DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
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COMPLEX ANALYSIS

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Solutions 01

1 . We want to use the geometric series (cf. Tutorials). In order to obtain a power series with center $a = -1$, we first have to rewrite f in the following way:

$$f(z) = \frac{1}{1-(z+1)+1} = \frac{1}{2} \cdot \frac{1}{1-(z+1)/2}.$$

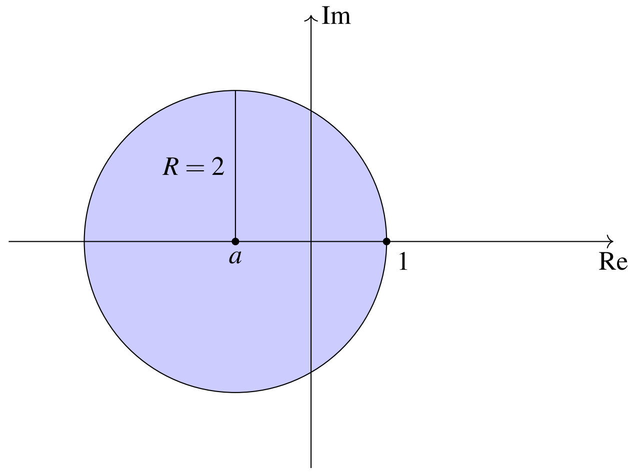
Now we can apply the geometric series to $(z+1)/2$ and obtain

$$\begin{aligned} f(z) &= \frac{1}{2}\sum_{n=0}^{\infty}((z+1)/2)^n \\ &= \sum_{n=0}^{\infty} 2^{-(n+1)}(z+1)^n. \end{aligned}$$

The geometric series used here converges for $|(z+1)/2| < 1$, i.e. for $|z+1| < 2$, hence the radius of convergence of the resulting power series is 2. Alternatively, we can compute with the ratio test

$$\lim_{n\to\infty}\left|\frac{2^{-(n+2)}}{2^{-(n+1)}}\right| = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}.$$

The domain of convergence is sketched below:



2 . i) The criterion of Cauchy-Hadamard states that

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$

By definition of the limes superior, there exists a subsequence of $(\sqrt[n]{|c_n|})_{n\in\mathbb{N}}$ which converges to $\frac{1}{R}$. If there are only finitely many n such that $\sqrt[n]{|c_n|} \geq \frac{1}{r}$, this implies that

$$\frac{1}{R} \leq \frac{1}{r},$$

and hence $R \geq r$.

ii) The assumption implies that there exists a subsequence of $(\sqrt[n]{|c_n|})_{n\in\mathbb{N}}$ which is bounded from below by $\frac{1}{r}$. Then by definition of the limes superior and Cauchy-Hadamard we get

$$\frac{1}{r} \leq \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{1}{R}$$

and hence $R \leq r$.

3 . i) For f we can use the ratio test to compute

$$\left| \frac{1/(n+1)^2}{1/n^2} \right| = \frac{n^2}{(n+1)^2}$$

which converges to 1 for $n \to \infty$. Hence the radius of convergence for f is $\frac{1}{1} = 1$.

For g we can use the representation

$$g(z) = \sum_{n=0}^{\infty} c_n z^n$$

where

$$c_n := \begin{cases} 1 & n = k! \text{ for a } k \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

In this form, we see that g is dominated by the geometric series and the dominated convergences theorem implies that g converges for $|z| < 1$. It is also easy to see that for $|z| \geq 1$

$$\lim_{n\to\infty} \left|z^{n!}\right| \neq 0,$$

hence g diverges for $|z| \geq 1$ and the radius of convergence is also 1.

ii) We already solved the boundary for g in part i). For f, we can use the well-known fact that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges to $\frac{\pi^2}{6}$ to conclude that f converges on the boundary.

4 . We can define

$$c_n := \begin{cases} 2^{n/2} & n \text{ even}, \\ 0 & n \text{ odd}, \end{cases}$$

to rewrite

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then the criterion of Cauchy-Hadamard yields

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$

To compute the limes superior, we have to determine all limit points of subsequences of $\sqrt[n]{|c_n|}$. These are obviously 0 for the subsequence of odd integers and $\sqrt{2}$ for the subsequence of even integers, hence the limes superior is $\sqrt{2}$ and the radius of convergence is $R = \frac{1}{\sqrt{2}}$.

For g we can use the the dominated convergence theorem and the geometric series since $|\cos(n)| \leq 1$ to obtain that g converges for all $z \in \mathbb{C}$ with $|z| < 1$.
For proving that the radius of converges is indeed 1, we need to show the hint:
We first show that the sequence $(\cos(n))_{n\in\mathbb{N}}$ does not converge to 0.
Suppose $(\cos(n))_{n\in\mathbb{N}}$ would converge to 0, then we can use the well-known formula

$$\cos(2n) = 2\cos(n)^2 - 1$$

and pass to the limit on both sides. This yields $0 = -1$, which is a contradiction.
Now the definition of convergence implies that there must exist an $\alpha > 0$ such that for any $N \in \mathbb{N}$ there exists an $n \geq N$ with $\cos(n) > \alpha$.

Now for $z \in \mathbb{C}$ with $|z| > 1$, we can use the hint to conclude that there exists a subsequence of $(\cos(n)z^n)_{n\in\mathbb{N}}$ which is bounded from below by $\alpha > 0$, hence

$$\lim_{n\to\infty} |\cos(n)z^n| \neq 0,$$

and g diverges. Altogether we find that the radius of convergence for g is 1.

Solutions 02

5 . i) By definition we obtain:

$$\binom{\sigma}{n+1} = \frac{\sigma\cdot(\sigma-1)\cdots(\sigma-n+1)\cdot(\sigma-n)}{(n+1)!} = \frac{\sigma\cdot(\sigma-1)\cdots(\sigma-n+1)}{n!}\cdot\frac{\sigma-n}{n+1}$$
$$= \frac{\sigma-n}{n+1}\cdot\binom{\sigma}{n}.$$

ii) For $\sigma\in\mathbb{N}$ and $n>\sigma$ we get $\binom{\sigma}{n}=0$, hence

$$f_\sigma(z) = \sum_{n=0}^{\sigma}\binom{\sigma}{n}z^n$$

is a polynomial and converges for all $z\in\mathbb{C}$.
Now we have to prove the binomial formula

$$(a+b)^\sigma = \sum_{n=0}^{\sigma}\binom{\sigma}{n}a^n b^{\sigma-n} \tag{1}$$

for any $\sigma\in\mathbb{N}$ and $a,b\in\mathbb{C}$. This can be proven by induction on σ:
For $\sigma=0$, we get

$$\sum_{n=0}^{0}\binom{0}{n}a^n b^{0-n} = 1 = (a+b)^0.$$

Now we assume that (1) holds for a $\sigma\in\mathbb{N}$. Then we get

$$
\begin{aligned}
(a+b)^{\sigma+1} &= (a+b)\cdot(a+b)^{\sigma} \\
&\overset{IH}{=} (a+b)\cdot\sum_{n=0}^{\sigma}\binom{\sigma}{n}a^n b^{\sigma-n} \\
&= \sum_{n=0}^{\sigma}\binom{\sigma}{n}a^{n+1}b^{\sigma-n} + \sum_{n=0}^{\sigma}\binom{\sigma}{n}a^n b^{\sigma+1-n} \\
&= a^{\sigma+1} + \sum_{n=0}^{\sigma-1}\binom{\sigma}{n}a^{n+1}b^{\sigma-n} + \sum_{n=1}^{\sigma}\binom{\sigma}{n}a^n b^{\sigma+1-n} + b^{\sigma+1} \\
&= a^{\sigma+1} + \sum_{n=1}^{\sigma}\underbrace{\left(\binom{\sigma}{n-1}+\binom{\sigma}{n}\right)}_{\overset{\text{Tut.}}{=}\binom{\sigma+1}{n}}a^n b^{\sigma+1-n} + b^{\sigma+1} \\
&= \sum_{n=0}^{\sigma+1}\binom{\sigma+1}{n}a^n b^{\sigma+1-n}.
\end{aligned}
$$

Using this formula for $a = z$ and $b = 1$ we obtain the desired result.

iii) Using the ratio test, we find

$$
\begin{aligned}
\frac{1}{R} &= \lim_{n\to\infty}\left|\binom{\sigma}{n+1}\Big/\binom{\sigma}{n}\right| \\
&\overset{i)}{=} \lim_{n\to\infty}\left|\frac{\sigma-n}{n+1}\right| \\
&= \lim_{n\to\infty}\left|\underbrace{\frac{\sigma}{n+1}}_{\to 0} - \underbrace{\frac{n}{n+1}}_{\to 1}\right| \\
&= 1.
\end{aligned}
$$

6 . Let $z \in \mathbb{C}$ with $|z| < \min\{R_1, R_2\}$. Then $f_1(z)$ and $f_2(z)$ converge and hence also $f_3(z) = (f_1 + f_2)(z)$ converges to $f_1(z) + f_2(z)$. Since this is true for all z with $|z| < \min\{R_1, R_2\}$, the radius of convergence R_3 must be at least this minimum.

Now suppose that $R_1 \neq R_2$ and without loss of generality we can assume $R_1 > R_2$. From the first part, we know that $R_3 \geq R_2$. Suppose now that $R_3 > R_2$. Then we can choose a $z \in \mathbb{C}$ such that $R_2 < |z| < R_3$ and $|z| < R_1$. This implies that f_1 and f_3 converge at z, which yields that

$$f_2(z) = (f_3 - f_1)(z) = f_3(z) - f_1(z)$$

converges. But this is a contradiction to the assumption $|z| > R_2$.
Therefore, $R_3 = R_2$ and we are done.

7 . i) Using the partial fraction decomposition as suggested by the hint, we obtain that $f_1(z) = \frac{1}{a-b}$ and $f_2(z) = \frac{1}{b-a}$. Then we can use the geometric series to compute

$$\begin{aligned} f(z) &= f_1(z)\frac{1}{1-az} + f_2(z)\frac{1}{1-bz} \\ &= f_1(z)\sum_{n=0}^{\infty}(az)^n + f_2(z)\sum_{n=0}^{\infty}(bz)^n \\ &= \sum_{n=0}^{\infty}\left(\frac{a^n}{a-b} + \frac{b^n}{b-a}\right)z^n \\ &= \sum_{n=0}^{\infty}\frac{a^n-b^n}{a-b}z^n. \end{aligned}$$

ii) We used the geometric series for az and bz, hence the radius of convergence of the first power series is $R_1 = \frac{1}{|a|}$ and of the second power series is $R_2 = \frac{1}{|b|}$. By exercise 6 we obtain that the radius of convergence of the power series computed in i) is

$$R_3 \geq \min\{|a|^{-1}, |b|^{-1}\}.$$

If $|a| \neq |b|$, we get immediately an equality here. If $|a| = |b|$, we have $b = \zeta \cdot a$ with $|\zeta| = 1, \zeta \neq 1$ and the coefficients can be rewritten as

$$\frac{a^n - b^n}{a-b} = a^{n-1}\frac{1-\zeta^n}{1-\zeta}.$$

Since $|\zeta| = 1$, we get that $0 < |1-\zeta^n| \leq 2$ for any $n \in \mathbb{N}$. Moreover, for any $\zeta \neq 1$ on the unit circle, there exist infinitely many n such that the real part of ζ^n is negative, hence $|1-\zeta^n| > 1$. This, together with the fact that the sequence $\sqrt[n]{c}$ converges to 1 for any constant $c > 0$, implies

$$\begin{aligned} \frac{1}{R_3} &= \limsup_{n\to\infty}\sqrt[n]{\frac{|a|^n}{|a|}\cdot\frac{|1-\zeta^n|}{|1-\zeta|}} \\ &= |a| \cdot \limsup_{n\to\infty}\sqrt[n]{|1-\zeta^n|} \\ &= |a|. \end{aligned}$$

Hence $R_3 = \frac{1}{|a|}$.

8 . i) We compute

$$
\begin{aligned}
(1-\alpha z-\beta z^2)f(z) &= \sum_{n=0}^{\infty} c_n z^n - \sum_{n=0}^{\infty} \alpha c_n z^{n+1} - \sum_{n=0}^{\infty} \beta c_n z^{n+2} \\
&= c_0 + c_1 z + \sum_{n=0}^{\infty} c_{n+2} z^{n+2} - \alpha c_0 z - \sum_{n=0}^{\infty} \alpha c_{n+1} z^{n+2} - \sum_{n=0}^{\infty} \beta c_n z^{n+2} \\
&= c_0 + (c_1 - \alpha c_0) z + \sum_{n=0}^{\infty} (c_{n+2} - \alpha c_{n+1} - \beta c_n) z^{n+2}.
\end{aligned}
$$

If we assume now that the coefficients satisfy the given relations, we see that the series in the above expression vanishes and inserting the values of c_0 and c_1 yields

$$(1-\alpha z-\beta z^2)f(z) = z.$$

If we assume that $f(z)$ satisfies the given relation, we can compare the coefficients to conclude that $c_0 = 0$, $c_1 = 1$ and $c_{n+2} = \alpha c_{n+1} + \beta c_n$.

ii) We define

$$f(z) = \frac{z}{1-z-z^2} = \frac{z}{(1-\Phi z)(1-\Phi' z)},$$

where Φ^{-1} and Φ'^{-1} are the zeros of $1-z-z^2$, i.e.

$$
\begin{aligned}
\Phi^{-1} &= -\frac{1-\sqrt{5}}{2}, \qquad & \Phi'^{-1} &= -\frac{1+\sqrt{5}}{2}, \\
\Phi &= \frac{1+\sqrt{5}}{2}, \qquad & \Phi' &= \frac{1-\sqrt{5}}{2}.
\end{aligned}
$$

Using exercise 7, we obtain that $f(z)$ can be expanded as a power series with radius of convergence given by

$$\min\{|\Phi|, |\Phi'|\} = |\Phi'| > 0.$$

Since $f(z)$ is a convergent power series, we can use part i) to conclude that its coefficients satisfy the recursive relation of the Fibonacci numbers, i.e. we have indeed

$$\sum_{n=0}^{\infty} c_n z^n = f(z) = \frac{z}{1-z-z^2.}$$

Moreover, we get from exercise 7 the closed form of the Fibonacci numbers

$$c_n = \frac{\Phi^n - \Phi'^n}{\Phi - \Phi'} = \frac{1}{\sqrt{5}}(\Phi^n - \Phi'^n).$$

Note that $\Phi' = 1 - \Phi$ and Φ is the well-known *golden ratio*.

Solutions 03

9 . i) Using the Cauchy product formula and the binomial formula from exercise 5ii), we obtain

$$\begin{aligned}
\exp(z_1)\cdot\exp(z_2) &= \left(\sum_{n=0}^{\infty}\frac{z_1^n}{n!}\right)\cdot\left(\sum_{n=0}^{\infty}\frac{z_2^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{z_1^k}{k!}\cdot\frac{z_2^{n-k}}{(n-k)!}\right)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\underbrace{\frac{n!}{k!(n-k)!}}_{=\binom{n}{k}}\cdot z_1^k z_2^{n-k}\\
&= \sum_{n=0}^{\infty}\frac{(z_1+z_2)^n}{n!}\\
&= \exp(z_1+z_2).
\end{aligned}$$

ii) Using i) and the Euler identity from the Tutorial, we obtain

$$\begin{aligned}
\sin(x+y) &= \frac{1}{2i}(\exp(i(x+y))-\exp(-i(x+y)))\\
&= \frac{1}{2i}(\exp(ix)\exp(iy)-\exp(-ix)\exp(-iy))\\
&= \frac{1}{2i}((\cos(x)+i\sin(x))(\cos(y)+i\sin(y))-(\underbrace{\cos(-x)}_{=\cos(x)}+i\underbrace{\sin(-x)}_{=-\sin(x)})(\cos(-y)+i\sin(-y)))\\
&= \frac{1}{2i}(\cos(x)\cos(y)-\sin(x)\sin(y)+i(\cos(x)\sin(y)+\cos(y)\sin(x))\\
&\quad -\cos(x)\cos(y)+\sin(x)\sin(y)+i(\cos(x)\sin(y)+\cos(y)\sin(x)))\\
&= \frac{1}{2i}\cdot 2i(\cos(x)\sin(y)+\cos(y)\sin(x))\\
&= \cos(x)\sin(y)+\cos(y)\sin(x).
\end{aligned}$$

Analogously, we get

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

10 . i) With the ratio test, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot n}{(n+1)\cdot(-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1,$$

hence $R = \frac{1}{1} = 1$. For $z = 1$, we get the well-known alternating harmonic series which converges to the value $\ln(2)$.
ii) Let $S_N := \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$ and $T_N := 1 + \sum_{n=1}^{N}\left(\frac{1}{4n-1} - \frac{1}{2n} + \frac{1}{4n+1}\right)$. Then we get

$$\begin{aligned}
S_{4N} + \frac{1}{2}S_{2N} &= \sum_{n=1}^{4N} \frac{(-1)^{n+1}}{n} + \sum_{n=1}^{2N} \frac{(-1)^{n+1}}{2n} \\
&= \sum_{n=1}^{N}\left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right) + \sum_{n=1}^{N}\left(\frac{1}{4n-2} - \frac{1}{4n}\right) \\
&= \sum_{n=1}^{N}\left(\frac{1}{4n-3} - \frac{1}{2n} + \frac{1}{4n-1}\right) \\
&= 1 + \sum_{n=1}^{N}\left(\frac{1}{4n+1} - \frac{1}{2n} + \frac{1}{4n-1}\right) \\
&= T_N.
\end{aligned}$$

Since

$$\lim_{N\to\infty} S_{4N} = \lim_{N\to\infty} S_{2N} = \lim_{N\to\infty} S_N = f(1),$$

we get

$$\tilde{f}(1) = \lim_{N\to\infty} T_N = \lim_{N\to\infty} S_{4N} + \frac{1}{2}\lim_{N\to\infty} S_{2N} = \frac{3}{2}f(1).$$

11 . i) We use induction on n and first compute

$$|d_1| = |c_1| \le M = \frac{1}{2}\cdot(2M)^1.$$

Now assume that

$$|d_n| \le \frac{1}{2}\cdot(2M)^n$$

for an $n \ge 1$. Then we get

$$|d_{n+1}| = \left| -\sum_{k=0}^{n} d_k \cdot c_{n+1-k} \right| \leq \sum_{k=0}^{n} |d_k| \cdot |c_{n+1-k}| \leq \sum_{k=0}^{n} \frac{1}{2}(2M)^k \cdot M^{n+1-k} = \frac{1}{2} \cdot M^{n+1} \underbrace{\sum_{k=0}^{n} 2^k}_{=2^{n+1}-1}$$

$$\leq \frac{1}{2} \cdot (2M)^{n+1}.$$

ii) The criterion of Cauchy-Hadamard states that

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|d_n|} \leq \limsup_{n\to\infty} \sqrt[n]{\frac{1}{2}} \cdot 2M = 2M,$$

Hence $R \geq 1/(2M) > 0$.

iii) Let $f(z)$ be an analytic function on U. Since $f(z) \neq 0$ for all $z \in U$, the function

$$\begin{aligned} 1/f : U &\longrightarrow \mathbb{C} \\ z &\longmapsto f(z)^{-1} \end{aligned}$$

is well-defined. Now we have to show that this function is analytic, i.e. for any $z_0 \in U$, we have to define a convergent power series $g_{z_o}(z)$ with center z_0 such that $g_{z_0}(z)$ converges to $1/f$ on a suitable domain. Since f is analytic we can expand $f(z)$ as a power series at z_0 with radius of convergence R_{z_0}, i.e. we can write

$$f(z) = f(z_0) \sum_{n=0}^{\infty} c_n (z-z_0)^n, \qquad \text{for } z \in \mathbb{C} \text{ with } |z-z_0| < R_{z_0},$$

with $c_0 = 1$ and $|c_n| \leq 1/R_{z_0}^n$ for almost all n. Since $f(z_0) \neq 0$, we can define

$$g_{z_0}(z) := \frac{1}{f(z_0)} \sum_{n=0}^{\infty} d_n (z-z_0)^n,$$

where d_n is the recursive sequence defined in the exercise. By part ii), we know that this is a convergent power series. Then we compute for $|z-z_0| < R_{z_0}/2$

$$\begin{aligned} f(z)g(z) &= \left(\sum_{n=0}^{\infty} c_n (z-z_0)^n \right) \cdot \left(\sum_{n=0}^{\infty} d_n (z-z_0)^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} d_k \cdot c_{n-k} \right) (z-z_0)^n \\ &= 1 + \sum_{n=1}^{\infty} \left(d_n + \underbrace{\sum_{k=0}^{n-1} d_k \cdot c_{n-k}}_{=-d_n} \right) (z-z_0)^n \\ &= 1, \end{aligned}$$

hence $g = 1/f$.

12 . i) Let $f(z) := \sum_{n=0}^{\infty} a_n z^n, g(z) := \sum_{n=0}^{\infty} b_n z^n \in \mathbb{C}\{z\}$. Then by exercise 6

$$f(z)+g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$$

is a convergent power series with center 0, i.e. it is an element of $\mathbb{C}\{z\}$. Now the addition is reduced to the adding the coefficients, hence the group structure with respect to $+$ is inherited from the group structure on $\mathbb{C}$.
From the lecture, we know that the Cauchy product

$$f(z)g(z) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k} \Big) z^n$$

is also convergent and hence an element of $\mathbb{C}\{z\}$. The commutativity, associativity and distributivity follow in exactly the same way as for polynomials. The unit is given by the power series with constant coefficient $c_0 = 1$ and all other coefficients $c_n = 0$.
ii) For $f, g \in \mathbb{C}\{z\}$ with $f(0) = 0 = g(0)$, we clearly have

$$(f+g)(0) = f(0)+g(0) = 0+0 = 0,$$

so $f + g \in \mathfrak{m}$. Hence, $\mathfrak{m}$ is an abelian group. For $f \in \mathfrak{m}$ and $g \in \mathbb{C}\{z\}$, we obtain

$$(f \cdot g)(0) = f(0) \cdot g(0) = 0 \cdot g(0) = 0,$$

so $f \cdot g \in \mathfrak{m}$ and $\mathfrak{m}$ is indeed an ideal.

iii) Let $f \in \mathbb{C}\{z\} \setminus \mathfrak{m}$. Then $f(0) \neq 0$ and we can write

$$f(z) = f(0) \sum_{n=0}^{\infty} c_n z^n,$$

with $c_0 = 1$. Using the coefficients d_n from exercise 11, we obtain

$$g(z) := \frac{1}{f(0)} \sum_{n=0}^{\infty} d_n z^n$$

and as in the solution of 11iii), it follows $f(z) \cdot g(z) = 1$. Hence, any element in $\mathbb{C}\{z\} \setminus \mathfrak{m}$ is a unit, i.e.

$$\mathbb{C}\{z\}^{\star} = \mathbb{C}\{z\} \setminus \mathfrak{m}.$$

Therefore, $\mathfrak{m}$ is the only maximal ideal of $\mathbb{C}\{z\}$ (see Tutorial). For the quotient field, we consider the map

$$\begin{aligned} \varphi : \mathbb{C}\{z\} &\longrightarrow \mathbb{C} \\ f(z) &\longmapsto f(0). \end{aligned}$$

An easy computation shows, that φ is a ring homomorphism. It is surjective, since for any $a \in \mathbb{C}$, the constant power series $f(z) = a$ is mapped to a. The kernel of φ is exactly given by $\mathfrak{m}$ and the homomorphism theorem gives

$$\mathbb{C}\{z\}/\mathfrak{m} \cong \mathbb{C}.$$

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Solutions 04

13 . i) Since $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$, we see that

$$|f(x)| \leq |x| \xrightarrow{x \to 0} 0,$$

hence f is continuous.
ii) Assume that there exists such a domain G and an analytic function F which extends f. Since $F(0) = 0$ but $F \neq 0$, the theorem of isolated zeros implies, that there must exist an $\varepsilon > 0$ such that $F(z) \neq 0$ for all z with $0 < |z| < \varepsilon$. But for any $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n\pi} < \varepsilon$ and we obtain

$$F\left(\frac{1}{n\pi}\right) = f\left(\frac{1}{n\pi}\right) = \frac{1}{n\pi} \cdot \sin(n\pi) = 0.$$

This is a contradiction, hence there is no such analytic function.

14 . We can use the series expansion from the lecture to obtain

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{(2n+1)}}{(2n+1)!} = z - \frac{z^3}{6} + \mathcal{O}(z^5).$$

With this, we obtain

$$\begin{aligned}
\sin^2(z) &= \left(z - \frac{z^3}{6} + \mathcal{O}(z^5)\right)^2 = z^2 - \frac{z^4}{3} + \mathcal{O}(z^6), \\
\sin^3(z) &= z^3 + \mathcal{O}(z^5), \\
\sin^4(z) &= z^4 + \mathcal{O}(z^6).
\end{aligned}$$

For $g(z)$, we can use the above results to compute

$$
\begin{aligned}
\exp(\sin(z)) &= \sum_{n=0}^{\infty} \frac{\sin^n(z)}{n!} \\
&= 1+\sin(z)+\frac{\sin^2(z)}{2}+\frac{\sin^3(z)}{6}+\frac{\sin^4(z)}{24}+\mathcal{O}(z^5) \\
&= 1+z-\frac{z^3}{6}+\frac{z^2}{2}-\frac{z^4}{6}+\frac{z^3}{6}+\frac{z^4}{24}+\mathcal{O}(z^5) \\
&= 1+z+\frac{z^2}{2}-\frac{z^4}{8}+\mathcal{O}(z^5).
\end{aligned}
$$

15 . First, we prove the hint: Let D be a disk with center a and radius r and let b and c be arbitrary points in D. Define

$$
\begin{aligned}
\gamma : [0,1] &\longrightarrow D \\
t &\longmapsto bt + c(1-t)
\end{aligned}
$$

This is well-defined, since

$$
\begin{aligned}
|\gamma(t)-a| &= |bt+c(1-t)-a| \\
&= |bt+c(1-t)-(at+a(1-t))| \\
&= |bt-at+c(1-t)-a(1-t)| \\
&\leq |b-a|\,t+|c-a|\,(1-t) \\
&< rt+r(1-t) \\
&= r.
\end{aligned}
$$

Therefore, there exists a path from b to c inside D and the disk D is path-connected.

Now let $U \subseteq \mathbb{C}$ be an open connected set and fix an arbitrary point $a \in U$. Define

$$
\begin{aligned}
U_1 &:= \{b \in U \mid \exists \text{ path from } a \text{ to } b\}, \\
U_2 &:= \{b \in U \mid \nexists \text{ path from } a \text{ to } b\}.
\end{aligned}
$$

Then we clearly have a decomposition $U = U_1 \dot\cup U_2$. Moreover, since the constant path from a to a always exists, the set $U_1 \neq \emptyset$. Now let $b \in U_1$ and $D \subseteq U$ be a disk around b which is contained in U. Then for an arbitrary point $c \in D$, there exists a path from b to c (since any disk is path-connected), and composing this path with the given path from a to b yields a path from a to c, hence $D \subseteq U_1$ and U_1 is an open set. Assume that $U_2 \neq \emptyset$, then a similar argument shows that U_2 would be an open set, too. But this is a contradiction, since U is connected. Hence $U_2 = \emptyset$ and therefore, U is path-connected.

16 . Define $G := \{z \in \mathbb{C} \mid |z-1| < 1\}$ and

$$a_0 := 1/\pi, \qquad a_n := \frac{1}{n\pi} \quad \text{for } n \geq 1.$$

Then clearly $0 < a_n \leq 1$ and hence $a_n \in G$ for all $n \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} a_n = 0 \in \partial G.$$

Define

$$\begin{aligned} f_n : G &\longrightarrow \mathbb{C} \\ z &\longmapsto \sin(1/z). \end{aligned}$$

This is clearly an analytic function on G and we get

$$f(a_0) = \sin(\pi) = 0, \qquad f(a_n) = \sin(n\pi) = 0$$

for all $n \geq 1$, i.e. $f(a_n) = 0$ for all $n \in \mathbb{N}$. Since $b := 2/\pi \in G$ satisfies

$$f(b) = \sin(\pi/2) = 1 \neq 0,$$

we get that f does not vanish identically on G. Note that the result does not contradict the Identity Theorem 1.17.

Solutions 05

17 . Suppose that $f, g \in \mathscr{A}(G)$ with $f \cdot g = 0$ and $f \neq 0$. Then there exists an open set $U \subseteq G$ such that $f(z) \neq 0$ for all $z \in U$. Since $f \cdot g = 0$, this implies $g(z) = 0$ for all $z \in U$. Since g is analytic, the Identity Theorem then implies $g = 0$, hence $\mathscr{A}(G)$ is an integral domain.

18 . i) Let $z \in \mathbb{C}$ such that $\sin(z) = 0$. Then we obtain

$$\begin{aligned} & & 0 = \sin(z) = \frac{1}{2i}(\exp(iz) - \exp(-iz)) \\ \Longrightarrow & & \exp(iz) = \exp(-iz) \\ \Longrightarrow & & \exp(-\Im(z)) = |\exp(iz)| = |\exp(-iz)| = \exp(\Im(z)) \\ \Longrightarrow & & \Im(z) = -\Im(z) = 0, \end{aligned}$$

hence all zeros of sin must be real numbers. These zeros are well-known:

$$\{\pi n \mid n \in \mathbb{Z}\}.$$

Analogously, we get for cos:

$$\{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}.$$

ii) Suppose that ω is a period of sin, then for each $z \in \mathbb{C}$ we have

$$\sin(z + \omega) = \sin(z).$$

In particular, we get for $z = 0$ that

$$\sin(\omega) = 0,$$

hence ω must be a zero of sin. Considering the zeros determined in i), we see that the odd multiples of π are no periods (e.g. $\sin(\pi/2) \neq \sin(-\pi/2)$), so $\omega = k \cdot 2\pi$ for a $k \in \mathbb{Z}$. For cos we look at $z = \pi/2$ to obtain

$$0 = \cos(\pi/2) = \cos(\pi/2 + \omega)$$

and then conclude in the same way that $\omega = k \cdot 2\pi$ for a $k \in \mathbb{Z}$.

19 . i) Suppose that $\widetilde{\log}$ is another branch of the logarithm, then

$$\widetilde{\log}(z) = \log(z) + 2\pi k i$$

for a $k \in \mathbb{N}$. Then for the corresponding power function $\widetilde{f}$ we obtain

$$\begin{aligned}\widetilde{f}(z) &= \exp(\alpha \cdot \widetilde{\log}(z)) \\ &= \exp(\alpha \cdot (\log(z) + 2\pi k i)) \\ &= \exp(\alpha \cdot \log(z)) \cdot \exp(k \cdot 2\pi i \alpha) \\ &= f(z) \cdot \exp(k \cdot 2\pi i \alpha).\end{aligned}$$

ii) For the principal branch of the logarithm, we obtain $\mathrm{Log}(i) = i\pi/2$ and hence

$$\begin{aligned}i^i &= \exp(i \cdot \mathrm{Log}(i)) = \exp(-\pi/2), \\ i^\pi &= \exp(\pi \cdot \mathrm{Log}(i)) = \exp(i\pi^2/2), \\ i^{-1} &= \exp(-1 \cdot \mathrm{Log}(i)) = \exp(-i\pi/2) = -i = \frac{1}{i}.\end{aligned}$$

20 . i) For $x \in [0, 2\pi] \setminus \{0, \pi/2, \pi, (3/2)\pi, 2\pi\}$, we know that $\sin(x) \neq 0 \neq \cos(x)$. Hence we can choose $a := |\sin(x)|$ and $b := |\cos(x)|$ for a fixed x. From the tutorial we know

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

Hence, for $z \in G_x$ we can write $u(y) := \sin(x)\cosh(y)$ for the real part of $\sin(z)$ and $v(y) := \cos(x)\sinh(y)$ for the imaginary part. Inserting this in the hyperbola equation, we obtain

$$\frac{\sin(x)^2\cosh(y)^2}{\sin(x)^2} - \frac{\cos(x)^2\sinh(y)^2}{\cos(x)^2} = \cosh(y)^2 - \sinh(y)^2.$$

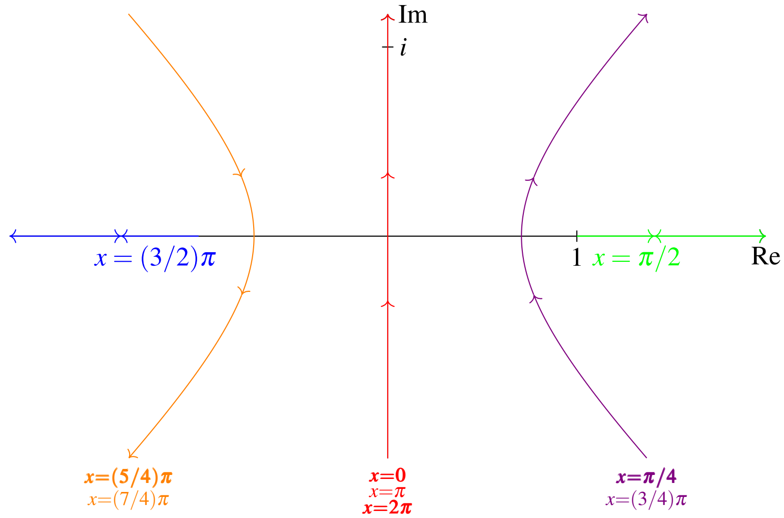
We have shown in the tutorial that this is equal to 1 for any y, hence the hyperbola equation is satisfied for this choice of a, b, u and v.

Analogously, we can define $a := \cosh(y) (= |\cosh(y)|)$ and $b := |\sinh(y)|$ for fixed $y \neq 0$, since these functions have no other zeros. For $z \in H_y$, we set $u(x) := \sin(x)\cosh(y)$ for the real part of $\sin(z)$ and $v(x) := \cos(x)\sinh(y)$ for the imaginary part. Inserting in the ellipse equation yields

$$\frac{\sin(x)^2\cosh(y)^2}{\cosh(y)^2} + \frac{\cos(x)^2\sinh(y)^2}{\sinh(y)^2} = \sin(x)^2 + \cos(x)^2 = 1$$

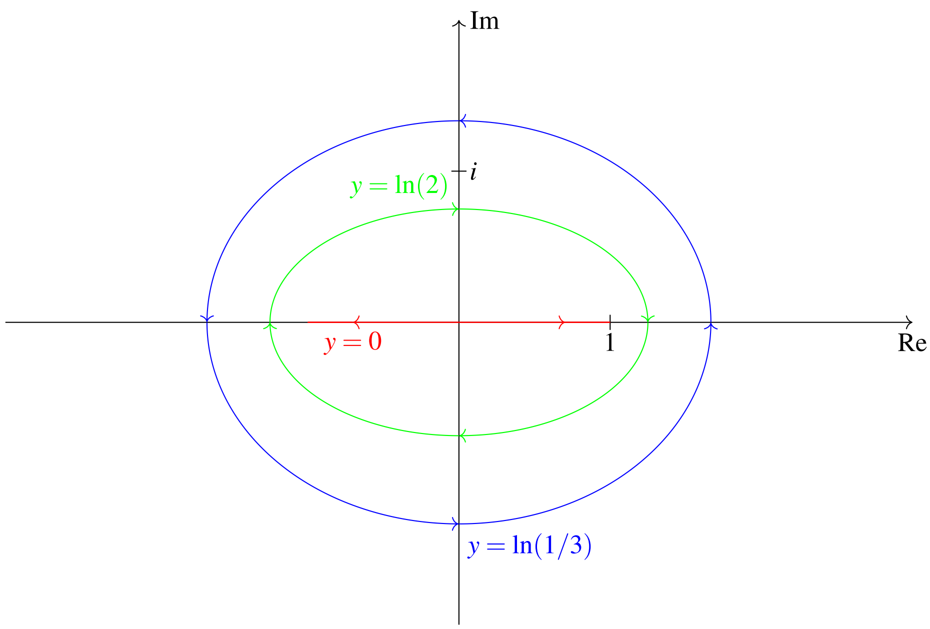
for all x, hence sin defines an ellipse here.

ii) $\sin(G_x)$:



The arrow tips indicate the direction of the graph for the bold value, whereas the graphs for the other values run in the opposite direction.

$\sin(H_y)$:



Solutions 06

21 . By definition of the path integral, we obtain

$$\int_{\gamma_1} \frac{dz}{z} = \int_0^\pi \frac{1}{\gamma_1(t)} \gamma_1'(t) dt = \int_0^\pi \frac{1}{e^{it}} \cdot i e^{it} dt = \int_0^\pi i dt = i\pi,$$
$$\int_{\gamma_2} \frac{dz}{z} = \int_0^\pi \frac{1}{\gamma_2(t)} \gamma_2'(t) dt = \int_0^\pi \frac{1}{e^{-it}} \cdot (-i) e^{-it} dt = \int_0^\pi (-i) dt = -i\pi.$$

22 . i) We use induction on n to show that the n-th derivative of f exists, is continuous and of the form

$$f^{(n)}(x) = \begin{cases} p_n(1/x) e^{-1/x^2} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

for some polynomial $p_n \in \mathbb{Q}[X]$.
For $n = 0$, this is the definition of f (for $x \to 0$, the quotient $-1/x^2 \to -\infty$ and hence $e^{-1/x^2} \to 0$ so f is continuous).
Now suppose that for some $n \in \mathbb{N}$, we have that $f^{(n)}$ is a continuous function of the form

$$f^{(n)}(x) = \begin{cases} p_n(1/x) e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then we compute

$$\lim_{h \to 0} \frac{f^{(n)}(h)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot p_n(1/h) e^{-1/h^2} = 0,$$

since the exponential function dominates any polynomial in the limit, in particular also $\frac{1}{h} \cdot p_n(1/h)$. Therefore, $f^{(n+1)}$ exists and $f^{(n+1)}(0) = 0$. Moreover, we can use the product rule to obtain for $x \neq 0$:

$$\begin{aligned} f^{(n+1)}(x) &= p_n(1/x) \cdot \frac{2}{x^3} \cdot e^{-1/x^2} + e^{-1/x^2} \cdot p_n'(1/x) \cdot \left(\frac{-1}{x^2}\right) \\ &= (2(1/x)^3 \cdot p_n(1/x) - (1/x)^2 \cdot p_n'(1/x)) \cdot e^{-1/x^2}. \end{aligned}$$

Since $p_n \in \mathbb{Q}[X]$, we also obtain that $p_n' \in \mathbb{Q}[X]$, so

$$p_{n+1} := 2X^3 \cdot p_n - X^2 \cdot p_n' \in \mathbb{Q}[X]$$

and

$$f^{(n+1)}(x) = \begin{cases} p_{n+1}(1/x)e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

ii) Suppose that $f : U \longrightarrow \mathbb{C}$ is such an extension on a suitable neighbourhood U of $\mathbb{R}$. Since U is open, there exists an $\varepsilon > 0$ such that the punctured disk $V := D_\varepsilon(0) \setminus \{0\}$ is completely contained in U. Now we can define

$$\begin{aligned} g : V &\longrightarrow \mathbb{C} \\ z &\longmapsto e^{-1/z^2} \end{aligned}$$

Since $-1/z^2$ is analytic on V and the exponential function is analytic on $\mathbb{C}$, g is also analytic on V. Hence f and g are two analytic functions on V which coincide on $I := (0, \varepsilon) \subset \mathbb{R}$. Since I has an accumulation point in V (e.g. $\varepsilon/2$), the Identity Theorem implies that $f = g$ on V.
Now let $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then we can define the sequence

$$z_n := i \cdot \frac{1}{N+n}, \qquad n \in \mathbb{N}.$$

By the choice of N, we get $z_n \in V$ for all n and clearly

$$\lim_{n\to\infty} z_n = 0.$$

But we find

$$f(z_n) = g(z_n) = e^{-1/z_n^2} = e^{(n+N)^2} \xrightarrow{n\to\infty} \infty,$$

whereas $f(0) = 0$. Hence f is not continuous and cannot be analytic, which is a contradiction.

23 . Since f and g are differentiable, their difference $f - g$ is also differentiable and satisfies the Cauchy-Riemann differential equations:

$$\begin{aligned} \frac{\partial Im(f-g)}{\partial y} &= \frac{\partial Re(f-g)}{\partial x} = \frac{\partial (Re(f) - Re(g))}{\partial x} = \frac{\partial 0}{\partial x} = 0, \\ \frac{\partial Im(f-g)}{\partial x} &= -\frac{\partial Re(f-g)}{\partial y} = -\frac{\partial (Re(f) - Re(g))}{\partial y} = -\frac{\partial 0}{\partial y} = 0. \end{aligned}$$

Therefore, $Im(f-g) = Im(f) - Im(g)$ must be constant.

24 . The Wirtinger derivatives at a point $z_0 = \gamma_j(t_0) \in \mathbb{C}$ are given by

$$\frac{\partial f}{\partial z}(z_0) = A(z_0)$$
$$\frac{\partial f}{\partial \bar{z}}(z_0) = B(z_0)$$

where $A, B : U \longrightarrow \mathbb{C}$ are continuous functions such that

$$f(z) = f(z_0) + A(z)(z - z_0) + B(z)(\bar{z} - \bar{z}_0).$$

Inserting $\gamma_j(t)$ in the above equation we obtain for the derivative of t:

$$\begin{aligned}(f \circ \gamma_j)'(t) = &(A \circ \gamma_j)'(t)(\gamma_j(t) - z_0) + (A \circ \gamma_j)(t)\gamma_j'(t) \\ &+ (B \circ \gamma_j)'(t)(\overline{\gamma_j}(t) - \overline{z_0}) + (B \circ \gamma_j)(t)\overline{\gamma_j}'(t)\end{aligned}$$

Now we can insert t_0 to obtain

$$(f \circ \gamma_j)'(t_0) = \frac{\partial f}{\partial z}(z_0) \cdot \gamma_j'(t_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \overline{\gamma_j}'(t_0). \tag{2}$$

Since f is differentiable at any z_0, $\partial f / \partial \bar{z}(z_0) = 0$, hence

$$(f \circ \gamma_j)'(t_0) = \frac{\partial f}{\partial z}(z_0)\gamma_j'(t_0)$$

and therefore we get that

$$\frac{(f \circ \gamma_2)'(t_0)}{(f \circ \gamma_1)'(t_0)} = \frac{\partial f / \partial z(z_0)\gamma_2'(t_0)}{\partial f / \partial z(z_0)\gamma_1'(t_0)} = \frac{\gamma_2'(t_0)}{\gamma_1'(t_0)}.$$

By the definition of $\sphericalangle$ we are done.

Solutions 07

25 . We choose the parametrization

$$\begin{aligned}\gamma : [0, 2\pi] &\longrightarrow \mathbb{C}\\ t &\longmapsto e^{it}\end{aligned}$$

and compute

$$\begin{aligned}-\int_{|\zeta|=1} \frac{\overline{f(\zeta)}}{\zeta^2} d\zeta &= -\int_0^{2\pi} \frac{\overline{f(e^{it})}}{e^{2it}} ie^{it} dt\\ &= \int_0^{2\pi} \overline{f(e^{it})}(-i)e^{-it} dt\\ &= \int_0^{2\pi} \overline{f(e^{it}) ie^{it}} dt = \overline{\int_0^{2\pi} f(e^{it}) ie^{it} dt}\\ &= \overline{\int_{|\zeta|=1} f(\zeta) d\zeta}.\end{aligned}$$

26 . i) If $|f(z)| = r$ is constant, then $f(G)$ is contained in the circle

$$S_r(0) = \{z \in \mathbb{C} \mid |z| = r\}$$

of radius r. If f would be a non-constant function, then the Open Mapping Theorem would imply that $f(G)$ is open which is a contradiction.

ii) Since $f(a) \neq 0$, $1/f$ is defined on U and is a holomorphic function (cf. Exercise 11). Moreover, $|1/f|$ assumes its maximum at $a \in U$ and hence the Maximum Principle shows that $1/f$ is constant, hence f is constant on U. By the Identity Theorem, f must be constant on G.

iii) In the lecture, it was shown that

$$f' = \frac{\partial f}{\partial x} = i \frac{\partial f}{\partial y}.$$

If $f' = 0$, we see that these real partial derivatives vanish, which implies that the function is constant.

27 . We know that

$$\frac{\partial \overline{f}}{\partial x} = \overline{\frac{\partial f}{\partial x}} \qquad\qquad \frac{\partial \overline{f}}{\partial y} = \overline{\frac{\partial f}{\partial y}},$$

hence we compute

$$\overline{\frac{\partial \overline{f}}{\partial \overline{z}}} = \overline{\frac{1}{2}\left(\frac{\partial \overline{f}}{\partial x} + i\frac{\partial \overline{f}}{\partial y}\right)} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial z}$$

and

$$\overline{\frac{\partial \overline{f}}{\partial z}} = \overline{\frac{1}{2}\left(\frac{\partial \overline{f}}{\partial x} - i\frac{\partial \overline{f}}{\partial y}\right)} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial \overline{z}}.$$

28 . Since $\{1, i\}$ is a basis of $\mathbb{C}$ as an $\mathbb{R}$-vector space and $j \circ T_{\mathbb{C}}$ and $T_{\mathbb{R}^2} \circ j$ are $\mathbb{R}$-linear maps, they are completely determined by their image on this basis. Hence we compute:

$$\begin{aligned}
(j \circ T_{\mathbb{C}})(1) &= j(f_z(z) + f_{\overline{z}}(z)) = j\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\right) = j\left(\frac{\partial f}{\partial x}\right) \\
&= j(u_x(x,y) + iv_x(x,y)) = \begin{pmatrix} u_x(x,y) \\ v_x(x,y) \end{pmatrix} = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = T_{\mathbb{R}^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (T_{\mathbb{R}^2} \circ j)(1) \\
(j \circ T_{\mathbb{C}})(i) &= j(f_z(z)i + f_{\overline{z}}(z)(-i)) = j\left(\frac{i}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\right) = j\left(\frac{\partial f}{\partial y}\right) \\
&= j(u_y(x,y) + iv_y(x,y)) = \begin{pmatrix} u_y(x,y) \\ v_y(x,y) \end{pmatrix} = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = T_{\mathbb{R}^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= (T_{\mathbb{R}^2} \circ j)(i)
\end{aligned}$$

Solutions 08

29 . It is clear that any constant function satisfies the required property. Suppose that f is a non-constant function such that $f \circ f = f$, then for any $z \in f(\mathbb{C})$ we have $f(z) = z$. Since f is holomorphic, $f(\mathbb{C})$ is open and non-empty, hence it contains an accumulation point. By the Identity Theorem, we get that $f(z) = z$ for all $z \in \mathbb{C}$, hence the only non-constant entire function satisfying $f \circ f = f$ is the identity.

30 . For any $s \in [0, 2\pi]$, $\gamma_s(0) = z_0$ and $\gamma_s'(0) = e^{is} \neq 0$, hence the properties of Exercise 24 are satisfied for each pair (γ_s, γ_0). Then we obtain that

$$\sphericalangle\left(\gamma_s{}'(0), \gamma_0{}'(0)\right) = arg(e^{is}) = s,$$

whereas

$$\sphericalangle\left((f \circ \gamma_s)'(0), (f \circ \gamma_0)'(0)\right) = arg\left(\frac{(f \circ \gamma_s)'(0)}{(f \circ \gamma_0)'(0)}\right).$$

Using (2), we obtain

$$(f \circ \gamma_s)'(0) = \frac{\partial f}{\partial z}(z_0) \cdot \gamma_s'(0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \overline{\gamma_s}'(0) = \frac{\partial f}{\partial z}(z_0) \cdot e^{is} + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot e^{-is},$$

hence

$$\begin{aligned}\sphericalangle\left((f \circ \gamma_s)'(0), (f \circ \gamma_0)'(0)\right) &= arg\left(e^{is} \cdot \frac{\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot e^{-2is}}{\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)}\right) \\ &= arg(e^{is}) + arg\left(\frac{\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot e^{-2is}}{\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)}\right).\end{aligned}$$

In order to obtain equality, we must have

$$arg\left(\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot e^{-2is}\right) = arg\left(\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\right)$$

which is independent of s. This can only be true for $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$, hence f must be holomorphic. If z_0 would be a zero of f', then

$$
\begin{aligned}
& 0=\frac{\partial f}{\partial z}(z_0)=\frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0)-i\frac{\partial f}{\partial y}(z_0)\right)\\
\implies\qquad & \frac{\partial u}{\partial x}(z_0)+i\frac{\partial v}{\partial x}(z_0)=-\frac{\partial v}{\partial y}(z_0)+i\frac{\partial u}{\partial y}(z_0).
\end{aligned}
$$

This, together with the Cauchy-Riemann differential equations, implies

$$0=\det(Jac(f)(z_0)),$$

which is a contradiction to the assumptions on f.

31 . We can use partial fraction decomposition to obtain

$$f(z)=\frac{1}{1-z}-\frac{1}{2-z}.$$

i) Since $|z|<1$ and hence also $\left|\frac{1}{2}z\right|<1$, we can use the geometric series to obtain the power series

$$f(z)=\sum_{n=0}^{\infty}z^n-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}z\right)^n=\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n.$$

ii) For $1<|z|<2$, we still obtain that $\left|\frac{1}{2}z\right|<1$ and we can use the geometric series for the second term as before. For the first term, we consider the following modification of the geometric series (cf. Tutorial):

$$-\sum_{n=1}^{\infty}\frac{1}{z^n}=\frac{1}{1-z}\qquad\text{for } |z|>1. \tag{3}$$

This enables us to compute

$$f(z)=-\sum_{n=1}^{\infty}z^{-n}-\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n=\sum_{n<0}-z^n+\sum_{n\geq 0}-\frac{1}{2^{n+1}}z^n.$$

iii) In this case, we have that both $|z|>1$ and $\left|\frac{1}{2}z\right|>1$, hence we can use (3) for both terms to obtain

$$f(z)=-\sum_{n=1}^{\infty}z^{-n}+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2}z\right)^{-n}=\sum_{n=1}^{\infty}\left(\frac{1}{2^{-n+1}}-1\right)z^{-n}=\sum_{n<0}\left(\frac{1}{2^{n+1}}-1\right)z^n.$$

54 . Suppose that $f(\mathbb{C})$ is not dense in $\mathbb{C}$, i.e. suppose that there exists a point $a\in\mathbb{C}$ and a radius $r>0$ such that $D_r(a)\cap f(\mathbb{C})=\emptyset$. Then for any $z\in\mathbb{C}$, we have

$$|a-f(z)|\geq r.$$

Therefore, the function

$$\begin{aligned} g : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{1}{a - f(z)} \end{aligned}$$

is holomorphic on $\mathbb{C}$ and

$$|g(z)| = \frac{1}{|a - f(z)|} \leq \frac{1}{r},$$

hence g is an entire, bounded function. By Liouville's Theorem, g must be constant. But this would imply that f is also constant, which is a contradiction.

Solutions 09

33 . We consider the function

$$\begin{aligned} g : \mathbb{C}^* &\longrightarrow \mathbb{C} \\ z &\longmapsto f(1/z) \end{aligned}$$

Since f is entire, g is holomorphic away from 0. We see that

$$\lim_{|z|\to 0} |g(z)| = \lim_{|z|\to 0} |f(1/z)| = \lim_{|z|\to\infty} |f(z)| = \infty,$$

hence g has a pole at $z = 0$. Therefore, g has a Laurent series expansion of the form

$$g(z) = \sum_{n=-k}^{\infty} c_n z^n = f(1/z),$$

so the Laurent series expansion of f is given by

$$f(z) = \sum_{n=-\infty}^{k} c_{-n} z^n.$$

Since f is entire, it is holomorphic at $z = 0$, hence the principal part of the Laurent series must be 0, so

$$f(z) = \sum_{n=0}^{k} c_{-n} z^n$$

is indeed a polynomial.

34 . We first compute

$$\begin{aligned}
\left|f(\rho\cdot e^{i\phi})\right|^2 &= f(\rho\cdot e^{i\phi})\overline{f(\rho\cdot e^{i\phi})}\\
&= \left(\sum_{m=0}^{\infty} c_m\rho^m\cdot e^{i\phi m}\right)\left(\sum_{m=0}^{\infty}\overline{c}_m\rho^m\cdot e^{-i\phi m}\right)\\
&= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} c_k\overline{c}_{m-k}e^{i\phi(2k-m)}\right)\rho^m.
\end{aligned}$$

Inserting this result in the integral given in the hint and using the absolute convergence, we find

$$\frac{1}{2\pi}\cdot\int_0^{2\pi}\left|f(\rho\cdot e^{i\phi})\right|^2 d\phi = \frac{1}{2\pi}\cdot\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} c_k\overline{c}_{m-k}\int_0^{2\pi} e^{i\phi(2k-m)}d\phi\right)\rho^m.$$

Now since

$$\int_0^{2\pi} e^{i\phi(2k-m)}d\phi = \begin{cases} 2\pi, & 2k=m,\\ 0, & 2k\neq m,\end{cases}$$

we obtain

$$\frac{1}{2\pi}\cdot\int_0^{2\pi}\left|f(\rho\cdot e^{i\phi})\right|^2 d\phi = \sum_{m=0}^{\infty}|c_m|^2\rho^{2m}.$$

Now inserting the assumptions of the exercise, we get

$$\begin{aligned}
M^2 &= \frac{1}{2\pi}\cdot\int_0^{2\pi} M^2 d\phi\\
&\geq \frac{1}{2\pi}\cdot\int_0^{2\pi}\left|f(\rho\cdot e^{i\phi})\right|^2 d\phi\\
&= \sum_{m=0}^{\infty}|c_m|^2\rho^{2m}\\
&= |c_n|^2\rho^{2m} + \sum_{\substack{m=0\\ m\neq n}}^{\infty}|c_m|^2\rho^{2m}\\
&= M^2\frac{\rho^{2m}}{r^{2m}} + \sum_{\substack{m=0\\ m\neq n}}^{\infty}|c_m|^2\rho^{2m}.
\end{aligned}$$

Since this inequality holds for any $\rho < r$, we can take the limit $\rho\to r$ to see that

$$M^2 \geq M^2 + \sum_{\substack{m=0\\ m\neq n}}^{\infty}|c_m|^2 r^{2m}$$

which implies that $|c_m| = 0$ for all $m\neq n$. Hence

$$f(z) = c_n \cdot z^n.$$

35 . i) Since f is entire, it is holomorphic on $D_r(0)$ for every $r \geq 0$. If f is non-constant, it assumes its maximum on the boundary of $D_r(0)$, hence

$$|f(z)| \leq M_f(r)$$

for any $z \in D_r(0)$. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be the power series expansion of f. Then for any $r \geq r_0$, the Cauchy inequalities imply

$$|c_n| \leq \frac{M_f(r)}{r^n} \leq \frac{\sqrt{r} \cdot \ln r}{r^n}$$

Since r^n dominates $\sqrt{r} \cdot \ln r$ for any $n > 0$, we see that

$$\lim_{r \to \infty} \frac{\sqrt{r} \cdot \ln r}{r^n} = 0$$

for $n > 0$. Hence $c_n = 0$ for $n > 0$, so f must be constant.

ii) The formula for the Laurent coefficients of f gives

$$\begin{aligned}
|c_n| &= \frac{1}{2\pi} \left| \int_{|z|=r} f(z) \cdot z^{-n-1} dz \right| \\
&\leq \frac{1}{2\pi} \int_{|z|=r} |f(z)| \, |z|^{-n-1} \, dz \\
&\leq \frac{1}{2\pi} \int_{|z|=r} M_f(r) r^{-n-1} dz \\
&= M_f(r) r^{-n} \\
&\leq |\ln(r)| \, r^{-n-1/2}
\end{aligned}$$

Taking the limit $r \to 0$, we find that for $n < 0$, $c_n = 0$. Hence f has a removable singularity at 0.

36 . i) Let $w_0 \in \mathbb{C}^*$ and $r = |w_0|$. Then there exists a branch of the logarithm log on $D_r(w_0)$. This is a holomorphic function on $D_r(w_0)$ and we set

$$g(w) := f\left(\frac{1}{2\pi i} \log(w)\right).$$

for $w \in D_r(w_0)$. First we have to check that this is well-defined. For this purpose, suppose that $\widetilde{\log}$ is another branch of the logarithm which is holomorphic on $D_r(w_0)$. Then

$$\widetilde{\log}(w) = \log(w) + k \cdot 2\pi i$$

for a $k \in \mathbb{Z}$. Hence

$$f\left(\frac{1}{2\pi i}\widetilde{\log}(w)\right) = f\left(\frac{1}{2\pi i}\log(w) + k\right) = f\left(\frac{1}{2\pi i}\log(w)\right),$$

since f is periodic with period 1. Since f is entire, g defines a holomorphic function on $D_r(w_0)$. Using another element $w_1 \in \mathbb{C}^*$ with $\rho = |w_1|$, we can define a function

$$\begin{aligned} g' : D_\rho(w_1) &\longrightarrow \mathbb{C} \\ w &\longmapsto f\left(\frac{1}{2\pi i}\log(w)\right) \end{aligned}$$

for a branch of logarithm log which is holomorphic on $D_\rho(w_1)$. Then $g' = g$ on the intersection of $D_r(w_0)$ and $D_\rho(w_1)$, since either we can use the same branch of logarithm or we obtain a summand of k which disappears due to the periodicity of f. Hence we obtain a holomorphic function g on the union of $D_r(w_0)$ and $D_\rho(w_1)$. Covering $\mathbb{C}^*$ with such balls, we obtain a holomorphic function

$$g : \mathbb{C}^* \longrightarrow \mathbb{C} \tag{4}$$

which satisfies

$$f(z) = g(e^{2\pi i z})$$

for any $z \in \mathbb{C}$. The formula for the coefficients of the Laurent series $g(w) = \sum_{n=-\infty}^{\infty} c_n w^n$ gives

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \cdot \int_{|\zeta|=1} \frac{g(\zeta)}{\zeta^{n+1}} d\zeta = \frac{1}{2\pi i} \cdot \int_0^{2\pi} \frac{g(e^{i\phi})}{e^{i\phi(n+1)}} i e^{i\phi} d\phi \\ &= \frac{1}{2\pi} \cdot \int_0^{2\pi} g(e^{i\phi}) e^{-in\phi} d\phi. \end{aligned}$$

ii) Using (4) for $z = \frac{\phi}{2\pi}$, we obtain

$$\begin{aligned} f(z) = g(e^{2\pi i z}) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \cdot \int_0^{2\pi} g(e^{i\phi}) e^{-in\phi} d\phi\right) e^{2\pi i n z} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \cdot \int_0^{2\pi} f\left(\frac{\phi}{2\pi}\right) e^{-in\phi} d\phi\right) e^{2\pi i n z}. \end{aligned}$$

Solutions 10

37 . By definition, the Bernoulli numbers are the coefficients of the Taylor series expansion of

$$f(z) := \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Multiplying this equation with

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!}$$

and using the Cauchy product on the right hand side yields

$$z = \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} \frac{B_n}{n!(N-n+1)!} \right) z^N = \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} \frac{(N+1)!}{n!(N+1-n)!} B_n \right) \frac{z^{N+1}}{(N+1)!}.$$

Comparing the coefficients then gives

$$0 = \sum_{n=0}^{N} \frac{(N+1)!}{n!(N+1-n)!} B_n = \sum_{n=0}^{N} \binom{N+1}{n} B_n.$$

38 . i) In Exercise 9 ii) we derived addition theorems for real arguments. Because of the Identity Theorem, they must also hold for complex arguments, hence we obtain

$$\begin{aligned} \sin(2z) &= 2\cos(z)\sin(z), \\ \cos(2z) &= \cos(z)^2 - \sin(z)^2. \end{aligned}$$

This enables us to compute

$$\begin{aligned} \cot(z) - 2 \cdot \cot(2z) &= \frac{\cos(z)}{\sin(z)} - 2 \cdot \frac{\cos(2z)}{\sin(2z)} = \frac{\cos(z)}{\sin(z)} - 2 \cdot \frac{\cos(z)^2 - \sin(z)^2}{2\cos(z)\sin(z)} \\ &= \frac{\cos(z)^2}{\cos(z)\sin(z)} - \frac{\cos(z)^2 - \sin(z)^2}{\cos(z)\sin(z)} = \frac{\sin(z)^2}{\cos(z)\sin(z)} = \frac{\sin(z)}{\cos(z)} \\ &= \tan(z) \end{aligned}$$

ii) Inserting the Taylor series expansion for cot in the formula proven in i), we get

$$\begin{aligned}
\tan(z) &= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \cdot \frac{2^{2k} \cdot B_{2k}}{(2k)!} \cdot z^{2k-1} - 2 \cdot \left(\frac{1}{2z} + \sum_{k=1}^{\infty} (-1)^k \cdot \frac{2^{2k} \cdot B_{2k}}{(2k)!} \cdot (2z)^{2k-1} \right) \\
&= \sum_{k=1}^{\infty} (-1)^k \cdot \left(\frac{2^{2k} \cdot B_{2k}}{(2k)!} - 2^{2k} \cdot \frac{2^{2k} \cdot B_{2k}}{(2k)!} \right) \cdot z^{2k-1} \\
&= \sum_{k=1}^{\infty} (-1)^k \cdot \frac{2^{2k}(1-2^{2k})B_{2k}}{(2k)!} \cdot z^{2k-1} \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} \cdot z^{2k-1}.
\end{aligned}$$

The radius of convergence of $\cot(z) - \frac{1}{z}$ is $R_1 = \pi$ by the lecture. Consequently, the radius of convergence of $-2\cot(2z) + \frac{1}{z}$ is $R_2 = \frac{\pi}{2}$ and by Exercise 6, we obtain that

$$R_3 = \min\{R_1, R_2\} = \frac{\pi}{2}$$

is the radius of convergence of $\tan(z)$.

39 . i) Since $|z| < 2$ and $n \geq 2$, we get that

$$\left| \frac{z^2}{n^2} \right| = \left(\frac{|z|}{n} \right)^2 < 1^2 = 1.$$

Moreover, we obtain

$$\frac{1}{z-n} + \frac{1}{z+n} = \frac{z+n+z-n}{z^2-n^2} = -\frac{2z}{n^2} \cdot \frac{1}{1-z^2/n^2}.$$

By the above computation, we can use the geometric series to obtain

$$\frac{1}{z-n} + \frac{1}{z+n} = -\frac{2z}{n^2} \sum_{k=0}^{\infty} \left(\frac{z^2}{n^2} \right)^k = -2 \cdot \sum_{k=0}^{\infty} \frac{z^{2k+1}}{n^{2k+2}} = -2 \cdot \sum_{k=1}^{\infty} \frac{z^{2k-1}}{n^{2k}}.$$

ii) Since $|z| > 1$, we also have $\left|z^2\right| > 1$. With the same computation as in i) and the modified geometric series (3) we obtain

$$\frac{1}{z-1} + \frac{1}{z+1} = -2z \cdot \frac{1}{1-z^2} = 2z \cdot \sum_{k=1}^{\infty} \frac{1}{z^{2k}} = 2 \cdot \sum_{k=1}^{\infty} \frac{1}{z^{2k-1}}.$$

iii) From the lecture we know that

$$
\begin{aligned}
\pi \cdot \cot(\pi z) &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) \\
&= \frac{1}{z} + \left(\frac{1}{z-1} + \frac{1}{z+1} \right) + \sum_{n=2}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right)
\end{aligned}
$$

For the first bracket we can use ii) and for the remaining sum we can use i) to obtain

$$
\pi \cdot \cot(\pi z) = \frac{1}{z} + 2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{z} \right)^{2k-1} - 2 \cdot \sum_{k=1}^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{n^{2k}} \right) \cdot z^{2k-1}
$$

40 . i) We compute

$$
\begin{aligned}
\frac{1}{2} \left(\cot\left(\frac{\pi z}{2} \right) + \tan\left(\frac{\pi z}{2} \right) \right) &= \frac{1}{2} \left(\frac{\cos(\pi z/2)}{\sin(\pi z/2)} + \frac{\sin(\pi z/2)}{\cos(\pi z/2)} \right) \\
&= \frac{1}{2} \left(\frac{\cos(\pi z/2)^2 + \sin(\pi z/2)^2}{\cos(\pi z/2) \sin(\pi z/2)} \right) \\
&= \frac{1}{2 \cos(\pi z/2) \sin(\pi z/2)}.
\end{aligned}
$$

Using the addition theorem for sin (cf. solution of Exercise 38 i)) this gives

$$
\frac{1}{2} \left(\cot\left(\frac{\pi z}{2} \right) + \tan\left(\frac{\pi z}{2} \right) \right) = \frac{1}{\sin(\pi z)}.
$$

ii) We compute with the addition theorems

$$
\cot\left(\frac{\pi(1-z)}{2} \right) = \frac{\cos(\pi(1-z)/2)}{\sin(\pi(1-z)/2)} = \frac{\cos(\pi/2)\cos(-\pi z/2) - \sin(\pi/2)\sin(-\pi z/2)}{\cos(\pi/2)\sin(-\pi z/2) + \sin(\pi/2)\cos(-\pi z/2)}.
$$

Now with

$$
\begin{aligned}
\cos(\pi/2) &= 0 & \sin(\pi/2) &= 1 \\
\cos(-z) &= \cos(z) & \sin(-z) &= -\sin(z)
\end{aligned}
$$

we obtain

$$
\cot\left(\frac{\pi(1-z)}{2} \right) = \frac{\sin(\pi z/2)}{\cos(\pi z/2)} = \tan\left(\frac{\pi z}{2} \right).
$$

With i), we are done.

iii) The pole set of $\frac{\pi}{\sin(\pi z)}$ is exactly the set of zeros of $\sin(\pi z)$ which is $\mathbb{Z}$ by Exercise 18. At the center $a = 0$ we have

$$\frac{\pi}{\sin(\pi z)} = \frac{\pi}{\pi z + \mathcal{O}(z^3)} = \frac{\pi}{\pi z} \cdot \frac{1}{1+\mathcal{O}(z^2)} = \frac{1}{z} \cdot (1+\mathcal{O}(z)) = \frac{1}{z} + \mathcal{O}(1).$$

Hence the principal part of $\frac{\pi}{\sin(\pi z)}$ at 0 is given by

$$H_0(z) = \frac{1}{z}$$

so the pole is of order 1. Since the function has period 2, the principal part at every even integer is given by

$$H_{2n}(z) = \frac{1}{z-2n}.$$

For the odd integers, we will use the equality

$$\sin(\pi z) = -\sin(\pi(z-1)).$$

Therefore, we find at the center $a = 1$

$$\begin{aligned} \frac{\pi}{\sin(\pi z)} &= -\frac{\pi}{\sin(\pi(z-1)} = -\frac{\pi}{\pi(z-1)+\mathcal{O}((z-1)^3)} = -\frac{1}{z-1} \cdot (1+\mathcal{O}(z-1)) \\ &= -\frac{1}{z-1} + \mathcal{O}(1), \end{aligned}$$

hence the principal part here is

$$H_1(z) = \frac{-1}{z-1}.$$

Therefore, the principal parts for the odd integers are

$$H_{2n+1}(z) = \frac{-1}{z-(2n+1)}$$

and we also obtain poles of order 1 here.
From the lecture we know

$$\begin{aligned} & \pi \cdot \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-(2n)^2} + \sum_{n=1}^{\infty} \frac{2z}{z^2-(2n-1)^2} \\ \Longrightarrow \quad & \pi \cdot \cot(\pi z/2) = \frac{2}{z} + \sum_{n=1}^{\infty} \frac{z}{(z/2)^2-n^2} = \frac{2}{z} + \sum_{n=1}^{\infty} \frac{4z}{z^2-(2n)^2}. \end{aligned}$$

Using Exercise 38 i) and part i), we obtain

$$
\begin{aligned}
\frac{\pi}{\sin(\pi z)} &= \frac{\pi}{2}\left(\cot\left(\frac{\pi z}{2}\right) + \cot\left(\frac{\pi z}{2}\right) - 2\cot(\pi z)\right) \\
&= \pi \cdot \cot\left(\frac{\pi z}{2}\right) - \pi \cdot \cot(\pi z) \\
&= \frac{2}{z} + \sum_{n=1}^{\infty} \frac{4z}{z^2 - (2n)^2} - \left(\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - (2n)^2} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - (2n-1)^2}\right) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - (2n)^2} - \sum_{n=1}^{\infty} \frac{2z}{z^2 - (2n-1)^2} \\
&= \frac{1}{z} + 2z \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2}.
\end{aligned}
$$

Solutions 11

41 . i) By definition, we have

$$E_p(z) = (1-z)\exp\left(\sum_{n=1}^{p} \frac{z^n}{n}\right).$$

For convenience we define the polynomial function

$$f(z) := \sum_{n=1}^{p} \frac{z^n}{n}.$$

Taking the derivative yields

$$\begin{aligned} E_p'(z) &= \exp(f(z)) \cdot (-1 + (1-z)f'(z)) \\ &= \exp(f(z)) \left(-1 + \sum_{n=1}^{p} z^{n-1} - \sum_{n=1}^{p} z^n\right) \\ &= \exp(f(z)) \left(\sum_{n=0}^{p-1} z^n - \sum_{n=0}^{p} z^n\right) \\ &= -z^p \exp(f(z)). \end{aligned}$$

With the power series of the exponential function, we hence obtain

$$E_p'(z) = -z^p \sum_{n=0}^{\infty} \frac{f(z)^n}{n!}.$$

Since $f(z)$ is a polynomial in z with positive coefficients, so is $f(z)^n$ for any $n \in \mathbb{N}$. Hence we can sort the terms and obtain

$$E_p'(z) = -z^p \sum_{n=0}^{\infty} \beta_n z^n = -\sum_{n=0}^{\infty} \beta_n z^{n+p}$$

with $\beta_n \geq 0$ for all n. Now let

$$E_p(z) = \sum_{n=0}^{\infty} (-a_n) z^n$$

be the Taylor series expansion of $E_p(z)$. Then clearly we have $a_0 = -1$ and we can differentiate to obatin

$$E_p'(z) = -\sum_{n=1}^{\infty} n a_n z^{n-1} = -\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n.$$

Comparing the coefficients of these two expansions, we get

$$\begin{aligned} a_n &= 0 && 1 \le n \le p, \\ a_n &= \frac{\beta_{n-p-1}}{n} \ge 0 && n \ge p+1. \end{aligned}$$

Altogether, we get

$$E_p(z) = 1 - \sum_{n=p+1}^{\infty} a_n z^n$$

with $a_n \ge 0$ for all $n \ge p+1$.

ii) Since $E_p(1) = 0$, we get with i) that

$$1 = \sum_{n=p+1}^{\infty} a_n.$$

For $|z| \le 1$, we also have $|z|^n \le |z|^{p+1}$ for all $n \ge p+1$. Hence we obtain

$$\left|E_p(z) - 1\right| = \left|\sum_{n=p+1}^{\infty} a_n z^n\right| \le \sum_{n=p+1}^{\infty} |a_n|\,|z|^n \le \sum_{n=p+1}^{\infty} a_n\,|z|^{p+1} = |z|^{p+1}.$$

42 . Since

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1}} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(-n)^2} = \frac{\pi^2}{6} < \infty,$$

the Weierstrass product theorem implies normal convergence of the product. Therefore, we can reorder the terms to obtain

$$
\begin{aligned}
\prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} E_1\left(\frac{z}{n}\right) &= \prod_{n=1}^{\infty}\left(E_1\left(\frac{z}{n}\right)E_1\left(\frac{z}{-n}\right)\right)\\
&= \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)\exp\left(\frac{z}{n}\right)\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\\
&= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\\
&= \frac{\sin(\pi z)}{\pi z},
\end{aligned}
$$

where the last equality was shown in the lecture.

43 . i) Let D be a divisor on U. Define

$$
\begin{aligned}
Z_1 &:= \{z \in supp(D) \mid D(z) > 0\},\\
Z_2 &:= \{z \in supp(D) \mid D(z) < 0\}.
\end{aligned}
$$

Then clearly $Z_1 \cap Z_2 = \emptyset$ and Z_i is a discrete set as a subset of $supp(D)$ for $i = 1,2$. Since unions of open sets are open and Z_i is discrete, we find that Z_i is also open in $supp(D)$ for $i = 1,2$, therefore the complement

$$Z_j = supp(D) \setminus Z_i, \qquad i,j \in 1,2, i \neq j,$$

is a closed subset of $supp(D)$. Since $supp(D)$ is closed in $\mathbb{C}$, this implies that Z_i is also closed in $\mathbb{C}$ for $i = 1,2$.
Now set

$$
\begin{aligned}
D_1 : U &\longrightarrow \mathbb{Z}\\
z &\longmapsto \begin{cases} D(z) & z \in Z_1,\\ 0 & z \notin Z_1,\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
D_2 : U &\longrightarrow \mathbb{Z}\\
z &\longmapsto \begin{cases} -D(z) & z \in Z_2,\\ 0 & z \notin Z_2.\end{cases}
\end{aligned}
$$

Then $D_i \geq 0$ and $supp(D_i) = Z_i$ for $i = 1,2$.

ii) By part i) it suffices to show the statement for non-negative divisors, since

$$(f/g) = (f) - (g)$$

by a corollary from the lecture.
So let $D \geq 0$ be such a divisor on $\mathbb{C}$. If $supp(D)$ is finite, we can define the polynomial function

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \prod_{a \in supp(D)} (z-a)^{D(a)} \end{aligned}$$

and hence $D = (f)$ by definition.
Now suppose that $supp(D)$ is infinite. First, we want to show that $supp(D)$ has no accumulation point in $\mathbb{C}$. Suppose $a \in \mathbb{C}$ would be such a point, then there exists a sequence of points $(a_n)_{n \in \mathbb{N}}$ in $supp(D)$ which converges to a. Since $supp(D)$ is closed, $a \in supp(D)$ and for any neighbourhood U of a, we find infinitely many $a_n \in U$. This is a contradiction to $supp(D)$ being discrete.
Now let $r > 0$ be arbitrary. Since $supp(D)$ has no accumulation point, there exists only finitely many elements in $supp(D) \cap D_r(0)$, hence the set

$$supp(D) = \bigcup_{n \in \mathbb{N}} (D_n(0) \cap supp(D))$$

is a countable union of finite sets and hence is countable. Moreover, for any $r > 0$ there exists an $a \in supp(D)$ with $|a| > r$. Hence by ordering $supp(D)$ with respect to $|\cdot|$, we get a sequence $(a_\nu)_{\nu \in \mathbb{N}}$ with

$$\lim_{\nu \to \infty} |a_\nu| = \infty.$$

Set $k_\nu := D(a_\nu)$. Then the solution of the Weierstrass problem provides a holomorphic function f which has exactly the zeros a_ν of order k_ν. Hence, by definition

$$(f) = D$$

and we are done.

44 . i) We show the statement by induction on n. For $n = 0$, we clearly have $\Gamma(z) = \Gamma(z)$. Now suppose that the statement is true for all $z \in RH(0)$ and a fixed $n \in \mathbb{N}$. From the lecture, we know that

$$\Gamma(z+1) = z \cdot \Gamma(z)$$

for any $z \in RH(0)$. In particular, we can apply this to $\Gamma(z+n)$ to obtain

$$\Gamma(z+n+1) = (z+n) \cdot \Gamma(z+n).$$

With the induction hypothesis, we obtain

$$
\begin{aligned}
\frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)(z+n)} &= \frac{1}{z(z+1)\cdots(z+n-1)}\cdot\frac{\Gamma(z+n+1)}{(z+n)} \\
&= \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \\
&= \Gamma(z).
\end{aligned}
$$

ii) Let $z \in \mathbb{C}$ be arbitrary and let $n \in \mathbb{N}$ such that $z+n \in RH(0)$. Define

$$
\Gamma(z) := \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}.
$$

By i) this is well-defined and independent of the choice of n. Moreover, $\Gamma(z+n)$ is holomorphic at z by the lecture. Clearly, the denominator is also holomorphic, so the quotient defines a meromorphic function on $\mathbb{C}$. The uniqueness of this extension now follows from the identity theorem.

iii) Let z be a pole of $\Gamma(z)$. Let n be such that $z+n \in RH(0)$, then z is a pole of

$$
\frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}.
$$

Since $\Gamma(z+n)$ is holomorphic, z cannot be a pole of this function and hence must be a zero of $p_n(z) := z(z+1)\cdots(z+n-1)$. The zeros of this function are exactly the numbers $0,\ldots,n-1$. Hence the set of poles of $\Gamma(z)$ is given by P. For the principal part at $-n$, we can use

$$
\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)} = \frac{1}{z+n}\cdot\underbrace{\frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)}}_{=:\psi(z)}.
$$

With

$$
\Psi(-n) = \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} = \frac{1}{(-1)^n\cdot n!} = \frac{(-1)^n}{n!}
$$

we get

$$
\Gamma(z) = \frac{1}{z+n}\cdot\psi(z) = \frac{1}{z+n}\left(\frac{(-1)^n}{n!}+O(z+n)\right) = \frac{(-1)^n}{n!}\cdot\frac{1}{z+n}+O(0),
$$

hence the principal part at $-n$ is given by

$$
H_{-n}(z) = \frac{(-1)^n}{n!}\cdot\frac{1}{z+n}
$$

and all the poles have order 1.

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Solutions 12

45 . Let $a \in \partial A$. We consider the neighbourhood $U := D_{1/2}(a)$ of a and the map

$$\begin{aligned} \rho : U &\longrightarrow \mathbb{R} \\ z := x+iy &\longmapsto |z|^2 - 1 = x^2 + y^2 - 1. \end{aligned}$$

Then ρ is continuously differentiable on U and for any $z \in A \cap U$, we have

$$\rho(z) = |z|^2 - 1 \leq 1 - 1 = 0.$$

On the other hand, if $z \in U \setminus (A \cap U)$, then clearly $|z| > 1$ hence

$$\rho(z) = |z|^2 - 1 > 1 - 1 = 0.$$

Together, we get

$$A \cap U = \{z \in U \mid \rho(z) \leq 0\}.$$

Now let $z \in U$, then $z = x + iy$ corresponds to the vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Then we compute the gradient of ρ at z:

$$grad(\rho)(z) = \begin{pmatrix} (\partial\rho/\partial x)(z) \\ (\partial\rho/\partial y)(z) \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \neq 0,$$

since $0 \notin U$. Therefore, A has a smooth boundary ∂A.

46 . i) Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-a)^n \\ 1/g(z) &= \sum_{n=-1}^{\infty} b_n (z-a)^n = \frac{b_{-1}}{z-a} + \sum_{n=0}^{\infty} b_n (z-a)^n \end{aligned}$$

be the Laurent series expansions of f and $1/g$ at a (note that the expansion for $1/g$ starts at $n = -1$, since g has a zero of order 1 at a). Then the Laurent series expansion of f/g is given by

$$\left(\frac{b_{-1}}{z-a}+\sum_{n=0}^{\infty} b_n(z-a)^n\right)\cdot\sum_{n=0}^{\infty} a_n(z-a)^n=\frac{b_{-1}}{z-a}\cdot a_0+\sum_{n=0}^{\infty} c_n(z-a)^n$$

for some coefficients $c_n \in \mathbb{C}$. Clearly, we have $a_0 = f(a)$. For b_{-1}, we consider the function

$$h(z):=\frac{z-a}{g(z)}=\sum_{n=0}^{\infty} b_{n-1}(z-a)^n.$$

Then

$$z-a=g(z)h(z)=g'(a)h(a)(z-a)+O((z-a)^2),$$

hence

$$b_{-1}=h(a)=\frac{1}{g'(a)}.$$

Therefore,

$$res\left(\frac{f}{g};a\right)=b_{-1}a_0=\frac{f(a)}{g'(a)}.$$

ii) First we see that the holomorphic function $g(z) := 1+z^4$ has the four zeros $\exp\left(\frac{i\pi}{4}\cdot k\right)$, where $k \in \{1,3,5,7\}$. g is a polynomial of degree four and we found four different zeros, the fundamental theorem of algebra implies that each of these zeros has order 1. Hence we are in the situation of part i) and find

$$res\left(1/g;\exp\left(\frac{i\pi}{4}\cdot k\right)\right)=\frac{1}{g'\left(\exp\left(\frac{i\pi}{4}\cdot k\right)\right)}=\frac{1}{4\exp\left(\frac{3i\pi}{4}\cdot k\right)}=\frac{1}{4}\exp\left(-\frac{3i\pi}{4}\cdot k\right).$$

Now we want to integrate $1/g(z)$ along the closed path consisting of the straight line from $-R$ to R and the semicircle described by

$$\begin{aligned}\gamma_R : [0,\pi] &\longrightarrow \mathbb{C}\\ t &\longmapsto Re^{it}.\end{aligned}$$

It is shown in the lecture, that the integral over γ_R vanishes for $R \to \infty$. For $R > 1$, the closed path described above contains the poles of $1/g$ which are contained in the upper half-plane, i.e. for $k = 1,3$. Then the residue theorem yields

$$\begin{aligned}2\pi i\cdot\left(res\left(1/g;\exp\left(\frac{i\pi}{4}\right)\right)+res\left(1/g;\exp\left(\frac{i\pi}{4}\cdot 3\right)\right)\right)&=\int_{-R}^{R}\frac{1}{1+x^4}dx+\int_{\gamma_R}\frac{1}{g(z)}dz\\ &\xrightarrow{R\to\infty}\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx,\end{aligned}$$

therefore, the value of the integral is given by

$$\frac{2\pi i}{4}\exp\left(-\frac{3i\pi}{4}\right)+\frac{2\pi i}{4}\exp\left(-\frac{3i\pi}{4}\cdot 3\right)=\frac{\pi i}{2}\left(-\frac{1+i}{\sqrt{2}}+\frac{1-i}{\sqrt{2}}\right)=\frac{\pi}{\sqrt{2}}$$

47 . i) By Fubini's theorem, we get

$$\int_{-\infty}^{\infty}e^{-x^2}dx\cdot\int_{-\infty}^{\infty}e^{-y^2}dy=\int_{-\infty}^{\infty}e^{-x^2}e^{-y^2}d(x,y)=\int_{-\infty}^{\infty}e^{-x^2}e^{-y^2}dx\wedge dy=\int_{-\infty}^{\infty}e^{-(x^2+y^2)}dx\wedge dy$$

Changing to polar coordinates, we obtain

$$\begin{aligned} x&=r\cos(\varphi) &&\Longrightarrow && dx=\cos(\varphi)dr-r\sin(\varphi)d\varphi,\\ y&=r\sin(\varphi) &&\Longrightarrow && dy=\sin(\varphi)dr+r\cos(\varphi)d\varphi,\end{aligned}$$

hence

$$\begin{aligned} dx\wedge dy&=(\cos(\varphi)dr-r\sin(\varphi)d\varphi)\wedge(\sin(\varphi)dr+r\cos(\varphi)d\varphi)\\ &=\cos(\varphi)\sin(\varphi)dr\wedge dr+r\cos(\varphi)^2dr\wedge d\varphi-r\sin(\varphi)^2d\varphi\wedge dr-r^2\sin(\varphi)\cos(\varphi)d\varphi\wedge d\varphi\\ &=r(\cos(\varphi)^2+\sin(\varphi)^2)dr\wedge d\varphi=rdr\wedge d\varphi.\end{aligned}$$

Therefore, we get

$$\int_{-\infty}^{\infty}e^{-(x^2+y^2)}dx^dy=\int_0^{2\pi}\int_0^{\infty}e^{-r^2}rdr\wedge d\varphi=2\pi\int_0^{\infty}e^{-r^2}rdr.$$

The remaining integral can be computed by substitution with $u=r^2$ (hence $du=2rdr$):

$$\int_0^{\infty}e^{-r^2}rdr=\frac{1}{2}\int_0^{\infty}e^{-u}du=\frac{1}{2}[-e^{-u}]_0^{\infty}=\frac{1}{2}(-0-(-1))=\frac{1}{2}.$$

Altogether, we get

$$\int_{-\infty}^{\infty}e^{-x^2}dx\cdot\int_{-\infty}^{\infty}e^{-y^2}dy=\pi,$$

and hence

$$\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}.$$

ii) The integration along the closed path from 0.1 splits into three integrals:

$$\int_{\gamma}e^{-z^2}dz=\int_0^R e^{-r^2}dr+\int_{\gamma_R}e^{-z^2}dz+\int_R^0\exp\left(-\left(re^{i\pi/4}\right)^2\right)\cdot e^{i\pi/4}dr,$$

where γ_R describes the arc from R to $Re^{i\pi/4}$. We start approximating the second integral by

$$\left|\int_{\gamma_R} e^{-z^2}dz\right| = \left|\int_0^{\pi/4} \exp(-(Re^{i\varphi})^2)i\cdot Re^{i\varphi}d\varphi\right| \le \int_0^{\pi/4}\left|\exp(-R^2e^{2i\varphi})i\cdot Re^{i\varphi}\right|d\varphi$$
$$= \int_0^{\pi/4} R\cdot\exp(-R^2\cos(2\varphi))d\varphi.$$

Similarly to the proof of Prop. 6.12, we can approximate $\cos(2\varphi) \ge 1-\frac{4}{\pi}\varphi$ for $\varphi \in [0,\pi/4]$, hence we get

$$\left|\int_{\gamma_R} e^{-z^2}dz\right| \le R\int_0^{\pi/4} e^{-R^2}\exp\left(\frac{4R^2\varphi}{\pi}\right)d\varphi = Re^{-R^2}\frac{\pi}{4R^2}(e^{R^2}-1) < \frac{\pi}{4R} \xrightarrow{R\to\infty} 0.$$

For the third integral, we compute

$$e^{i\pi/4} = \cos(\pi/4)+i\sin(\pi/4) = \frac{1+i}{\sqrt{2}}$$

and hence

$$\begin{aligned}
\int_R^0 \exp\left(-\left(re^{i\pi/4}\right)^2\right)\cdot e^{i\pi/4}dr &= -\frac{1+i}{\sqrt{2}}\int_0^R \exp\left(-r^2e^{i\pi/2}\right)dr \\
&= -\frac{1+i}{\sqrt{2}}\int_0^R \exp(-ir^2)dr \\
&= -\frac{1+i}{\sqrt{2}}\int_0^R \cos(-r^2)+i\sin(-r^2)dr \\
&= -\frac{1}{\sqrt{2}}\left(\int_0^R\cos(r^2)dr+\int_0^R\sin(r^2)dr\right)-\frac{i}{\sqrt{2}}\left(\int_0^R\cos(r^2)dr-\int_0^R\sin(r^2)dr\right).
\end{aligned}$$

Since the function e^{-z^2} is holomorphic in the complex plane, Cauchy's integral theorem implies

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\left(\int_0^R\cos(r^2)dr+\int_0^R\sin(r^2)dr\right)+\frac{i}{\sqrt{2}}\left(\int_0^R\cos(r^2)dr-\int_0^R\sin(r^2)dr\right) \\
&\quad = \int_0^R e^{-r^2}dr = \frac{1}{2}\int_{-R}^R e^{-r^2}dr \xrightarrow{R\to\infty} \frac{\sqrt{\pi}}{2},
\end{aligned}$$

where the value in the limit was computed in part i). Taking the limit on the left hand side and comparing the real and imaginary parts, we obtain

$$\int_0^\infty \cos(r^2)dr = \int_0^\infty \sin(r^2)dr = \frac{\sqrt{\pi}}{2\sqrt{2}} = \sqrt{\frac{\pi}{8}}.$$

48 . i) We consider the Weierstrass elementary factors E_1 and the canonical product for the sequence $(a_\nu)_{\nu\in\mathbb{N}_{\geq 1}}$ with $a_\nu := -\nu$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$, the Weierstrass product theorem gives the function

$$f(z) := \prod_{n=1}^{\infty} E_1\left(\frac{z}{-n}\right) = \prod_{n=1}^{\infty} \frac{1+(z/n)}{e^{z/n}},$$

which is holomorphic on $\mathbb{C}$ and has exactly the zero $-n$, $n \geq 1$, of order 1. Multyplying with $z \cdot e^{-Cz}$ preserves the holomorphicity but adds a zero of order 1 at $z = 0$. Then we see that the resulting function satisfies

$$z \cdot e^{Cz} \cdot f(z) = \frac{1}{\gamma(z)}.$$

Hence, $\gamma(z)$ is meromorphic on $\mathbb{C}$ and has the pole set

$$P = \{-n \mid n \in \mathbb{N}\}.$$

Moreover, we find that

$$\gamma(z) = \frac{1}{z} \cdot \lim_{N\to\infty} \exp\left(-z \cdot \sum_{n=1}^{N}\left(\frac{1}{n}\right) - \ln(N)\right) \prod_{n=1}^{N} \frac{e^{z/n}}{1+(z/n)}.$$

We compute

$$\begin{aligned}
\exp\left(-z \cdot \sum_{n=1}^{N}\left(\frac{1}{n}\right) - \ln(N)\right) \prod_{n=1}^{N} \frac{e^{z/n}}{1+(z/n)} &= e^{z\ln(N)} \exp\left(\sum_{n=1}^{N} \frac{-z}{n}\right) \prod_{n=1}^{N} \frac{e^{z/n}}{1+(z/n)} \\
&= N^z \prod_{n=1}^{N} \frac{e^{-z/n}e^{z/n}}{1+(z/n)} \\
&= N^z \prod_{n=1}^{N} \frac{n}{n+z} \\
&= \frac{N^z N!}{(z+1)(z+2)\cdots(z+N)}.
\end{aligned}$$

Inserting this in the above equation, we obtain

$$\gamma(z) = \frac{1}{z} \lim_{N\to\infty} \frac{N^z N!}{(z+1)(z+2)\cdots(z+N)}.$$

ii) We compute

$$
\begin{aligned}
\gamma(z+1) &= \frac{1}{z+1} \lim_{N\to\infty} \frac{N \cdot N^z N!}{(z+2)(z+3)\cdots(z+N+1)} \\
&= z \cdot \frac{1}{z} \lim_{N\to\infty} \frac{(N+1)^z (N+1)!}{(z+1)(z+2)\cdots(z+N+1)} \cdot \left(\frac{N}{N+1}\right)^{z+1}.
\end{aligned}
$$

Since $\lim_{N\to\infty} \frac{N}{N+1} = 1$, we can rewrite this as

$$
\begin{aligned}
\gamma(z+1) &= z \cdot \frac{1}{z} \lim_{N\to\infty} \frac{(N+1)^z (N+1)!}{(z+1)(z+2)\cdots(z+N+1)} \cdot \left(\lim_{N\to\infty} \frac{N}{N+1}\right)^{z+1} \\
&= z \cdot \gamma(z) \cdot 1^{z+1} \\
&= z \cdot \gamma(z).
\end{aligned}
$$

Now we see that

$$\gamma(z) = \frac{1}{z} \cdot (1 + O(z)) = \frac{1}{z} + O(0),$$

Hence it has principal part $\frac{1}{z}$ at center $a = 0$. This coincides with the principal part of $\Gamma(z)$ at center $a = 0$ and since both functions satisfy the same functional equation relating the principal parts at $-n$ and $-n+1$, we conclude that $\gamma(z)$ and $\Gamma(z)$ have the same principal part at any pole.

iii) We see that $|z+n| \geq Re(z+n)$ for any $n \geq 0$ and

$$|N^z| = \left|e^{z \cdot \log N}\right| = e^{Re(z) \cdot \log N} = N^{Re(z)}.$$

Hence

$$
\begin{aligned}
\left|\frac{N^z \cdot N!}{z(z+1)\cdots(z+N)}\right| &= \frac{|N^z| \cdot N!}{|z| \cdot |z+1| \cdots |z+N|} \\
&\leq \frac{N^{Re(z)} \cdot N!}{Re(z) \cdot (Re(z)+1) \cdots (Re(z)+N)}
\end{aligned}
$$

and therefore

$$|\gamma(z)| \leq \gamma(Re(z)).$$

The function $\gamma(x)$ on the real interval $[1,2]$ is continuous and hence bounded, therefore, $\gamma(z)$ is bounded on $B_{1,2}$. Similarly, we know from the proof of Prop. 5.25 that $|\Gamma(z)| \leq \Gamma(Re(z))$ for $Re(z) > 1$, which is also bounded on $B_{1,2}$ by continuity on the real interval $[1,2]$. Therefore, the difference $g(z) = \Gamma(z) - \gamma$ is bounded on $B_{1,2}$. Moreover, we get

$$\frac{g(z+1)}{z} = \frac{\Gamma(z+1) - \gamma(z+1)}{z} = \frac{\Gamma(z+1)}{z} - \frac{\gamma(z+1)}{z} = \Gamma(z) - \gamma(z) = g(z).$$

Considering the entire function g on $D_1(0) \cap B_{0,1}$, it is bounded as a holomorphic function on a compact set. For $z \in B_{0,1} \setminus D_1(0)$ we get

$$|g(z)| = \left| \frac{g(z+1)}{z} \right| = \frac{|g(z+1)|}{|z|} \leq |g(z+1)|.$$

Since g is bounded on $B_{1,2}$, it is therefore also bounded on $B_{0,1}$.
Now let $z \in \mathbb{C}$ be arbitrary. We compute

$$\begin{aligned} S(z+1) &= g(z+1) \cdot g(1-(z+1)) = g(z) \cdot z \cdot g(-z) = -g(z) \cdot (-z) \cdot g(-z) \\ &= -g(z)g(1-z) = -S(z). \end{aligned}$$

For $z \in B_{1,2}$ we have $1-z \in B_{0,1}$, hence $g(z)$ and $g(1-z)$ are bounded, so $S(z)$ is bounded on $B_{1,2}$. By the functional equation we also obtain that $S(z)$ is bounded in $B_{0,1}$. Moreover, we get that

$$S(z+2) = -S(z+1) = S(z),$$

so $S(z)$ has period 2.

iv) Part iii) implies that $S(z)$ is bounded on $\mathbb{C}$. Since it is also holomorphic, the theorem of Liouville implies that $S(z)$ is constant. The constant is given by

$$S(1) = g(1) \cdot g(0) = 0 \cdot g(0) \cdot g(0) = 0,$$

hence we can use Exercise 17 to conclude that if

$$g(z) \cdot g(1-z) = 0$$

for all $z \in \mathbb{C}$, then $g(z)$ or $g(1-z)$ is the zero function. In any case, $g(z) = 0$ and hence $\Gamma(z) = \gamma(z)$.

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Solutions 13

49 . i) Let $\phi(z) := -z^8 - 1$ and $f(z) := -3z^2$. Then we get for $|z| = 1$:

$$|\phi(z)| = \left|z^8 + 1\right| \le |z|^8 + 1 = 2 < 3 = \left|3z^2\right| = |f(z)|.$$

Applying the theorem of Rouché, we get that $f(z)$ has the same number of zeros in $D_1(0)$ as

$$(f - \phi)(z) = z^8 - 3z^2 + 1 = p(z).$$

Clearly, f has two zeros in the unit disk. By the fundamental theorem of algebra, we find that $p(z)$ has eight zeros in $\mathbb{C}$. Altogether, we see that $p(z)$ has six zeros with $|z| > 1$.

ii) For $\phi(z) := 7z - 2$ and $f(z) := 3z^4$, we get for $|z| = 3/2$:

$$|\varphi(z)| = |7z - 2| \le 7|z| + 2 = 25/2 < 3^5/2^4 = 3\left|z^4\right| = |f(z)|.$$

Hence, the theorem of Rouché implies that

$$q(z) = (f - \phi)(z)$$

has four zeros in $D_{3/2}(0)$. Now defining $\phi(z) := -3z^4 - 2$ and $f(z) = -7z$ we see that for $|z| = 1$, we get that

$$q(z) = (f - \phi)(z)$$

has one zero in $D_1(0)$. Hence $q(z)$ has three zeros for $1 < |z| < 3/2$.

50 . Let G be a star-like domain with respect to a and let $\gamma : [0,1] \to G$ be a closed path. Without loss of generality, we can assume that $\gamma(0) = \gamma(1) = a$. Since $\gamma(t) \in G$ for any t and G is star-like, the line

$$\{(1-s)\gamma(t) + sa \mid s \in [0,1]\}$$

is contained in G. Hence, we can define

$$\begin{aligned}\Phi : [0,1]\times[0,1] &\longrightarrow G\\ (t,s) &\longmapsto (1-s)\gamma(t)+sa.\end{aligned}$$

Clearly, Φ is continuous. Moreover, we have $\Phi(-,0)=\gamma$ and $\Phi(-,1)=a$. Now for any $s\in]0,1[$, we have that

$$\gamma_s(0)=(1-s)\gamma(0)+sa=(1-s)\gamma(1)+sa=\gamma_s(1),$$

hence γ_s is a closed path. Altogether, Φ is a homotopy and γ and the constant path a are homotopic. Since γ was arbitrary, G is simply connected.

51 . i) Let $z\in G$ and let γ_z be any path from z_0 to z. Note that any continuous function on a compact set can be approximated uniformly by polynomials according to Stone-Weierstrass. In particular, polynomials are continuously differentiable, so any path is homotopic to a continuously differential path. Hence, we can define

$$F(z):=\int_{\gamma_z} f(\zeta)d\zeta.$$

This is well-defined on G, since the integral coincides by assumption for any two paths from z_0 to z. Now fix $z_1\in G$ and let U be a neighbourhood of z_1. Since f is holomorphic, there exists a primitive F_1 of f on U. Then for any $z\in U$, we have

$$F(z)=\int_{\gamma_z} f(\zeta)d\zeta=\int_{\gamma_{z_1}} f(\zeta)d\zeta+\int_{z_1}^{z} f(\zeta)d\zeta=\int_{\gamma_{z_1}} f(\zeta)d\zeta+F_1(z)-F_1(z_1),$$

where the integral from z to z_1 does not depend on the path, since the other to integrals are independent of the paths by assumption. We see that the first and the third summand are constant and since F_1 is a primitive of f, it is holomorphic on U. Altogether, we see that F is holomorphic on U. By definition $F(z)=0$ for any $z\in U$, hence we obtain

$$0=F'(z)=F_1'(z)=f(z)$$

for any $z\in U$. Since $z_1\in G$ was arbitrary, we conclude $f=0$ on G.

ii) Suppose $[\gamma]=[\gamma']$. Since f is holomorphic and γ and γ' are homotopic, we can apply Theorem 7.5 from the lecture to obtain

$$\int_{\gamma} f(z)dz=\int_{\gamma'} f(z)dz,$$

hence T_f is indeed well-defined.

iii) Let $f:G\longrightarrow \mathbb{C}$ be a holomorphic function and

$$\begin{aligned}\gamma_1 : [0,1] &\longrightarrow G\\ t &\longmapsto e^{2\pi i t}.\end{aligned}$$

Then the residue theorem yields

$$T_f([\gamma_1]) = \int_{\gamma_1} f(z)dz = \int_{\partial D_1(0)} f(z)dz = 2\pi i \cdot res(f;0).$$

Now let $a \in \mathbb{C}$ be arbitrary. Define

$$\begin{aligned}f : G &\longrightarrow \mathbb{C}\\ z &\longmapsto \frac{a}{2\pi i z}.\end{aligned}$$

Then $res(f;0) = \frac{a}{2\pi i}$ and hence

$$T_f(\gamma_1) = a.$$

Therefore, $\mathbb{C} = \bigcup_{f \in \mathscr{O}(G)} T_f(\pi_1(G,x_0))$.
Let $[\gamma_n] \in \pi_1(G,x_0)$ be given by $\gamma_n(t) = e^{n \cdot 2\pi i t}$. Then the residue theorem implies for any $f \in \mathscr{O}(G)$

$$T_f([\gamma_n]) = \int_{\gamma_n} f(z)dz = n \cdot \int_{\gamma_1} f(z)dz = n \cdot 2\pi i \cdot res(f;0),$$

where the second equality is proven in the tutorials. Hence $T_f = 0$ if and only if $res(f;0) = 0$.

52 . Let $z_0 \in G$ be arbitrary. Since holomorphicity is a local property, it suffices to show that f is holomorphic on a disk $D_r(z_0)$ for $r > 0$ such that $D_r(z_0) \subseteq G$. Without loss of generality, we can assume $z_0 = 0$ (if $f(z - z_0)$ is holomorphic, then so is $f(z)$). So let $G = D_r(0)$. For $z = x + iy \in G$, define $\gamma_z : [0, x+y] \to G$ by

$$\gamma_z(t) = \begin{cases} t & 0 \le t \le x, \\ x + (t-x)i & x \le t \le x+y. \end{cases}$$

This is a continuous path which describes two sides of a rectangle. Note that the integral over a path is independent of the parametrization as long as the orientation of the path is preserved, therefore we can choose γ_z to be a function defined on $[0, x+y]$ instead of $[0,1]$. Hence we can define

$$F(z) := \int_{\gamma_z} f(\zeta)d\zeta.$$

We also define

$$\eta_z(t) = \begin{cases} it & 0 \le t \le y, \\ iy + (t-y) & y \le t \le y+x. \end{cases}$$

Then the assumption on f implies

$$F(z) = \int_{\eta_z} f(\zeta) d\zeta.$$

We compute

$$\begin{aligned} \frac{\partial F}{\partial y}(z) = frac\partial\partial y \int_{\gamma_z} f(z)dz = \frac{\partial}{\partial y}\left(\int_0^x f(t)dt + \int_x^{x+y} f(x+(t-x)i)idt\right) \\ = \frac{\partial}{\partial y}\int_0^x f(t)dt + i\frac{\partial}{\partial y}\int_0^y f(x+it)dt = if(x+iy) \\ = if(z) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F}{\partial x}(z) = frac\partial\partial x \int_{\eta_z} f(z)dz = \frac{\partial}{\partial x}\left(\int_0^y f(it)idt + \int_y^{y+x} f(iy+(t-y))dt\right) \\ = i\frac{\partial}{\partial x}\int_0^y f(it)dt + \frac{\partial}{\partial x}\int_0^x f(iy+t)dt = f(x+iy) \\ = f(z). \end{aligned}$$

By the assumption on f, the partial derivatives are continuous. Decomposing the above equations into the real and imaginary parts, we see that F satisfies the Cauchy-Riemann differential equations. Hence F is holomorphic and also its derivative

$$F'(z) = \frac{\partial F}{\partial x}(z) = f(z).$$

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COMPLEX ANALYSIS

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Solutions 12

53 . If $g = 0$ is the zero function, then the assumption implies that $f(z) = 0$ for any $z \in \mathbb{C}$, hence $f = g$. Now suppose that $g \neq 0$ and let N be the set of zeros of g and $G := \mathbb{C} \setminus N$. Then we can define

$$\begin{aligned} h : G &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{f(z)}{g(z)}. \end{aligned}$$

This is a holomorphic function on G satisfying

$$|h(z)| = \frac{|f(z)|}{|g(z)|} \leq 1.$$

Since h is a quotient of holomorphic functions, it is meromorphic on $\mathbb{C}$ with isolated singularities at the points in N. Let $a \in N$ and let $r > 0$ be such that $D_r^*(a) \cap N = \emptyset$. Then h is holomorphic and bounded on $D_r^*(a)$ by 1 (note that this bound does not depend on the chosen singularity a), hence it has a removable singularity by Riemann's theorem. Extending h to an entire function $\widehat{h}$, we find that $\widehat{h}$ is bounded on $\mathbb{C}$ by 1, hence it is constant by the theorem of Liouville. Therefore, $h(z) = \lambda$ with $|\lambda| \leq 1$.

54 . See Exercise 32.

56 . The function f has singularities at $a = 0$ and at each zero of $\cos(1/z)$. The zeros of $\cos(1/z)$ satisfy

$$\frac{1}{z} = \frac{\pi}{2} + k \cdot \pi \qquad \Longleftrightarrow \qquad z = \frac{1}{\pi/2 + k \cdot \pi}$$

for $k \in \mathbb{Z}$. For any $r > 0$, there exists a $k \in \mathbb{Z}$ such that $\frac{1}{\pi/2+k\cdot\pi} < r$, hence any neighbourhood of 0 contains another singularity of f. Therefore, $a = 0$ is no isolated singularity of f.