 and 1.4.4, namely

$$\omega = e_1^* \wedge \dots \wedge e_n^*.$$

It is called *the volume form* of the oriented euclidean vector space.

If $e \in V$ is a unit vector, $\|e\| = 1$, then the $(n-1)$-form $i_e\omega = \omega(e, \dots)$ restricts to a volume form on the hyperplane $e^\perp$ orthogonal to e, and thus determines an orientation on $e^\perp$. The induced volume form $i_e\omega|_{e^\perp}$ is the volume form on $e^\perp$ associated to the induced scalar product and orientation. If $(e, e_2, \dots, e_n)$ is a positively oriented ONB of V, then $(e_2, \dots, e_n)$ is a positively oriented ONB of $e^\perp$ and

$$i_e\omega = e_2^* \wedge \dots \wedge e_n^*.$$

Replacing the unit vector e by $-e$ yields the reversed orientation on $(-e)^\perp = e^\perp$.

Star operator. We keep assuming that V is a n-dim vector space equipped with a scalar product $\langle\cdot,\cdot\rangle_V$ and an orientation. Then there are natural *isometric linear isomorphisms*

$$\Lambda_k V \xrightarrow{\star} \Lambda_{n-k} V$$

for $0 \leq k \leq n$, characterized by the property that

$$a \wedge \star b = \langle a, b\rangle_{\Lambda_k V}\, \phi$$

for $a, b \in \Lambda_k V$, where $\phi \in \Lambda_n V$ denotes the unit n-vector positive with respect to the orientation. Indeed, with respect to a positively oriented ONB (e_i), we have $\phi = e_1 \wedge \ldots \wedge e_n$ and such operators can be defined by

$$e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)} \overset{\star}{\mapsto} e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}$$

for all even permutations $\sigma \in S_n$. They are unique, because an $(n-k)$-vector is determined by its wedge products with all k-vectors. Note that $\phi = \star 1$.

Combining these operators for all grades k, we obtain the *grade reflecting* isometric linear isomorphism

$$\Lambda_* V \xrightarrow{\star} \Lambda_* V$$

called the *Hodge star operator*. It satisfies

$$\star\star|_{\Lambda_k V} = (-1)^{k(n-k)}\, \mathrm{id}_{\Lambda_k V}.$$

Dually, on forms, we have the star operators

$$\Lambda^k V \xrightarrow{\star} \Lambda^{n-k} V$$

satisfying

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle_{\Lambda^k V}\, \omega$$

for $\alpha, \beta \in \Lambda^k V$.

We note that the interior product of a vector with the volume form can be written in terms of the star operator as

$$i_v \omega = \star \underbrace{\langle v, \cdot \rangle_V}_{\in V^*}.$$

Cross product. The following structure is special to dimension 3. Suppose that $(V, \langle \cdot, \cdot \rangle_V)$ is a 3-dim euclidean vector space equipped with an orientation. Let ω denote its distinguished volume form. Then there is unique alternating bilinear map

$$V \times V \xrightarrow{\times} V, \quad (u, v) \mapsto u \times v,$$

called the *cross product* on V, so that

$$\langle u \times v, w \rangle = \omega(u, v, w)$$

for $u, v, w \in V$. Indeed, each linear form $\omega(u, v, \cdot)$ can be written as the scalar product $\langle u \times v, \cdot \rangle$ with a vector denoted $u \times v$, and the resulting map $(u, v) \mapsto u \times v$ is alternating bilinear.

If (e_1, e_2, e_3) is a positively oriented ONB, then

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1 \quad \text{and} \quad e_3 \times e_1 = e_2.$$

For general vectors $u = \sum u_i e_i$ and $v = \sum v_i e_i$ one obtains

$$u \times v = \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} e_1 + \begin{pmatrix} u_3 & u_1 \\ v_3 & v_1 \end{pmatrix} e_2 + \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} e_3.$$