SOME PROPERTIES OF FACTORIZABLE HOPF ALGEBRAS

H.-J. SCHNEIDER

Abstract. A direct proof without modular category theory is given of a recent theorem of Etingof and Gelaki [EG] on the dimensions of irreducible representations. Factorizable Hopf algebras are characterized in terms of their Drinfeld double, and their character rings and the group-like elements of their duals are described.

1. Introduction

Etingof and Gelaki [EG] recently obtained a fundamental result in the representation theory of the Drinfeld double of a semisimple Hopf algebra H over an algebraically closed field of characteristic 0. Using the theory of modular ribbon categories, they showed that the dimension of any simple $D(H)$- module divides the dimension of H.

In this note, a direct proof of their result is given. More generally let (A, R) be a quasitriangular and semisimple Hopf algebra. Assume that (A, R) is factorizable in the sense of [RS] (see section 2). Then by Theorem 3.2, the square of the dimension of any simple A-module divides the dimension of A. The theorem of Etingof and Gelaki is an immediate corollary by taking $A = D(H)$ with the usual R-matrix of the double. The main ingredients of the short proof of Theorem 3.2 are the class equation of Kac and Zhu (see [L] for an elementary proof), and Drinfeld's central element construction in [D]. Drinfeld's construction is reviewed in section 2. Here it is essential to note (see Theorem 2.1) that Drinfeld's proof of [D, 3.3] shows more than what is actually stated in [D, 3.3].

Moreover this paper contains two new results about factorizable Hopf algebras over an arbitrary field. Let (A, R) be a factorizable Hopf algebra. It is not difficult to see that Drinfeld's construction defines an algebra isomorphism between a character ring $C(A)$ and the center $Z(A)$ of A. By Theorem 2.3, this isomorphism induces a group isomorphism between the group-like elements of the linear dual A^* and the

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central group-like elements of A. This last result answers a question of Sonia Natale. When A is the double of a finite-dimensional Hopf algebra H, the group isomorphism in 2.3 was already obtained by Radford [R93, Propositions 9 and 10].

In section 4, a new characterization of factorizable Hopf algebras is given. Let (A, R) be a quasitriangular and finite-dimensional Hopf algebra. If (A, R) is factorizable, then it is stated in [RS] (without an explicit proof) that the double $D(A)$ is isomorphic to a 2-cocycle twist of the tensor product Hopf algebra $A \otimes A$. Tsang and Zhu recently rediscovered this result in the case when A is the double of another Hopf algebra H. Their proof uses the special structure of the double $D(H)$. In Theorem 4.3, a stronger result is obtained in a more conceptual way: (A, R) is factorizable if and only if the double $D(A)$ is isomorphic (in a specific way) to a 2-cocycle twist of the tensor product $A \otimes A$. The proof of 4.3 follows from the theorem on Hopf modules.

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2. Central and group-like elements

In this paper, algebras and coalgebras are defined over the groundfield k; comultiplication and counit of a coalgebra and the antipode of a Hopf algebra will be denoted by Δ, ϵ and S. If A is an algebra, the dual vector space A^* is an (A, A)-bimodule with $(af)(b) = f(ba)$ and $(fa)(b) = f(ab)$ for all $a, b \in A$.

Let A be a Hopf algebra and R an invertible element in $A \otimes A$. Elements t in $A \otimes A$ will be written symbolically as $t = t^1 \otimes t^2$. Following Drinfeld, the pair (A, R) is called *quasitriangular* if

(1) $x_2 R^1 \otimes x_1 R^2 = R^1 x_1 \otimes R^2 x_2$, for all $x \in A$.
(2) $\Delta(R^1) \otimes R^2 = R^1 \otimes r^1 \otimes R^2 r^2$.
(3) $R^1 \otimes \Delta(R^2) = R^1 r^1 \otimes r^2 \otimes R^2$.

Here $R = r$ and the symbolic notations $R = R^1 \otimes R^2 = r^1 \otimes r^2$ (to indicate two different summation indices) and $\Delta(x) = x_1 \otimes x_2$ are used. Define $R_{21} = R^2 \otimes R^1 = \tau(R)$ where $\tau : R \otimes R \to R \otimes R$ is the usual twist map, and

$$b := R_{21} R = r^2 R^1 \otimes r^1 R^2.$$

Note that $b\Delta(x) = \Delta(x)b$ for all $x \in A$ since R and R_{21}^{-1} both satisfy (1).

The quasitriangular Hopf algebra (A, R) is called *factorizable* [RS] if the map

$$\Phi_R = \Phi : A^* \to A, \qquad \Phi(f) := b^1 f(b^2) \text{ for all } f \in A^*,$$

is an isomorphism of vector spaces; A^* is the k-linear dual of A. Important examples of factorizable Hopf algebras are the Drinfeld doubles of arbitrary finite-dimensional Hopf algebras [RS, 2.10] (see 2.4 1) below).

In [D], Drinfeld introduced a basic construction of central elements of a quasitriangular Hopf algebra. As in [D] let

$$C(A) := \{f \in A^* \mid \text{ for all } x, y \in A : f(xy) = f(yS^2(x))\}.$$

Let $Z(A)$ denote the center of A. The next crucial result is due to Drinfeld [D, 1.2 and 3.3]. However the formulation below in (b) is more general; [D, 3.3] just says that the map Φ defines by restriction an algebra homomorphism on $C(A)$. For completeness the short proof will be repeated (in a different notation).

Theorem 2.1 (Drinfeld [D])**.** *Let (A, R) be a quasitriangular Hopf algebra and $\Phi = \Phi_R : A^* \to A$. Then for all $g \in C(A)$ and $f \in A^*$,*
(a) $\Phi(g) \in Z(A)$, and
(b) $\Phi(fg) = \Phi(f)\Phi(g)$.

Proof. (a) For all $g \in C(A)$ and $x \in A$,

$$\begin{aligned} x\Phi(g) &= x_1 b^1 g(S^{-1}(x_3)x_2 b^2) \\ &= b^1 x_1 g(b^2 x_2 S(x_3)) \quad \text{since } g \in C(A), \text{ and } \Delta(x)b = b\Delta(x) \\ &= \Phi(g)x. \end{aligned}$$

(b) For all $f \in A^*$ and $g \in C(A)$,

$$\begin{aligned} \Phi(fg) &= R^2 r^1 (fg)(R^1 r^2) \\ &= R^2 r^1 f(R^1_1 r^2_1) g(R^1_2 r^2_2) \\ &= R^2 r^2 s^1 t^1 f(R^1 t^2) g(r^1 s^2) \quad \text{by (2) and (3) with R = r = s = t} \\ &= R^2 r^2 s^1 g(r^1 s^2) t^1 f(R^1 t^2) \\ &= R^2 \Phi(g) t^1 f(R^1 t^2) \\ &= \Phi(f)\Phi(g) \qquad \text{since } \Phi(g) \text{ is central by (a).} \end{aligned}$$

□

Lemma 2.2. *Let A be a finite-dimensional unimodular Hopf algebra. Then $dim(C(A)) = dim(Z(A))$.*

Proof. Let $\lambda \in A^*$ be a non-zero left integral. Then $A \to A^*$, $a \mapsto \lambda a$, is bijective. Since A is unimodular, for all $a, b \in A$, $\lambda(ab) = \lambda(bS^2(a))$ [OS, 3.2, 2)a)]. Hence for all $a, x, y \in A$, $(\lambda a)(yS^2(x)) = \lambda(ayS^2(x)) = \lambda(xay)$. Thus $\lambda a \in C(A)$ if and only if for all $x \in A$, $\lambda(xa) = \lambda(ax)$, that is $a \in Z(A)$. □

If C is a coalgebra, then $G(C) = \{c \in C \mid \Delta(c) = c \otimes c, \epsilon(c) = 1\}$ will denote the set of all group-like elements of C.

Theorem 2.3. *Let (A, R) be a factorizable Hopf algebra. Then $\Phi_R : A^* \to A$ induces by restriction*

(a) an algebra isomorphism $C(A) \to Z(A)$, and

(b) a group isomorphism $G(A^*) \to G(A) \cap Z(A)$.

Proof. (a) By [R94, Proposition 3], A is unimodular (for another proof see 4.4 below). By Theorem 2.1, Φ_R defines an injective algebra map $C(A) \to Z(A)$ which is bijective by Lemma 2.2.

(b) Let $f \in G(A^*) = \mathrm{Alg(A,k)}$. Then $f \in C(A)$ since $fS^2 = f$, fS being the inverse of f in the group $G(A^*)$. Hence $\Phi(f) \in Z(A)$ by (a), and

$$\Phi(f) = R^2 r^1 f(R^1 r^2) = R^2 f(R^1) r^1 f(r^2) = uv,$$

where $u := R^2 f(R^1), v := r^1 f(r^2)$. It follows easily from (3) and (2) that u and v are group-like elements of A. Thus Φ defines an injective group homomorphism from $G(A^*)$ to $G(A) \cap Z(A)$.

To prove surjectivity of this map take $g \in G(A) \cap Z(A)$. By Lemma 2.2 λg^{-1} is in $C(A)$ since g^{-1} is central. Then for all $p \in A^*$, $p(\lambda g^{-1}) = p(g)\lambda g^{-1}$ in A^*, since for all $x \in A$,

$$(p(\lambda g^{-1}))(x) = p(x_1)\lambda(g^{-1}x_2) = ((pg)\lambda)(g^{-1}x) = p(g)(\lambda g^{-1})(x),$$

since λ is a left integral. Hence it follows from Theorem 2.1 (b) that for all $p \in A^*$,

$$\Phi(p)\Phi(\lambda g^{-1}) = \Phi(p(\lambda g^{-1})) = p(g)\Phi(\lambda g^{-1}).$$

Therefore the left A-module generated by $\Phi(\lambda g^{-1})$ is one-dimensional (since $\Phi : A^* \to A$ is bijective). Hence there is an algebra homomorphism $\chi : A \to k$ with $p(g) = \chi(\Phi(p))$ for all $p \in A^*$.

It remains to show that $\Phi(\chi) = g$ or equivalently $\chi(\Phi(p)) = p(\Phi(\chi))$ for all $p \in A^*$. Define $u := R^1\chi(R^2), v := r^2\chi(r^1)$. For all $p \in A^*$,

$$\chi(\Phi(p)) = \chi(R^2)\chi(r^1)p(R^1 r^2) = p(uv), \text{ and}$$

$$p(\Phi(\chi)) = p(R^2 r^1 \chi(R^1)\chi(r^2)) = p(vu).$$

Thus the claim follows from the equality $uv = vu$. Since Φ defines an algebra homomorphism $C(A) \to Z(A)$, $\Phi(\chi) = S(\Phi(\chi S))$; hence

$$uv = \Phi(\chi) = S(r^1)\chi(S(r^2))S(R^2)\chi(S(R^1)) = vu,$$

since $R = S(R^1) \otimes S(R^2)$ by [K, VIII.2.4]. □

Remark 2.4. 1) Let H be a finite-dimensional Hopf algebra and $A = D(H) = H^{*cop} \otimes H$ the Drinfeld double of H. Elements $p \otimes h$, $p \in H^*, h \in H$ of $D(H)$ will be simply denoted by ph. Then H^{*cop} and H

are sub Hopf algebras of $D(H)$ and comultiplication and multiplication in $D(H)$ are given by

$$\Delta(ph) = p_2h_1 \otimes p_1h_2, \text{ and } hp = p(S^{-1}(h_3) - h_1)h_2,$$

for all $h \in H, p \in H^*$ [K, IX.4.1].

Let (e_i) and (f_i) be dual bases in H and H^*. Then $D(H)$ is quasitriangular with R-matrix $R = \sum_i e_i \otimes f_i$ [K, IX.4.2]. Thus $b = \sum_{i,j} f_je_i \otimes e_jf_i$ and $\Phi(F) = \sum_{i,j} f_je_iF(e_jf_i)$ for all $F \in D(H)^*$. Since the elements f_je_i, $1 \leq i,j \leq n$, and $S(f_i)S(e_j)$, $1 \leq i,j \leq n$, are bases of $D(H)$, ΦS is injective. Hence $(D(H), R)$ is factorizable (cf. [RS, Theorem 2.10], [R94, p. 226]).

2) The group isomorphism in Theorem 2.3 can be described explicitly in case $A = D(H)$ with R-matrix as in 1). For any $\chi \in G(A^*) =$ Alg(A,k), $\Phi(\chi) = uv$ with $u = \sum_i f_i\chi(e_i)$ and $v = \sum_i e_i\chi(f_i)$. Thus $u \in G(A^*)$ is the restriction of χ on H^*, and the group-like element $v \in G(H)$ is defined by $p(v) = \chi(p)$ for all $p \in H^*$. Therefore the isomorphism $G(D(H)^*) \to G(D(H)) \cap Z(D(H))$ of Theorem 2.3 is the isomorphism constructed by Radford [R93, Propositions 9 and 10].

3. A direct proof of a theorem of Etingof-Gelaki

Throughout this section the field k is algebraically closed of characteristic 0. The Theorem of Etingof-Gelaki [EG, 1.4] will follow from Drinfeld's central element construction in section 2 and the following result by Kac-Zhu (the class equation, see [L] for a more elementary proof).

Theorem 3.1 (Kac-Zhu). *Let A be a semisimple Hopf algebra and e a primitive idempotent in the character algebra $C(A)$. Then $dim(A^*e)$ divides $dim(A)$.*

Theorem 3.2. *Let (A, R) be a factorizable Hopf algebra. If A is a semisimple algebra and V a simple left A-module, then $dim(V)^2$ divides $dim(A)$.*

Proof. By assumption, $A \cong M_{d_1}(k) \times \cdots \times M_{d_t}(k)$ is isomorphic to a product of full matrix rings $M_{d_i}(k)$, and $Z(A) \cong k \times \cdots \times k$. Let E_i be the primitive idempotent in $Z(A)$ corresponding to the simple module V. Thus $AE_i \cong M_{d_i}(k)$ has dimension d_i^2, and $d_i = \dim(V)$. By Theorem 2.1, Φ_R induces an algebra isomorphism $C(A) \to Z(A)$ (Note that here unimodularity of A is very easy to see since A is semisimple). Let e_i be the primitive idempotent in $C(A)$ with $\Phi(e_i) = E_i$. Then by Theorem 2.1 (b), $\Phi(A^*e_i) = \Phi(A^*)\Phi(e_i) = AE_i$. Since Φ is bijective, $\dim(A^*e_i) = \dim(AE_i) = (\dim(V))^2$. Therefore $(\dim(V))^2$ divides $\dim(A)$ by Theorem 3.1. $\square$

Corollary 3.3 (Etingof-Gelaki [EG]). *(1) If H is a semisimple Hopf algebra and V a simple left $D(H)$-module, then dim(V) divides dim(H).*

(2) If (H, R) is a quasitriangular Hopf algebra and H is semisimple, then for any simple left H-module, dim(V) divides dim(H).

Proof. (1) Since the characteristic of k is 0, $D(H)$ is semisimple by [LR]. Hence by Theorem 3.2 applied to the factorizable Hopf algebra $(D(H), R)$, $(\dim(V))^2$ divides $\dim(D(H)) = (\dim(H))^2$. Thus $\dim(V)$ divides $\dim(H)$.

(2) follows from (1) as in [EG] since H is an epimorphic image of $D(H)$. □

Remark 3.4. The original proof of Corollary 3.3 (1) uses the theory of modular categories. In particular, in the first line of the proof of [EG, Lemma 1.2], the authors need a trace formula in modular categories. For a proof they refer to [Ki] who refers to Turaev's book [T]. The lucky reader may find a proof in [T, II,3.2.2.(ii)] making use of various twistings in the graphical calculus.

However, in the case of the modular category of all finite-dimensional representations of a semisimple factorizable Hopf algebra (A, R) (in [EG] A is the Drinfeld double of a semisimple Hopf algebra H), this formula is in fact a special case of the class equation 3.1 in the version of Lorenz [L].

Let $V_i, 0 \leq i \leq m$, with $V_0 := k$, be a complete set of representatives of the isomorphism classes of the simple left A-modules. For all i, let χ_i be the character of V_i, and $\chi_{i^*} = S(\chi_i)$ be the character of $(V_i)^*$. By definition, the matrix (s_{ij}) in [EG] is given by

$$s_{ij} = \chi_i(\Phi(\chi_{j^*})), 0 \leq i, j \leq m,$$

where $\Phi = \Phi_R$. The trace formula needed in the proof of [EG, Lemma 1.2] then is:

$$\text{For all j, } \sum_i s_{ji} s_{ij^*} = \dim(A).$$

To prove this formula, let E_i in $Z(A)$ be the central primitive idempotent of $V_i, 0 \leq i \leq m$. For all i, define $e_i := \Phi^{-1}(E_i)$, and let μ_i be the character of the simple $C(A)$-module $C(A)e_i$. Since Φ defines an algebra isomorphism between $C(A)$ and $Z(A)$, $\pi_i := \mu_i \Phi^{-1}$ is the character of the simple $Z(A)$-module $Z(A)E_i$. Note that for any $a \in Z(A)$, left multiplication with a on V_i is multiplication with $\pi_i(a)$, hence $\chi_i(a) = \pi_i(a)\dim(V_i)$ (For, since V_i is simple, left multplication with a on V_i is multiplication with some scalar; since $AE_i \cong V_i^{d_i}, d_i := \dim(V_i)$, and $Z(A)E_i$ is one-dimensional, this scalar is $\pi_i(a)$).

Hence for all i, j,

$$s_{ij} = \chi_i(\Phi(\chi_{j^*})) = \pi_i(\Phi(\chi_{j^*}))\dim(V_i) = \mu_i(\chi_{j^*})\dim(V_i),$$

since $\Phi(\chi_{j^*})$ is in the center of A by Theorem 2.1.

Moreover it is not difficult to see that for all i, j, $s_{ij} = s_{ji}$, and $s_{ij} = s_{i^*j^*}$, hence $s_{ij^*} = s_{ji^*}$ (cf. [Ki]).

Therefore one obtains

$$\sum_i s_{ji}s_{ij^*} = \sum_i s_{ji}s_{ji^*} = \sum_i \mu_j(\chi_{i^*})\mu_j(\chi_i)(\dim(V_j))^2.$$

Finally, as in the proof of Theorem 3.2, by Theorem 2.1 (b), $\dim(A^*e_j) = (\dim(V_j))^2$. Thus the trace formula is equivalent to the equality

$$\sum_i \mu_j(\chi_{i^*})\mu_j(\chi_i) = \frac{\dim(A)}{\dim(A^*e_j)},$$

which is exactly the formula

$$\omega(\widetilde{e}) = d$$

(for left modules) in the proof of the class equation in [L], where $\widetilde{e} = e_j = \sum_i \mu_j(\chi_{i^*})\chi_i$ and $\omega = \mu_j$, since here $C(A)$ is commutative.

4. The Drinfeld double of a factorizable Hopf algebra

If (A, R) is a finite-dimensional quasitriangular Hopf algebra, define $\pi_R : D(A) \to A$ by $\pi_R(fa) := f(R^1)R^2a$ for all $f \in A^*, a \in A$. (See 2.4 for notations concerning $D(A)$). This map appears in [D, Proof of Proposition 6.2] where it was observed that π_R is a Hopf algebra homomorphism. Since $\widetilde{R} := (R_{21})^{-1} = S^{-1}(R^2) \otimes R^1 = R^2 \otimes S(R^1)$ is another R-matrix for A [K, VIII.2], that is $(A, \widetilde{R})$ is again quasitriangular, there is always a second projection $\pi_{\widetilde{R}} : D(A) \to A$.

Finally, if $\varrho : A \to B$ is a homomorphism of bialgebras, then $A^{co\varrho} := \{x \in A \mid x_1 \otimes \varrho(x_2) = x \otimes 1\}$.

Lemma 4.1. *Let (A, R) be a finite-dimensional quasitriangular Hopf algebra. Define $\pi := \pi_R, \widetilde{\pi} := \pi_{\widetilde{R}}, \Phi := \Phi_R$, and let $i : D(A)^{co\pi} \to D(A)$ be the inclusion map. Then*

(1) $\Psi : A^* \to D(A)^{co\pi}$, $\Psi(f) := f_2S(\pi(f_1))$, is bijective.

(2) $S\Phi = \widetilde{\pi}i\Psi : A^* \xrightarrow{\Psi} D(A)^{co\pi} \xrightarrow{i} D(A) \xrightarrow{\widetilde{\pi}} A$.

Proof. (1) The map Ψ is well-defined since for all $f \in A^*$,

$$\Psi(f)_1 \otimes \pi(\Psi(f)_2) = f_4S(\pi(f_1)) \otimes \pi(f_3S(\pi(f_2)) = \Psi(f) \otimes 1,$$

since π is a Hopf algebra homomorphism. The inverse of Ψ is given by $\mathrm{id} \otimes \varepsilon$ since for all $f \in A^*$, $(\mathrm{id} \otimes \varepsilon)\Psi(f) = f$, and if $x = \sum_i f_ia_i \in$

$D(A)^{co\pi}$, $f_i \in A^*$ and $a_i \in A$ for all i, then $\sum_i f_{i2}a_{i1} \otimes \pi(f_{i1}a_{i2}) = x \otimes 1$, hence

$$x = \sum_i f_{i2}a_{i1}S\pi(a_{i2})S\pi(f_{i1})) = \Psi(\sum_i f_i\varepsilon(a_i)).$$

(2) For any $f \in A^*$,

$$\begin{aligned}
\widetilde{\pi}(\Psi(f)) &= \widetilde{\pi}(f_2 f_1(R^1)S(R^2)) \\
&= f_2(S^{-1}(r^2))r^1 f_1(R^1)S(R^2) \\
&= S(R^2 S^{-1}(r^1)f(R^1 S^{-1}(r^2)) \\
&= S(R^2 r^1 f(R^1 r^2)) \quad \text{since } S^{-1}(r^1) \otimes S^{-1}(r^2) = r^1 \otimes r^2 \\
&= S(\Phi(f)).
\end{aligned}$$

□

Let H be a bialgebra with comultiplication Δ and augmentation ε. An invertible element $F \in H \otimes H$ is called a *2-cocycle for H* or a *gauge transformation* if

$$F_{12}(\Delta \otimes \mathrm{id})(F) = F_{23}(\mathrm{id} \otimes \Delta)(F) \text{ in } H \otimes H \otimes H, \text{ and}$$
$$\varepsilon(F^1)F^2 = 1 = F^1\varepsilon(F^2).$$

If F is a 2-cocycle for H, then the twisted bialgebra H^F is defined as follows. $H^F = H$ as an algebra with the new comultiplication

$$\Delta_F(x) := F\Delta(x)F^{-1} \text{ for all } x \in H,$$

and the old augmentation ε. If H is a Hopf algebra, then H^F is a Hopf algebra with antipode $S_F(x) := vS(x)v^{-1}$ for all $x \in H$, where $v := F^1S(F^2), v^{-1} = S(G^1)G^2, G := F^{-1}$ (see [K, XV.3 and XV.6]).

Theorem 2.9 in [RS] describes the Drinfeld double of a factorizable Hopf algebra A as a 2-cocycle twist of the usual (componentwise) tensor product bialgebra $A \otimes A$ without giving an explicit proof.

In the case when $A = D(H)$ is itself the Drinfeld double of another Hopf algebra H, Tsang and Zhu [TZ, Theorem 2] recently rediscovered this result. Their proof depends on the structure of A as the double of H.

In the next theorem a stronger result is shown by using the concept of Hopf modules in the next lemma. Factorizable Hopf algebras can in fact be characterized by the property that their double is isomorphic (in a specific way) to a 2-cocycle twist of the usual tensor product bialgebra.

Lemma 4.2. *Let A, B and D be bialgebras and $\varphi : D \to A, \psi : D \to B$ bialgebra maps. Define $\delta : D \to A \otimes B$ by $\delta(x) := \varphi(x_1) \otimes \psi(x_2)$ for all $x \in D$.*

(1) If (D,R) is a quasitriangular Hopf algebra, then

$$F := 1 \otimes \psi(S(R^1)) \otimes \varphi(R^2) \otimes 1$$

is a 2-cocycle for $A \otimes B$ with componentwise bialgebra structure, and $\delta : D \to (A \otimes B)^F$ is a bialgebra map.

(2) Assume that B is a Hopf algebra and that there is a bialgebra homomorphism $\gamma : B \to D$ with $\psi\gamma = id_B$. Then $\delta : D \to A \otimes B$ is bijective if and only if the restriction of φ defines an isomorphism between $D^{co\psi}$ and A.

Proof. (1) can be checked directly.

(2) It is easy to see that $\delta : D \to A \otimes B$ is a map of right (B,B)-Hopf modules. Here D is a right B-module by restriction via $\gamma : B \to D$, and a right B-comodule by $(\mathrm{id} \otimes \psi)\Delta$; $A \otimes B$ is a right B-comodule by $\mathrm{id} \otimes \Delta$, and the right B-module structure is given by $(a \otimes b) \cdot c := a\varphi(\gamma(c_1)) \otimes bc_2$ for all $a \in A$ and $b, c \in B$. Hence by the theorem on Hopf modules [M, 1.9], δ is bijective if and only if $D^{coB} = D^{co\psi} \to (A \otimes B)^{coB} \cong A, x \mapsto \varphi(x)$, is bijective. ☐

Theorem 4.3. *Let (A,R) be a finite-dimensional quasitriangular Hopf algebra. Define $F_R := 1 \otimes R^2 \otimes R^1 \otimes 1$ and $\delta_R : D(A) \to A \otimes A, x \mapsto \delta_R(x) := \widetilde{\pi}(x_1) \otimes \pi(x_2)$ with $\pi := \pi_R$ and $\widetilde{\pi} := \pi_{\widetilde{R}}$. Then F_R is a 2-cocycle for $A \otimes A$ with componentwise bialgebra structure,*

$$\delta_R : D(A) \to (A \otimes A)^{F_R}$$

is a Hopf algebra homomorphism, and the following are equivalent:

(1) δ_R is bijective.

(2) (A,R) is factorizable.

Proof. Let (e_i) and (f_i) be dual bases of A and A^*. Apply Lemma 4.2 with $D := D(A)$ and R-matrix $\sum_i e_i \otimes f_i$, $\varphi := \widetilde{\pi}$, $\psi := \pi$, and $\gamma : A \to D(A)$ the inclusion map. The 2-cocycle of Lemma 4.2 (1) then is

$$\begin{aligned}
\sum_i 1 \otimes \pi(S(e_i)) \otimes \widetilde{\pi}(f_i) \otimes 1 &= \sum_i 1 \otimes S(e_i) \otimes f_i(R^2)S(R^1) \otimes 1 \\
&= 1 \otimes S(\sum_i e_i f_i(R^2)) \otimes S(R^1) \otimes 1 \\
&= 1 \otimes S(R^2) \otimes S(R^1) \otimes 1 \\
&= F_R.
\end{aligned}$$

By Lemma 4.2 (2) and Lemma 4.1, conditions (1) and (2) are both equivalent to the bijectivity of the map $D(A)^{co\pi} \to A, x \mapsto \widetilde{\pi}(x)$. ☐

Remark 4.4. Radford [R94, Proposition 3] showed that a factorizable Hopf algebra is unimodular. Another proof of this result follows from the isomorphism $D(A) \cong (A \otimes A)^{F_R}$ in Theorem 4.3 and the unimodularity of the double [R93, Theorem4]. Let Λ be a left integral in A. Then $\Lambda \otimes \Lambda$ is a left integral in $(A \otimes A)^{F_R}$. Since $D(A)$ is unimodular, $\Lambda \otimes \Lambda$ is a right integral. Hence Λ is a right integral in A.

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Mathematisches Institut, Universität München, Theresienstrasse 39, D-80333 München, Germany

E-mail address: `hanssch@rz.mathematik.uni-muenchen.de`