POINTED HOPF ALGEBRAS

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Abstract. This a survey on pointed Hopf algebras over algebraically closed fields of characteristic 0. We propose to classify pointed Hopf algebras A by first determining the graded Hopf algebra $\operatorname{gr} A$ associated to the coradical filtration of A. The A_0-convariants elements form a braided Hopf algebra R in the category of Yetter-Drinfeld modules over the coradical $A_0 = \Bbbk\Gamma$, Γ the group of group-like elements of A, and $\operatorname{gr} A \simeq R\#A_0$. We call the braiding of the primitive elements of R the infinitesimal braiding of A. If this braiding is of Cartan type [AS2], then it is often possible to determine R, to show that R is generated as an algebra by its primitive elements and finally to compute all deformations or liftings, that is pointed Hopf algebras such that $\operatorname{gr} A \simeq R\#\Bbbk\Gamma$. In the last Chapter, as a concrete illustration of the method, we describe explicitly all finite-dimensional pointed Hopf algebras with abelian group $G(A)$ and infinitesimal braiding of type A_n (up to some exceptional cases). In other words, we compute all the liftings of type A_n; this result is our main new contribution in this paper.

Contents

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Introduction

A Hopf algebra A over a field $\Bbbk$ is called *pointed* [Sw], [M1], if all its simple left or right comodules are one-dimensional. The coradical A_0 of A is the sum of all its simple subcoalgebras. Thus A is pointed if and only if A_0 is a group algebra.

We will always assume that the field $\Bbbk$ is algebraically closed of characteristic 0 (although several results of the paper hold over arbitrary fields).

It is easy to see that A is pointed if it is generated as an algebra by group-like and skew-primitive elements. In particular, group algebras, universal enveloping algebras of Lie algebras and the q-deformations of the universal enveloping algebras of semisimple Lie algebras are all pointed.

An essential tool in the study of pointed Hopf algebras is the *coradical filtration*

$$A_0 \subset A_1 \subset \cdots \subset A, \bigcup_{n\geq 0} A_n = A$$

of A. It is dual to the filtration of an algebra by the powers of the Jacobson radical. For pointed Hopf algebras it is a Hopf algebra filtration, and the associated graded Hopf algebra $\operatorname{gr} A$ has a Hopf algebra projection onto $A_0 = \Bbbk\Gamma, \Gamma = G(A)$ the group of all group-like elements of A. By a theorem of Radford [Ra], $\operatorname{gr} A$ is a biproduct

$$\operatorname{gr} A \cong R\#\Bbbk\Gamma,$$

where R is a graded braided Hopf algebra in the category of left Yetter-Drinfeld modules over $\Bbbk\Gamma$ [AS2].

This decomposition is an analog of the theorem of Cartier–Kostant–Milnor–Moore on the semidirect product decomposition of a cocommutative Hopf algebra into an infinitesimal and a group algebra part.

The vector space $V = P(R)$ of the primitive elements of R is a Yetter-Drinfeld submodule. We call its braiding

$$c : V \otimes V \to V \otimes V$$

the **infinitesimal braiding** of A. The infinitesimal braiding is the key to the structure of pointed Hopf algebras.

The subalgebra $\mathfrak{B}(V)$ of R generated by V is a braided Hopf subalgebra. As an algebra and coalgebra, $\mathfrak{B}(V)$ only depends on the infinitesimal braiding of V. In his thesis [N] published in 1978, Nichols studied Hopf algebras of the form $\mathfrak{B}(V)\#\Bbbk\Gamma$ under the name of bialgebras of type one. We call $\mathfrak{B}(V)$ the *Nichols algebra* of V. These Hopf algebras were found independently later by Woronowicz [Wo] and other authors.

Important examples of Nichols algebras come from quantum groups [Dr1]. If $\mathfrak{g}$ is a semisimple Lie algebra, $U_q^{\geq 0}(\mathfrak{g})$, q not a root of unity, and the finite-dimensional Frobenius-Lusztig kernels $\mathfrak{u}_q^{\geq 0}(\mathfrak{g})$, q a root of unity of order N, are both of the form $\mathfrak{B}(V)\#\Bbbk\Gamma$ with $\Gamma = \mathbb{Z}^\theta$ resp. $(\mathbb{Z}/(N))^\theta, \theta \geq 1$. ([L3], [Ro1], [Sbg], and [L2], [Ro1], [Mu]). (Here, there are some technical conditions on N).

In general, the classification problem of pointed Hopf algebras has three parts:

(1) Structure of the Nichols algebras $\mathfrak{B}(V)$.

(2) The lifting problem: Determine the structure of all pointed Hopf algebras A with $G(A) = \Gamma$ such that $\operatorname{gr} A \cong \mathfrak{B}(V)\#\Bbbk\Gamma$.

(3) Generation in degree one: Decide which Hopf algebras A are generated by group-like and skew-primitive elements, that is $\operatorname{gr} A$ is generated in degree one.

We conjecture that all finite-dimensional pointed Hopf algebras over an algebraically closed field of characteristic 0 are indeed generated by group-like and skew-primitive elements.

In the sequel, we describe the steps of this program in detail and explain the positive results obtained so far in this direction. It is not our intention to give a complete survey on all aspects of pointed Hopf algebras.

We will mainly report on recent progress in the classification of pointed Hopf algebras with *abelian* group of group-like elements.

If the group Γ is abelian, and V is a finite-dimensional Yetter-Drinfeld module, then the braiding is given by a family of scalars $q_{ij} \in \Bbbk, 1 \le i \le \theta$, in the form

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \text{ where } x_1, \dots, x_\theta \text{ is a basis of } V.$$

Moreover there are elements $g_1, \dots, g_\theta \in \Gamma$, and characters $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ such that $q_{ij} = \chi_j(g_i)$. The group acts on x_i via the character χ_i, and x_i is a g_i-homogeneous element with respect to the coaction of Γ. We introduced braidings of Cartan type [AS2] where

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}}, 1 \le i, j \le \theta, \text{ and } (a_{ij}) \text{ is a generalized Cartan matrix.}$$

If (a_{ij}) is a Cartan matrix of finite type, then the algebras $\mathfrak{B}(V)$ can be understood as twisting of the Frobenius-Lusztig kernels $\mathfrak{u}^{\ge 0}(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra.

By deforming the quantum Serre relations for simple roots which lie in two different connected components of the Dynkin diagram, we define finite-dimensional pointed Hopf algebras $\mathfrak{u}(\mathcal{D})$ in terms of a "linking datum $\mathcal{D}$ of finite Cartan type" [AS4]. They generalize the Frobenius-Lusztig kernels $\mathfrak{u}(\mathfrak{g})$ and are liftings of $\mathfrak{B}(V)\#\Bbbk\Gamma$.

In some cases linking data of finite Cartan type are general enough to obtain complete classification results.

For example, if $\Gamma = (\mathbb{Z}/(p))^s$, p a prime > 17 and $s \ge 1$, we have determined the structure of all finite-dimensional Hopf algebras A with $G(A) \simeq \Gamma$. They are all of the form $\mathfrak{u}(\mathcal{D})$ [AS4].

Similar data allow a classification of infinite-dimensional pointed Hopf algebras A with abelian group $G(A)$, without zero divisors and finite Gelfand-Kirillov dimension, in the case when the infinitesimal braiding is "positive" [AS5]. But the general case is more involved. We also have to deform the root vector relations of the $\mathfrak{u}(\mathfrak{g})'s$.

The structure of pointed Hopf algebras A with *non-abelian* group $G(A)$ is widely unknown. One basic open problem is to decide which finite groups appear as groups of group-like elements of finite-dimensional pointed Hopf algebras which are link-indecomposable in the sense of [M2]. In our formulation, this problem is the main part of following question: given a finite group Γ, determine all Yetter-Drinfeld modules V over Γ such that $\mathfrak{B}(V)$ is finite dimensional. On the one hand, there are a number of severe constraints on V [Gñ3]. See also the exposition in [A, 5.3.10]. On the other hand, it is very hard to prove the finiteness of the dimension, and in fact this was done only for a few examples [MiS], [FK], [FP] which are again related to root systems. The examples over the symmetric groups in [FK] were introduced to describe the cohomology ring of the flag variety. At this stage, the main difficulty is to decide when certain Nichols algebras over non-abelian groups, for example the symmetric groups $\mathbb{S}_n$, are finite-dimensional.

The last Chapter can serve as a concrete illustration of the theory explained in this paper. We describe explicitly all finite-dimensional pointed Hopf algebras with abelian group $G(A)$ and infinitesimal braiding of type A_n (up to some exceptional cases). The main results in this Chapter are new, and complete proofs are given. The only cases which were known before are the easy case A_1 [AS1], and A_2 [AS3].

The new relations concern the root vectors $e_{i,j}, 1 \leq i < j \leq n+1$. The relations $e_{i,j}^N = 0$ in $\mathfrak{u}_q^{\geq 0}(sl_n)$, q a root of unity of order N, are replaced by

$$e_{i,j}^N = u_{i,j} \text{ for a family } u_{i,j} \in \Bbbk\Gamma, 1 \leq i < j \leq n+1,$$

depending on a family of free parameters in $\Bbbk$.

To study the relations between a filtered object and its associated graded object is a basic technique in modern algebra. We would like to stress that finite dimensional pointed Hopf algebras enjoy a remarkable rigidity; it is seldom the case that one is able to describe precisely all the liftings of a graded object, as in this context.

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Conventions. As said above, our ground field $\Bbbk$ is algebraically closed field of characteristic 0. Throughout, "Hopf algebra" means "Hopf algebra with bijective antipode".

We denote by $\tau : V \otimes W \to W \otimes V$ the usual transposition, that is $\tau(v \otimes w) = w \otimes v$.

We use Sweedler's notation for the comultiplication and coaction; but, to avoid confusions, we use the following variant for the comultiplication of a braided Hopf algebra R: $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$.

1. Braided Hopf algebras

1.1. **Braided categories.**

Braided Hopf algebras play a central rôle in this paper. Although we have tried to minimize the use of categorical language, we briefly and informally recall the notion of a braided category which is the appropriate setting for braided Hopf algebras.

Braided categories were introduced in [JS]. We refer to [Ka, Ch. XI, Ch. XIII] for a detailed exposition. There is a hierarchy of categories with a tensor product functor:

a). A *monoidal* or *tensor* category is a collection $(\mathcal{C}, \otimes, a, \mathbb{I}, \ell, r)$, where

- $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor,
- $\mathbb{I}$ is an object of $\mathcal{C}$, and

- $a : V \otimes (W \otimes U) \to (V \otimes W) \otimes U$, $\ell : V \to V \otimes \mathbb{I}$, $r : V \to \mathbb{I} \otimes V$ are natural isomorphisms;

such that the so-called "pentagon" and "triangle" axioms are satisfied, see [Ka, Ch. XI, (2.6) and (2.9)]. These axioms essentially express that the tensor product of a finite number of objects is well-defined, regardless of the place where parentheses are inserted; and that $\mathbb{I}$ is a unit for the tensor product.

b). A *braided (tensor)* category is a collection $(\mathcal{C}, \otimes, a, \mathbb{I}, \ell, r, c)$, where

- $(\mathcal{C}, \otimes, a, \mathbb{I}, \ell, r)$ is a monoidal category and
- $c_{V,W} : V \otimes W \to W \otimes V$ is a natural isomorphism;

such that the so-called "hexagon" axioms are satisfied, see [Ka, Ch. XIII, (1.3) and (1.4)]. A very important consequence of the axioms of braided category, is the following equality for any objects V, W, U:

$$(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}), \tag{1.1}$$

see [Ka, Ch. XIII, (1.8)]. For simplicity we have omitted the associativity morphisms.

c). A *symmetric* category is a braided category where $c_{V,W}c_{W,V} = \mathrm{id}_{W\otimes V}$ for all objects V, W. Symmetric categories have been studied since the pioneering work of Mac Lane.

d). A left dual of an object V of a monoidal category, is a collection $(V^*, \mathrm{ev}_V, b_V)$, where V^* is another object and $\mathrm{ev}_V : V^* \otimes V \to \mathbb{I}$, $b_V : \mathbb{I} \to V \otimes V^*$ are morphisms such that the compositions

$$V \longrightarrow \mathbb{I} \otimes V \xrightarrow{b_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} V \otimes \mathbb{I} \longrightarrow V$$

and

$$V^* \longrightarrow V^* \otimes \mathbb{I} \xrightarrow{\mathrm{id}_{V^*} \otimes b_V} V^* \otimes V \otimes V^* \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{V^*}} \mathbb{I} \otimes V^* \longrightarrow V^*$$

are, respectively, the identity of V and V^*. A braided category is *rigid* if any object V admits a left dual [Ka, Ch. XIV, Def. 2.1].

1.2. **Braided vector spaces and Yetter-Drinfeld modules.**

We begin with the fundamental

Definition 1.1. Let V be a vector space and $c : V \otimes V \to V \otimes V$ a linear isomorphism. Then (V, c) is called a *braided vector space*, if c is a solution of the *braid equation*, that is

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c). \tag{1.2}$$

It is well-known that the braid equation is equivalent to the *quantum Yang-Baxter equation*:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.3}$$

Here we use the standard notation: $R_{13} : V\otimes V\otimes V \to V\otimes V\otimes V$ is the map given by $\sum_j r_j \otimes \mathrm{id} \otimes r^j$, if $R = \sum_j r_j \otimes r^j$. Similarly for R_{12}, R_{23}.

The equivalence between solutions of (1.2) and solutions of (1.3) is given by the equality $c = \tau \circ R$. For this reason, some authors call (1.2) the quantum Yang-Baxter equation.

An easy and for this paper important example is given by a family of non-zero scalars $q_{ij} \in \Bbbk, i, j \in I$, where V is a vector space with basis $x_i, i \in I$. Then

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \text{for all } i, j \in I$$

is a solution of the braid equation.

Examples of braided vector spaces come from braided categories. In this article, we are mainly concerned with examples related to the notion of Yetter-Drinfeld modules.

Definition 1.2. Let H be a Hopf algebra. A (left) *Yetter-Drinfeld module* V over H is simultaneously a left H-module and a left H-comodule satisfying the compatibility condition

$$\delta(h.v) = h_{(1)} v_{(-1)} \mathcal{S} h_{(3)} \otimes h_{(2)}.v_{(0)}, \qquad v \in V,\, h \in H. \tag{1.4}$$

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter-Drinfeld modules over H; the morphisms in this category preserve both the action and the coaction by H. The category ${}^H_H\mathcal{YD}$ is a braided monoidal category; indeed the tensor product of two Yetter-Drinfeld modules is again a Yetter-Drinfeld module, with the usual tensor product module and comodule structure. The compatibility condition (1.4) is not difficult to verify.

For any two Yetter-Drinfeld-modules M and N, the braiding $c_{M,N} : M \otimes N \to N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = m_{(-1)}.n \otimes m_{(0)}, \qquad m \in M,\, n \in N, \tag{1.5}$$

The subcategory of ${}^H_H\mathcal{YD}$ consisting of finite dimensional Yetter-Drinfeld modules is rigid. Namely, if $V \in {}^H_H\mathcal{YD}$ is finite-dimensional, the dual $V^* = \operatorname{Hom}(V, \Bbbk)$ is in ${}^H_H\mathcal{YD}$ with the following action and coaction:

- $(h \cdot f)(v) = f(\mathcal{S}(v))$ for all $h \in H$, $f \in V^*$, $v \in V$.
- If $f \in V^*$, then $\delta(f) = f_{(-1)} \otimes f_{(0)}$ is determined by the equation
$$f_{(-1)} f_{(0)}(v) = \mathcal{S}^{-1}(v_{-1}) f(v_0), \qquad v \in V.$$

Then the usual evaluation and coevaluation maps are morphisms in ${}^H_H\mathcal{YD}$.

Let V, W be two finite-dimensional Yetter-Drinfeld modules over H. We shall consider the isomorphism $\Phi : W^* \otimes V^* \to (V \otimes W)^*$ given by

$$\Phi(\varphi \otimes \psi)(v \otimes w) = \psi(v)\varphi(w), \qquad \varphi \in W^*, \psi \in V^*, v \in V, w \in W. \tag{1.6}$$

Remark 1.3. We see that a Yetter-Drinfeld module is a braided vector space. Conversely, a braided vector space (V, c) can be realized as a Yetter-Drinfeld module over some Hopf algebra H if and only if c is *rigid* [Tk1]. If this is the case, it can be realized in many different ways.

We recall that a Hopf bimodule over a Hopf algebra H is simultaneously a bimodule and a bicomodule satisfying all possible compatibility conditions. The category ${}^H_H\mathcal{M}^H_H$ of all Hopf bimodules over H is a braided category. The category ${}^H_H\mathcal{YD}$ is equivalent, as a braided category, to the category of Hopf bimodules. This was essentially first observed in [Wo] and then independently in [AnDe, Appendix], [Sbg], [Ro1].

If H is a finite dimensional Hopf algebra, then the category ${}_H^H\mathcal{YD}$ is equivalent to the category of modules over the double of H [Mj1]. The braiding in ${}_H^H\mathcal{YD}$ corresponds to the braiding given by the "canonical" R-matrix of the double. In particular, if H is a semisimple Hopf algebra then ${}_H^H\mathcal{YD}$ is a semisimple category. Indeed, it is known that the double of a semisimple Hopf algebra is again semisimple.

The case of Yetter-Drinfeld modules over group algebras is especially important for the applications to pointed Hopf algebras. If $H = \Bbbk\Gamma$, where Γ is a group, then an H-comodule V is just a Γ-graded vector space: $V = \oplus_{g\in\Gamma} V_g$, where $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$. We will write ${}_\Gamma^\Gamma\mathcal{YD}$ for the category of Yetter-Drinfeld modules over $\Bbbk\Gamma$, and say that $V \in {}_\Gamma^\Gamma\mathcal{YD}$ is a Yetter-Drinfeld module over Γ (when the field is fixed).

Remark 1.4. Let Γ be a group, V a left $\Bbbk\Gamma$-module, and a left $\Bbbk\Gamma$-comodule with grading $V = \oplus_{g\in\Gamma} V_g$. We define a linear isomorphism $c : V \otimes V \to V \otimes V$ by

$$c(x \otimes y) = gy \otimes x, \text{ for all } x \in V_g,\ g \in \Gamma,\ y \in V. \tag{1.7}$$

Then

a) $V \in {}_\Gamma^\Gamma\mathcal{YD}$ if and only if $gV_h \subset V_{ghg^{-1}}$ for all $g, h \in \Gamma$.

b) If $V \in {}_\Gamma^\Gamma\mathcal{YD}$, then (V, c) is a braided vector space.

c) Conversely, if V is a faithful Γ-module (that is, if for all $g \in \Gamma, gv = v$ for all $v \in V$, implies $g = 1$), and if (V, c) is a braided vector space, then $V \in {}_\Gamma^\Gamma\mathcal{YD}$.

Proof. a) is clear from the definition.

By applying both sides of the braid equation to elements of the form $x \otimes y \otimes z, x \in V_g, y \in V_h, z \in V$, it is easy to see that (V, c) is a braided vector space if and only if

$$c(gy \otimes gz) = ghz \otimes gy, \text{ for all } g, h \in \Gamma,\ y \in V_h,\ z \in V. \tag{1.8}$$

Let us write $gy = \sum_{a\in\Gamma} x_a$, where $x_a \in V_a$ for all $a \in \Gamma$. Then $c(gy \otimes gz) = \sum_{a\in\Gamma} agz \otimes x_a$. Hence (1.8) means that $agz = ghz$, for all $z \in V$ and $a \in \Gamma$ such that the homogeneous component x_a is not zero. This proves b) and c). □

Remark 1.5. If Γ is abelian, a Yetter-Drinfeld module over $H = \Bbbk\Gamma$ is nothing but a Γ-graded Γ-module.

Assume that Γ is abelian and furthermore that the action of Γ is diagonalizable (this is always the case if Γ is finite). That is, $V = \oplus_{\chi\in\widehat{\Gamma}} V^\chi$, where $V^\chi = \{v \in V \mid gv = \chi(g)v \text{ for all } g \in \Gamma\}$. Then

$$V = \oplus_{g\in\Gamma,\chi\in\widehat{\Gamma}} V_g^\chi, \tag{1.9}$$

where $V_g^\chi = V^\chi \cap V_g$. Conversely, any vector space with a decomposition (1.9) is a Yetter-Drinfeld module over Γ. The braiding is given by

$$c(x \otimes y) = \chi(g) y \otimes x, \text{ for all } x \in V_g,\ g \in \Gamma,\ y \in V^\chi,\ \chi \in \widehat{\Gamma}.$$

It is useful to characterize abstractly those braided vector spaces which come from Yetter-Drinfeld modules over groups or abelian groups. The first part of the following definition is due to M. Takeuchi.

Definition 1.6. Let (V, c) be a finite dimensional braided vector space.

- (V, c) is of *group type* if there exists a basis $x_1, \dots, x_\theta$ of V and elements $g_i(x_j) \in V$ for all i, j such that

$$c(x_i \otimes x_j) = g_i(x_j) \otimes x_i, \qquad 1 \leq i, j \leq \theta; \tag{1.10}$$

  necessarily $g_i \in GL(V)$.

- (V, c) is of *finite group type* (resp. of *abelian group type*) if it is of group type and the subgroup of $GL(V)$ generated by $g_1, \dots, g_\theta$ is finite (resp. abelian).

- (V, c) is of *diagonal type* if V has a basis $x_1, \dots, x_\theta$ such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad 1 \leq i, j \leq \theta, \tag{1.11}$$

  for some q_{ij} in $\Bbbk$. The matrix (q_{ij}) is called the *matrix* of the braiding.

- If (V, c) is of diagonal type, then we say that it is *indecomposable* if for all i and j, there exists a sequence $i = i_1,\ i_2,\ \dots,\ i_t = j$ of elements of $\{1, \dots, \theta\}$ such that $q_{i_s, i_{s+1}} q_{i_{s+1}, i_s} \neq 1$, $1 \leq s \leq t-1$. Otherwise, we say that the matrix is decomposable. We can also refer then to the components of the matrix.

If $V \in {}^\Gamma_\Gamma\mathcal{YD}$ is finite-dimensional with braiding c, then (V, c) is of group type by (1.4). Conversely, assume that (V, c) is a finite-dimensional braided vector space of group type. Let Γ be the subgroup of $GL(V)$ generated by $g_1, \dots, g_\theta$. Define a coaction by $\delta(x_i) = g_i \otimes x_i$ for all i. Then V is a Yetter-Drinfeld module over Γ with braiding c by Remark 1.4 , c).

A braided vector space of diagonal type is clearly of abelian group type; it is of finite group type if the q_{ij}'s are roots of one.

1.3. **Braided Hopf algebras.**

The notion of "braided Hopf algebra" is one of the basic features of braided categories. We will deal in this paper only with braided Hopf algebras in categories of Yetter-Drinfeld modules, mainly over a group algebra.

Let H be a Hopf algebra. First, the tensor product in ${}^H_H\mathcal{YD}$ allows to define algebras and coalgebras in ${}^H_H\mathcal{YD}$. Namely, an algebra in the category ${}^H_H\mathcal{YD}$ is an associative algebra (R, m), where $m : R \otimes R \to R$ is the product, with unit $u : \Bbbk \to R$, such that R is a Yetter-Drinfeld module over H and both m and u are morphisms in ${}^H_H\mathcal{YD}$.

Similarly, a coalgebra in the category ${}^H_H\mathcal{YD}$ is a coassociative coalgebra (R, Δ), where $\Delta : R \to R \otimes R$ is the coproduct, with counit $\varepsilon : R \to \Bbbk$, such that R is a Yetter-Drinfeld module over H and both Δ and ε are morphisms in ${}^H_H\mathcal{YD}$.

Let now R, S be two algebras in ${}^H_H\mathcal{YD}$. Then the braiding $c : S \otimes R \to R \otimes S$ allows to provide the Yetter-Drinfeld module $R \otimes S$ with a "twisted" version of algebra in ${}^H_H\mathcal{YD}$. Namely, the product in $R \otimes S$ is $m_{R \otimes S} := (m_R \otimes m_S)(\mathrm{id} \otimes c \otimes \mathrm{id})$:

$$\begin{array}{ccc} R \otimes S \otimes R \otimes S & \longrightarrow & R \otimes S \\ {\scriptstyle \mathrm{id} \otimes c \otimes \mathrm{id}} \Big\downarrow & & \Big\downarrow {\scriptstyle =} \\ R \otimes R \otimes S \otimes S & \xrightarrow{m_R \otimes m_S} & R \otimes S. \end{array}$$

We shall denote this algebra by $R \underline{\otimes} S$. The difference with the usual tensor product algebra is the presence of the braiding c instead of the usual transposition τ.

Definition 1.7. A *braided bialgebra in* ${}^H_H\mathcal{YD}$ is a collection $(R, m, u, \Delta, \varepsilon)$, where

- (R, m, u) is an algebra in ${}^H_H\mathcal{YD}$.
- (R, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$.
- $\Delta : R \to R \underline{\otimes} R$ is a morphism of algebras.
- $u : \Bbbk \to R$ and $\varepsilon : R \to \Bbbk$ are morphisms of algebras.

We say that it is a *braided Hopf algebra in* ${}^H_H\mathcal{YD}$ if in addition:

- The identity is convolution invertible in $\mathrm{End}\,(R)$; its inverse is the antipode of R.

A *graded* braided Hopf algebra in ${}^H_H\mathcal{YD}$ is a braided Hopf algebra R in ${}^H_H\mathcal{YD}$ provided with a grading $R = \oplus_{n \geq 0} R(n)$ of Yetter-Drinfeld modules, such that R is a graded algebra and a graded coalgebra.

Remark 1.8. There is a non-categorical version of braided Hopf algebras, see [Tk1]. Any braided Hopf algebra in ${}^H_H\mathcal{YD}$ gives rise to a braided Hopf algebra in the sense of [Tk1] by forgetting the action and coaction, and preserving the multiplication, comultiplication and braiding. For the converse see [Tk1, Th. 5.7]. Analogously, one can define graded braided Hopf algebras in the spirit of [Tk1].

Let R be a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$. The dual $S = R^*$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with multiplication $\Delta_R^* \Phi$ and comultiplication $\Phi^{-1}\mu_R^*$, *cf.* (1.6); this is R^{*bop} in the notation of [AG, Section 2].

In the same way, if $R = \oplus_{n\geq 0} R(n)$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ with finite-dimensional homogeneous components, then the graded dual $S = R^* = \oplus_{n\geq 0} R(n)^*$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$.

1.4. **Examples. The quantum binomial formula.**

We shall provide many examples of braided Hopf algebras in Chapter 2. Here we discuss a very simple class of braided Hopf algebras.

We first recall the well-known quantum binomial formula. Let U and V be elements of an associative algebra over $\Bbbk[q]$, q an indeterminate, such that $VU = qUV$. Then

$$(U+V)^n = \sum_{1\leq i\leq n} \binom{n}{i}_q U^i V^{n-i}, \qquad \text{if } n \geq 1. \tag{1.12}$$

Here

$$\binom{n}{i}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}, \quad \text{where } (n)_q! = \prod_{1\leq i\leq n} (i)_q, \quad \text{and } (i)_q = \sum_{0\leq j\leq i-1} q^j.$$

By specialization, (1.12) holds for $q \in \Bbbk$. In particular, if U and V are elements of an associative algebra over $\Bbbk$, and q is a root of 1 of order dividing n, such that $VU = qUV$ then

$$(U+V)^n = U^n + V^n. \tag{1.13}$$

Example 1.9. Let $(q_{ij})_{1\leq i,j\leq\theta}$ be a matrix such that

$$q_{ij}q_{ji} = 1, \quad 1 \leq i,j \leq \theta,\ i \neq j. \tag{1.14}$$

Let N_i be the order of q_{ii}, when this is finite.

Let R be the algebra presented by generators $x_1, \dots, x_\theta$ with relations

(1.15) $x_i^{N_i} = 0, \quad \text{if } \operatorname{ord} q_{ii} < \infty.$

(1.16) $x_i x_j = q_{ij} x_j x_i, \quad 1 \leq i < j \leq \theta.$

Given a group Γ and elements $g_1, \dots, g_\theta$ in the center of Γ, and characters $\chi_1, \dots, \chi_\theta$ of Γ, there exists a unique structure of Yetter-Drinfeld module over Γ on R, such that

$$x_i \in R^{\chi_i}_{g_i}, \qquad 1 \leq i \leq \theta.$$

Note that the braiding is determined by

$$c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i, \text{ where } q_{ij} = \chi_j(g_i), \quad 1 \leq i,j \leq \theta.$$

Furthermore, R is a braided Hopf algebra with the comultiplication given by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$. To check that the comultiplication preserves (1.15) one uses (1.13); the verification for (1.16) is easy. We know [AS1] that $\dim R$ is infinite unless all the orders of q_{ii}'s are finite; in this last case, $\dim R = \prod_{1\leq i\leq\theta} N_i$. We also have $P(R) = \oplus_{1\leq i\leq\theta} \Bbbk x_i$.

1.5. **Biproducts, or bosonizations.**

Let A, H be Hopf algebras and $\pi : A \to H$ and $\iota : H \to A$ Hopf algebra homomorphisms. Assume that $\pi\iota = \mathrm{id}_H$, so that π is surjective, and ι is injective. By analogy with elementary group theory, one would like to reconstruct A from H and the kernel of π as a semidirect product. However, the natural candidate for the kernel of π is the algebra of coinvariants

$$R := A^{\mathrm{co}\,\pi} = \{a \in A : (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$$

which is *not*, in general, a Hopf algebra. Instead, R is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with the following structure:

- The action $\cdot$ of H on R is the restriction of the adjoint action (composed with ι).
- The coaction is $(\pi \otimes \mathrm{id})\Delta$.
- R is a subalgebra of A.
- The comultiplication is $\Delta_R(r) = r_{(1)}\iota\pi\mathcal{S}(r_{(2)}) \otimes r_{(3)}$, for all $r \in R$.

Given a braided Hopf algebra R in ${}^H_H\mathcal{YD}$, one can consider the *bosonization* or *biproduct* of R by H [Ra], [Mj2]. This is a usual Hopf algebra $R\#H$, with underlying vector space $R \otimes H$, whose multiplication and comultiplication are given by

$$\begin{aligned}(r\#h)(s\#f) &= r(h_{(1)} \cdot s)\#h_{(2)}f,\\ \Delta(r\#h) &= r^{(1)}\#(r^{(2)})_{(-1)}h_{(1)} \otimes (r^{(2)})_{(0)}\#h_{(2)}.\end{aligned} \tag{1.17}$$

The maps $\pi : R\#H \to H$ and $\iota : H \to R\#H$, $\pi(r\#h) = \epsilon(r)h$, $\iota(h) = 1\#h$, are Hopf algebra homomorphisms; we have $R = \{a \in R\#H : (\mathrm{id} \otimes\pi)\Delta(a) = a \otimes 1\}$.

Conversely, if A and H are Hopf algebras as above and $R = A^{\mathrm{co}\,\pi}$ then $A \simeq R\#H$.

Let $\vartheta : A \to R$ be the map given by $\vartheta(a) = a_{(1)}\iota\pi\mathcal{S}(a_{(2)})$. Then

$$\vartheta(ab) = a_{(1)}\vartheta(b)\iota\pi\mathcal{S}(a_{(2)}), \tag{1.18}$$

for all $a, b \in A$, and $\vartheta(\iota(h)) = \varepsilon(h)$ for all $h \in H$; therefore, for all $a \in A$, $h \in H$, we have $\vartheta(a\iota(h)) = \vartheta(a)\varepsilon(h)$ and

$$\vartheta(\iota(h)a) = h \cdot \vartheta(a). \tag{1.19}$$

Notice also that ϑ induces a coalgebra isomorphism $A/A\iota(H)^+ \simeq R$. In fact, the isomorphism $A \to R\#H$ can be expressed explicitly as

$$a \mapsto \vartheta(a_{(1)})\#\pi(a_{(2)}), \qquad a \in A.$$

If A is a Hopf algebra, the adjoint representation ad of A on itself is given by

$$\mathrm{ad}\, x(y) = x_{(1)}y\mathcal{S}(x_{(2)}).$$

If R is a braided Hopf algebra in ${}^H_H\mathcal{YD}$, then there is also a braided adjoint representation ad_c of R on itself defined by

$$\mathrm{ad}_c x(y) = \mu(\mu \otimes \mathcal{S}_R)(\mathrm{id} \otimes c)(\Delta_R \otimes \mathrm{id})(x \otimes y),$$

where μ is the multiplication and $c \in \operatorname{End}(R \otimes R)$ is the braiding. Note that if $x \in \mathcal{P}(R)$ then the braided adjoint representation of x is just

$$\operatorname{ad}_c x(y) = \mu(\operatorname{id} - c)(x \otimes y) =: [x, y]_c. \tag{1.20}$$

For any $x, y \in R$, we call $[x, y]_c := \mu(\operatorname{id} - c)(x \otimes y)$ a *braided commutator.*

When $A = R\#H$, then for all $b, d \in R$,

$$\operatorname{ad}_{(b\#1)}(d\#1) = (\operatorname{ad}_c b(d))\#1. \tag{1.21}$$

1.6. **Some properties of braided Hopf algebras.**

In this Section, we first collect several useful facts about braided Hopf algebras in the category of Yetter-Drinfeld modules over an abelian group Γ. We begin with some identities on braided commutators.

In the following two Lemmata, R denotes a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Let $a_1, a_2, \dots \in R$ be elements such that $a_i \in R^{\chi_i}_{g_i}$, for some $\chi_i \in \widehat{\Gamma}$, $g_i \in \Gamma$.

Lemma 1.10. *(a).*

$$[[a_1, a_2]_c, a_3]_c + \chi_2(g_1) a_2 [a_1, a_3]_c = [a_1, [a_2, a_3]_c]_c + \chi_3(g_2)[a_1, a_3]_c a_2. \tag{1.22}$$

(b). If $[a_1, a_2]_c = 0$ and $[a_1, a_3]_c = 0$ then $[a_1, [a_2, a_3]_c]_c = 0$.

(c). If $[a_1, a_3]_c = 0$ and $[a_2, a_3]_c = 0$ then $[[a_1, a_2]_c, a_3]_c = 0$.

(d). Assume that $\chi_1(g_2)\chi_2(g_1)\chi_2(g_2) = 1$. Then

$$[[a_1, a_2]_c, a_2]_c = \chi_2(g_1)^{-1}\chi_1(g_2)[a_2, [a_2, a_1]_c]_c \tag{1.23}$$

Proof. Left to the reader. $\square$

The following technical Lemma will be used at a crucial point in Section 6.1.

Lemma 1.11. *Assume that $\chi_2(g_2) \neq -1$ and*

$$\chi_1(g_2)\chi_2(g_1)\chi_2(g_2) = 1, \tag{1.24}$$

$$\chi_2(g_3)\chi_3(g_2)\chi_2(g_2) = 1. \tag{1.25}$$

If

$$[a_2, a_2]_c, a_1]_c = 0, \tag{1.26}$$

$$[[a_2, a_2]_c, a_3]_c = 0, \tag{1.27}$$

$$[a_1, a_3]_c = 0, \tag{1.28}$$

then

$$[[[a_1, a_2]_c, a_3]_c, a_2]_c = 0. \tag{1.29}$$

Proof. We compute:

$$\begin{aligned}[[[a_1,a_2]_c,a_3]_c,a_2]_c &= a_1a_2a_3a_2 - \chi_2(g_1)\, a_2a_1a_3a_2 - \chi_3(g_1)\chi_3(g_2)\, a_3a_1a_2^2 \\ &+ \chi_3(g_1)\chi_3(g_2)\chi_2(g_1)\, a_3a_2a_1a_2 - \chi_2(g_1)\chi_2(g_2)\chi_2(g_3)\, a_2a_1a_2a_3 \\ &+ \chi_2(g_1)^2\chi_2(g_2)\chi_2(g_3)\, a_2^2a_1a_3 + \chi_2(g_1)\chi_2(g_2)\chi_2(g_3)\chi_3(g_1)\chi_3(g_2)\, a_2a_3a_1a_2 \\ &- \chi_2(g_1)^2\chi_2(g_2)\chi_2(g_3)\chi_3(g_1)\chi_3(g_2)\, a_2a_3a_2a_1.\end{aligned}$$

We index consecutively the terms in the right-hand side by roman numbers: $(I),\dots,(VIII)$. Then $(II)+(VII)=0$, by (1.25) and (1.28). Now,

$$\begin{aligned}(I) &= \frac{1}{\chi_3(g_2)(1+\chi_2(g_2))}a_1a_2^2a_3 + \frac{\chi_2(g_2)\chi_3(g_2)}{1+\chi_2(g_2)}a_1a_3a_2^2 \\ &= \frac{1}{\chi_3(g_2)(1+\chi_2(g_2))}a_1a_2^2a_3 + \frac{\chi_2(g_2)\chi_3(g_2)\chi_3(g_1)}{1+\chi_2(g_2)}a_3a_1a_2^2 \\ &= (I\,a)+(I\,b),\end{aligned}$$

by (1.27) and (1.28). By the same equations (1.27) and (1.28), we also have

$$\begin{aligned}(VIII) &= -\frac{\chi_2(g_1)^2\chi_2(g_2)\chi_2(g_3)\chi_3(g_1)}{1+\chi_2(g_2)}a_2^2a_3a_1 - \frac{\chi_2(g_1)^2\chi_2(g_2)\chi_2(g_3)\chi_3(g_1)\chi_3(g_2)^2}{1+\chi_2(g_2)}a_3a_2^2a_1 \\ &= -\frac{\chi_2(g_1)^2\chi_2(g_2)\chi_2(g_3)}{1+\chi_2(g_2)}a_2^2a_1a_3 - \frac{\chi_2(g_1)^2\chi_2(g_2)\chi_2(g_3)\chi_3(g_1)\chi_3(g_2)^2}{1+\chi_2(g_2)}a_3a_2^2a_1 \\ &= (VIII\,a)+(VIII\,b).\end{aligned}$$

We next use (1.26) to show that

$$(I\,a)+(V)+(VI)+(VIII\,a)=0,$$

$$(I\,b)+(III)+(IV)+(VIII\,b)=0.$$

In the course of the proof of these equalities, we need (1.24) and (1.25). This finishes the proof of (1.29). □

Let H be a Hopf algebra. Then the existence of an integral for finite-dimensional braided Hopf algebras implies

Lemma 1.12. *Let $R = \bigoplus_{n=0}^{N} R(n)$ be a finite-dimensional graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ with $R(N) \neq 0$. There exists $\lambda \in R(N)$ which is a left integral on R and such that $R(i) \otimes R(N-i) \to \Bbbk, \quad x \otimes y \mapsto \lambda(xy)$, is a non-degenerate pairing, for all $0 \leq i \leq N$. In particular,*

$$dim R(i) = dim R(N-i).$$

Proof. This is essentially due to Nichols [N, 1.5]. In this formulation, one needs the existence of non-zero integrals on R; this follows from [FMS]. See [AG, Prop. 3.2.2] for details. □

1.7. **The infinitesimal braiding of Hopf algebras whose coradical is a Hopf subalgebra.**

For the convenience of the reader, we first recall in this Section some basic definitions from coalgebra theory.

Definition 1.13. Let C be a coalgebra.

- $G(C) := \{x \in C \setminus \{0\} \mid \Delta(x) = x \otimes x\}$ is the set of all group-like elements of C.
- If $g, h \in G(C)$, then $x \in C$ is (g,h)-*skew primitive* if $\Delta(x) = x \otimes h + g \otimes x$. The space of all (g,h)-skew primitive elements of C is denoted by $\mathcal{P}(C)_{g,h}$. If C is a bialgebra or a braided bialgebra, and $g = h = 1$, then $P(C) = \mathcal{P}(C)_{1,1}$ is the space of *primitive* elements.
- The *coradical* of C is $C_0 := \sum D$, where D runs through all the simple subcoalgebras of C; it is the largest cosemisimple subcoalgebra of C. In particular, $\Bbbk G(C) \subseteq C_0$.
- C is *pointed* if $\Bbbk G(C) = C_0$.
- The *coradical filtration* of C is the ascending filtration $C_0 \subseteq C_1 \subseteq \cdots \subseteq C_j \subseteq C_{j+1} \subseteq \dots$, defined by $C_{j+1} := \{x \in C \mid \Delta(x) \in C_j \otimes C + C \otimes C_0\}$. This is a coalgebra filtration: $\Delta C_j \subseteq \sum_{0 \leq i \leq j} C_i \otimes C_{j-i}$; and it is exhaustive: $C = \bigcup_{n \geq 0} C_n$.
- A *graded coalgebra* is a coalgebra G provided with a grading $G = \oplus_{n \geq 0} G(n)$ such that $\Delta G(j) \subseteq \sum_{0 \leq i \leq j} G(i) \otimes G(j-i)$ for all $j \geq 0$.
- A *coradically graded* coalgebra [CM] is a graded coalgebra $G = \oplus_{n \geq 0} G(n)$ such that its coradical filtration coincides with the standard ascending filtration arising from the grading: $(\operatorname{gr} C)_n = \oplus_{m \leq n} \operatorname{gr} C(m)$. A *strictly graded* coalgebra [Sw] is a coradically graded coalgebra G such that $G(0)$ is one-dimensional.
- The graded coalgebra associated to the coalgebra filtration of C is $\operatorname{gr} C = \oplus_{n \geq 0} \operatorname{gr} C(n)$, where $\operatorname{gr} C(n) := C_n / C_{n-1}$, $n > 0$, $\operatorname{gr} C(0) := C_0$. It is a coradically graded coalgebra.

We shall need a basic technical fact on pointed coalgebras.

Lemma 1.14. [M1, 5.3.3]. *A morphism of pointed coalgebras which is injective in the first term of the coalgebra filtration, is injective.* □

Let now A be a Hopf algebra. We shall assume in what follows that the coradical A_0 is not only a subcoalgebra but a Hopf subalgebra of A; this is the case if A is pointed.

To study the structure of A, we consider its coradical filtration; because of our assumption on A, it is also an algebra filtration [M1]. Therefore, the associated graded coalgebra $\operatorname{gr} A$ is a graded Hopf algebra. Furthermore, $H := A_0 \simeq \operatorname{gr} A(0)$ is a Hopf subalgebra of $\operatorname{gr} A$; and the projection $\pi : \operatorname{gr} A \to \operatorname{gr} A(0)$ with kernel $\oplus_{n>0} \operatorname{gr} A(n)$, is a Hopf algebra map and a retraction of the inclusion. We can then apply the general remarks of Section 1.5. Let R be the algebra of coinvariants of π; R is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $\operatorname{gr} A$ can be reconstructed from R and H as a bosonization $\operatorname{gr} A \simeq R\#H$.

The braided Hopf algebra R is graded, since it inherits the gradation from $\operatorname{gr} A$: $R = \oplus_{n\geq 0} R(n)$, where $R(n) = \operatorname{gr} A(n) \cap R$. Furthermore, R is strictly graded; this means,

(a). $R(0) = \Bbbk 1$ (hence the coradical is trivial, *cf.* [Sw, Chapter 11]).

(b). $R(1) = P(R)$ (the space of primitive elements of R).

It is in general not true that a braided Hopf algebra R satisfying **(a)** and **(b)**, also satisfies

(c). R is generated as an algebra over $\Bbbk$ by $R(1)$.

A braided graded Hopf algebra satisfying **(a)**, **(b)** and **(c)** is called a Nichols algebra. In the next chapter we will discuss this notion in detail. Notice that the subalgebra R' of R generated by $R(1)$, a Hopf subalgebra of R, is indeed a Nichols algebra.

Definition 1.15. The braiding

$$c : V \otimes V \to V \otimes V,$$

of $V := R(1) = P(R)$ is called the *infinitesimal braiding* of A.

The graded braided Hopf algebra R is called the *diagram* of A.

The dimension of $V = P(R)$ is called the *rank* of A.

2. Nichols algebras

Let H be a Hopf algebra. In this Chapter, we discuss a functor $\mathfrak{B}$ from the category ${}^H_H\mathcal{YD}$ to the category of braided Hopf algebras in ${}^H_H\mathcal{YD}$; given a Yetter-Drinfeld module V, the braided Hopf algebra $\mathfrak{B}(V)$ is called the *Nichols algebra* of V.

The structure of a Nichols algebra appeared first in the paper "Bialgebras of type one" [N] of Nichols and was rediscovered later by several authors. In our language, a bialgebra of type one is just a bosonization $\mathfrak{B}(V)\#H$. Hence Nichols algebras are the H-coinvariant elements of bialgebras of type one, also called quantum symmetric algebras in [Ro2]. Several years after [N], Woronowicz defined Nichols algebras in his approach to "quantum differential calculus" [Wo]; again, they appeared as the invariant part of his "algebra of quantum differential forms". Lusztig's algebras $\mathfrak{f}$ [L3], defined by the non-degeneracy of a certain invariant bilinear form, are Nichols algebras. In fact Nichols algebras can always be defined by the non-degeneracy of an invariant bilinear form [AG]. The algebras $\mathfrak{B}(V)$ are called bitensor algebras in [Sbg].

In a sense, Nichols algebras are similar to symmetric algebras; indeed, both notions coincide in the trivial braided category of vector spaces, or more generally in any symmetric category (*e. g.* in the category of super vector spaces). But when the braiding is not a symmetry, a Nichols algebra could have a much richer structure. We hope that this will be clarified in the examples. On the other hand, Nichols algebras are also similar to universal enveloping algebras. However, in spite of the efforts of several authors, it is not clear to us how to achieve a compact, functorial definition of a "braided Lie algebra" from a Nichols algebra.

We believe that Nichols algebras are very interesting objects of an essentially new nature.

2.1. **Definition of Nichols algebras.**
We now present one of the main notions of this survey.

Definition 2.1. Let V be a Yetter-Drinfeld module over H. A braided graded Hopf algebra $R = \oplus_{n\geq 0} R(n)$ in ${}^H_H\mathcal{YD}$ is called a *Nichols algebra* of V if $\Bbbk \simeq R(0)$ and $V \simeq R(1)$ in ${}^H_H\mathcal{YD}$, and

(2.1) $P(R) = R(1)$,

(2.2) R is generated as an algebra by $R(1)$.

The dimension of V will be called the *rank* of R.

We need some preliminaries to show the existence and uniqueness of the Nichols algebra of V in ${}^H_H\mathcal{YD}$.

Let V be a Yetter-Drinfeld module over H. Then the tensor algebra $T(V) = \bigoplus_{n\geq 0} T(V)(n)$ of the vector space V admits a natural structure of a Yetter-Drinfeld module, since ${}^H_H\mathcal{YD}$ is a braided category. It is then an algebra in ${}^H_H\mathcal{YD}$. There exists a unique algebra map $\Delta : T(V) \to T(V)\underline{\otimes}T(V)$ such that $\Delta(v) = v\otimes 1 + 1\otimes v$, for all $v \in V$. For example, if $x, y \in V$, then

$$\Delta(xy) = 1\otimes xy + x\otimes y + x_{(-1)}\cdot y\otimes x_{(0)} + yx\otimes 1.$$

With this structure, $T(V)$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ with counit $\varepsilon : T(V) \to \Bbbk$, $\varepsilon(v) = 0$, if $v \in V$. To show the existence of the antipode, one notes that the coradical of the coalgebra $T(V)$ is $\Bbbk$, and uses a result of Takeuchi [M1, 5.2.10]. Hence all the braided bialgebra quotients of $T(V)$ in ${}^H_H\mathcal{YD}$ are braided Hopf algebras in ${}^H_H\mathcal{YD}$.

Let us consider the class $\mathfrak{S}$ of all $I \subset T(V)$ such that

- I is a homogeneous ideal generated by homogeneous elements of degree ≥ 2,
- I is also a coideal, *i. e.* $\Delta(I) \subset I\otimes T(V) + T(V)\otimes I$.

Note that we do *not* require that the ideals I are Yetter-Drinfeld submodules of $T(V)$. Let then $\widetilde{\mathfrak{S}}$ be the subset of $\mathfrak{S}$ consisting of all $I \in \mathfrak{S}$ which are Yetter-Drinfeld submodules of $T(V)$. The ideals

$$I(V) = \sum_{I\in\mathfrak{S}} I, \qquad \widetilde{I}(V) = \sum_{J\in\widetilde{\mathfrak{S}}} J$$

are the largest elements in $\mathfrak{S}$, respectively $\widetilde{\mathfrak{S}}$.

If $I \in \mathfrak{S}$ then $R := T(V)/I = \oplus_{n\geq 0} R(n)$ is a graded algebra and a graded coalgebra with

$$R(0) = \Bbbk, \qquad V \simeq R(1) \subset P(R).$$

If actually $I \in \widetilde{\mathfrak{S}}$, then R is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$.

We can show now existence and uniqueness of Nichols algebras.

Proposition 2.2. *Let $\mathfrak{B}(V) := T(V)/\widetilde{I}(V)$. Then the following hold:*

1. *$V = P(\mathfrak{B}(V))$, hence $\mathfrak{B}(V)$ is a Nichols algebra of V.*
2. *$I(V) = \widetilde{I}(V)$.*
3. *Let $R = \oplus_{n\geq 0} R(n)$ be a graded Hopf algebra in ${}^H_H\mathcal{YD}$ such that $R(0) = \Bbbk 1$ and R is generated as an algebra by $V := R(1)$. Then there exists a surjective map of graded Hopf algebras $R \to \mathfrak{B}(V)$, which is an isomorphism of Yetter-Drinfeld modules in degree 1.*
4. *Let $R = \oplus_{n\geq 0} R(n)$ be a Nichols algebra of V. Then $R \simeq \mathfrak{B}(V)$ as braided Hopf algebras in ${}^H_H\mathcal{YD}$.*
5. *Let $R = \bigoplus_{n\geq 0} R(n)$ be a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ with $R(0) = \Bbbk 1$ and $R(1) = P(R) = V$. Then $\mathfrak{B}(V)$ is isomorphic to the subalgebra $\Bbbk\langle V\rangle$ of R generated by V.*

Proof. 1. We have to show the equality $V = P(\mathfrak{B}(V))$. Let us consider the inverse image X in $T(V)$ of all homogeneous primitive elements of $\mathfrak{B}(V)$ in degree $n \geq 2$. Then X is a graded Yetter-Drinfeld submodule of $T(V)$, and for all $x \in X$, $\Delta(x) \in x\otimes 1 + 1\otimes x + T(V)\otimes \widetilde{I}(V) + \widetilde{I}(V)\otimes T(V)$. Hence the ideal generated by $\widetilde{I}(V)$ and X is in $\widetilde{\mathfrak{S}}$, and $X \subset \widetilde{I}(V)$ by the maximality of $\widetilde{I}(V)$. Hence the image of X in $\mathfrak{B}(V)$ is zero. This proves our claim since the primitive elements form a graded submodule.

2. We have to show that the surjective map $\mathfrak{B}(V) \to T(V)/I(V)$ is bijective. This follows from 1. and Lemma 1.14.

3. The kernel I of the canonical projection $T(V) \to R$ belongs to $\widetilde{\mathfrak{S}}$; hence $I \subseteq \widetilde{I}(V)$.

4. follows again from Lemma 1.14, as in 2.

5. follows from 4. $\square$

If U is a braided subspace of $V \in {}^H_H\mathcal{YD}$, that is a subspace such that $c(U\otimes U) \subset U\otimes U$, where c is the braiding of V, we can define $\mathfrak{B}(U) := T(U)/I(U)$ with the obvious meaning of $I(U)$. Then the description stated in Proposition 2.8 also applies to $\mathfrak{B}(U)$.

Corollary 2.3. *The assignment $V \mapsto \mathfrak{B}(V)$ is a functor from ${}^H_H\mathcal{YD}$ to the category of braided Hopf algebras in ${}^H_H\mathcal{YD}$.*

If U is a Yetter-Drinfeld submodule of V, or more generally if U is a braided subspace of V, then the canonical map $\mathfrak{B}(U) \to \mathfrak{B}(V)$ is injective.

Proof. If $\phi : U \to V$ is a morphism in ${}^H_H\mathcal{YD}$, then $T(\phi) : T(U) \to T(V)$ is a morphism of braided Hopf algebras. Since $T(\phi)(I(U))$ is a coideal and a Yetter-Drinfeld submodule of $T(V)$, the ideal generated by $T(\phi)(\widetilde{I}(U))$ is contained in $\widetilde{I}(V)$. Hence by Proposition 2.2, $\mathfrak{B}$ is a functor.

The second part of the claim follows from Proposition 2.2, 5. □

The duality between conditions (2.1) and (2.2) in the definition of Nichols algebra, emphasized by Parts 3 and 5 of Proposition 2.2, is explicitly stated in the following

Lemma 2.4. *Let $R = \oplus_{n\geq 0} R(n)$ be a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$; suppose finite-dimensional homogeneous components and $R(0) = \Bbbk 1$. Let $S = \oplus_{n\geq 0} R(n)^*$ be the graded dual of R. Then $R(1) = P(R)$ if and only if S is generated as an algebra by $S(1)$.*

Proof. See for instance [AS2, Lemma 5.5]. □

Example 2.5. Let F be a field of positive characteristic p. Let S be the (usual) Hopf algebra $F[x]/\langle x^{p^2}\rangle$ with $x \in P(S)$. Then $x^p \in P(S)$. Hence $R = S^*$ satisfies (2.2) but not (2.1).

Example 2.6. Let $S = \Bbbk[X] = \oplus_{n\geq 0} S(n)$ be a polynomial algebra in one variable. We consider S as a braided Hopf algebra in ${}^H_H\mathcal{YD}$, where $H = \Bbbk\Gamma$, Γ an infinite cyclic group with generator g, with action, coaction and comultiplication given by

$$\delta(X^n) = g^n \otimes X^n, \quad g \cdot X = qX, \quad \Delta(X) = X \otimes 1 + 1 \otimes X.$$

Here $q \in \Bbbk$ is a root of 1 of order N. That is, S is a so-called quantum line. Then S satisfies (2.2) but not (2.1) since X^N is also primitive. Hence the graded dual $R = S^d = \oplus_{n\geq 0} S(n)^*$ is a braided Hopf algebra satisfying (2.1) but not (2.2).

However, in characteristic 0 we do not know any finite dimensional example of a braided Hopf algebra satisfying (2.1) but not (2.2).

Conjecture 2.7. [AS2, Conjecture 1.4] *Any finite dimensional braided Hopf algebra in ${}^H_H\mathcal{YD}$ satisfying* (2.1) *also satisfies* (2.2). *(Recall that the base field $\Bbbk$ has characteristic zero.)*

The compact description of $\mathfrak{B}(V)$ in Lemma 2.2 shows that it depends only on the algebra and coalgebra structure of $T(V)$. Since the comultiplication of the tensor algebra was defined using the "twisted" multiplication of $T(V)\underline{\otimes}T(V)$, we see that $\mathfrak{B}(V)$ *depends as an algebra and coalgebra only on the braiding of V*. The explicit formula for the comultiplication of $T(V)$ leads to the following alternative description of $\mathfrak{B}(V)$.

2.2. **The braid group.**

Let us recall that the braid group $\mathbb{B}_n$ is presented by generators $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i \leq n-2,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i, , \quad 1 \leq i, j \leq n-2, \ |i-j| > 1.$$

Here are some basic well-known facts about the braid group.

There is a natural projection $\pi : \mathbb{B}_n \to \mathbb{S}_n$ sending σ_i to the transposition $\tau_i := (i, i+1)$ for all i. The projection π admits a set-theoretical section $s : \mathbb{S}_n \to \mathbb{B}_n$ determined by

$$s(\tau_i) = \sigma_i, \quad 1 \leq i \leq n-1,$$
$$s(\tau\omega) = s(\tau)s(\omega), \quad \text{if } \ell(\tau\omega) = \ell(\tau) + \ell(\omega).$$

Here ℓ denotes the lenght of an element of $\mathbb{S}_n$ with respect to the set of generators $\tau_1, \dots, \tau_{n-1}$. The map s is called the Matsumoto section. In other words, if $\omega = \tau_{i_1} \dots \tau_{i_M}$ is a reduced expression of $\omega \in \mathbb{S}_n$, then $s(\omega) = \sigma_{i_1} \dots \sigma_{i_M}$.

Let $q \in \mathbb{k}$, $q \neq 0$. The quotient of the group algebra $\mathbb{k}(\mathbb{B}_n)$ by the two-sided ideal generated by the relations

$$(\sigma_i - q)(\sigma_i + 1), \quad 1 \leq i \leq n-1,$$

is the so-called *Hecke algebra* of type A_n, denoted by $\mathcal{H}_q(n)$.

Using the section s, the following distinguished elements of the group algebra $\mathbb{k}\mathbb{B}_n$ are defined:

$$\mathfrak{S}_n := \sum_{\sigma \in \mathbb{S}_n} s(\sigma), \qquad \mathfrak{S}_{i,j} := \sum_{\sigma \in X_{i,j}} s(\sigma);$$

here $X_{i,j} \subset \mathbb{S}_n$ is the set of all (i,j)-shuffles. The element $\mathfrak{S}_n$ is called the quantum symmetrizer.

Given a braided vector space (V, c), there are representations of the braid groups $\rho_n : \mathbb{B}_n \to \operatorname{Aut}(V^{\otimes n})$ for any $n \geq 0$, given by

$$\rho_n(\sigma_i) = \mathrm{id} \otimes \dots \otimes \mathrm{id} \otimes c \otimes \mathrm{id} \otimes \dots \otimes \mathrm{id},$$

where c acts in the tensor product of the i and $i+1$ copies of V. By abuse of notation, we shall denote by $\mathfrak{S}_n$, $\mathfrak{S}_{i,j}$ also the corresponding endomorphisms $\rho(\mathfrak{S}_n)$, $\rho(\mathfrak{S}_{i,j})$ of $V^{\otimes n} = T^n(V)$.

If $C = \bigoplus_{n \geq 0} C(n)$ is a graded coalgebra with comultiplication Δ, we denote by $\Delta_{i,j} : C(i+j) \to C(i) \otimes C(j)$, $i, j \geq 0$, the (i,j)-graded component of the map Δ.

Proposition 2.8. *Let $V \in {}^H_H\mathcal{YD}$. Then*

$$\Delta_{i,j} = \mathfrak{S}_{i,j}, \tag{2.3}$$

$$\mathfrak{B}(V) = \bigoplus_{n \geq 0} T^n(V)/ker(\mathfrak{S}_n). \tag{2.4}$$

Proof. See for instance [Sbg]. □

This description of the relation of $\mathfrak{B}(V)$ does not mean that the relations are known. In general it is very hard to compute the kernels of the maps $\mathfrak{S}_n$ in concrete terms. For any braided vector space (V, c), we may define $\mathfrak{B}(V)$ by (2.4).

2.3. **Invariance under twisting.**
Twisting is a method to construct new Hopf algebras by "deforming" the comultiplication; originally due to Drinfeld [Dr2], it was adapted to Hopf algebras in [Re].

Let A be a Hopf algebra and $F \in A \otimes A$ be an invertible element. Let $\Delta_F := F\Delta F^{-1} : A \to A \otimes A$; it is again an algebra map. If

$$(1 \otimes F)(\mathrm{id} \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes \mathrm{id})(F), \tag{2.5}$$

$$(\mathrm{id} \otimes \varepsilon)(F) = (\varepsilon \otimes \mathrm{id})(F) = 1, \tag{2.6}$$

then A_F (the same algebra, but with comultiplication Δ_F) is again a Hopf algebra. We shall say that A_F is obtained from A via twisting by F; F is a cocycle in a suitable sense.

There is a dual version of the twisting operation, which amounts to a twist of the multiplication [DT]. Let A be a Hopf algebra and let $\sigma : A \times A \to \Bbbk$ be an invertible 2-cocycle[1], that is

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}),$$
$$\sigma(1, 1) = 1,$$

for all $x, y, z \in A$. Then A_σ – the same A but with the multiplication $\cdot_\sigma$ below – is again a Hopf algebra, where

$$x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}).$$

Assume now that H is a Hopf algebra, R is a braided Hopf algebra in ${}^H_H\mathcal{YD}$, and $A = R\#H$. Let $\pi : A \to H$ and $\iota : H \to A$ be the canonical projection and injection. Let $\sigma : H \times H \to \Bbbk$ be an invertible 2-cocycle, and define $\sigma_\pi : A \times A \to \Bbbk$ by

$$\sigma_\pi := \sigma(\pi \otimes \pi);$$

σ_π is an invertible 2-cocycle, with inverse $(\sigma^{-1})_\pi$. The maps $\pi : A_{\sigma_\pi} \to H_\sigma$, $\iota : H_\sigma \to A_{\sigma_\pi}$ are still Hopf algebra maps. Because the comultiplication is not changed, the space of coinvariants of π is R; this is a subalgebra of A_{σ_π} that we denote R_σ; the multiplication in R_σ is given by

$$x \cdot_\sigma y = \sigma(x_{(-1)}, y_{(-1)})x_{(0)}y_{(0)}, \qquad x, y \in R = R_\sigma. \tag{2.7}$$

Equation (2.7) follows easily using (1.17). Clearly, R_σ is a Yetter-Drinfeld Hopf algebra in ${}^{H_\sigma}_{H_\sigma}\mathcal{YD}$. The coaction of H_σ on R_σ is the same as the coaction of H on R, since the comultiplication was not altered. The explicit formula for the action of H_σ on R_σ can be written down; we shall do this only in the setting we are interested in.

Let $H = \Bbbk\Gamma$ be a group algebra; an invertible cocycle $\sigma : H \times H \to \Bbbk$ is uniquely determined by its restriction $\sigma : \Gamma \times \Gamma \to \Bbbk^\times$, a group cocycle with respect to the trivial action.

[1] Here "invertible" means that the associated linear map $\sigma : A \otimes A \to \Bbbk$ is invertible with respect to the convolution product.

Lemma 2.9. *Let Γ be an abelian group and let R be a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Let $\sigma : \Gamma \times \Gamma \to \Bbbk^{\times}$ be a cocycle. Let S be the subalgebra of R generated by $P(R)$. In the case $y \in S^{\eta}_{h}$, for some $h \in \Gamma$ and $\eta \in \widehat{\Gamma}$, the action of $H = H_\sigma$ on R_σ is*

$$g \rightharpoonup_{\sigma} y = \sigma(g,h)\sigma^{-1}(h,g)\eta(g)y, \qquad g \in \Gamma. \tag{2.8}$$

Hence, the braiding c_σ in R_σ is given in this case by

$$c_\sigma(x \otimes y) = \sigma(g,h)\sigma^{-1}(h,g)\eta(g)\, y \otimes x, \qquad x \in R_g, g \in \Gamma. \tag{2.9}$$

Therefore, for such x and y, we have

$$[x,y]_{c_\sigma} = \sigma(g,h)[x,y]_c. \tag{2.10}$$

Proof. To prove (2.8), it is enough to assume $y \in P(R)^{\eta}_{h}$.

Let $A = R\#H$; in A_{σ_π}, we have

$$\begin{aligned} g\cdot_\sigma y &= \sigma(g,\pi(y))\, g\, \sigma^{-1}(g,1) + \sigma(g,h)\, gy\, \sigma^{-1}(g,1) + \sigma(g,h)\, gh\, \sigma^{-1}(g,\pi(y)) \\ &= \sigma(g,h)gy; \\ y\cdot_\sigma g &= \sigma(\pi(y),g)\, g\, \sigma^{-1}(1,g) + \sigma(h,g)\, yg\, \sigma^{-1}(1,g) + \sigma(h,g)\, hg\, \sigma^{-1}(\pi(y),g) \\ &= \sigma(h,g)yg; \end{aligned}$$

hence

$$g\cdot_\sigma y = \sigma(g,h)gy = \sigma(g,h)\eta(g)yg = \sigma(g,h)\sigma^{-1}(h,g)\eta(g)y\cdot_\sigma g,$$

which is equivalent to (2.8). Now (2.9) follows at once, and (2.10) follows from (2.7) and (2.9):

$$[x,y]_{c_\sigma} = x\cdot_\sigma y - \cdot_\sigma c_\sigma(x \otimes y) = \sigma(g,h)xy - \sigma(g,h)\sigma^{-1}(h,g)\eta(g)\,\sigma(h,g)yx = \sigma(g,h)[x,y]_c.$$

□

The proof of the following Lemma is clear, since the comultiplication of a Hopf algebra is not changed by twisting.

Lemma 2.10. *Let H be a Hopf algebra and let R be a braided Hopf algebra in ${}^{H}_{H}\mathcal{YD}$. Let $\sigma : H \times H \to \Bbbk$ be an invertible 2-cocycle. Then R is a Nichols algebra if and only if R_σ is a Nichols algebra in ${}^{H_\sigma}_{H_\sigma}\mathcal{YD}$.* □

3. Types of Nichols algebras

We now discuss several examples of Nichols algebras. We are interested in explicit presentations, *e. g.* by generators and relations, of $\mathfrak{B}(V)$, for braided vector spaces in suitable classes, for instance, those of group type. We would also like to determine when $\mathfrak{B}(V)$ has finite dimension, or polynomial growth.

3.1. **Symmetries and braidings of Hecke type.** We begin with the simplest class of braided vector spaces.

Example 3.1. Let $\tau : V \otimes V \to V \otimes V$ be the usual transposition; the braided vector space (V, τ) can be realized as a Yetter-Drinfeld module over any Hopf algebra H, with trivial action and coaction. Then $\mathfrak{B}(V) = \operatorname{Sym}(V)$, the symmetric algebra of V.

The braided vector space $(V, -\tau)$, which can be realized *e. g.* in ${}^{\mathbb{Z}}_{\mathbb{Z}}\mathcal{YD}$, has $\mathfrak{B}(V) = \Lambda(V)$, the exterior algebra of V.

Example 3.2. Let $V = \oplus_{i\in\mathbb{Z}/2} V(i)$ be a super vector space and let $c : V \otimes V \to V \otimes V$ be the supersymmetry:

$$c(v \otimes w) = (-1)^{i.j} w \otimes v \qquad v \in V(i),\ w \in V(j).$$

Clearly, V can be realized as a Yetter-Drinfeld module over $\mathbb{Z}/2$. Then $\mathfrak{B}(V) \simeq \operatorname{Sym}(V(0)) \otimes \Lambda(V(1))$, the super-symmetric algebra of V.

The simple form of $\mathfrak{B}(V)$ in these examples can be explained in the following context.

Definition 3.3. We say that a braided vector space (V, c) is of *Hecke-type* with label $q \in \Bbbk$, $q \neq 0$, if

$$(c - q)(c + 1) = 0.$$

In this case, the representation of the braid group $\rho_n : \mathbb{B}_n \to \operatorname{Aut}(V^{\otimes n})$ factorizes through the Hecke algebra $\mathcal{H}_q(n)$, for all $n \geq 0$; *cf.* Section 2.2.

If $q = 1$, one says that c is a *symmetry.* Then ρ_n factorizes through the symmetric group $\mathbb{S}_n$, for all $n \geq 0$. The categorical version of symmetries is that of symmetric categories, see Section 1.1.

Proposition 3.4. *Let (V, c) be a braided vector space of Hecke-type with label q, wich is either 1 or not a root of 1. Then $\mathfrak{B}(V)$ is a quadratic algebra; that is, the ideal $I(V)$ is generated by $I(V)(2) = \operatorname{Ker}\mathfrak{S}_2$.*

Moreover, $\mathfrak{B}(V)$ is a Koszul algebra and its Koszul dual is the Nichols algebra $\mathfrak{B}(V^*)$ corresponding to the braided vector space $(V^*, q^{-1}c^t)$.

A nice exposition on Koszul algebras is [BGS, Chapter 2].

Proof. The argument for the first claim is taken from [AA, Prop. 3.3.1]. The image of the quantum symmetrizer $\mathfrak{S}_n$ in the Hecke algebra $\mathcal{H}_q(n)$ is $[n]_q! M_\varepsilon$, where M_ε satsfies the following properties:

$$M_\varepsilon^2 = M_\varepsilon, \quad M_\varepsilon c_i = c_i M_\varepsilon = q M_\varepsilon, \qquad 1 \leq i \leq n-1.$$

See for instance [HKW]. Now, we have to show that $\operatorname{Ker}\mathfrak{S}_n = T^n(V) \cap I$, where I is the ideal generated by $\operatorname{Ker}\mathfrak{S}_2 = \operatorname{Ker}(c+1) = \operatorname{Im}(c-q)$; but clearly $T^n(V) \cap I = \sum_i I^{n,i}$, where

$$I^{n,i} = T^{i-1}(V) \otimes \operatorname{Im}(c-q) \otimes T^{n-i-1}(V) = \operatorname{Im}(c_i - q).$$

It follows that $T^n(V) \cap I \subseteq \operatorname{Ker} \mathfrak{S}_n$, a fact that we already know from the general theory. But moreover, $T^n(V) \cap I$ is a $\mathcal{H}_q(n)$-submodule of $T^n(V)$ since

$$c_j(c_i - q) = (c_j - q)(c_i - q) + q(c_i - q).$$

This computation also shows that the action of $\mathcal{H}_q(n)$ on the quotient module $T^n(V)/T^n(V) \cap I$ is via the character that sends σ_i to q; hence M_ε acts on $T^n(V)/T^n(V) \cap I$ by an automorphism, and *a fortiori* $T^n(V) \cap I \supseteq \operatorname{Ker} \mathfrak{S}_n$. Having shown the first claim, the second claim is essentially a result from [Gu, Wa]; see also the exposition in [AA, Sections 3.3 and 3.4]. □

Example 3.5. Let $q \in \Bbbk^\times$, q is not a root of 1. The braided vector space $(V, q\tau)$ can be realized in ${}^{\mathbb{Z}}_{\mathbb{Z}}\mathcal{YD}$. It can be shown that $\mathfrak{B}(V) = T(V)$, the tensor algebra of V, for all q in an open set. Problem: Determine this open set.

It would be interesting to know whether other conditions on the minimal polynomial of a braiding have consequences on the structure of the corresponding Nichols algebra. The first candidate should be a braiding of BMW-type.

3.2. **Braidings of diagonal type.**

In this Section, (V, c) denotes a finite dimensional braided vector space of diagonal type; that is, V has a basis $x_1, \dots, x_\theta$ such that (1.11) holds for some non-zero q_{ij} in $\Bbbk$. Our first goal is to determine polynomial relations on the generators $x_1, \dots, x_\theta$ that should hold in $\mathfrak{B}(V)$. We look at polynomial expressions in these generators which are homogeneous of degree ≥ 2, and give rise to primitive elements in any braided Hopf algebra containing V inside its primitive elements. For related material, see [Kh].

Lemma 3.6. *Let R be a braided Hopf algebra in ${}^H_H\mathcal{YD}$, for some Hopf algebra H, such that $V \subseteq P(R)$ as braided vector spaces.*

(a). *If q_{ii} is a root of 1 of order $N > 1$ for some $i \in \{1, \dots, \theta\}$, then $x_i^N \in P(R)$.*

(b). *Let $1 \leq i, j \leq \theta, i \neq j$, such that $q_{ij}q_{ji} = q_{ii}^r$, where $0 \leq -r < \operatorname{ord} q_{ii}$ (which could be infinite). Then $(\operatorname{ad}_c x_i)^{1-r}(x_j)$ is primitive in R.*

Proof. (a) and (b) are consequences of the quantum binomial formula, see *e. g.* [AS2, Appendix] for (b). □

We apply these first remarks to $\mathfrak{B}(V)$ and see how conditions on the Nichols algebra induce conditions on the braiding.

Lemma 3.7. *Let $R = \mathfrak{B}(V)$.*

(a). *If q_{ii} is a root of 1 of order $N > 1$ then $x_i^N = 0$. In particular, if $\mathfrak{B}(V)$ is an integral domain, then $q_{hh} = 1$ or it is not a root of 1, for all h.*

(b). *If $i \neq j$, then $(\operatorname{ad}_c x_i)^r(x_j) = (r)!_{q_{ii}} \prod_{0 \leq k \leq r} \left(1 - q_{ii}^k q_{ij} q_{ji}\right) x_i^r x_j$.*

(c). *If* $i \neq j$ *and* $q_{ij}q_{ji} = q_{ii}^r$, *where* $0 \leq -r < \operatorname{ord} q_{ii}$ *(which could be infinite), then* $(\operatorname{ad}_c x_i)^{1-r}(x_j) = 0$.

(d). *If* $\mathfrak{B}(V)$ *has finite growth, then for all* $i \neq j$, *there exists* $r_{ij} > 0$ *such that* $(\operatorname{ad}_c x_i)^{r_{ij}}(x_j) = 0$.

(e). *If* $\mathfrak{B}(V)$ *is a domain of finite growth and there exists* k *such that* $q_{kk} = 1$, *then* $q_{ii} = 1$, *for all* $1 \leq i \leq \theta$, *and* $q_{ij}q_{ji} = 1$ *for all* $1 \leq i, j \leq \theta, i \neq j$.

(f). *If* $\mathfrak{B}(V)$ *has finite growth and* $i \neq j$, *then*
$\min\{r > 0 \mid (\operatorname{ad}_c x_i)^r(x_j) = 0\} = \min\left(\{\operatorname{ord} N_{ii}\} \bigcup \{k > 0 \mid q_{ij}q_{ji}q_{ii}^k = 1\}\right).$

Proof. Parts (a) and (c) follow from Lemma 3.6. Part (b) is apparently well-known and appears in [Ro2] without proof. Part(d) is [Ro2, Lemma 20]. The rest follows from these parts. □

We now discuss how the twisting operation, *cf.* Section 2.3, affects Nichols algebras of diagonal type.

Definition 3.8. We shall say that two braided vector spaces (V, c) and (W, d) of diagonal type, with matrices (q_{ij}) and $(\widehat{q}_{ij})$, are *twist-equivalent* if $\dim V = \dim W$ and, for all i, j, $q_{ii} = \widehat{q}_{ii}$ and

$$q_{ij}q_{ji} = \widehat{q}_{ij}\widehat{q}_{ji}. \tag{3.1}$$

Proposition 3.9. *Let* (V, c) *and* (W, d) *be two twist-equivalent braided vector spaces of diagonal type, with matrices* (q_{ij}) *and* $(\widehat{q}_{ij})$*; say with respect to basis* $x_1, \dots x_\theta$, *resp.* $\widehat{x}_1, \dots \widehat{x}_\theta$. *Then there exists a linear isomorphism* $\psi : \mathfrak{B}(V) \to \mathfrak{B}(W)$ *such that*

$$\psi(x_i) = \widehat{x}_i, \qquad 1 \leq i \leq \theta. \tag{3.2}$$

Proof. Let Γ be the free abelian group of rank θ, with basis $g_1, \dots, g_\theta$. We define characters $\chi_1, \dots, \chi_\theta$, $\widehat{\chi}_1, \dots, \widehat{\chi}_\theta$ of Γ by

$$\chi_i(g_j) = q_{ji}, \qquad \widehat{\chi}_i(g_j) = \widehat{q}_{ji}, \qquad 1 \leq i, j \leq \theta.$$

We consider V, W as Yetter-Drinfeld modules over Γ by declaring $x_i \in V_{g_i}^{\chi_i}$, $\widehat{x}_i \in V_{g_i}^{\widehat{\chi}_i}$. Hence, $\mathfrak{B}(V), \mathfrak{B}(W)$ are braided Hopf algebras in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

Let $\sigma : \Gamma \times \Gamma \to \Bbbk^\times$ be the unique bilinear form such that

$$\sigma(g_i, g_j) = \begin{cases} \widehat{q}_{ij}q_{ij}^{-1}, & i \leq j, \\ 1, & i > j; \end{cases} \tag{3.3}$$

it is a group cocycle. We claim that $\varphi : W \to \mathfrak{B}(V)_\sigma(1)$, $\varphi(\widehat{x}_i) = x_i$, $1 \leq i \leq \theta$, is an isomorphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. It clearly preserves the coaction; for the action, we assume $i \leq j$ and compute

$$\begin{aligned} g_j \cdot_\sigma x_i &= \sigma(g_j, g_i)\sigma^{-1}(g_i, g_j)\chi_i(g_j)x_i \\ &= (\widehat{q}_{ij})^{-1}q_{ij}q_{ji}x_i = \widehat{q}_{ji}x_i, \end{aligned}$$

and also

$$\begin{aligned} g_i \cdot_\sigma x_j &= \sigma(g_i, g_j)\sigma^{-1}(g_j, g_i)\chi_j(g_i)x_j \\ &= \widehat{q}_{ij} q_{ij}^{-1} q_{ij} x_j = \widehat{q}_{ij} x_j, \end{aligned}$$

where we have used (2.8) and the hypothesis (3.1). This proves the claim. By Lemma 3.6, φ extends to an isomorphism $\varphi : \mathfrak{B}(W) \to \mathfrak{B}(V)_\sigma$; $\psi = \varphi^{-1}$ is the map we are looking for. $\square$

Remarks 3.10. (i). The map ψ defined in the proof is much more than just linear; by (2.7) and (2.10), we have

$$\psi(xy) = \sigma(g,h)\psi(x)\psi(y), \qquad x \in \mathfrak{B}(V)_g, \quad y \in \mathfrak{B}(V)_h; \tag{3.4}$$

$$\psi([x,y]_c) = \sigma(g,h)[\psi(x), \psi(y)]_c, \qquad x \in V_g^\chi, \quad y \in \mathfrak{B}(V)_h^\eta. \tag{3.5}$$

(ii). A braided vector space (V, c) of diagonal type, with matrix (q_{ij}), is twist-equivalent to (W, d), with a *symmetric* matrix $(\widehat{q}_{ij})$.

3.3. **Braidings of diagonal type but not Cartan.**
In the next Chapter, we shall concentrate on braidings of Cartan type. There are a few examples of Nichols algebras $\mathfrak{B}(V)$ of finite group type and rank 2, which are *not* of Cartan type, but where we know that the dimension is finite. We now list them all, following [N, Gñ3]. The braided vector space is necessarily of diagonal type; we shall give the matrix Q of the braiding, the constraints on their entries and the dimension d of $\mathfrak{B}(V)$. Below, ω, resp. ζ, denotes an arbitrary primitive third root of 1, resp. different from 1.

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & -1 \end{pmatrix}; \quad q_{11}^{-1} = q_{12}q_{21} \neq 1; \qquad d = 4\operatorname{ord}(q_{12}q_{21}). \tag{3.6}$$

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & \omega \end{pmatrix}; \quad q_{11}^{-1} = q_{12}q_{21} \neq \pm 1, \omega^{-1}; \qquad d = 9\operatorname{ord}(q_{11})\operatorname{ord}(q_{12}q_{21}\omega). \tag{3.7}$$

$$\begin{pmatrix} -1 & q_{12} \\ q_{21} & \omega \end{pmatrix}; \quad q_{12}q_{21} = -1; \qquad d = 108. \tag{3.8}$$

$$\begin{pmatrix} -1 & q_{12} \\ q_{21} & \omega \end{pmatrix}; \quad q_{12}q_{21} = \omega; \qquad d = 72. \tag{3.9}$$

$$\begin{pmatrix} -1 & q_{12} \\ q_{21} & \omega \end{pmatrix}; \quad q_{12}q_{21} = -\omega; \qquad d = 36. \tag{3.10}$$

$$(3.11)\quad \begin{pmatrix} -1 & q_{12} \\ q_{21} & \zeta \end{pmatrix}; \quad q_{12}q_{21} = \zeta^{-2}; \qquad\qquad d = 4\operatorname{ord}(\zeta)\operatorname{ord}(-\zeta^{-1}).$$

3.4. **Braidings of finite non-abelian group type.**

We begin with a class of examples studied in [MiS].

Let Γ be a group and $T \subset \Gamma$ a subset such that for all $g \in \Gamma, t \in T, gtg^{-1} \in T$. Thus T is a union of conjugacy classes of Γ. Let $\phi : \Gamma \times T \to \Bbbk \setminus \{0\}$ be a function such that for all $g, h \in \Gamma$ and $t \in T$,

$$(3.12)\quad \phi(1,t) = 1,$$

$$(3.13)\quad \phi(gh,t) = \phi(g, hth^{-1})\phi(h,t).$$

We can then define a Yetter-Drinfeld module $V = V(\Gamma, T, \phi)$ over Γ with $\Bbbk$-basis x_t, $t \in T$, and action and coaction of W given by

$$(3.14)\quad gx_t = \phi(g,t)x_{gtg^{-1}},$$

$$(3.15)\quad \delta(x_t) = t \otimes x_t$$

for all $g \in \Gamma, t \in T$.

Conversely, if the function ϕ defines a Yetter-Drinfeld module on the vector space V by (3.14), (3.15), then ϕ satisfies (3.12), (3.13).

Note that the braiding c of $V(\Gamma, T, \phi)$ is determined by

$$c(x_s \otimes x_t) = \phi(s,t)x_{sts_{-1}} \otimes x_t \text{ for all } s, t \in T,$$

hence by the values of ϕ on $T \times T$.

The main examples come from the theory of Coxeter groups ([BL, Chapitre IV]). Let S be a subset of the group W of elements of order 2. For all $s, s' \in S$ let $m(s,s')$ be the order of ss'. (W, S) is called a *Coxeter system* and W a *Coxeter group* if W is generated by S with defining relations $(ss')^{m(s,s')} = 1$ for all $s, s' \in S$ such that $m(s,s')$ is finite.

Let (W, S) be a Coxeter system. For any $g \in W$ there is a sequence $(s_1, \dots, s_q)$ of elements in S with $g = s_1 \cdot \dots \cdot s_q$. If q is minimal among all such representations, then $q = l(g)$ is called the *length* of g, and $(s_1, \dots, s_q)$ is a *reduced representation* of g.

Definition 3.11. Let (W, S) be a Coxeter system, and $T = \{gsg^{-1} \mid g \in W, s \in S\}$. Define $\phi : W \times T \to \Bbbk \setminus \{0\}$ by

$$(3.16)\qquad \phi(g,t) = (-1)^{l(g)} \text{ for all } g \in W, t \in T.$$

This ϕ satisfies (3.12) and (3.13). Thus we have associated to each Coxeter group the Yetter-Drinfeld module $V(W, T, \phi) \in {}^W_W\mathcal{YD}$.

The functions ϕ satisfying (3.12), (3.13) can be constructed up to a diagonal change of the basis from characters of the centralizers of elements in the conjugacy classes. This is a special case of the description of the simple modules in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ (see [W] and also [L4]); the equivalent classification of the simple Hopf bimodules over Γ was obtained in [DPR] (over $\Bbbk$) and then in [Ci] (over any field).

Let t be an element in Γ. We denote by $\mathcal{O}_t$ and Γ^t the conjugacy class and the centralizer of t in Γ. Let U be any left $\Bbbk\Gamma^t$-module. It is easy to see that the induced representation $V = \Bbbk\Gamma \otimes_{\Bbbk\Gamma^t} U$ is a Yetter-Drinfeld module over Γ with the induced action of Γ and the coaction

$$\delta : V \to \Bbbk\Gamma \otimes V, \quad \delta(g \otimes u) = gtg^{-1} \otimes g \otimes u \text{ for all } g \in \Gamma, u \in U.$$

We will denote this Yetter-Drinfeld module over Γ by $M(t, U)$.

Assume that Γ is finite. Then $V = M(t, U)$ is a simple Yetter-Drinfeld module if U is a simple representation of Γ^t, and each simple module in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ has this form. If we take from each conjugacy class one element t and non-isomorphic simple Γ^t-modules, any two of these simple Yetter-Drinfeld modules are non-isomorphic.

Let $s_i, 1 \le i \le \theta$, be a complete system of representatives of the residue classes of Γ^t. We define $t_i = s_i t s_i^{-1}$ for all $1 \le i \le \theta$. Thus

$$\Gamma/\Gamma^t \to \mathcal{O}_t, \quad s_i\Gamma^t \mapsto t_i, 1 \le i \le \theta,$$

is bijective, and as a vector space, $V = \bigoplus_{1 \le i \le \theta} s_i \otimes U$. For all $g \in \Gamma$ and $1 \le i \le \theta$, there is a uniquely determined $1 \le j \le \theta$ with $s_j^{-1} g s_i \in \Gamma^t$, and the action of g on $s_i \otimes u, u \in U$, is given by

$$g s_i \otimes u = s_j \otimes (s_j^{-1} g s_i) u.$$

In particular, if U is a one-dimensional Γ^t-module with basis u and action $hu = \chi(g)u$ for all $h \in \Gamma^t$ defined by the character $\chi : \Gamma^t \to \Bbbk \setminus \{0\}$, then V has a basis $x_i = s_i \otimes u, 1 \le i \le \theta$, and the action and coaction of Γ are given by

$$g x_i = \chi(s_j^{-1} g s_i) x_j \quad \text{ and } \delta(x_i) = x_j,$$

if $s_j^{-1} g s_i \in \Gamma^t$. Note that $g t_i g^{-1} = t_j$. Hence the module we have constructed is $V(\Gamma, T, \phi)$, where T is the conjugacy class of t, and ϕ is given by $\phi(g, t_i) = \chi(s_j^{-1} g s_i)$.

We now construct another example of a function ϕ satisfying (3.12), (3.13).

Definition 3.12. Let T be the set of all transpositions in the symmetric group $\mathbb{S}_n$. Define $\phi : \mathbb{S}_n \times T \to \Bbbk \setminus \{0\}$ for all $g \in \mathbb{S}_n, 1 \le i < j \le n$, by

$$\chi(g, (ij)) = \begin{cases} 1 & , \text{ if } g(i) < g(j), \\ -1 & , \text{ if } g(i) > g(j). \end{cases} \tag{3.17}$$

Let $t = (12)$. The centralizer of t in $\mathbb{S}_n$ is $\langle (34), (45), \dots, (n-1, n) \rangle \cup \langle (34), (45), \dots, (n-1, n) \rangle (12)$. Let χ be the character of $(\mathbb{S}_n)^t$ with $\chi((ij)) = 1$ for all $3 \le i < j \le n$, and $\chi((12)) = -1$. Then the function ϕ definied by (3.17) is given by the character χ as described above.

Up to base change we have found all functions ϕ satisfying (3.12), (3.13) for $\Gamma = \mathbb{S}_n$, where T is the conjugacy class of all transpositions, and $\phi(t,t) = -1$ for all $t =\in T$. The case $\phi(t,t) = 1$ for some $t \in T$ would lead to a Nichols algebra $\mathfrak{B}(V)$ of infinite dimension.

To determine the structure of $\mathfrak{B}(V)$ for the Yetter-Drinfeld modules defined by the functions ϕ in (3.16) and (3.17) seems to be a fundamental and very hard combinatorial problem. Only a few partial results are known [MiS], [FK], [FP].

We want to describe some of the relations of $\mathfrak{B}(V)$. Let us first recall an important tool introduced by Nichols [N, 3.3] to deal with $\mathfrak{B}(V)$ over group algebras $\Bbbk\Gamma$ without knowing the relations explicitly (see [MiS, 2.4]). Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ be of finite dimension θ. We choose a basis $x_i \in V_{g_i}$ with $g_i \in \Gamma, 1 \leq i \leq \theta$, of Γ-homogeneous elements. Let $I \in \mathfrak{S}$ and $R = T(V)/I$ (see Section 2.1). Then R is a graded Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with $R(0) = \Bbbk 1$ and $R(1) = V$. For all $1 \leq i \leq \theta$ let $\sigma_i : R \to R$ be the algebra automorphism given by the action of g_i.

Proposition 3.13. *1) For all $1 \leq i \leq \theta$, there exists a uniquely determined (id, σ_i)-derivation $D_i : R \to R$ with $D_i(x_j) = \delta_{i,j}$ (Kronecker δ) for all j.*
2) $I = I(V)$, that is $R = \mathfrak{B}(V)$, if and only if $\bigcap_{i=1}^{\theta} ker(D_i) = \Bbbk 1$. □

These skew derivations are very useful to find relations of $\mathfrak{B}(V)$. We consider some special cases.

Example 3.14. Let $W = \mathbb{S}_n, n \geq 2$, and $T = \{(ij) \mid 1 \leq i < j \leq n\}$ the set of all transpositions. Define χ by (3.16) and let $V = V(W,T,\phi)$. Then the following relations hold in $\mathfrak{B}(V)$ for all $1 \leq i < j \leq n, 1 \leq k < l \leq n$:

$$x_{(ij)}^2 = 0. \tag{3.18}$$

$$\text{If } \{i,j\} \cap \{k,l\} = \emptyset, \text{ then } \quad x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)} = 0. \tag{3.19}$$

$$\text{If } i < j < k, \text{ then } \quad x_{(ij)}x_{(jk)} + x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)} = 0, \tag{3.20}$$

$$x_{(jk)}x_{(ij)} + x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)} = 0.$$

Example 3.15. Let $W = \mathbb{S}_n, n \geq 2$, and $T = \{(ij) \mid 1 \leq i < j \leq n\}$ the set of all transpositions. Define ϕ by (3.17) and let $V = V(W,T,\phi)$. Then the following relations hold in $\mathfrak{B}(V)$ for all $1 \leq i < j \leq n, 1 \leq k < l \leq n$:

$$x_{(ij)}^2 = 0. \tag{3.21}$$

$$\text{If } \{i,j\} \cap \{k,l\} = \emptyset, then \quad x_{(ij)}x_{(kl)} - x_{(kl)}x_{(ij)} = 0. \tag{3.22}$$

$$\text{If } i < j < k, then \quad x_{(ij)}x_{(jk)} - x_{(jk)}x_{(ik)} - x_{(ik)}x_{(ij)} = 0, \tag{3.23}$$

$$x_{(jk)}x_{(ij)} - x_{(ik)}x_{(jk)} - x_{(ij)}x_{(ik)} = 0.$$

The algebras $\widetilde{\mathfrak{B}}(V)$ generated by all $x_{(ij)}, 1 \leq i < j \leq n$, with the quadratic relations in the examples 3.14 resp. 3.15 are braided Hopf algebras in the category of Yetter-Drinfeld modules over $\mathbb{S}_n$. $\widetilde{\mathfrak{B}}(V)$ in example 3.15 is the algebra $\mathcal{E}_n$ introduced by Fomin and Kirillov in [FK] to describe the cohomology ring of the flag variety. We believe that indeed the quadratic relations in the examples 3.14 and 3.15 are defining relations for $\mathfrak{B}(V)$, that is $\widetilde{\mathfrak{B}}(V) = \mathfrak{B}(V)$ in these cases.

It was noted in [MiS] that the conjecture in [FK] about the "Poincaré-duality" of the dimensions of the homogeneous components of the algebras $\mathcal{E}_n$ (in case they are finite-dimensional) follows from the braided Hopf algebra structure as a special case of Lemma 1.12.

Another result about the algebras $\mathcal{E}_n$ by Fomin and Procesi [FP] says that $\mathcal{E}_{n+1}$ is a free module over $\mathcal{E}_n$, and $P_{\mathcal{E}_n}$ divides $P_{\mathcal{E}_{n+1}}$, where P_A denotes the Hilbert series of a graded algebra A. The proof in [FP] used the relations in Example 3.15.

This result is in fact a special case of a very general splitting theorem for braided Hopf algebras in [MiS, Theorem3.2] which is an application of the fundamental theorem for Hopf modules in the braided situation. This splitting theorem generalizes the main result of [Gñ2].

In [MiS] some partial results are obtained about the structure of the Nichols algebras of Coxeter groups. In particular

Theorem 3.16. [MiS] *Let (W, S) be a Coxeter system, T the set of all W-conjugates of elements in S, ϕ defined by (3.16), $V = V(W, T, \phi)$ and $R = \mathfrak{B}(V)$. For all $g \in W$, choose a reduced representation $g = s_1 \cdots s_q$, $s_1, \cdots, s_q \in S$, of g, and define*

$$x_g = x_{s_1} \cdots x_{s_q}.$$

Then the subalgebra of R generated by all $x_s, s \in S$ has the $\Bbbk$-basis $x_g, g \in W$. For all $g \in W$, the g-homogeneous component R_g of R is isomorphic to R_1.

If R is finite-dimensional, then W is finite and $dim(R) = ord(W) dim(R_1)$. □

This theorem holds for more general functions ϕ, in particular for S_n and ϕ defined in (3.17).

Let (W, S) be a Coxeter system and $V = V(W, T, \phi)$ as in Theorem [MiS]. Then $\mathfrak{B}(V)$ was computed in [MiS] in the following cases:

- $W = \mathbb{S}_3, S = \{(12), (23)\}$: The relations of $\mathfrak{B}(V)$ are the quadratic relations in Example 3.14, and $\dim \mathfrak{B}(V) = 12$.
- $W = \mathbb{S}_4, S = \{(12), (23), (34)\}$: The relations of $\mathfrak{B}(V)$ are the quadratic relations in Example 3.14, and $\dim \mathfrak{B}(V) = 24 \cdot 24$.
- $W = D_4$, the dihedral group of order 8, $S = \{t, t'\}$, where t, t' are generators of D_4 of order 2 such that tt' is of order 4. There are quadratic relations and relations of order 4 defining $\mathfrak{B}(V)$, and $\dim \mathfrak{B}(V) = 64$.

In all three cases the integral, which is the longest non-zero word in the generators x_t, can be described in terms of the longest element in the Coxeter group. In all the other cases it is not known whether $\mathfrak{B}(V)$ is finite-dimensional.

In [FK] it is shown that

- $\dim(\mathcal{E}_3) = 12$.

- $\dim(\mathcal{E}_4) = 24 \cdot 24$.
- $\dim(\mathcal{E}_5)$ is finite by using a computer program.

Again, for the other cases $n > 5$ it is not known whether $\mathcal{E}_n$ is finite-dimensional.

In [Gñ3, 5.3.2] another example of a finite-dimensional Nichols algebra of a braided vector space (V, c) of finite group type is given with $\dim(V) = 4$ and $\dim(\mathfrak{B}(V)) = 72$. The defining relations of $\mathfrak{B}(V)$ are quadratic and of order 6.

By a result of Montgomery [M2], any pointed Hopf algebra B can be decomposed as a crossed product

$$B \simeq A \#_\sigma \Bbbk G, \quad \sigma \text{ a 2-cocycle}$$

of A, its link-indecomposable component containing 1 (a Hopf subalgebra) and a group algebra $\Bbbk G$. However, the structure of such link-indecomposable Hopf algebras A, in particular in the case when A is finite-dimensional and the group of its group-like elements $G(A)$ is non-abelian, is not known. To define *link-indecomposable pointed Hopf algebras*, we recall the definition of the *quiver* of A in [M2]. The vertices of the quiver of A are the elements of the group $G(A)$; for $g, h \in G(A)$, there exists an arrow from h to g if $P_{g,h}(A)$ is non-trivial, that is if $\Bbbk(g - h) \subsetneqq P_{g,h}(A)$. The Hopf algebra A is called link-indecomposable, if its quiver is connected as an undirected graph.

Definition 3.17. Let Γ be a finite group and $V \in {}^\Gamma_\Gamma\mathcal{YD}$. V is called *link-indecomposable* if the group Γ is generated by the elements g with $V_g \neq 0$.

By [MiS, 4.2], $V \in {}^\Gamma_\Gamma\mathcal{YD}$ is link-indecomposable if and only if the Hopf algebra $\mathfrak{B}(V)\#\Bbbk\Gamma$ is link-indecomposable.

Thus by the examples constructed above, there are link-indecomposable, finite-dimensional pointed Hopf algebras A with $G(A)$ isomorphic to $S_n, 3 \leq n \leq 5$, or to D_4.

Question 3.18. Which finite groups are isomorphic to $G(A)$ for some finite-dimensional, link-indecomposable pointed Hopf algebra A? Are there finite groups which do not occur in this form?

Finally, let us come back to the simple Yetter-Drinfeld modules $V = M(t, U) \in {}^\Gamma_\Gamma\mathcal{YD}$, where $t \in \Gamma$ and U is a simple left Γ^t-module of dimension > 1. In this case, strong restrictions are known for $\mathfrak{B}(V)$ to be finite-dimensional. By Schur's lemma, t acts as a scalar q on U.

Proposition 3.19. [Gñ3, 3.1] *Assume that* $\dim \mathfrak{B}(V)$ *is finite. If* $\dim U \geq 3$, *then* $q = -1$; *and if* $\dim U = 2$, *then* $q = -1$ *or* q *is a root of unity of order three.* □

In the proof of Proposition 3.19, a result of Lusztig on braidings of Cartan type (see [AS2, Theorem 3.1]) is used. In a similar way Graña showed

Proposition 3.20. [Gñ3, 3.2] *Let* Γ *be a finite group of odd order, and* $V \in {}^\Gamma_\Gamma\mathcal{YD}$. *Assume that* $\mathfrak{B}(V)$ *is finite-dimensional. Then the multiplicity of any simple Yetter-Drinfeld module over* Γ *as a direct summand in* V *is at most 2.*

In particular, up to isomorphism there are only finitely many Yetter-Drinfeld modules $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that $\mathfrak{B}(V)$ is finite-dimensional. □

The second statement in Proposition 3.20 was a conjecture in a preliminary version of [AS2].

3.5. **Braidings of (infinite) group type.**

We briefly mention Nichols algebras over a free abelian group of finite rank with a braiding which is not diagonal.

Example 3.21. Let $\Gamma = \langle g \rangle$ be a free group in one generator. Let $\mathcal{V}(t,2)$ be the Yetter-Drinfeld module of dimension 2 such that $\mathcal{V}(t,2) = \mathcal{V}(t,2)_g$ and the action of g on $\mathcal{V}(t,2)$ is given, in a basis x_1, x_2, by

$$g \cdot x_1 = tx_1, \qquad g \cdot x_2 = tx_2 + x_1.$$

Here $t \in \Bbbk^{\times}$. Then:

(a). If t is not a root of 1, then $\mathfrak{B}(\mathcal{V}(t,2)) = T(\mathcal{V}(t,2))$.

(b). If $t = 1$, then $\mathfrak{B}(\mathcal{V}(1,2)) = \Bbbk < x_1, x_2 | x_1x_2 = x_2x_1 + x_1^2 >$; this is the well-known Jordanian quantum plane.

Example 3.22. More generally, if $t \in \Bbbk^{\times}$, let $\mathcal{V}(t,\theta)$ be the Yetter-Drinfeld module of dimension $\theta \geq 2$ such that $\mathcal{V}(t,\theta) = \mathcal{V}(t,\theta)_g$ and the action of g on $\mathcal{V}(t,\theta)$ is given, in a basis $x_1, \dots, x_\theta$, by

$$g \cdot x_1 = tx_1, \qquad g \cdot x_j = tx_j + x_{j-1}, \quad 2 \leq j \leq \theta.$$

Note there is an inclusion of Yetter-Drinfeld modules $\mathcal{V}(t,2) \hookrightarrow \mathcal{V}(t,\theta)$; hence, if t is not a root of 1, $\mathfrak{B}(\mathcal{V}(t,\theta))$ has exponential growth.

Question 3.23. Compute $\mathfrak{B}(\mathcal{V}(1,\theta))$; does it have finite growth?

4. Nichols algebras of Cartan type

We now discuss fundamental examples of Nichols algebras of diagonal type that come from the theory of quantum groups.

We first need to fix some notation. Let $A = (a_{ij})_{1 \leq i,j \leq \theta}$ be a generalized symmetrizable Cartan matrix [K]; let $(d_1, \dots, d_\theta)$ be positive integers such that $d_i a_{ij} = d_j a_{ji}$. Let g be the Kac-Moody algebra corresponding to the Cartan matrix A. Let $\mathcal{X}$ be the set of connected components of the Dynkin diagram corresponding to it. For each $I \in \mathcal{X}$, we let $\mathfrak{g}_I$ be the Kac-Moody Lie algebra corresponding to the generalized Cartan matrix $(a_{ij})_{i,j \in I}$ and $\mathfrak{n}_I$ be the Lie subalgebra of $\mathfrak{g}_I$ spanned by all its positive roots. We omit the subindex I when $I = \{1, \dots, \theta\}$. We assume that for each $I \in \mathcal{X}$, there exist c_I, d_I such that $I = \{j : c_I \leq j \leq d_I\}$; that is, after reordering the Cartan matrix is a matrix of blocks corresponding to the connected components. Let $I \in \mathcal{X}$ and $i \sim j$ in I; then $N_i = N_j$, hence $N_I := N_i$ is well defined. Let Φ_I, resp. Φ_I^+, be the root system, resp. the subset of positive roots, corresponding to the Cartan matrix $(a_{ij})_{i,j \in I}$; then $\Phi = \bigcup_{I \in \mathcal{X}} \Phi_I$, resp. $\Phi^+ = \bigcup_{I\in\mathcal{X}} \Phi_I^+$ is the root system, resp. the subset of positive roots, corresponding to the Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$. Let $\alpha_1, \dots, \alpha_\theta$ be the set of simple roots.

Let $\mathcal{W}_I$ be the Weyl group corresponding to the Cartan matrix $(a_{ij})_{i,j\in I}$; we identify it with a subgroup of the Weyl group $\mathcal{W}$ corresponding to the Cartan matrix (a_{ij}). We fix a reduced decomposition of the longest element $\omega_{0,I}$ of $\mathcal{W}_I$ in terms of simple reflections. Then we obtain a reduced decomposition of the longest element $\omega_0 = s_{i_1} \dots s_{i_P}$ of $\mathcal{W}$ from the expression of ω_0 as product of the $\omega_{0,I}$'s in some fixed order of the components, say the order arising from the order of the vertices. Therefore $\beta_j := s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j})$ is a numeration of Φ^+.

Example 4.1. Let $q \in \Bbbk$, $q \neq 0$, and consider the braided vector space $(\mathbb{V}, c)$, where $\mathbb{V}$ is a vector space with a basis $x_1, \dots, x_\theta$ and the braiding c is given by

$$c(x_i \otimes x_j) = q^{d_i a_{ij}} x_j \otimes x_i, \tag{4.1}$$

Theorem 4.2. [L3] *Let $(\mathbb{V}, c)$ be a braided vector space with braiding matrix (4.1). If q is not a root of 1, then*

$$\mathfrak{B}(\mathbb{V}) = \Bbbk\langle x_1, \dots, x_\theta | \operatorname{ad}_c(x_i)^{1-a_{ij}} x_j = 0, \quad i \neq j\rangle.$$

□

The Theorem says that $\mathfrak{B}(\mathbb{V})$ is the well-known "positive part" $U_q^+(g)$ of the quantum enveloping algebra of g.

To state the following important Theorem, we recall the definition of braided commutators (1.20). Lusztig defined root vectors $X_\alpha \in \mathfrak{B}(\mathbb{V})$, $\alpha \in \Phi^+$ [L2]. One can see from [L1, L2] that, up to a non-zero scalar, each root vector can be written as an iterated braided commutator in some sequence $X_{\ell_1}, \dots, X_{\ell_a}$ of simple root vectors such as $[[X_{\ell_1}, [X_{\ell_2}, X_{\ell_3}]_c]_c, [X_{\ell_4}, X_{\ell_5}]_c]_c$. See also [Ri].

Theorem 4.3. [L1, L2, L3, Ro1, Mu]. *Let $(\mathbb{V}, c)$ be a braided vector space with braiding matrix (4.1). Assume that q is a root of 1 of odd order N; and that 3 does not divide N if there exists $I \in \mathcal{X}$ of type G_2.*

The algebra $\mathfrak{B}(\mathbb{V})$ is finite dimensional if and only if (a_{ij}) is a finite Cartan matrix.

If this happens, then $\mathfrak{B}(\mathbb{V})$ can be presented by generators X_i, $1 \le i \le \theta$, and relations

$$\operatorname{ad}_c(X_i)^{1-a_{ij}}(X_j) = 0, \qquad i \neq j, \tag{4.2}$$

$$X_\alpha^N = 0, \qquad \alpha \in \Phi^+. \tag{4.3}$$

Moreover, the following elements constitute a basis of $\mathfrak{B}(\mathbb{V})$:

$$X_{\beta_1}^{h_1} X_{\beta_2}^{h_2} \dots X_{\beta_P}^{h_P}, \qquad 0 \le h_j \le N-1, \quad 1 \le j \le P.$$

□

The Theorem says that $\mathfrak{B}(\mathbb{V})$ is the well-known "positive part" $\mathfrak{u}_q^+(g)$ of the so-called Frobenius-Lusztig kernel of g.

Motivated by the preceding Theorems and results, we introduced the following notion in [AS2] (see also [FG]).

Definition 4.4. Let (V, c) a braided vector space of diagonal type with basis $x_1, \dots, x_\theta$, and matrix (q_{ij}), that is

$$c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i, \text{ for all } 1 \le i, j \le \theta.$$

We shall say that (V, c) is *of Cartan type* if for all i, j, q_{ij} is a root of unity, $q_{ii} \ne 1$ and there exists $a_{ij} \in \mathbb{Z}$ such that

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}}, \text{ for all } 1 \le i, j \le \theta. \tag{4.4}$$

The integers a_{ij} are uniquely determined when chosen in the following way:

(4.5) $a_{ii} = 2$;

(4.6) $-\operatorname{ord} q_{ii} < a_{ij} \le 0, \quad i \ne j.$

It follows that (a_{ij}) is a generalized Cartan matrix (GCM) in the sense of the book [K]. We shall adapt the terminology from generalized Cartan matrices and Dynkin diagrams to braidings of Cartan type. For instance, we shall say that (V, c) is of *finite Cartan type* if it is of Cartan type and the corresponding GCM is actually of finite type, *i. e.* a Cartan matrix associated to a finite dimensional semisimple Lie algebra. We shall say that a Yetter-Drinfeld module V is *of Cartan type* if the matrix (q_{ij}) as above is of Cartan type.

To formulate our first main result, we need one more definition from [AS2] for braidings which are close to the braidings of the Frobenius-Lusztig kernels.

Definition 4.5. Let (V, c) be a braided vector space of Cartan type with Cartan matrix (a_{ij}). We say that (V, c) is of *FL-type* if there exist positive integers $d_1, \dots, d_\theta$ such that

$$\text{For all } i, j, \ d_i a_{ij} = d_j a_{ji} \text{ (thus } (a_{ij}) \text{ is symmetrizable).} \tag{4.7}$$

$$\text{There exists } q \in \Bbbk \text{ such that } q_{ij} = q^{d_i a_{ij}} \text{ for all } i, j. \tag{4.8}$$

We call (V, c) *locally of FL-type* if any principal 2×2 submatrix of (q_{ij}) defines a braiding of FL-type.

We now fix for each $\alpha \in \Phi^+$ such a representation of X_α as an iterated braided commutator. For a general braided vector space (V, c) of finite Cartan type, we define root vectors x_α in the tensor algebra $T(V)$, $\alpha \in \Phi^+$, as the same formal iteration of braided commutators in the elements $x_1, \dots, x_\theta$ instead of $X_1, \dots, X_\theta$ but with respect to the braiding c given by the general matrix (q_{ij}).

Theorem 4.6. [AS2, AS4]. *Let (V, c) be a braided vector space of Cartan type. We also assume that q_{ij} has odd order for all i, j.*

(i). Assume that (V,c) is locally of FL-type and that, for all i, the order of q_{ii} is relatively prime to 3 whenever $a_{ij} = -3$ for some j, and is different from 3, 5, 7, 11, 13, 17. If $\mathfrak{B}(V)$ is finite dimensional, then (V,c) is of finite Cartan type.

(ii). If (V,c) is of finite Cartan type, then $\mathfrak{B}(V)$ is finite dimensional, and if moreover 3 does not divide the order of q_{ii} for all i in a connected component of the Dynkin diagram of type G_2, then

$$\dim \mathfrak{B}(V) = \prod_{I\in\mathcal{X}} N_I^{\dim \mathfrak{n}_I},$$

where $N_I = \operatorname{ord}(q_{ii})$ for all $i \in I$ and $I \in \mathcal{X}$. The Nichols algebra $\mathfrak{B}(V)$ is presented by generators x_i, $1 \le i \le \theta$, and relations

$$\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0, \qquad i \neq j, \tag{4.9}$$

$$x_\alpha^{N_I} = 0, \qquad \alpha \in \Phi_I^+,\ I \in \mathcal{X}. \tag{4.10}$$

Moreover, the following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \qquad 0 \le h_j \le N_I - 1, \text{ if } \beta_j \in I, \quad 1 \le j \le P.$$

□

Let $\widehat{\mathfrak{B}}(V)$ be the braided Hopf algebra in ${}_\Gamma^\Gamma\mathcal{YD}$ generated by $x_1, \dots, x_\theta$ with relations (4.9), where the x_i's are primitive. Let $\mathcal{K}(V)$ be the subalgebra of $\widehat{\mathfrak{B}}(V)$ generated by $x_\alpha^{N_I}$, $\alpha \in \Phi_I^+$, $I \in \mathcal{X}$; it is a Yetter-Drinfeld submodule of $\widehat{\mathfrak{B}}(V)$.

Theorem 4.7. [AS4] *$\mathcal{K}(V)$ is a braided Hopf subalgebra in ${}_\Gamma^\Gamma\mathcal{YD}$ of $\widehat{\mathfrak{B}}(V)$.* □

5. Classification of pointed Hopf algebras by the lifting method

5.1. Lifting of Cartan type.

We propose to subdivide the classification problem for finite-dimensional pointed Hopf algebras into the following problems:

(a). Determine all braided vector spaces V of group type such that $\mathfrak{B}(V)$ is finite dimensional.

(b). Given a finite group Γ, determine all realizations of braided vector spaces V as in (a) as Yetter-Drinfeld modules over Γ.

(c). The lifting problem: For $\mathfrak{B}(V)$ as in (a), compute all Hopf algebras A such that $\operatorname{gr} A \simeq \mathfrak{B}(V)\#H$.

(d). Investigate whether any finite dimensional pointed Hopf algebra is generated as an algebra by its group-like and skew-primitive elements.

Problem (a) was discussed in Chapters 3 and 4. We have seen the very important class of braidings of finite Cartan type and some isolated examples where the Nichols algebra is finite-dimensional. But the general case of problem (a) seems to require completely new ideas.

Problem (b) is of computational nature. For braidings of finite Cartan type with Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$ and an abelian group Γ we have to compute elements $g_1,\dots,g_\theta\in\Gamma$ and characters $\chi_1,\dots,\chi_\theta\in\widehat{\Gamma}$ such that

$$\chi_i(g_j)\chi_j(g_i)=\chi_i(g_i)^{a_{ij}},\ \text{for all } 1\le i,j\le\theta. \tag{5.1}$$

To find these elements one has to solve a system of quadratic congruences in several unknowns. In many cases they do not exist. In particular, if $\theta>2(\mathrm{ord}\Gamma)^2$, then the braiding cannot be realized over the group Γ. We refer to [AS2, Section 8] for details.

Problem (d) is the subject of Section 5.4.

We will now discuss the lifting problem (c).

The coradical filtration $\Bbbk\Gamma=A_0\subset A_1\subset\dots$ of a pointed Hopf algebra A is stable under the adjoint action of the group. For abelian groups Γ and finite-dimensional Hopf algebras, the following stronger result holds. It is the starting point of the lifting procedure, and we will use it several times.

If M is a $\Bbbk\Gamma$-module, we denote by $M^\chi=\{m\in M\mid gm=\chi(g)m \text{ for all } g\in\Gamma\}$, $\chi\in\widehat{\Gamma}$, the isotypic component of type χ.

Lemma 5.1. *Let A be a finite-dimensional Hopf algebra with abelian group $G(A)=\Gamma$ and diagram R. Let $V=R(1)\in{}^{\Gamma}_{\Gamma}\mathcal{YD}$ with basis $x_i\in V_{g_i}^{\chi_i}, g_i\in\Gamma,\chi_i\in\widehat{\Gamma}, 1\le i\le\theta$.*

(a). The isotypic component of trivial type of A_1 is A_0. Therefore, $A_1=A_0\oplus(\oplus_{\chi\neq\varepsilon}(A_1)^\chi)$ and

$$\oplus_{\chi\neq\varepsilon}(A_1)^\chi\xrightarrow{\simeq}A_1/A_0\xleftarrow{\simeq}V\#\Bbbk\Gamma. \tag{5.2}$$

(b). For all $g\in\Gamma$, $\chi\in\widehat{\Gamma}$ with $\chi\neq\varepsilon$,

$$\mathcal{P}_{g,1}(A)^\chi\neq 0\iff \text{there is some } 1\le\ell\le\theta: g=g_\ell,\chi=\chi_\ell; \tag{5.3}$$

$$\mathcal{P}_{g,1}(A)^\varepsilon=\Bbbk(1-g). \tag{5.4}$$

Proof. (a) follows from [AS1, Lemma 3.1] and implies (b). See [AS1, Lemma 5.3]. □

We assume that A is a finite-dimensional pointed Hopf algebra with abelian group $G(A)=\Gamma$, and that

$$\mathrm{gr}A\simeq\mathfrak{B}(V)\#\Bbbk\Gamma,$$

where $V\in{}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a given Yetter-Drinfeld module with basis $x_i\in V_{g_i}^{\chi_i}, g_1\dots,g_\theta\in\Gamma,\chi_1,\dots,\chi_\theta\in\widehat{\Gamma}, 1\le i\le\theta$.

We first lift the basis elements x_i. Using (5.2), we choose $a_i\in\mathcal{P}(A)_{g_i,1}^{\chi_i}$ such that the canonical image of a_i in A_1/A_0 is x_i (which we identify with $x_i\#1$), $1\le i\le\theta$. Since the elements x_i together with Γ generate $\mathfrak{B}(V)\#\Bbbk\Gamma$, it follows from a standard argument that $a_1,\dots,a_\theta$ and the elements in Γ generate A as an algebra.

Our aim is to find relations between the $a_i's$ and the elements in Γ which define a quotient Hopf algebra of the correct dimension $\dim\mathfrak{B}(V)\cdot\mathrm{ord}(\Gamma)$. The idea is to "lift" the relations between the $x_i's$ and the elements in Γ in $\mathfrak{B}(V)\#\Bbbk\Gamma$.

We now assume moreover that V is of finite Cartan type with Cartan matrix (a_{ij}) with respect to the basis $x_1, \dots, x_\theta$, that is (5.1) holds. We also assume

(5.5) $\operatorname{ord}(\chi_j(g_i))$ is odd for all i, j,

(5.6) $N_i = \operatorname{ord}(\chi_i(g_i))$ is prime to 3 for all $i \in I, I \in \mathcal{X}$ of type G_2.

We fix a presentation $\Gamma = \langle y_1 \rangle \oplus \cdots \oplus \langle y_\sigma \rangle$, and denote by M_ℓ the order of y_ℓ, $1 \le \ell \le \sigma$. Then Theorem 4.6 and formulas (1.17) imply that $\mathfrak{B}(V)\#\Bbbk\Gamma$ can be presented by generators h_ℓ, $1 \le \ell \le \sigma$, and x_i, $1 \le i \le \theta$ with defining relations

(5.7) $h_\ell^{M_\ell} = 1, \quad 1 \le \ell \le \sigma;$

(5.8) $h_\ell h_t = h_t h_\ell, \quad 1 \le t < \ell \le \sigma;$

(5.9) $h_\ell x_i = \chi_i(y_\ell) x_i h_\ell, \quad 1 \le \ell \le \sigma, \quad 1 \le i \le \theta;$

(5.10) $x_\alpha^{N_I} = 0, \qquad \alpha \in \Phi_I^+, I \in \mathcal{X};$

(5.11) $\operatorname{ad}(x_i)^{1-a_{ij}}(x_j) = 0, \qquad i \ne j,$

and where the Hopf algebra structure is determined by

(5.12) $\Delta(h_\ell) = h_\ell \otimes h_\ell, \quad 1 \le \ell \le \sigma;$

(5.13) $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad 1 \le i \le \theta.$

Thus A is generated by the elements $a_i, 1 \le i \le \theta$, and $h_l, 1 \le l \le \sigma$. By our previous choice, relations (5.7), (5.8), (5.9) and (5.12), (5.13) all hold in A with the $x_i's$ replaced by the $a_i's$.

The remaining problem is to lift the quantum Serre relations (5.11) and the root vector relations (5.10). We will do this in the next two Sections.

5.2. **Lifting the quantum Serre relations.**

We divide the problem into two cases.

- Lifting of the "quantum Serre relations" $x_i x_j - \chi_j(g_i) x_j x_i = 0$, when $i \ne j$ are in different components of the Dynkin diagram.
- Lifting of the "quantum Serre relations" $\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0$, when $i \ne j$ are in the same component of the Dynkin diagram.

The first case is settled in the next Lemma from [AS4].

Lemma 5.2. *Assume that $1 \le i, j \le \theta, i < j$ and $i \nsim j$. Then*

$$a_i a_j - \chi_j(g_i) a_j a_i = \lambda_{ij}(1 - g_i g_j), \tag{5.14}$$

where λ_{ij} is a scalar in $\Bbbk$ which can be chosen such that

(5.15) *λ_{ij} is arbitrary if $g_i g_j \ne 1$ and $\chi_i \chi_j = \varepsilon$, but 0 otherwise.*

Proof. It is easy to check that $a_ia_j - \chi_j(g_i)a_ja_i \in \mathcal{P}(A)^{\chi_i\chi_j}_{g_ig_j,1}$. Suppose that $\chi_i\chi_j \neq \varepsilon$ and $a_ia_j - \chi_j(g_i)a_ja_i \neq 0$. Then by (5.3), $\chi_i\chi_j = \chi_l$ and $g_ig_j = g_l$ for some $1 \leq l \leq \theta$.

Substituting g_l and χ_l in $\chi_i(g_l)\chi_l(g_i) = \chi_i(g_i)^{a_{il}}$ and using $\chi_i(g_j)\chi_j(g_i) = 1$ (since $a_{ij} = 0$, since i and j lie in different components), we get $\chi_i(g_i)^2 = \chi_i(g_i)^{a_{il}}$.

Thus we have shown that $a_{il} \equiv 2 \mod N_i$, and in the same way $a_{jl} \equiv 2 \mod N_j$. Since $i \not\sim j$, a_{il} or a_{jl} must be 0, and we obtain the contradiction $N_i = 2$ or $N_i = 2$.

Therefore $\chi_i\chi_j = \varepsilon$, and the claim follows from (5.4), or $a_ia_j - \chi_j(g_i)a_ja_i = 0$, and the claim is trivial. □

Lemma 5.2 motivates the following notion.

Definition 5.3. [AS4] We say that two vertices i and j *are linkable* (or that i *is linkable to* j) if

(5.16) $i \not\sim j$,

(5.17) $g_ig_j \neq 1$ and

(5.18) $\chi_i\chi_j = \varepsilon$.

The following elementary properties are easily verified:

(5.19) If i is linkable to j, then $\chi_i(g_j)\chi_j(g_i) = 1, \quad \chi_j(g_j) = \chi_i(g_i)^{-1}$.

(5.20) If i and k, resp. j and ℓ, are linkable, then $a_{ij} = a_{k\ell}$, $a_{ji} = a_{\ell k}$.

(5.21) A vertex i can not be linkable to two different vertices j and h.

A *linking datum* is a collection $(\lambda_{ij})_{1\leq i<j\leq\theta, i\not\sim j}$ of elements in $\Bbbk$ such that λ_{ij} is arbitrary if i and j are linkable but 0 otherwise. Given a linking datum, we say that two vertices i and j *are linked* if $\lambda_{ij} \neq 0$.

The notion of a linking datum encodes the information about lifting of relations in the first case.

Definition 5.4. The collection $\mathcal{D}$ formed by a finite Cartan matrix (a_{ij}), and $g_1, \dots, g_\theta \in \Gamma, \chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ satisfying (5.1), (5.5) and (5.6), and a linking datum $(\lambda_{ij})_{1\leq i<j\leq\theta, i\not\sim j}$ will be called a *linking datum of finite Cartan type* for Γ. We define the Yetter-Drinfeld module $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ of $\mathcal{D}$ as the vector space with basis $x_1, \dots, x_\theta$ with $x_i \in V^{\chi_i}_{g_i}$ for all i.

If $\mathcal{D}$ is a linking datum of finite Cartan type for Γ, we define the Hopf algebra $\mathfrak{u}(\mathcal{D})$ by generators $a_i, 1 \leq i \leq \theta$, and $h_l, 1 \leq l \leq \sigma$ and the relations (5.7),(5.8),(5.9),(5.10), the quantum Serre relations (5.11) for $i \neq j$ and $i \sim j$, (5.12),(5.13) with the x_i's replaced by the a_i's, and the lifted quantum Serre relations (5.14).

In the definition of $\mathfrak{u}(\mathcal{D})$ we could always assume that the linking datum contains only elements $\lambda_{ij} \in \{0, 1\}$ (by multiplying the generators a_i with non-zero scalars).

Theorem 5.5. [AS4] *Let Γ be a finite abelian group and $\mathcal{D}$ a linking datum of finite Cartan type for Γ with Yetter-Drinfeld module V. Then $\mathfrak{u}(\mathcal{D})$ is a finite-dimensional pointed Hopf algebra with $\operatorname{gr}\mathfrak{u}(\mathcal{D}) \simeq \mathfrak{B}(V)\#\Bbbk\Gamma$.* □

The proof of the Theorem is by induction on the number of irreducible components of the Dynkin diagram. In the induction step a new Hopf algebra is constructed by twisting the multiplication of the tensor product of two Hopf algebras by a 2-cocycle. The 2-cocycle is defined in terms of the linking datum.

Note that the Frobenius-Lusztig kernel $\mathfrak{u}_q(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ is a special case of $\mathfrak{u}(\mathcal{D})$. Here the Dynkin diagram of $\mathcal{D}$ is the disjoint union of two copies of the Dynkin diagram of $\mathfrak{g}$, and corresponding points are linked pairwise. But many other linkings are possible, for example 4 copies of A_3 linked in a circle [AS4, 5.13]. See [D] for a combinatorical description of all linkings of Dynkin diagrams.

Let us now turn to the second case. Luckily it turns out that (up to some small order exceptions) in the second case the Serre relations simply hold in the lifted situation without any change.

Theorem 5.6. [AS4, Theorem 6.8]. *Let $I \in \mathcal{X}$. Assume that $N_I \neq 3$. If I is of type B_n, C_n or F_4, resp. G_2, assume further that $N_I \neq 5$, resp. $N_I \neq 7$. Then the quantum Serre relations hold for all $i, j \in I, i \neq j, i \sim j$.* □

5.3. **Lifting the root vector relations.**

Assume first that the root α is simple and corresponds to a vertex i. It is not difficult to see, using the quantum binomial formula, that $a_i^{N_i}$ is a $(g_i^{N_i}, 1)$-skew-primitive. By Lemma 5.1, we have

$$a_i^{N_i} = \mu_i\left(1 - g_i^{N_i}\right), \tag{5.22}$$

for some scalar μ_i; this scalar can be chosen so that

(5.23) μ_i is arbitrary if $g_i^{N_i} \neq 1$ and $\chi_i^{N_i} = 1$ but 0 otherwise.

Now, if the root α is *not* simple then $a_i^{N_i}$ is not necessarily a skew-primitive, but a skew-primitive "modulo root vectors of shorter lenght".

In general, we define the root vector a_α for $\alpha \in \Phi_I^+, I \in \mathcal{X}$, by replacing the x_i by a_i in the formal expression for x_α as a braided commutator in the simple root vectors. Then $a_\alpha^{N_I}, \alpha \in I$, should be an element u_α in the group algebra of the subgroup generated by the N_I-th powers of the elements in Γ.

Finally, the Hopf algebra generated by $a_i, 1 \leq i \leq \theta$, and $h_l, 1 \leq l \leq \sigma$ with the relations (5.7),(5.8),(5.9) (with a_i instead of x_i),

- the lifted root vector relations $a_\alpha^{N_I} = u_\alpha, \alpha \in \Phi_I^+, I \in \mathcal{X}$,
- the quantum Serre relations (5.11) for $i \neq j$ and $i \sim j$ (with a_i instead of x_i),
- the lifted quantum Serre relations (5.14),

should have the correct dimension $\dim(\mathfrak{B}(V)) \cdot \operatorname{ord}(\Gamma)$.

We carried out all the steps of this program in the following cases:

(a) All connected components of the Dynkin diagram are of type A_1 [AS1].

(b) The Dynkin diagram is of type A_2, and $N > 3$ is odd [AS3].

(c) The Dynkin diagram is arbitrary, but we assume $g_i^{N_i} = 1$ for all i [AS4].

(d) The Dynkin diagram is of type A_n, any $n \geq 2$, and $N > 3$, see Section 6 of this paper.

The cases $A_2, N = 3$ and B_2, N odd and $\neq 5$, were recently done in [BDR]. Here N denotes the common order of $\chi_i(g_i)$ for all i when the Dynkin diagram is connected.

5.4. **Generation in degree one.**

Let us now discuss step (d) of the Lifting method.

It is not difficult to show that our conjecture 2.7 about Nichols algebras, in the setting of $H = \Bbbk\Gamma$, is equivalent to

Conjecture 5.7. [AS3]. *Any pointed finite dimensional Hopf algebra over $\Bbbk$ is generated by group-like and skew-primitive elements.*

We have seen in Section 2.1 that the corresponding conjecture is false when the Hopf algebra is infinite-dimensional or when the Hopf algebra is finite-dimensional and the characteristic of the field is > 0. A strong indication that the conjecture is true is given by:

Theorem 5.8. [AS4]. *Let A be a finite-dimensional pointed Hopf algebra with coradical $\Bbbk\Gamma$ and diagram R, that is*

$$\operatorname{gr} \mathcal{A} \simeq R\#\Bbbk\Gamma.$$

Assume that $R(1)$ is a Yetter-Drinfeld module of finite Cartan type with braiding $(q_{ij})_{1\leq i,j\leq\theta}$. For all i, let $q_i = q_{ii}, N_i = ord(q_i)$. Assume that $ord(q_{ij})$ is odd and N_i is not divisible by 3 and > 7 for all $1 \leq i, j \leq \theta$.

1. *For any $1 \leq i \leq \theta$ contained in a connected component of type B_n, C_n or F_4 resp. G_2, assume that N_i is not divisible by 5 resp. by 5 or 7.*

2. *If i and j belong to different components, assume $q_iq_j = 1$ or $\operatorname{ord}(q_iq_j) = N_i$.*

Then R is generated as an algebra by $R(1)$, that is A is generated by skew-primitive and group-like elements. □

Let us discuss the idea of the proof of Theorem 5.8. At one decisive point, we use our previous results about braidings of Cartan type of rank 2.

Let S be the graded dual of R. By the duality principle in Lemma 2.4, S is generated in degree one since $P(R) = R(1)$. Our problem is to show that R is generated in degree one, that is S is a Nichols algebra.

Since S is generated in degree one, there is a surjection of graded braided Hopf algebras $S \to \mathfrak{B}(V)$, where $V = S(1)$ has the same braiding as $R(1)$. But we know the defining relations of $\mathfrak{B}(V)$, since it is of finite Cartan type. So have to show that these relations also hold in S.

In the case of a quantum Serre relation $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0$, $i \neq j$, we consider the Yetter-Drinfeld submodule W of S generated by x_i and $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j)$ and assume that $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) \neq 0$. The assumptions (1) and (2) of the Theorem guarantee that W also is of Cartan type, but not of finite Cartan type. Thus $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0$ in S.

Since the quantum Serre relations hold in S, the root vector relations follow automatically from the next Lemma which is a consequence of Theorem 4.7.

Lemma 5.9. [AS4, Lemma 7.5] *Let $S = \oplus_{n\geq 0} S(n)$ be a finite-dimensional graded Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that $S(0) = \Bbbk 1$. Assume that $V = S(1)$ is of Cartan type with basis $(x_i)_{1\leq i,j\leq\theta}$ as described in the beginning of this Section. Assume the Serre relations*

$$(\mathrm{ad}_c x_i)^{1-a_{ij}} x_j = 0 \text{ for all } 1 \leq i,j \leq \theta, i \neq j \text{ and } i \sim j.$$

Then the root vector relations

$$x_\alpha^{N_I} = 0, \quad \alpha \in \Phi_I^+, \quad I \in \mathcal{X},$$

hold in S. □

5.5. **Applications.**

As a special case of the theory explained above we obtain a complete answer to the classification problem in a significant case.

Theorem 5.10. [AS4] *Let p be a prime > 17, $s \geq 1$, and $\Gamma = (\mathbb{Z}/(p))^s$. Up to isomorphism there are only finitely many finite-dimensional pointed Hopf algebras A with $G(A) \simeq \Gamma$. They all have the form*

$$A \simeq \mathfrak{u}(\mathcal{D}), \text{ where } \mathcal{D} \text{ is a linking datum of finite Cartan type for } \Gamma.$$

□

If we really want to write down all these Hopf algebras we still have to solve the following serious problems:

- Determine all Yetter-Drinfeld modules V over $\Gamma = (\mathbb{Z}/(p))^s$ of finite Cartan type.
- Determine all the possible linkings for the modules V over $(\mathbb{Z}/(p))^s$ in (a).

By [AS2, Proposition 8.3], $\dim V \leq 2s\dfrac{p-1}{p-2}$, for all the possible V in (a). This proves the finiteness statement in Theorem 5.10.

Note that we have a precise information about the dimension of the Hopf algebras in 5.10:

$$\dim \mathfrak{u}(\mathcal{D}) = p^{s|\phi^+|},$$

where $|\phi^+|$ is the number of the positive roots of the root system of rank $\theta \leq 2s\dfrac{p-1}{p-2}$ of the Cartan matrix of $\mathcal{D}$.

For arbitrary finite abelian groups Γ, there usually are infinitely many non-isomorphic pointed Hopf algebras of the same finite dimension. The first examples were found in 1997 independently in [AS1], [BDG], [G]. Now it is very easy to construct lots of examples by lifting. Using [AS3, Lemma 1.2] it is possible to decide when two liftings are non-isomorphic.

But we have a bound on the dimension of A:

Theorem 5.11. [AS4] *For any finite (not necessarily abelian) group Γ of odd order there is a natural number $n(\Gamma)$ such that*

$$dim A \leq n(\Gamma)$$

for any finite-dimensional pointed Hopf algebra A with $G(A) = \Bbbk\Gamma$. $\square$

Remark 5.12. As a corollary of the Theorem and its proof, we get the complete classification of all finite dimensional pointed Hopf algebras with coradical of prime dimension p, $p \neq 2, 5, 7$. By [AS2, Theorem 1.3], the only possibilities for the Cartan matrix of $\mathcal{D}$ with Γ of odd prime order p are

(a) A_1 and $A_1 \times A_1$,

(b) A_2, if $p = 3$ or $p \equiv 1 \mod 3$,

(c) B_2, if $p \equiv 1 \mod 4$,

(d) G_2, if $p \equiv 1 \mod 3$,

(e) $A_2 \times A_1$ and $A_2 \times A_1$, if $p = 3$.

The Nichols algebras over $\mathbb{Z}/(p)$ for these Cartan matrices are listed in [AS2, Theorem 1.3]. Hence we obtain from Theorem 5.10 for $p \neq 2, 5, 7$ the bosonizations of the Nichols algebras, the liftings in case (a), that is quantum lines and quantum planes [AS1], and the liftings of type A_2 [AS3] in case (b).

This result was also obtained by Musson [Mus], using the lifting method and [AS2].

The case $p = 2$ was already done in [N]. In this case the dimension of the pointed Hopf algebras with 2-dimensional coradical is not bounded.

Let us mention briefly some classification results for Hopf algebras of special order which can be obtained by the methods we have described. Let $p > 2$ be a prime. Then all pointed Hopf algebras A of dimension $p^n, 1 \leq n \leq 5$ are known. If the dimension is p or p^2, then A is a group algebra or a Taft Hopf algebra. The cases of dimension p^3 and p^4 were treated in [AS1] and [AS3], and the classification of dimension p^5 follows from [AS4] and [Gñ1]. Independently and by other methods, the case p^3 was also solved in [CD] and [SvO].

See [A] for a discussion of what is known on classification of finite dimensional Hopf algebras.

To formulate a classification result for infinite-dimensional Hopf algebras, we now assume that $\Bbbk$ is the field of complex numbers and we introduce a notion from [AS5]. The collection $\mathcal{D}$ formed by a *free abelian group* Γ of finite rank, a finite Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$, $g_1,\dots,g_\theta\in\Gamma$, $\chi_1,\dots,\chi_\theta\in\widehat{\Gamma}$, and a linking datum $(\lambda_{ij})_{1\le i<j\le\theta, i\not\sim j}$, will be called a *positive datum* if

$$\chi_i(g_j)\chi_j(g_i)=\chi_i(g_i)^{d_i a_{ij}},\ \text{and}\ \chi_i(g_i)>0,\ \text{for all}\ 1\le i,j,\le\theta,$$

where $d_1,\dots,d_\theta$ are integers between 1 and 3 such that $d_i a_{ij}=d_j a_{ji}$.

If $\mathcal{D}$ is a positive datum we define the Hopf algebra $U(\mathcal{D})$ by generators $a_i, 1\le i\le\theta$, and $h_l^{\pm}, 1\le l\le\sigma$ and the relations $h_m^{\pm}h_l^{\pm}=h_l^{\pm}h_l^{\pm}, h_l^{\pm}h_l^{\mp}=1$, for all $1\le l,m\le\sigma$, defining the free abelian group of rank σ, and (5.9), the quantum Serre relations (5.11) for $i\ne j$ and $i\sim j$, (5.12),(5.13) (with a_i instead of x_i), and the lifted quantum Serre relations (5.14).

If (V,c) is a finite-dimensional braided vector space, we will say that the *braiding is positive* if it is diagonal, and all the entries of the matrix (q_{ij}) of the braiding are positive, and $q_{ii}\ne 1$ for all i.

The next theorem follows from a result of Rosso [Ro2, Theorem 21] and the theory described in the previous Sections.

Theorem 5.13. [AS5] *Let A be a pointed Hopf algebra with abelian group $\Gamma=G(A)$ and diagram R. Assume that $R(1)$ has finite dimension θ and positive braiding. Then the following are equivalent:*

(a). A is a domain of finite Gelfand-Kirillov dimension.

(b). The group Γ is free abelian of finite rank, and

$$A\simeq U(\mathcal{D}),\ \textit{where}\ \mathcal{D}\ \textit{is a positive datum for}\ \Gamma.$$

□

It is likely that the positivity assumption on the infinitesimal braiding in the last theorem is related to the existence of a compact involution.

6. Pointed Hopf algebras of type A_n

In this Chapter, we develop from scratch, *i. e.* without using Lusztig's results, the classification of all finite dimensional pointed Hopf algebras whose infinitesimal braiding is of type A_n. The main results of this Chapter are new.

6.1. Nichols algebras of type A_n.

Let N be an integer, $N>2$, and let q be a root of 1 of order N. For the case $N=2$, see [AnDa].

Let q_{ij}, $1\le i,j\le n$, be roots of 1 such that

$$q_{ii}=q,\quad q_{ij}q_{ji}=\begin{cases}q^{-1}, & \text{if } |i-j|=1,\\ 1, & \text{if } |i-j|\ge 2.\end{cases}\tag{6.1}$$

for all $1 \leq i, j \leq n$. For convenience, we denote

$$B^{i,j}_{p,r} := \prod_{i \leq \ell \leq j-1,\, p \leq h \leq r-1} q_{\ell,h},$$

for any $1 \leq i < j \leq n+1$, $1 \leq p < r \leq n+1$. Then we have the following identities, whenever $i < s < j$, $p < t < r$:

$$B^{i,s}_{p,r} B^{s,j}_{p,r} = \prod_{i \leq \ell \leq s-1, p \leq h \leq r-1} q_{\ell,h} \prod_{s \leq \ell \leq j-1, p \leq h \leq r-1} q_{\ell,h} = B^{i,j}_{p,r}; \tag{6.2}$$

$$B^{i,j}_{p,t} B^{i,j}_{t,r} = B^{i,j}_{p,r}; \tag{6.3}$$

also,

$$B^{i,j}_{j,j+1} B^{j,j+1}_{i,j} = \prod_{i \leq \ell \leq j-1} q_{\ell,j} \prod_{i \leq h \leq j-1} q_{j,h} = q^{-1}; \tag{6.4}$$

$$B^{i,j}_{i,j} = q. \tag{6.5}$$

We consider in this Section a vector space $V = V_n$ with a basis $x_1, \dots, x_n$ and braiding determined by:

$$c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i, \quad 1 \leq i, j \leq n;$$

that is, V is of type A_n.

Remark 6.1. Let Γ be a group, $g_1, \dots, g_n$ in the center of Γ, and $\chi_1, \dots, \chi_n$ in $\widehat{\Gamma}$ such that

$$q_{ij} = \langle \chi_j, g_i \rangle, \quad 1 \leq i, j \leq n.$$

Then V can be realized as a Yetter-Drinfeld module over Γ by declaring

$$x_i \in V^{\chi_i}_{g_i}, \qquad 1 \leq i \leq n. \tag{6.6}$$

For example, we could consider $\Gamma = (\mathbb{Z}/P)^n$, where P is divisible by the orders of all the q_{ij}'s; and take $g_1, \dots, g_n$ as the canonical basis of Γ.

We shall consider a braided Hopf algebra R provided with an inclusion of braided vector spaces $V \to P(R)$. We identify the elements $x_1, \dots, x_n$ with their images in R. Distinguished examples of such R are the tensor algebra $T(V)$ and the Nichols algebra $\mathfrak{B}(V)$. Additional hypotheses on R will be stated when needed.

We introduce the family $(e_{ij})_{1 \leq i < j \leq n+1}$ of elements of R as follows:

(6.7) $e_{i,i+1} := x_i;$

(6.8) $e_{i,j} := [e_{i,j-1}, e_{j-1,j}]_c, \quad 1 \leq i < j \leq n+1,\ j - i \geq 2.$

The braiding between elements of this family is given by:

(6.9) $c(e_{i,j} \otimes e_{p,r}) = B^{i,j}_{p,r} e_{p,r} \otimes e_{i,j}, \quad 1 \leq i < j \leq n+1,\ 1 \leq p < r \leq n+1.$

In particular,

$$e_{i,j} = e_{i,j-1}e_{j-1,j} - B^{i,j-1}_{j-1,j}e_{j-1,j}e_{i,j-1}.$$

Remark 6.2. When V is realized as a Yetter-Drinfeld module over Γ as in Remark 6.1, we have $e_{i,j} \in R^{\chi_{i,j}}_{g_{i,j}}$, where

$$\chi_{i,j} = \prod_{i\le \ell\le j-1} \chi_\ell, \qquad g_{i,j} = \prod_{i\le \ell\le j-1} g_\ell, \qquad 1 \le i < j \le n+1. \tag{6.10}$$

Lemma 6.3. *(a). If R is finite dimensional or $R \simeq \mathfrak{B}(V)$, then*

$$e^N_{i,i+1} = 0, \quad \textit{if} \quad 1 \le i \le n. \tag{6.11}$$

(b). Assume that $R \simeq \mathfrak{B}(V)$. Then

$$[e_{i,i+1}, e_{p,p+1}]_c = 0, \quad \textit{that is} \quad e_{i,i+1}e_{p,p+1} = q_{ip}e_{p,p+1}e_{i,i+1}, \quad \textit{if} \quad 1 \le i < p \le n, p-i \ge 2. \tag{6.12}$$

(c). Assume that $R \simeq \mathfrak{B}(V)$. Then

$$[e_{i,i+1}, [e_{i,i+1}, e_{i+1,i+2}]_c]_c = 0, \quad \textit{if} \quad 1 \le i < n; \tag{6.13}$$

$$[e_{i+1,i+2}, [e_{i+1,i+2}, e_{i,i+1}]_c]_c = 0, \quad \textit{if} \quad 1 \le i < n; \tag{6.14}$$

that is

$$e_{i,i+1}e_{i,i+2} = B^{i,i+1}_{i,i+2}e_{i,i+2}e_{i,i+1}, \tag{6.15}$$

$$e_{i,i+2}e_{i+1,i+2} = B^{i,i+2}_{i+1,i+2}e_{i+1,i+2}e_{i,i+2}. \tag{6.16}$$

Proof. (a). This follows from Lemma 3.6 (a), use $c(e^N_{i,i+1} \otimes e^N_{i,i+1}) = e^N_{i,i+1} \otimes e^N_{i,i+1}$ in the finite dimensional case.

(b) and (c). By Lemma 3.6 (b), the elements

$$[e_{i,i+1}, e_{p,p+1}]_c, \quad [e_{i,i+1}, [e_{i,i+1}, e_{i+1,i+2}]_c]_c \quad \text{and } [e_{i+1,i+2}, [e_{i+1,i+2}, e_{i,i+1}]_c]_c$$

are primitive. Since they are homogeneous of degree 2, respectively of degree 3, they should be 0. To derive (6.16) from (6.14), use (1.23). □

Lemma 6.4. *Assume that* (6.12) *holds in R. Then*

$$[e_{i,j}, e_{p,r}]_c = 0, \qquad \textit{if} \quad 1 \le i < j < p < r \le n+1. \tag{6.17}$$

$$[e_{p,r}, e_{i,j}]_c = 0, \qquad \textit{if} \quad 1 \le i < j < p < r \le n+1. \tag{6.18}$$

$$[e_{i,p}, e_{p,j}]_c = e_{i,j}, \qquad \textit{if} \quad 1 \le i < p < j \le n+1. \tag{6.19}$$

Proof. (6.17). For $j = i+1$ and $r = p+1$, this is (6.12); the general case follows recursively using (1.22). (6.18) follows from (6.17), since $B^{i,j}_{p,r}B^{p,r}_{i,j} = 1$ in this case.

(6.19). By induction on $j-p$; if $p=j-1$ then (6.19) is just (6.8). For $p<j$, we have

$$e_{i,j+1}=[e_{i,j},e_{j,j+1}]_c=[[e_{i,p},e_{p,j}]_c,e_{j,j+1}]_c=[e_{i,p},[e_{p,j},e_{j,j+1}]_c]_c=[e_{i,p},e_{p,j+1}]_c$$

by (1.22), since $[e_{i,p},e_{j,j+1}]_c=0$ by (6.17).

□

Lemma 6.5. *Assume that* (6.12) *holds in* R. *Then for any* $1\le i<j\le n+1$,

$$\Delta(e_{i,j})=e_{i,j}\otimes 1+1\otimes e_{i,j}+(1-q^{-1})\sum_{i<p<j}e_{i,p}\otimes e_{p,j}. \tag{6.20}$$

Proof. We proceed by induction on $j-i$. If $j-i=1$, the formula just tells that the x_i's are primitive. For the inductive step, we compute

$$\begin{aligned}\Delta(e_{i,j}e_{j,j+1})&=\left(e_{i,j}\otimes 1+1\otimes e_{i,j}+(1-q^{-1})\sum_{i<p<j}e_{i,p}\otimes e_{p,j}\right)(e_{j,j+1}\otimes 1+1\otimes e_{j,j+1})\\&=e_{i,j}e_{j,j+1}\otimes 1+B^{i,j}_{j,j+1}e_{j,j+1}\otimes e_{i,j}+(1-q^{-1})\sum_{i<p<j}B^{p,j}_{j,j+1}e_{i,p}e_{j,j+1}\otimes e_{p,j}\\&\quad+e_{i,j}\otimes e_{j,j+1}+1\otimes e_{i,j}e_{j,j+1}+(1-q^{-1})\sum_{i<p<j}e_{i,p}\otimes e_{p,j}e_{j,j+1};\end{aligned}$$

and also

$$\begin{aligned}\Delta(e_{j,j+1}e_{i,j})&=(e_{j,j+1}\otimes 1+1\otimes e_{j,j+1})\left(e_{i,j}\otimes 1+1\otimes e_{i,j}+(1-q^{-1})\sum_{i<p<j}e_{i,p}\otimes e_{p,j}\right)\\&=e_{j,j+1}e_{i,j}\otimes 1+e_{j,j+1}\otimes e_{i,j}+(1-q^{-1})\sum_{i<p<j}e_{j,j+1}e_{i,p}\otimes e_{p,j}\\&\quad+B^{j,j+1}_{i,j}e_{i,j}\otimes e_{j,j+1}+1\otimes e_{j,j+1}e_{i,j}+(1-q^{-1})\sum_{i<p<j}(B^{i,p}_{j,j+1})^{-1}e_{i,p}\otimes e_{j,j+1}e_{p,j}.\end{aligned}$$

Hence

$$\begin{aligned}\Delta(e_{i,j+1})&=e_{i,j+1}\otimes 1+1\otimes e_{i,j+1}+(1-B^{ij}_{j,j+1}B^{j,j+1}_{ij})e_{i,j}\otimes e_{j,j+1}\\&\quad+(1-q^{-1})\sum_{i<p<j}\left(B^{p,j}_{j,j+1}B^{i,p}_{j,j+1}-B^{i,j}_{j,j+1}\right)e_{j,j+1}e_{i,p}\otimes e_{p,j}\\&\quad+(1-q^{-1})\sum_{i<p<j}e_{i,p}\otimes\left(e_{p,j}e_{j,j+1}-B^{i,j}_{j,j+1}(B^{i,p}_{j,j+1})^{-1}e_{j,j+1}e_{p,j}\right)\\&=e_{i,j+1}\otimes 1+1\otimes e_{i,j+1}+(1-q^{-1})\sum_{i<p<j+1}e_{i,p}\otimes e_{p,j+1};\end{aligned}$$

by (6.4), (6.17) and the hypothesis. □

Remark 6.6. Let Γ be a group with $g_1, \dots, g_n$ in the center of Γ, $\chi_1, \dots, \chi_n$ in $\widehat{\Gamma}$, as in 6.1. Let R be a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that (6.12) holds in R. It follows from (6.20) and the reconstruction formulas for the bosonization (1.17) that

$$\Delta_{R\#\Bbbk\Gamma}(e_{i,j}) = e_{i,j} \otimes 1 + g_{i,j} \otimes e_{i,j} + (1 - q^{-1}) \sum_{i<p<j} e_{i,p}g_{p,j} \otimes e_{p,j}. \tag{6.21}$$

Lemma 6.7. *Assume that* (6.12), (6.13), (6.14) *hold in* R. *Then*

$$[e_{i,j}, e_{p,r}]_c = 0, \qquad \text{if} \quad 1 \le i < p < r < j \le n+1; \tag{6.22}$$

$$[e_{i,j}, e_{i,r}]_c = 0, \qquad \text{if} \quad 1 \le i < j < r \le n+1; \tag{6.23}$$

$$[e_{i,j}, e_{p,j}]_c = 0, \qquad \text{if} \quad 1 \le i < p < j \le n+1. \tag{6.24}$$

Proof. (a). We prove (6.22) by induction on $j - i$. If $j - i = 3$, then

$$[e_{i,i+3}, e_{i+1,i+2}]_c = [[e_{i,i+2}, e_{i+2,i+3}]_c, e_{i+1,i+2}]_c = [[[e_{i,i+1}, e_{i+1,i+2}]_c, e_{i+2,i+3}]_c, e_{i+1,i+2}]_c = 0,$$

by Lemma 1.11. If $j - i > 3$ we argue by induction on $r - p$. If $r - p = 1$, then there exists an index h such that either $i < h < p < r = p+1 < j$ or $i < p < r = p+1 < h < j$. In the first case, by (6.19), we have

$$[e_{i,j}, e_{p,r}]_c = [[e_{i,h}, e_{h,j}]_c, e_{p,p+1}]_c = 0;$$

the last equality follows from Lemma 1.10 (c), because of (6.17) and the induction hypothesis. In the second case, we have

$$[e_{i,j}, e_{p,r}]_c = [[e_{i,h}, e_{h,j}]_c, e_{p,p+1}]_c = 0;$$

the last equality follows from Lemma 1.10 (c), because of the induction hypothesis and (6.18). Finally, if $r - p > 1$ then

$$[e_{i,j}, e_{p,r}]_c = [e_{i,j}, [e_{p,r-1}, e_{r-1,r}]_c]_c = 0$$

by Lemma 1.10 (b) and the induction hypothesis.

(b). We prove (6.23) by induction on $r - i$. If $r - i = 2$, then the claimed equality is just (6.13). If $r - i > 2$ we argue by induction on $j - i$. If $j - i = 1$ we have

$$[e_{i,i+1}, e_{i,r}]_c = [e_{i,i+1}, [e_{i,r-1}, e_{r-1,r}]_c]_c = [[e_{i,i+1}, e_{i,r-1}]_c, e_{r,r-1}]_c = 0$$

by (1.22), since $[e_{i,i+1}, e_{r-1,r}]_c = 0$ by (6.17). If $j - i > 2$, we have

$$[e_{i,j}, e_{i,r}]_c = [[e_{i,j-1}, e_{j-1,j}]_c, e_{i,r}]_c = 0$$

by Lemma 1.10 (c), because of the induction hypothesis and (6.22).

The proof of (6.24) is analogous to the proof of (6.23), using (6.14) instead of (6.13). $\square$

Lemma 6.8. *Assume that* (6.12), (6.13), (6.14) *hold in* R. *Then*

$$[e_{i,j}, e_{p,r}]_c = B^{pj}_{jr}(q-1)e_{ir}e_{pj}, \qquad \textit{if} \quad 1 \le i < p < j < r \le n+1. \tag{6.25}$$

Proof. We compute:

$$\begin{aligned}
[e_{i,j}, e_{p,r}]_c &= [[e_{i,p}, e_{p,j}]_c, e_{p,r}]_c \\
&= [e_{i,p}, [e_{p,j}, e_{p,r}]_c]_c + B^{p,j}_{p,r}[e_{i,p}, e_{p,r}]_c e_{p,j} - B^{i,p}_{p,j} e_{p,j}[e_{i,p}, e_{p,r}]_c \\
&= B^{p,j}_{p,r} e_{i,r} e_{p,j} - B^{i,p}_{p,j} e_{p,j} e_{i,r} \\
&= \left(B^{p,j}_{p,r} - B^{i,p}_{p,j}(B^{i,r}_{p,j})^{-1}\right) e_{i,r} e_{p,j} \\
&= \left(B^{p,j}_{p,r} - (B^{p,r}_{p,j})^{-1}\right) e_{i,r} e_{p,j} \\
&= B^{pj}_{jr}(q-1)e_{ir}e_{pj}.
\end{aligned}$$

Here, the first equality is by (6.19); the second, by Lemma 1.10 (a); the third, by (6.23) and by (6.19); the fourth, by (6.22). $\square$

Lemma 6.9. *Assume that* (6.12), (6.13) *and* (6.14) *hold in* R. *For any* $1 \le i < j \le n+1$ *we have*

$$\Delta(e^N_{i,j}) = e^N_{i,j} \otimes 1 + 1 \otimes e^N_{i,j} + (1-q^{-1})^N \sum_{i<p<j} \left(B^{p,j}_{i,p}\right)^{N(N-1)/2} e^N_{i,p} \otimes e^N_{p,j}. \tag{6.26}$$

Proof. By (6.20), and using several times the quantum binomial formula (1.13), we have

$$\begin{aligned}
\Delta(e^N_{i,j}) &= \left(e_{i,j} \otimes 1 + (1-q^{-1}) \sum_{i<p<j} e_{i,p} \otimes e_{p,j}\right)^N + 1 \otimes e^N_{i,j} \\
&= e^N_{i,j} \otimes 1 + (1-q^{-1})^N \left(\sum_{i<p<j} e_{i,p} \otimes e_{p,j}\right)^N + 1 \otimes e^N_{i,j} \\
&= e^N_{i,j} \otimes 1 + (1-q^{-1})^N \sum_{i<p<j} (e_{i,p} \otimes e_{p,j})^N + 1 \otimes e^N_{i,j} \\
&= e^N_{i,j} \otimes 1 + (1-q^{-1})^N \sum_{i<p<j} \left(B^{p,j}_{i,p}\right)^{N(N-1)/2} e^N_{i,p} \otimes e^N_{p,j} + 1 \otimes e^N_{i,j}.
\end{aligned}$$

Here, in the first equality we use that $(1\otimes e_{i,j})(e_{i,j}\otimes 1) = q(e_{i,j}\otimes 1)(1\otimes e_{i,j})$ and $(1\otimes e_{i,j})(e_{i,p}\otimes e_{p,j}) = q(e_{i,p}\otimes e_{p,j})(1\otimes e_{i,j})$, this last by (6.24); in the second, we use $(e_{i,p}\otimes e_{p,j})(e_{i,j}\otimes 1) = q(e_{i,j}\otimes 1)(e_{i,p}\otimes e_{p,j})$, which follows from (6.23); the third, that $(e_{i,p}\otimes e_{p,j})(e_{i,s}\otimes e_{s,j}) = q^2(e_{i,s}\otimes e_{s,j})(e_{i,p}\otimes e_{p,j})$ for $p < s$, which follows from (6.23) and (6.24); the fourth, from $(e_{i,p}\otimes e_{p,j})^h = \left(B^{p,j}_{i,p}\right)^{h(h-1)/2} e^h_{i,p} \otimes e^h_{p,j}$. $\square$

Remark 6.10. Let Γ be a group with $g_1, \dots, g_n$ in the center of Γ, $\chi_1, \dots, \chi_n$ in $\widehat{\Gamma}$, as in 6.1. Let R be a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that (6.12), (6.13) and (6.14) hold in R. By (6.26) and the reconstruction formulas (1.17), we have

$$\Delta_{R\#\Bbbk\Gamma}(e_{i,j}^N) = e_{i,j}^N \otimes 1 + g_{i,j}^N \otimes e_{i,j}^N + (1-q^{-1})^N \sum_{i<p<j} \left(B_{i,p}^{p,j}\right)^{N(N-1)/2} e_{i,p}^N g_{p,j}^N \otimes e_{p,j}^N. \tag{6.27}$$

Lemma 6.11. *Assume that* $R = \mathfrak{B}(V)$. *Then*

$$e_{i,j}^N = 0, \qquad 1 \le i < j \le n+1. \tag{6.28}$$

Proof. This follows from Lemma 6.9 by induction on $j-i$, the case $j-i=1$ being Lemma 6.3 (c). $\square$

Lemma 6.12. *Assume that* (6.12), (6.13), (6.14), (6.28), *hold in* R. *Assume, furthermore, that* R *is generated as an algebra by the elements* $x_1, \dots, x_n$. *Then the algebra* R *is spanned as a vector space by the elements*

$$e_{1,2}^{\epsilon_{1,2}} e_{1,3}^{\epsilon_{1,3}} \dots e_{1,n+1}^{\epsilon_{1,n+1}} e_{2,3}^{\epsilon_{2,3}} \dots e_{2,n+1}^{\epsilon_{2,n+1}} \dots e_{n,n+1}^{\epsilon_{n,n+1}}, \qquad \textit{with } \epsilon_{i,j} \in \{0, 1, \dots, N-1\}. \tag{6.29}$$

Proof. We order the family (e_{ij}) by

$$e_{1,2} \prec e_{1,3} \prec \dots e_{1,n+1} \prec e_{2,3} \prec \dots e_{2,n+1} \prec \dots e_{n,n+1};$$

this induces an ordering in the monomials (6.29). If M is an ordered monomial, we set $\sigma(M) := e_{r,s}$ if $e_{r,s}$ is the first element appearing in M. Let B be the subspace generated by the monomials in (6.29). We show by induction on the length that, for any ordered monomial M and for any i, $e_{i,i+1}M \in B$ and it is 0 or a combination of monomials N with $\sigma(N) \succeq \min\{e_{i,i+1}, \sigma(M)\}$, length of $N \le$ length of $M+1$. The statement is evident if the length of M is 0; so that assume that the length is positive. Write $M = e_{p,q}M'$ where $e_{p,q} \preceq M'$. We have several cases:

If $i < p$ or $i = p$ and $i+1 < q$, $e_{i,i+1} \prec e_{p,q}$ and we are done.

If $i = p$ and $i+1 = q$, then the claim is clear.

If $p < q < i$ then $e_{i,i+1}e_{p,q} = B_{p,q}^{i,i+1} e_{p,q}e_{i,i+1}$ by (6.18); hence $e_{i,i+1}M = e_{i,i+1}e_{p,q}M' = B_{p,q}^{i,i+1} e_{p,q}e_{i,i+1}M'$; by the inductive hypothesis and the fact that $e_{p,q} \preceq \min\{e_{i,i+1}, \sigma(M')\}$, the claim follows.

If $p < i = q$ then $e_{i,i+1}e_{p,i} = (B_{i,i+1}^{p,i})^{-1}(e_{p,i}e_{i,i+1} - e_{p,i+1})$ by (6.19); again, the inductive hypothesis and $e_{p,q} \preceq \min\{e_{i,i+1}, \sigma(M')\}$ imply that $e_{p,i}e_{i,i+1}M'$ has the form we want. To see that $e_{p,i+1}M$" satisfies the claim when $e_{p,i} = \sigma(M')$, we use $e_{p,i+1}e_{p,i} = (B_{p,i+1}^{p,i})_{-1}e_{p,i}e_{p,i+1}$ by (6.23).

If $p < i < q$ then $e_{i,i+1}e_{p,q} = (B_{i,i+1}^{p,q})^{-1}e_{p,q}e_{i,i+1}$ by (6.22) or (6.24); we then argue as in the two preceding cases.

Therefore, $B = R$ since it is a left ideal containing 1. $\square$

We shall say that the elements $e_{1,2}, e_{1,3}, \dots, e_{1,n+1} e_{2,3} \dots e_{2,n+1} \dots e_{n,n+1}$, in this order, form a *PBW-basis* for R if the monomials (6.29) form a basis of R. Then we have:

Theorem 6.13. *The elements $e_{1,2}, e_{1,3}, \dots, e_{1,n+1} e_{2,3} \dots e_{2,n+1} \dots e_{n,n+1}$, in this order, form a PBW basis for $\mathfrak{B}(V_n)$. In particular,*

$$\dim \mathfrak{B}(V_n) = N^{\frac{n(n+1)}{2}}.$$

Proof. We proceed by induction on n. The case $n = 1$ is clear, see [AS1, Section 3] for details. We assume the statement for $n-1$. We consider V_n as a Yetter-Drinfeld module over $\Gamma = (\mathbb{Z}/P)^n$, as explained in Remark 6.1. Let $Z_n = \mathfrak{B}(V_n)\#\Bbbk\Gamma$. Let $i_n : V_{n-1} \to V_n$ be given by $x_i \mapsto x_i$ and $p_n : V_n \to V_{n-1}$ by $x_i \mapsto x_i$, $1 \le i \le n-1$ and $x_n \mapsto 0$. The splitting of Yetter-Drinfeld modules $\operatorname{id}_{V_{n-1}} = p_n i_n$ gives rise to a splitting of Hopf algebras $\operatorname{id}_{Z_{n-1}} = \pi_n \iota_n$, where $\iota_n : Z_{n-1} \to Z_n$ and $\pi_n : Z_n \to Z_{n-1}$ are respectively induced by i_n, p_n. Let

$$R_n = Z_n^{\operatorname{co} \pi_n} = \{z \in Z_n : (\operatorname{id} \otimes \pi_n)\Delta(z) = z \otimes 1\}.$$

Then R_n is a braided Hopf algebra in the category ${}^{Z_{n-1}}_{Z_{n-1}}\mathcal{YD}$; we shall denote by c_{R_n} the corresponding braiding of R_n. We have $Z_n \simeq R_n \# Z_{n-1}$ and in particular $\dim Z_n = \dim R_n \dim Z_{n-1}$.

For simplicity, we denote $h_i = e_{i,n+1}$, $1 \le i \le n$. We have $h_i h_j = B^{i,n+1}_{j,n+1} h_j h_i$, for $i < j$, by (6.24). We claim that $h_1, \dots, h_n$ are linearly independent primitive elements of the braided Hopf algebra R_n.

Indeed, it follows from (6.8) that $\pi_n(h_i) = 0$; by (6.20), we conclude that $h_i \in R_n$. We prove by induction on $j = n+1-i$ that h_i is a primitive element of R_n, the case $j = 1$ being clear. Assume the statement for j. Now

$$\begin{aligned} x_{i-1} \rightharpoonup h_i &= x_{i-1} h_i + g_{i-1} h_i \mathcal{S}(x_{i-1}) = x_{i-1} h_i - g_{i-1} h_i g_{i-1}^{-1} x_{i-1} \\ &= x_{i-1} h_i - B^{i-1,i}_{i,n+1} h_i x_{i-1} = [x_{i-1}, h_i]_c = h_{i-1}. \end{aligned}$$

So

$$\begin{aligned} \Delta_{R_n}(h_{i-1}) &= \Delta_{R_n}(x_{i-1} \rightharpoonup h_i) = x_{i-1} \rightharpoonup \Delta(h_i) \\ &= g_{i-1} \rightharpoonup 1 \otimes x_{i-1} \rightharpoonup h_i + x_{i-1} \rightharpoonup h_i \otimes 1 = h_{i-1} \otimes 1 + 1 \otimes h_{i-1}. \end{aligned}$$

We prove also by induction on $j = n+1-i$ that $h_i \neq 0$ using (6.20) and the induction hypothesis on Z_{n-1}. Since h_i is homogeneous of degree j (with respect to the grading of Z_n), we conclude that $h_1, \dots, h_n$ are linearly independent.

We next claim that $c_{R_n}(h_i \otimes h_j) = B^{i,n+1}_{j,n+1} h_j \otimes h_i$, for any $i > j$.

By (6.27), the coaction of Z_{n-1} on R_n satisfies

$$\delta(h_i) = g_{i,n+1} \otimes e_{i,n+1} + (1 - q^{-1}) \sum_{i<p<n+1} e_{i,p} g_{p,n+1} \otimes e_{p,n+1}.$$

If $j < i$, we compute the action on R_n:

$$\begin{aligned} e_{i,p} \rightharpoonup h_j &= e_{i,p} h_j + g_{i,p} h_j \mathcal{S}(e_{i,n+1}) + (1 - q^{-1}) \sum_{i<t<p} e_{i,t} g_{t,p} h_j \mathcal{S}(e_{t,p}) \\ &= (B^{j,n+1}_{i,p})^{-1} h_j e_{i,p} + B^{i,p}_{j,n+1} h_j g_{i,p} \mathcal{S}(e_{i,n+1}) \\ &\quad + h_j (1 - q^{-1}) \sum_{i<t<p} B^{t,p}_{j,n+1} (B^{j,n+1}_{i,t})^{-1} e_{i,t} g_{t,p} \mathcal{S}(e_{t,p}) \\ &= B^{i,p}_{j,n+1} h_j e_{i,p_{(1)}} \mathcal{S}(e_{i,p_{(2)}}) \\ &= 0, \end{aligned}$$

by (6.24). Thus

$$c_{R_n}(h_i \otimes h_j) = g_{i,n+1} \rightharpoonup h_j \otimes h_i = B^{i,n+1}_{j,n+1} h_j \otimes h_i.$$

We next claim that the dimension of the subalgebra of R_n spanned by $h_1, \dots, h_n$ is $\geq N^n$.

We already know that

$$\Delta(h_j^{m_j}) = \sum_{0 \leq i_j \leq m_j} \binom{m_j}{i_j}_q h_j^{i_j} \otimes h_j^{m_j - i_j}, \qquad m_j \leq N.$$

Let us denote $\mathbf{m} = (m_1, \dots, m_j, \dots, m_n)$, $\mathbf{1} = (1, \dots, 1, \dots, 1)$, $\mathbf{N} = (N, \dots, N)$. We consider the partial order $\mathbf{i} \leq \mathbf{m}$, if $i_j \leq m_j$, $j = 1, \dots, n$. We set $h^{\mathbf{m}} := h_n^{m_n} \dots h_j^{m_j} \dots h_1^{m_1}$. ¿From the preceding claim, we deduce that

$$\Delta(h^{\mathbf{m}}) = h^{\mathbf{m}} \otimes 1 + 1 \otimes h^{\mathbf{m}} + \sum_{0 \leq \mathbf{i} \leq \mathbf{m}, \quad 0 \neq \mathbf{i} \neq \mathbf{m}} c_{\mathbf{m},\mathbf{i}} h^{\mathbf{i}} \otimes h^{\mathbf{m}-\mathbf{i}}, \quad \mathbf{m} \leq \mathbf{N} - \mathbf{1};$$

where $c_{\mathbf{m},\mathbf{i}} \neq 0$ for all $\mathbf{i}$. We then argue recursively as in the proof of [AS1, Lemma 3.3] to conclude that the elements $h^{\mathbf{m}}$, $\mathbf{m} \leq \mathbf{N} - \mathbf{1}$, are linearly independent; hence the dimension of the subalgebra of R_n spanned by $h_1, \dots, h_n$ is $\geq N^n$, as claimed.

We can now finish the proof of the Theorem. Since $\dim Z_n \leq N^{\frac{n(n+1)}{2}}$ by Lemma 6.12 and $\dim Z_{n-1} = N^{\frac{n(n-1)}{2}}$ by the induction hypothesis, we have $\dim R_n \leq N^n$. By what we have just seen, this dimension is exactly N^n. Therefore, $\dim Z_n = N^{\frac{n(n+1)}{2}}$; in presence of Lemma 6.12, this implies the Theorem. ☐

Theorem 6.14. *The Nichols algebra $\mathfrak{B}(V)$ can be presented by generators $e_{i,i+1}$, $1 \leq i \leq n$, and relations* (6.12), (6.13), (6.14) *and* (6.28).

Proof. Let $\mathfrak{B}'$ be the algebra presented by generators $e_{i,i+1}$, $1 \le i \le n$, and relations (6.12), (6.13), (6.14) and (6.28). We claim that $\mathfrak{B}'$ is is a braided Hopf algebra with the $e_{i,i+1}$'s primitive. Indeed, the claim follows without difficulty; use Lemma 6.9 for relations (6.28).

By Lemma 6.12, we see that the monomials (6.29) span $\mathfrak{B}'$ as vector space, and in particular that $\dim \mathfrak{B}' \le N^{\frac{n(n+1)}{2}}$. By Lemmas 6.3 and 6.11, there is a surjective algebra map $\psi : \mathfrak{B}' \to \mathfrak{B}(V)$. By Theorem 6.13, ψ is an isomorphism. □

6.2. **Lifting of Nichols algebras of type A_n.**

We fix in this Section a finite *abelian* group Γ such that our braided vector space V can be realized in ${}_\Gamma^\Gamma\mathcal{YD}$, as in Remark 6.1. That is, we have $g_1, \dots, g_n$ in Γ, $\chi_1, \dots, \chi_n$ in $\widehat{\Gamma}$, such that $q_{ij} = \langle \chi_j, g_i \rangle$ for all i,j, and V can be realized as a Yetter-Drinfeld module over Γ by (6.6).

We also fix a finite dimensional pointed Hopf algebra A such that $G(A)$ is isomorphic to Γ, and the infinitesimal braiding of A is isomorphic to V as Yetter-Drinfeld module over Γ. That is, $\operatorname{gr} A \simeq R\#\Bbbk\Gamma$, and the subalgebra R' of R generated by $R(1)$ is isomorphic to $\mathfrak{B}(V)$. We choose elements $a_i \in (A_1)_{g_i}^{\chi_i}$ such that $\pi(a_i) = x_i$, $1 \le i \le n$.

We shall consider, more generally, Hopf algebras H provided with

- a group isomorphism $\Gamma \to G(H)$;
- elements $a_1, \dots, a_n$ in $\mathcal{P}(H)_{g_i,1}^{\chi_i}$.

Further hypotheses on H will be stated when needed. The examples of such H we are thinking of are the Hopf algebra A, and any bosonization $R\#\Bbbk\Gamma$, where R is any braided Hopf algebra in ${}_\Gamma^\Gamma\mathcal{YD}$ provided with a monomorphism of Yetter-Drinfeld modules $V \to P(R)$; so that $a_i := x_i\#1$, $1 \le i \le n$. This includes notably the Hopf algebras $T(V)\#\Bbbk\Gamma$, $\widehat{\mathfrak{B}}(V)\#\Bbbk\Gamma$, $\mathfrak{B}(V)\#\Bbbk\Gamma$.

Here $\widehat{\mathfrak{B}}(V)$ be the braided Hopf algebra in ${}_\Gamma^\Gamma\mathcal{YD}$ generated by $x_1, \dots, x_\theta$ with relations (6.12), (6.13) and (6.14).

We introduce inductively the following elements of H:

(6.30) $E_{i,i+1} := a_i;$

(6.31) $E_{i,j} := \operatorname{ad}(E_{i,j-1})(E_{j-1,j}), \quad 1 \le i < j \le n+1,\ j-i \ge 2.$

Assume that $H = R\#\Bbbk\Gamma$ as above. Then, by the relations between braided commutators and the adjoint (1.21), the relations (6.12), (6.13) and (6.14) translate respectively to

$$\operatorname{ad} E_{i,i+1}(E_{p,p+1}) = 0; \qquad 1 \le i < p \le n, \qquad p-i \ge 2; \tag{6.32}$$

$$(\operatorname{ad} E_{i,i+1})^2(E_{i+1,i+2}) = 0, \qquad 1 \le i < n; \tag{6.33}$$

$$(\operatorname{ad} E_{i+1,i+2})^2(E_{i,i+1}) = 0, \qquad 1 \le i < n. \tag{6.34}$$

Remark 6.15. Relations (6.32), (6.33) and (6.34) can be considered, more generally, in any H as above. If these relations hold in H, then we have a Hopf algebra map $\pi_H : \widehat{\mathfrak{B}}(V)\#\Bbbk\Gamma \to H$. On the other hand, we know by Remark 6.10 that the comultiplication of the elements E_{ij}^N is given by (6.27). Hence, the same formula is valid in H, provided that relations (6.32), (6.33) and (6.34) hold in it. In particular, the subalgebra of H generated by the elements E_{ij}^N, $g_{i,j}^N$, $1 \le i < j \le n+1$, is a Hopf subalgebra of H.

Lemma 6.16. *Relations* (6.32), (6.33) *and* (6.34) *hold in A if $N > 3$.*

Proof. This is a particular case of Theorem 5.6; we include the proof for completeness. We know, by Lemma 2.10, that

$$\operatorname{ad} E_{i,i+1}(E_{p,p+1}) \in \mathcal{P}_{g_i g_p,1}(A)^{\chi_i\chi_p}, \qquad 1 \le i < p \le n, \qquad p - i \ge 2,$$
$$(\operatorname{ad} E_{i,i+1})^2(E_{p,p+1}) \in \mathcal{P}_{g_i^2 g_p,1}(A)^{\chi_i^2\chi_p}, \qquad 1 \le i, p \le n, \qquad |p - i| = 1.$$

Assume that $\operatorname{ad} E_{i,i+1}(E_{p,p+1}) \ne 0$, and $\chi_i\chi_p \ne \varepsilon$, where $1 \le i < p \le n$, $p - i \ge 2$. By Lemma 5.1, there exists ℓ, $1 \le \ell \le n$, such that $g_i g_p = g_\ell$, $\chi_i\chi_p = \chi_\ell$. But then

$$q = \chi_\ell(g_\ell) = \chi_i(g_i)\chi_i(g_p)\chi_p(g_i)\chi_p(g_p) = q^2.$$

Hence $q = 1$, a contradiction.

Assume next that $\operatorname{ad} E_{i,i+1}^2(E_{p,p+1}) \ne 0$, $|p-i| = 1$. and $\chi_i^2\chi_p \ne \varepsilon$. By Lemma 5.1 , there exists ℓ, $1 \le \ell \le n$, such that $g_i^2 g_p = g_\ell$, $\chi_i^2\chi_p = \chi_\ell$. But then

$$q = \chi_\ell(g_\ell) = \chi_i(g_i)^4\chi_i(g_p)^2\chi_p(g_i)^2\chi_p(g_p) = q^3.$$

Hence $q = \pm 1$, a contradiction (we assumed $N > 2$).

It remains to exclude the cases $\chi_i\chi_p = \varepsilon, |p-i| \ge 2$, and $\chi_i^2\chi_p = \varepsilon, |p - i| = 1$. The first case leads to the contradiction $N = 3$. In the second case it follows from the connectivity of A_n that N would divide 2 which is also impossible. □

Lemma 6.17. *If $H = A$, then $E_{i,j}^N \in \Bbbk\Gamma^N$, for any $1 \le i < j \le n+1$.*

Proof. We first show that $E_{i,j}^N \in \Bbbk\Gamma$, $1 \le i < j \le n+1$. (For our further purposes, this is what we really need).

Let $i < j$. We claim that there exists no ℓ, $1 \le \ell \le n$, such that $g_{i,j}^N = g_\ell$, $\chi_{i,j}^N = \chi_\ell$. Indeed, otherwise we would have

$$q = \chi_\ell(g_\ell) = \chi_{i,j}(g_{i,j})^{N^2} = q^{N^2} = 1.$$

By Lemma 6.16 and Remark 6.15, we have

$$\Delta(E_{i,j}^N) = E_{i,j}^N \otimes 1 + g_{i,j}^N \otimes E_{i,j}^N + (1 - q^{-1})^N \sum_{i<p<j} \left(B_{i,p}^{p,j}\right)^{N(N-1)/2} E_{i,p}^N g_{p,j}^N \otimes E_{p,j}^N. \tag{6.35}$$

We proceed by induction on $j - i$. If $j - i = 1$, then, by Lemma 5.1 , either $E_{i,i+1}^N \in \Bbbk\Gamma$ or $E_{i,i+1}^N \in \mathcal{P}_{g_i^N,1}(A)^{\chi_i^N}$ and $\chi_i^N \ne \varepsilon$, hence $g_i^N = g_l, \chi_i^N = \chi_l$ for some l; but this last possibility contradicts the claim above. Assume then that $j - i > 1$. By the recursion hypothesis, $\Delta(E_{i,j}^N) = E_{i,j}^N \otimes 1 + g_{i,j}^N \otimes E_{i,j}^N + u$, for some $u \in \Bbbk\Gamma \otimes \Bbbk\Gamma$. In particular, we see that $E_{i,j}^N \in (A_1)^{\chi_i^N}$. Then, by Lemma 5.1, either $\chi_i^N = \varepsilon$ and hence $E^N_{i,i+1} \in \Bbbk\Gamma$, or else $\chi_i^N \neq \varepsilon$, which implies $u = 0$ and $E^N_{i,i+1} \in \mathcal{P}_{g_i^N,1}(A)^{\chi_i^N}$. Again, this last possibility contradicts the claim above.

Finally, let C be the subalgebra of A generated by the elements E^N_{ij}, $g^N_{i,j}$, $1 \leq i < j \leq n+1$, which is a Hopf subalgebra of H. Since $E^N_{i,j} \in \Bbbk\Gamma \cap C$, we conclude that $E^N_{i,j} \in C_0 = \Bbbk\Gamma^N$. $\square$

To solve the lifting problem, we see from Lemma 6.17 that we first have to answer a combinatorial question in the group algebra of an abelian group. To simplify the notation we define

$$h_{ij} = g^N_{i,j}, \qquad C^j_{i,p} = (1-q^{-1})^N \left(B^{p,j}_{i,p}\right)^{N(N-1)/2}.$$

We are looking for families $(u_{ij})_{1\leq i<j\leq n+1}$ of elements in $\Bbbk\Gamma$ such that

$$\Delta(u_{ij}) = u_{ij} \otimes 1 + h_{i,j} \otimes u_{ij} + \sum_{i<p<j} C^j_{i,p} u_{i,p}\, h_{p,j} \otimes u_{p,j}. \tag{6.36}$$

The coefficients $C^j_{i,p}$ satisfy the rule

$$C^j_{is} C^j_{st} = C^t_{is} C^j_{it}, \text{ for all } 1 \leq i < s < t < j \leq n+1. \tag{6.37}$$

This follows from (6.2) and (6.3) since

$$B^{sj}_{is} B^{tj}_{st} = B^{st}_{is} B^{tj}_{is} B^{tj}_{st} = B^{st}_{it} B^{tj}_{it}.$$

Theorem 6.18. *Let Γ be a finite abelian group and $h_{ij} \in \Gamma$, $1 \leq i < j \leq n+1$, a family of elements such that*

$$h_{ij} = h_{i,p}\, h_{p,j}, \qquad \textit{if} \quad i<p<j. \tag{6.38}$$

Let $C^j_{i,p} \in \Bbbk^\times$, $1 \leq i < p < j \leq n+1$, be a family of elements satisfying (6.37). Then the solutions $(u_{ij})_{1\leq i<j\leq n+1}$ of (6.36), $u_{ij} \in \Bbbk\Gamma$ for all $i<j$, have the form $(u_{ij}(\gamma))_{1\leq i<j\leq n+1}$ where $\gamma = (\gamma_{ij})_{1\leq i<j\leq n+1}$ is an arbitrary family of scalars $\gamma_{ij} \in \Bbbk$ such that

$$\textit{for all } 1 \leq i < j \leq n+1,\ \gamma_{ij} = 0 \textit{ if } h_{ij} = 1, \tag{6.39}$$

and where the elements $u_{ij}(\gamma)$ are defined by induction on $j-i$ by

$$u_{ij}(\gamma) = \gamma_{ij}(1-h_{ij}) + \sum_{i<p<j} C^j_{ip}\gamma_{ip} u_{pj}(\gamma) \textit{ for all } 1 \leq i < j \leq n+1. \tag{6.40}$$

Proof. We proceed by induction on k. We claim that the solutions $u_{ij} \in \Bbbk\Gamma, 1 \leq i < j \leq n+1, j-i \leq k$, of (6.36) for all $i<j$ with $j-i \leq k$ are given by arbitrary families of scalars $\gamma_{ij}, 1 \leq i < j \leq n+1, j-i \leq k$ such that

$$u_{ij} = \gamma_{ij}(1-h_{ij}) + \sum_{i<p<j} C^j_{ip}\gamma_{ip} u_{pj} \text{ for all } 1 \leq i < j \leq n+1 \text{ with } j-i \leq k.$$

Suppose $k = 1$. For any $1 \le i < n, j = i+1$, $u_{i,i+1}$ is a solution of (6.36) if and only if $u_{i,i+1}$ is $(h_{i,i+1}, 1)$-primitive in $\Bbbk\Gamma$, that is $u_{i,i+1} = \gamma_{i,i+1}(1 - h_{i,i+1})$ for some $\gamma_{i,i+1} \in \Bbbk$. We may assume that $\gamma_{i,i+1} = 0$, if $h_{i,i+1} = 1$.

For the induction step, let $k > 2$. We assume that $\gamma_{ab} \in \Bbbk, 1 \le a < b \le n+1, b-a \le k-1$, is a family of scalars with $\gamma_{ab} = 0$, if $h_{ab} = 1$, and that the family $u_{ab} \in \Bbbk\Gamma, 1 \le a < b \le n+1, b-a \le k-1$, defined inductively by the γ_{ab} by (6.40) is a solution of (6.36). Let $1 \le i < j \le n+1$, and $j - i = k$. We have to show that

$$\Delta(u_{ij}) = u_{ij} \otimes 1 + h_{ij} \otimes u_{ij} + \sum_{i<p<j} C^j_{ip} u_{ip} h_{pj} \otimes u_{pj} \tag{6.41}$$

is equivalent to

$$u_{ij} = \gamma_{ij}(1 - h_{ij}) + \sum_{i<p<j} C^j_{ip} \gamma_{ip} u_{pj} \text{ for some } \gamma_{ij} \in \Bbbk. \tag{6.42}$$

We then may define $\gamma_{ij} = 0$ if $h_{ij} = 1$.

We denote

$$z_{ij} := u_{ij} - \sum_{i<p<j} C^j_{ip} \gamma_{ip} u_{pj}.$$

Then (6.42) is equivalent to

$$\Delta(z_{ij}) = z_{ij} \otimes 1 + h_{ij} \otimes z_{ij}. \tag{6.43}$$

For all $i < p < j$ we have $\Delta(u_{pj}) = u_{pj} \otimes 1 + h_{pj} \otimes u_{pj} + \sum_{p<s<j} C^j_{ps} u_{ps} h_{sj} \otimes u_{sj}$, since $j - p < k$. Using this formula for $\Delta(u_{pj})$ we compute

$$\begin{aligned}
&\Delta(z_{ij}) - z_{ij} \otimes 1 - h_{ij} \otimes z_{ij} \\
&= \Delta(u_{ij}) - \sum_{i<p<j} C^j_{ip} \gamma_{ip} \Delta(u_{pj}) - z_{ij} \otimes 1 - h_{ij} \otimes z_{ij} \\
&= \Delta(u_{ij}) - \sum_{i<p<j} C^j_{ip} \gamma_{ip} (u_{pj} \otimes 1 + h_{pj} \otimes u_{pj} + \sum_{p<s<j} C^j_{ps} u_{ps} h_{sj} \otimes u_{sj}) \\
&- (u_{ij} - \sum_{i<p<j} C^j_{ip} \gamma_{ip} u_{pj}) \otimes 1 - h_{ij} \otimes (u_{ij} - \sum_{i<p<j} C^j_{ip} \gamma_{ip} u_{pj}) \\
&= \Delta(u_{ij}) - u_{ij} \otimes 1 - h_{ij} \otimes u_{ij} \\
&+ \sum_{i<p<j} C^j_{ip} \gamma_{ip} h_{ij} \otimes u_{pj} - \sum_{i<p<j} C^j_{ip} \gamma_{ip} h_{pj} \otimes u_{pj} - \sum_{i<p<s<j} C^j_{ip} C^j_{ps} \gamma_{ip} u_{ps} h_{sj} \otimes u_{sj}.
\end{aligned}$$

Therefore, (6.41) and (6.42) are equivalent if and only if the identity

$$\sum_{i<p<j} C^j_{ip}\gamma_{ip}h_{ij}\otimes u_{pj} - \sum_{i<p<j} C^j_{ip}\gamma_{ip}h_{pj}\otimes u_{pj} - \sum_{i<p<s<j} C^j_{ip}C^j_{ps}\gamma_{ip}u_{ps}h_{sj}\otimes u_{sj} = -\sum_{i<p<j} C^j_{ip}u_{ip}h_{pj}\otimes u_{pj} \tag{6.44}$$

holds.

To prove (6.44) we use (6.40) for all $i<p$, where $i<p<j$, that is $u_{ip}=\gamma_{ip}(1-h_{ip})+\sum_{i<s<p} C^p_{is}\gamma_{is}u_{sp}$. Then

$$\begin{aligned}
&\sum_{i<p<j} C^j_{ip}h_{pj}u_{ip}\otimes u_{pj} + \sum_{i<p<j} C^j_{ip}\gamma_{ip}h_{ij}\otimes u_{pj} - \sum_{i<p<j} C^j_{ip}\gamma_{ip}h_{pj}\otimes u_{pj}\\
&= \sum_{i<p<j} (C^j_{ip}h_{pj}(\gamma_{ip}(1-h_{ip}) + \sum_{i<s<p} C^p_{is}\gamma_{is}u_{sp}) + C^j_{ip}\gamma_{ip}(h_{ij}-h_{pj}))\otimes u_{pj}\\
&= \sum_{i<p<j} C^j_{ip}h_{pj}\sum_{i<s<p} C^p_{is}\gamma_{is}u_{sp}\otimes u_{pj}, \text{ since } h_{pj}(1-h_{ip}) = h_{pj}-h_{ij} \text{ by (6.38)},\\
&= \sum_{i<s<p<j} C^j_{is}C^j_{sp}\gamma_{is}u_{sp}h_{pj}\otimes u_{pj}, \text{ since } C^j_{ip}C^p_{is} = C^j_{is}C^j_{sp} \text{ by (6.37)}.
\end{aligned}$$

This proves (6.44) by interchanging s and p. □

Remarks 6.19. 1) Let $\gamma=(\gamma_{ij})_{1\leq 1qi<j\leq n+1}$ be an arbitrary family of scalars. Then it is easy to see that the family $u_{ij}(\gamma)\in \Bbbk\Gamma, 1\leq i<j\leq n+1$, can be defined explicitly as follows:

$$u_{ij}(\gamma) = \sum_{i\leq p<j} \phi^j_{ip}(\gamma)(1-h_{pj}) \text{ for all } i<j,$$

where

$$\phi^j_{ip}(\gamma) = \sum_{i=i_1<\cdots<i_k=p,k\geq 1} C^j_{i_1,i_2}\dots C_{i_{k-1},i_k}\gamma_{i_1,i_2}\dots\gamma_{i_{k-1},i_k}\gamma_{pj} \text{ for all } i\leq p<j$$

is a polynomial of degree p in the free variables $(\gamma_{ij})_{1\leq i<j\leq n+1}$.

2) Let $\gamma=(\gamma_{ij})_{1\leq i<j\leq n+1}$ and $\widetilde{\gamma}=(\widetilde{\gamma}_{ij})_{1\leq i<j\leq n+1}$ be families of scalars in $\Bbbk$ satisfying (6.39). Assume that for all $i<j$, $u_{ij}(\gamma)=u_{ij}(\widetilde{\gamma})$. Then $\gamma=\widetilde{\gamma}$. This follows easily be induction on $j-i$ from (6.40).

Lemma 6.20. *Assume the situation of Theorem 6.18. Let $\gamma=(\gamma_{ij})_{1\leq i<j\leq n+1}$ be a family of scalars in $\Bbbk$ satisfying* (6.39) *and define $u_{ij}=u_{ij}(\gamma)$ for all $1\leq i<j\leq n+1$ by* (6.40).

1) *The following are equivalent:*
   (a) *For all $i<j, u_{ij}=0$ if $\chi^N_{ij}\neq\varepsilon$.*

(b) *For all* $i < j, \gamma_{ij} = 0$ *if* $\chi_{ij}^N \neq \varepsilon$.

2) *Assume that* $h_{ij} = g_{ij}^N$ *for all* $i < j$. *Then the following are equivalent:*

(a) *For all* $i < j, u_{ij} = 0$ *if* $\chi_{ij}^N(g_l) \neq 1$ *for some* $1 \leq l \leq n$.

(b) *For all* $i < j, \gamma_{ij} = 0$ *if* $\chi_{ij}^N(g_l) \neq 1$ *for some* $1 \leq l \leq n$.

(c) *The elements* $u_{ij}, 1 \leq i < j \leq n+1$, *are central in* $\widehat{\mathfrak{B}}(V)\#\Bbbk\Gamma$.

Proof. 1) follows by induction on $j - i$.

Suppose $j = i+1$. Then $u_{i,i+1} = \gamma_{i,i+1}(1 - h_{i,i+1})$. If $h_{i,i+1} = 1$, then both $u_{i,i+1}$ and $\gamma_{i,i+1}$ are 0. If $h_{i,i+1} \neq 1$, then $u_{i,i+1} = 0$ if and only if $\gamma_{i,i+1} = 0$.

The induction step follows in the same way from (6.40), since for all $i < p < j$, if $\chi_{ij}^N \neq \varepsilon$, then $\chi_{ip}^N \neq \varepsilon$ or $\chi_{pj}^N \neq \varepsilon$, hence by induction $\gamma_{ip} = 0$ or $u_{pj} = 0$, and $u_{ij} = \gamma_{ij}(1 - h_{ij})$.

2) Suppose that for all $i < p < j$, u_{pj} is central in $\widehat{\mathfrak{B}}(V)\#\Bbbk\Gamma$, and let $1 \leq l \leq n$. Then

$$h_{ij}x_l = x_l\chi_l(h_{ij})h_{ij},$$

and we obtain from (6.40)

$$u_{ij}x_l = x_l\gamma_{ij}(1 - \chi_l(h_{ij})h_{ij}) + x_l \sum_{i<p<j} C_{ip}^j \gamma_{ip} u_{pj}.$$

Hence u_{ij} is central in $\widehat{\mathfrak{B}}(V)\#\Bbbk\Gamma$ if and only if $\gamma_{ij} = \gamma_{ij}\chi_l(h_{ij})$ for all $1 \leq l \leq n$. Since the braiding is of type A_n and the order of $q = \chi_l(g_l)$ is N,

$$\chi_l(h_{ij}) = \chi_l(g_{ij}^N) = \chi_{ij}^{-N}(g_l),$$

and the equivalence of (b) and (c) follows by induction on $j - i$. The equivalence of (a) and (b) is shown as in 1). □

6.3. **Classification of pointed Hopf algebras of type A_n.**

Using the previous results we will now determine exactly which finite-dimensional pointed Hopf algebras are of type A_n. We will find a big new class of deformations of $\mathfrak{u}_q^{\geq 0}(sl_n)$.

As before, we fix a natural number n, a finite abelian group Γ, an integer $N > 2$, a root of unity q of order N, $g_1, \dots, g_n \in \Gamma, \chi_1, \dots, \chi_n \in \widehat{\Gamma}$ such that $q_{ij} = \chi_j(g_i)$ for all i, j satisfy (6.1), and $V \in {}_\Gamma^\Gamma\mathcal{YD}$ with basis $x_i \in V_{g_i}^{\chi_i}, 1 \leq i \leq n$.

We recall that $\widehat{\mathfrak{B}}(V)$ is the braided Hopf algebra in ${}_\Gamma^\Gamma\mathcal{YD}$ generated by $x_1, \dots, x_n$ with the quantum Serre relations (6.12), (6.13) and (6.14).

In $\widehat{\mathfrak{B}}(V)$ we consider the iterated braided commutators $e_{i,j}, 1 \leq i < j \leq n+1$ defined inductively by (6.8) beginning with $e_{i,i+1} = x_i$ for all i.

Let $\mathbb{A}$ be the set of all families $(a_{i,j})_{1 \leq i < j \leq n+1}$ of integers $a_{i,j} \geq 0$ for all $1 \leq i < j \leq n+1$. For any $a \in \mathbb{A}$ we define

$$e^a := (e_{1,2})^{a_{1,2}}(e_{1,3})^{a_{1,3}} \dots (e_{n,n+1})^{a_{n,n+1}},$$

where the order in the product is the lexicographic order of the index pairs. We begin with the PBW-theorem for $\widehat{\mathfrak{B}}(V)$.

Theorem 6.21. *The elements $e^a, a \in \mathbb{A}$, form a basis of the $\Bbbk$-vector space $\widehat{\mathfrak{B}}(V)$.*

Proof. This can be shown similarly to the proof of Theorem 6.13. For general finite Cartan type the theorem can be derived from the PBW-basis of $U_q(\mathfrak{g})$ (see [L3]) by changing the group and twisting as described in [AS4, Section 4.2]. $\square$

The following commutation rule for the elements e_{ij}^N is crucial.

Lemma 6.22. *For all $1 \leq i < j \leq n+1, 1 \leq s < t \leq n+1$,*

$$[e_{i,j}, e_{s,t}^N]_c = 0, \text{ that is } e_{i,j}e_{s,t}^N = \chi_{s,t}^N(g_{i,j})e_{s,t}^N e_{i,j}.$$

Proof. Since $e_{i,j}$ is a linear combination of elements of the form $x_{i_1} \dots x_{i_k}$ with $k = j - i$ and $g_{i_1} \dots g_{i_k} = g_{ij}$, it is enough to consider the case when $j = i+1$.

To show $[e_{i,i+1}, e_{s,t}^N]_c = 0$, we will distinguish several cases.

First assume that $(i, i+1) < (s,t)$. If $i+1 < s$ resp. $i = s$ and $i+1 < t$, then $[e_{i,i+1}, e_{s,t}]_c = 0$ by (6.17) resp. (6.23), and the claim follows.

If $i+1 = s$, we denote $x = e_{i+1,t}, y = e_{i,i+1}, z = e_{it}$ and $\alpha = \chi_{i+1,t}(g_i), \beta = \chi_{i+1,t}(g_{i,t})$. Then

$$yx = \alpha xy + z, \text{ by (6.19), and } zx = \beta xz, \text{ by (6.24).}$$

Moreover, $\alpha = \chi_{i+1,t}(g_i) \neq \beta = \chi_{i+1,t}(g_{i,t}) = \chi_{i+1,t}(g_i)\chi_{i+1,t}(g_{i+1,t})$, and $\alpha^N = \beta^N$, since $\chi_{i+1,t}(g_{i+1,t}) = q$ by (6.5). Therefore it follows from [AS4, Lemma 3.4] that $yx^N = \alpha^N x^N y$, which was to be shown.

The claim is clear if $i = s$, and $i+1 = t$, since $\chi_i^N(g_i) = 1$.

It remains to consider the case when $(i, i+1) > (s,t)$. If $s < i$ and $t = i+1$, resp. $s < i$ and $i+1 < t$, resp. $s = i$ and $t < i+1$, then $e_{s,t}e_{i,i+1} = \chi_i(g_{s,t})e_{i,i+1}e_{s,t}$ by (6.24), resp.(6.22),resp.(6.23). Hence in all cases, $e_{i,i+1}e_{s,t}^N = \chi_i^{-N}(g_{s,t})e_{s,t}^N e_{i,i+1}$. This proves the claim $[e_{i,i+1}, e_{s,t}^N]_c = 0$, since $\chi_i^{-N}(g_{s,t}) = \chi_{s,t}^N(g_i)$. $\square$

We want to compute the dimension of certain quotient algebras of $\mathfrak{B}(V)\#\Bbbk\Gamma$. Since this part of the theory works for any finite Cartan type, we now consider more generally a left $\Bbbk\Gamma$-module algebra R over any abelian group Γ and assume that there are integers P and $N_i > 1$, elements $y_i \in R$, $h_i \in \Gamma$, $\eta_i \in \widehat{\Gamma}$, $1 \leq i \leq P$, such that

(6.45) $g \cdot y_i = \eta_i(g)y_i$, for all $g \in \Gamma, 1 \leq i \leq P$.

(6.46) $y_i y_j^{N_j} = \eta_j^{N_j}(h_i)y_j^{N_j}y_i$ for all $1 \leq i,j \leq P$.

(6.47) The elements $y_1^{a_1} \dots y_P^{a_P}, a_1, \dots, a_P \geq 0$, form a $\Bbbk$ − basis of R.

Let $\mathbb{L}$ be the set of all $l = (l_i)_{1 \le i \le P} \in \mathbb{N}^P$ such that $0 \le l_i < N_i$ for all $1 \le i \le P$. For $a = (a_i)_{1\le i \le P} \in \mathbb{N}^P$, we define

$$y^a = y_1^{a_1} \dots y_P^{a_P}, \text{ and } aN = (a_i N_i)_{1 \le i \le P}.$$

Then by (6.46), (6.47), the elements

$$y^l y^{aN}, \qquad l \in \mathbb{L},\ a \in \mathbb{N}^P,$$

form a $\Bbbk$-basis of R.

In the application to $\mathfrak{B}(V)\#\Bbbk\Gamma$, P is the number of positive roots, and the y_i play the role of the root vectors $e_{i,j}$.

To simplify the notation in the smash product algebra $R\#\Bbbk\Gamma$, we identify $r \in R$ with $r\#1$ and $v \in \Bbbk\Gamma$ with $1\#v$. For $1 \le i \le P$, let $\widetilde{\eta}_i : \Bbbk\Gamma \to \Bbbk\Gamma$ be the algebra map defined by $\widetilde{\eta}_i(g) = \eta_i(g)g$ for all $g \in \Gamma$. Then

$$v y_i = y_i \widetilde{\eta}_i(v) \text{ for all } v \in \Bbbk\Gamma.$$

We fix a family $u_i, 1 \le i \le P$, of elements in $\Bbbk\Gamma$, and denote

$$u^a := \prod_{1 \le i \le P} u_i^{a_i}, \text{ if } a = (a_i)_{1 \le i \le P} \in \mathbb{N}^P.$$

Let M be a free right $\Bbbk\Gamma$-module with basis $m(l), l \in \mathbb{L}$. We then define a right $\Bbbk\Gamma$-linear map

$$\varphi : R\#\Bbbk\Gamma \to M \text{ by } \varphi(y^l y^{aN}) := m(l)u^a \text{ for all } l \in \mathbb{L}, a \in \mathbb{N}^P.$$

Lemma 6.23. *Assume that*

- u_i *is central in* $R\#\Bbbk\Gamma$*, for all* $1 \le i \le P$*, and*
- $u_i = 0$ *if* $\eta_i^{N_i}(h_j) \ne 1$ *for some* $1 \le j \le P$*.*

Then the kernel of φ *is a right ideal of* $R\#\Bbbk\Gamma$ *containing* $y_i^{N_i} - u_i$ *for all* $1 \le i \le P$*.*

Proof. By definition, $\varphi(y_i^{N_i}) = m(0)u_i = \varphi(u_i)$.

To show that the kernel of φ is a right ideal, let

$$z = \sum_{l \in \mathbb{L}, a \in \mathbb{N}^P} y^l y^{aN} v_{l,a}, \text{ where } v_{l,a} \in \Bbbk\Gamma, \text{for all } l \in \mathbb{L}, a \in \mathbb{N}^P,$$

be an element with $\varphi(z) = 0$. Then $\varphi(z) = \sum_{l,a} m(l)u^a v_{l,a} = 0$, hence

$$\sum_{a \in \mathbb{N}^P} u^a v_{l,a} = 0, \text{for all } l \in \mathbb{L}.$$

Fix $1 \le i \le P$. We have to show that $\varphi(zy_i) = 0$.

For any $l \in \mathbb{L}$, we have the basis representation

$$y^l y_i = \sum_{t \in \mathbb{L}, b \in \mathbb{N}^P} \alpha_{t,b}^l y^t y^{bN}, \text{ where } \alpha_{t,b}^l \in \Bbbk \text{ for all } t \in \mathbb{L}, b \in \mathbb{N}^P.$$

Since u^a is central in $R\#\Bbbk\Gamma$,

$$u^a = \widetilde{\eta}_i(u^a) \text{ for all } a \in \mathbb{N}^P. \tag{6.48}$$

For any $a = (a_i)_{1\le i\le P} \in \mathbb{N}^P$ and any family $(g_i)_{1\le i\le P}$ of elements in Γ we define $\eta^{aN}((g_i)) = \prod_i {\eta_i}^{a_iN_i}(g_i)$. Then by (6.46), for all $a, b \in \mathbb{N}^P$,

$$y^{aN}y_i = y_iy^{aN}\eta^{aN}(g^a), \text{ and } y^{bN}y^{aN} = y^{(a+b)N}\eta^{aN}(g^b), \tag{6.49}$$

for some families of elements g^a, g^b in Γ.

By a reformulation of our assumption,

$$u^a\eta^{aN}((g_i)) = u^a \text{ for any } a \in \mathbb{N}^P \text{ and family}(g_i) \text{ in } \Gamma. \tag{6.50}$$

Using (6.49) we now can compute

$$\begin{aligned} zy_i &= \sum_{l,a} y^l y^{aN} v_{l,a} y_i = \sum_{l,a} y^l y^{aN} y_i \widetilde{\eta}_i(v_{l,a}) \\ &= \sum_{l,a} y^l y_i y^{aN} \eta^{aN}(g^a)\widetilde{\eta}_i(v_{l,a}) \\ &= \sum_{l,a}\sum_{t,b} \alpha^l_{t,b} y^t y^{bN} e^{aN} \eta^{aN}(g^a)\widetilde{\eta}_i(v_{l,a}) \\ &= \sum_{l,a}\sum_{t,b} \alpha^l_{t,b} y^t y^{(a+b)N} \eta^{aN}(g^a)\eta^{aN}(g^b)\widetilde{\eta}_i(v_{l,a}). \end{aligned}$$

Therefore

$$\begin{aligned} \varphi(zy_i) &= \sum_t m(t) \sum_{l,a,b} \alpha^l_{t,b} u^{a+b}\eta^{aN}(g^a)\eta^{aN}(g^b)\widetilde{\eta}_i(v_{l,a}) \\ &= \sum_t m(t) \sum_{l,a,b} \alpha^l_{t,b} u^{a+b}\widetilde{\eta}_i(v_{l,a}), \text{ by (6.50)}, \\ &= \sum_t m(t) \sum_{b,l} \alpha^l_{t,b} u^b \sum_a u^a\widetilde{\eta}_i(v_{l,a}) \\ &= \sum_t m(t) \sum_{b,l} \alpha^l_{t,b} u^b \widetilde{\eta}_i(\sum_a u^a v_{l,a}), \text{ by (6.48)}, \\ &= 0, \text{ since } \sum_a u^a v_{l,a} = 0. \end{aligned}$$

□

Theorem 6.24. *Let $u_i, 1 \le i \le P$, be a family of elements in $\Bbbk\Gamma$, and I the ideal in $R\#\Bbbk\Gamma$ generated by all $y_i^{N_i} - u_i, 1 \le i \le P$. Let $A = (R\#\Bbbk\Gamma)/I$ be the quotient algebra. Then the following are equivalent:*

1) The residue classes of $y^lg, l \in \mathbb{L}, g \in \Gamma$, form a $\Bbbk$-basis of A.

2) u_i is central in $R\#\Bbbk\Gamma$ for all $1 \le i \le P$, and $u_i = 0$ if $\eta_i^{N_i} \neq \varepsilon$.

Proof. 1) $\Rightarrow$ 2) : For all i and $g \in \Gamma$, $gy_i^{N_i} = \eta_i^{N_i}(g)y_i^{N_i}g$, hence $u_ig = gu_i \equiv \eta_i^{N_i}(g)u_ig \mod I$. Since by assumption, $\Bbbk\Gamma$ is a subspace of A, we conclude that $u_i = \eta_i^{N_i}(g)u_i$, and $u_i = 0$ if $\eta_i^{N_i} \neq \varepsilon$.

Similarly, for all $1 \le i, j \le n$, $y_iy_j^{N_j} = \eta_j^{N_j}(h_i)y_j^{N_j}y_i$ by (6.46), hence $y_iu_j \equiv \eta_j^{N_j}(h_i)u_jy_i \mod I$. Since we already know that $u_i = 0$ if $\eta_j^{N_j} \neq \varepsilon$, we see that $y_iu_j \equiv u_jy_i \mod I$. On the other hand $u_jy_i = y_i\widetilde{\eta}_i(u_j)$. Then our assumption in 1) implies that $\widetilde{\eta}_i(u_j) = u_j$. In other words, u_j is central in $R\#\Bbbk\Gamma$.

2) $\Rightarrow$ 1): Let J be the right ideal of $R\#\Bbbk\Gamma$ generated by all $y_i^{N_i} - u_i, 1 \le i \le P$. For any $1 \le i \le P$ and $g \in \Gamma$,

$$g(y_i^{N_i} - u_i) = y_i^{N_i}g\eta_i^{N_i}(g) - gu_i = (y_i^{N_i} - u_i)\eta_i^{N_i}(g)g,$$

since $gu_i = u_i\eta_i^{N_i}(g)g$ by 2).

And for all $1 \le i, j \le P$,

$$y_i(y_j^{N_j} - u_j) = \eta_j^{N_j}(h_i)y_j^{N_j}y_i - y_iu_j = (y_j^{N_j} - u_j)\eta_j^{N_j}(h_i)y_i,$$

since by 2) $y_iu_j = u_jy_i = u_j\eta_j^{N_j}(h_i)y_i$.

This proves $J = I$.

It is clear that the images of all $y^lg, l \in \mathbb{L}, g \in \Gamma$, generate the vector space A. To show linear independency, suppose

$$\sum_{l\in\mathbb{L},g\in\Gamma} \alpha_{l,g}y^lg \in I, \text{ with } \alpha_{l,g} \in \Bbbk \text{ for all } l \in \mathbb{L}, g \in \Gamma.$$

Since $I = J$, we obtain from Lemma 6.23 that $\varphi(I) = 0$. Therefore,

$$0 = \varphi(\sum_{l\in\mathbb{L},g\in\Gamma} \alpha_{l,g}y^lg) = \sum_{l\in\mathbb{L},g\in\Gamma} \alpha_{l,g}m(l)g,$$

hence $\alpha_{l,g} = 0$ for all l, g. □

We come back to A_n. Our main result in this Chapter is

Theorem 6.25. *(i). Let $\gamma = (\gamma_{i,j})_{1\le i<j\le n+1}$ be any family of scalars in $\Bbbk$ such that for any $i < j$, $\gamma_{i,j} = 0$ if $g_{i,j}^N = 1$ or $\chi_{i,j}^N \neq \varepsilon$. Define $u_{i,j} = u_{i,j}(\gamma) \in \Bbbk\Gamma, 1 \le i < j \le n+1$, by (6.40). Then*

$$A_\gamma := (\widehat{\mathfrak{B}}(V)\#\Bbbk\Gamma)/(e_{i,j}^N - u_{i,j} \mid 1 \le i < j \le n+1)$$

is a pointed Hopf algebra of dimension $N^{\frac{n(n+1)}{2}}\, ord(\Gamma)$ with $gr A_\gamma \simeq \mathfrak{B}(V)\#\Bbbk\Gamma$.

(ii). Conversely, if A is a finite-dimensional pointed Hopf algebra with $gr A \simeq \mathfrak{B}(V)\#\Bbbk\Gamma$, and $N > 3$, or if the infinitesimal braiding of A is of type A_n with $N > 7$ and not divisible by 3, then A is isomorphic to a Hopf algebra A_γ in (i).

Proof. (i). By Lemma 6.20, the elements $u_{i,j}$ are central in $\widehat{U} := \widehat{\mathfrak{B}}(V)\#\Bbbk\Gamma$, and $u_{i,j} = 0$ if $g_{i,j}^N = 1$ or $\chi_{i,j}^N \neq \varepsilon$. Hence the residue classes of the elements $e^l g, l \in \mathbb{A}, 0 \leq l_{i,j} < N$ for all $1 \leq i < j \leq n+1, g \in \Gamma$, form a basis of A_γ by Theorem 6.24. By Theorem 6.18, the $u_{i,j}$ satisfy (6.36). The ideal I of $\widehat{U}$ generated by all $e_{i,j}^N - u_{i,j}$ is a biideal, since

$$\begin{aligned}\Delta(e_{i,j}^N - u_{i,j}) =& (e_{i,j}^N - u_{i,j}) \otimes 1 + g_{i,j}^N \otimes (e_{i,j}^N - u_{i,j}) \\ &+ \sum_{i<p<j} C_{i,p}^j((e_{i,p}^N - u_{i,p})g_{p,j}^N \otimes e_{p,j}^N + u_{i,p}g_{p,j}^N \otimes (e_{p,j}^N - u_{p,j})) \in I \otimes \widehat{U} + \widehat{U} \otimes I,\end{aligned}$$

by (6.35) and (6.36).

Since A_γ is generated by group-like and skew-primitive elements, and the group-like elements form a group, A_γ is a Hopf algebra.

For all $1 \leq i \leq n$, let $a_i \in \mathrm{gr}(A_\gamma)(1)$ be the residue class of $x_i \in (A_\gamma)_1$. Define root vectors $a_{i,j} \in \mathrm{gr}(A_\gamma), 1 \leq i < j \leq N+1$ inductively as in (6.30) and (6.31). Then $a_{i,j}^N = 0$ in $\mathrm{gr}(A_\gamma)$ since $e_{i,j}^N = 0$ in A_γ. Therefore, by Theorem 6.14, there is a surjective Hopf algebra map

$$\mathfrak{B}(V)\#\Bbbk\Gamma \to \mathrm{gr}(A_\gamma) \text{ mapping } x_i\#g \text{ onto } a_i g, 1 \leq i \leq n, g \in \Gamma.$$

This map is an isomorphism, since $\dim(\mathrm{gr}(A_\gamma)) = \dim(A_\gamma) = N^{\frac{n(n+1)}{2}} |\Gamma| = \dim(\mathfrak{B}(V)\#\Bbbk\Gamma)$ by Theorem 6.13.

(ii). As in Section 6.2, we choose elements $a_i \in (A_1)_{g_i}^{\chi_i}$ such that $\pi(a_i) = x_i, 1 \leq i \leq n$. By assumption resp. by Theorem 6.16, there is a Hopf algebra map

$$\phi : \widehat{\mathfrak{B}}(V)\#\Bbbk\Gamma \to A, \phi(x_i\#g) = a_i g, 1 \leq i \leq n, g \in \Gamma.$$

By Theorem 5.8, A is generated in degree one, hence ϕ is surjective. We define the root vector $E_{i,j} \in A, 1 \leq i < j \leq n+1$, by (6.30), (6.31). By Lemma 6.17, $E_{i,j}^N =: u_{i,j} \in \Bbbk\Gamma$ for all $1 \leq i < j \leq n+1$. Then for all $g \in \Gamma$ and $i < j$, $gE_{i,j}^N = \chi_{i,j}^N(g)E_{i,j}^N g$, hence $gu_{i,j} = \chi_{i,j}^N(g)u_{i,j}g$, and $u_{i,j} = \chi_{i,j}^N(g)u_{i,j}$. By (6.35) and Theorem 6.18 we therefore know that $u_{i,j} = u_{i,j}(\gamma)$ for all $i < j$, for some family $\gamma = (\gamma_{i,j})_{1 \leq i \leq J \leq n+1}$ of scalars in $\Bbbk$ such that for all $i < j$, $\gamma_{i,j} = 0$ if $g_{i,j}^N = 1$ or $\chi_{i,j}^N = \varepsilon$. Hence ϕ indices a surjective Hopf algebra map $A_\gamma \to A$ which is an isomorphism since $\dim(A_\gamma) = N^{\frac{n(n+1)}{2}} \mathrm{ord}(\Gamma) = \dim(A)$ by 1). □

Remark 6.26. Up to isomorphism, A_γ does not change if we replace each x_i by a non-zero scalar multiple of itself. Hence in the definition of A_γ we may always assume that

$$\gamma_{i,i+1} = 0 \text{ or } 1 \text{ for all } 1 \leq i \leq n.$$

We close the paper with a very special case of Theorem 6.25. We obtain a large class of non-isomorphic Hopf algebras which have exactly the same infinitesimal braiding as $\mathfrak{u}_q^{\geq 0}(sl_n)$. Here q has order N, but the group is $\prod_{i=1}^n \mathbb{Z}/(Nm_i)$ and not $(\mathbb{Z}/(N))^n$ as for $\mathfrak{u}_q^{\geq 0}(sl_n)$.

Example 6.27. Let N be > 2, q a root of unity of order N, and $m_1, \dots, m_n$ integers > 1 such that $m_i \neq m_j$ for all $i \neq j$. Let Γ be the commutative group generated by $g_1, \dots, g_n$ with relations $g_i^{Nm_i} = 1$, $1 \leq i \leq n$. Define $\chi_1, \dots, \chi_n \in \widehat{\Gamma}$ by

$$\chi_j(g_i) = q^{a_{ij}}, \text{ where } a_{ii} = 2 \text{ for all } i, a_{ij} = -1 \text{ if } |i-j| = 1, a_{ij} = 0 \text{ if } |i-j| \geq 2.$$

Then $\chi_{i,j}^N = \varepsilon$ and $g_{i,j}^N \neq 1$ for all $i < j$. Thus for any family $\gamma = (\gamma_{i,j})_{1 \leq i < j \leq n+1}$ of scalars in $\Bbbk$, A_γ in Theorem 6.25 has infinitesimal braiding of type A_n.

Moreover, if $\gamma, \widetilde{\gamma}$ are arbitrary such families with $\gamma_{i,i+1} = 1 = \widetilde{\gamma}_{i,i+1}$ for all $1 \leq i \leq n$, then

$$A_\gamma \ncong A_{\widetilde{\gamma}}, \text{ if } \gamma \neq \widetilde{\gamma}.$$

Proof. We let $\widetilde{x}_i$ and $\widetilde{e}_{i,j}$ denote the elements of $A_{\widetilde{\gamma}}$ corresponding to x_i and $e_{i,j}$ in A_γ as above, for all i and $i < j$. Suppose $\phi : A_\gamma \to A_{\widetilde{\gamma}}$ is a Hopf algebra isomorphism. By Lemma [AS3, Lemma 1.2] there are a permutation $\sigma \in \mathbb{S}_n$ and non-zero scalars $\alpha_1, \dots, \alpha_n \in \Bbbk$ such that $\phi(g_i) = g_{\sigma(i)}$ and $\phi(x_i) = \alpha_i \widetilde{x}_{\sigma(i)}$ for all i. Since $\operatorname{ord}(g_i) = m_i N \neq m_j N = \operatorname{ord}(g_j)$ for all $i \neq j$, σ must be the identity, and ϕ induces the identity on Γ by restriction. In particular, $1 - g_i^N = \phi(x_i^N) = \alpha_i^N \widetilde{x}_i^N = \alpha_i^N(1 - g_i^N)$, and $\alpha_i^N = 1$ for all i. Therefore we obtain for all $i < j$,

$$u_{i,j}(\gamma) = \phi(e_{i,j}^N) = \alpha_i^N \alpha_{i+1}^N \dots \alpha_{j-1}^N \widetilde{e}_{i,j}^N = u_{i,j}(\widetilde{\gamma}),$$

and by Remark 6.19, 2), $\gamma = \widetilde{\gamma}$. $\square$

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