DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
WINTER TERM 2021/22

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Problems 01

1. Give a counterexample which shows: A function of two real variables with continuous partial derivatives of first order does not necessarily have partial derivatives of arbitrary order.

2. Give a counterexample which shows: Not each function of one real variable, which has derivatives of arbitrary order, expands into a convergent power series.

3. Show for a domain $G \subset \mathbb{C}^n$ and its complex vector space of holomorphic functions the equivalence:

$$dim_{\mathbb{C}}\mathscr{O}(G) < \infty \iff dim_{\mathbb{C}}\mathscr{O}(G) = 1 \iff n = 0$$

4. For an open set $U \subset \mathbb{C}^n$ denote by $\mathscr{E}(U)$ the ring of smooth functions on U, i.e. functions with partial derivatives of arbitrary order. Choose an exhaustion $(U_\nu)_{\nu\in\mathbb{N}}$ of U by relatively compact open subsets.

i) For each $\nu, k \in \mathbb{N}$ define the seminorm

$$p_{\nu,k} : \mathscr{E}(U) \to \mathbb{R}$$

as

$$p_{\nu,k}(f) := sup \left\{ \left| \frac{D^{|j|} f}{\partial z_1^{j_1} \ldots \partial z_n^{j_n}}(z) \right| : \ j = (j_1, \ldots j_n) \text{ with } |j| \le k \text{ and } z \in U_\nu \right\}$$

Show: The family

$$(p_{\nu,k})_{\nu,k\in\mathbb{N}}$$

defines a Fréchet topology on the complex vector space $\mathscr{E}(U)$.

ii) Show: The inclusion $\mathscr{O}(U) \subset \mathscr{E}(U)$ is an inclusion of Fréchet spaces.

Discussion: Thursday, 28.10.2021, 12.15 pm.

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Problems 02

5. i) Prove Hartogs' "Kugelsatz" in its form for polydiscs: Show for two concentric polydiscs

$$\Delta_1 \subset\subset \Delta_2 \subset \mathbb{C}^n,\ n \geq 2:$$

Each holomorphic function

$$f \in \mathscr{O}(\Delta_2 \setminus \overline{\Delta}_1)$$

extends uniquely to a holomorphic function $\tilde{f} \in \mathscr{O}(\Delta_2)$.

ii) Consider an open set $U \subset \mathbb{C}^n,\ n \geq 2$, a point $a \in U$, and a holomorphic function

$$f \in \mathscr{O}(U \setminus \{a\}).$$

The point a is named an *isolated singularity* of f if f does not extend to a holomorphic function $\tilde{f} \in \mathscr{O}(U)$.

Show: No holomorphic function $f \in \mathscr{O}(U \setminus \{a\})$ has an isolated singularity in a.

iii) Consider an open set $U \subset \mathbb{C}^n,\ n \geq 2$.

Show: No holomorphic function $f \in \mathscr{O}(U)$ has an isolated zero $a \in U$.

6. Show: Each Fréchet space is metrizable with the distance $d(f,g)$ introduced in the lecture.

7. For an open set $U \subset \mathbb{C}^n$ show: The topology of compact convergence on the vector space $\mathscr{C}(U)$ is Hausdorff.

Hint. You may use the following fact: For each $f \in \mathscr{C}(U)$ with $f \neq 0$ exists p_ν with $p_\nu(f) \neq 0$.

8. Consider an open set $U \subset \mathbb{C}^n$. A subset $A \subset U$ is an *analytic subset of U* if for each point $x \in U$ exist an open neighbourhood $V \subset U$ of x and finitely many holomorphic functions $f_1, ..., f_k \in \mathscr{O}(V)$ satisfying

$$A \cap V = \{z \in V : \ f_1(z) = ... = f_k(z) = 0\}$$

For a domain $G \subset \mathbb{C}^n$ and an analytic subset $A \subset G$ show:

If a point $a \in A$ has an open neighbourhood $V \subset G$ with $V \subset A$ then

$$A = G.$$

Discussion: Thursday, 4.11.2021, 12.15 pm.

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Problems 03

9. Consider an open set $U \subset \mathbb{C}^n$, an analytic subset $A \subset U$ and a point $x \in A$. Then A has in x a *Remmert-Stein codimension* $\geq r$, expressed as

$$codim_x\, A \geq r,$$

if there exists a r-dimensional plane $E \subset \mathbb{C}^n$ passing through x such that the point $x \in A$ is an isolated point in $A \cap E$. One defines

$$codim_x\, A = r \text{ and } dim_x\, A := n - r$$

if the pair (A, x) satisfies

$$codim_x\, A \geq r \text{ but not } codim_x\, A \geq r + 1.$$

For a domain $G \subset \mathbb{C}^n$ and an analytic subset $A \subset G,\ A \neq G$, show for all points $x \in A$:

$$codim_x\, A \geq 1$$

10. Consider an open set $U \subset \mathbb{C}^n$ and an analytic set $A \subset U$.

Show: i) The presheaf

$$V \mapsto \mathscr{I}_A(V) := \{f \in \mathscr{O}_U(V) :\ f|A \cap V = 0\},\ V \subset U \text{ open},$$

defines after sheafification a sheaf $\mathscr{I}_A$ of rings on U.

ii) The sheaf $\mathscr{I}_A \subset \mathscr{O}_U$ is a subsheaf of ideals (*Ideal sheaf of* A).

iii) For each $k \in \mathbb{N}^*$ the sheaf

$$\mathscr{I}_A^k \text{ (k-times product)}$$

is a subsheaf of ideals of $\mathscr{O}(U)$. And the restriction satisfies for each open subset $W \subset (U \setminus A)$

$$\mathscr{I}_A^k | W = \mathscr{O}_U | W$$

11. i) Show: The singleton

$$A := \{0 \in \mathbb{C}\}$$

is an analytic set in $\mathbb{C}$.

ii) Denote by

$$R := \mathbb{C}\{z\}$$

the ring of convergent power series in one complex variable and by

$$\mathfrak{m} := < z > \subset R$$

the ideal generated by $z \in R$. Show:

The ideal $\mathfrak{m} \subset R$ is the unique maximal ideal of R.

12. Consider the analytic set A from Problem 11. Describe the ideal sheaf

$$\mathscr{I}_A \subset \mathscr{O}_{\mathbb{C}} \text{ and the quotient sheaves } \mathscr{O}_{\mathbb{C}}/\mathscr{I}_A^k,\ k \in \mathbb{N}^*.$$

Discussion: Thursday, 11.11.2021, 12.15 pm.

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Problems 04

13. Consider a presheaf of Abelian groups $\mathscr{F}$ on a topological space X. For the presheaf $\hat{\hat{\mathscr{F}}}$ show:

i) $\hat{\hat{\mathscr{F}}}$ is a sheaf.

ii) For each $x \in X$ the induced morphism on the level of stalks

$$\mathscr{F}_x \to \hat{\hat{\mathscr{F}}}_x$$

is an isomorphism.

14. Consider a morphism of sheaves on a topological space X

$$f : \mathscr{F} \to \mathscr{G}.$$

i) Show for each open $U \subset X$: If for each $x \in U$ the induced map on stalks

$$f_x : \mathscr{F}_x \to \mathscr{G}_x$$

is injective, then the map of sections

$$f_U : \mathscr{F}(U) \to \mathscr{G}(U)$$

is injective.

ii) Construct an example with an open $U \subset X$ and f_x surjective for all $x \in U$, but

$$f_U : \mathscr{F}(U) \to \mathscr{G}(U)$$

not surjective.

15. Derive as a consequence of Weierstrass' product theorem from complex analysis of one variable

$$\mathscr{M}(\mathbb{C}) = Q(\mathscr{O}(\mathbb{C})) \text{ (quotient field).}$$

16. For $r \in \mathbb{R}^*_+$ denote by

$$\Delta(r) := \{z \in \mathbb{C} : \ |z| < r\}$$

the 1-dimensional disc with radius r, and for $k \in \mathbb{N}$ by

$$\Delta^k(r) := \Delta(r) \times ... \times \Delta(r) \subset \mathbb{C}^k$$

the k-dimensional polydisc with each component of its polyradius $= r$. Consider an open set $U \subset \mathbb{C}^n$ and an analytic set $A \subset U$ with

$$codim_a\, A \geq 1$$

for all $a \in A$.

i) Show that w.l.o.g the geometric situation around a given point $a \in A$ is as follows, see Figure 0.1: There exists $r \in \mathbb{R}^*_+$ with

- $$a = 0 \in \Delta^n(r) \subset U$$
- $$A \cap \Delta^n(r) = \{z \in \Delta^n(r) : \ f_1(z) = ... = f_m(z) = 0\}$$
  for suitable $f_1, ..., f_m \in \mathcal{O}(\Delta^n(r))$
- The projection
  $$p : \Delta^n(r) \to \Delta^{n-1}(r), \ z = (z_1, ..., z_n) \mapsto z' := (z_1, ..., z_{n-1}),$$
  satisfies
  $$p^{-1}(0) \cap A = \{a\}$$
- For given $0 < \rho < r$ exists $0 < \varepsilon < r$ such that
  $$R := \left\{z = (z', z_n) : \ z' \in \Delta^{n-1}(\varepsilon), \ |z_n| = \rho\right\}$$
  satisfies
  $$R \subset U \setminus A$$
- For each $z' \in \Delta^{n-1}(\varepsilon)$ the 1-dimensional fibre $p^{-1}(z')$ intersects A in a discrete set.

ii) For a bounded holomorphic function

$$f \in \mathcal{O}(\Delta^n(r) \setminus A)$$

show for each $z = (z', z_n) \in \Delta^{n-1}(\varepsilon) \times \Delta(\rho)$:

$$\tilde{f}(z) := \frac{1}{2\pi i} \cdot \int_{|\zeta|=\rho} \frac{f(z',\zeta)}{\zeta - z_n} \, d\zeta$$

is well-defined. The resulting function $\tilde{f}$ is holomorphic on $\Delta^{n-1}(\varepsilon) \times \Delta(\rho)$.

iii) Show:

$$\overline{U \setminus A} = U,$$

and each holomorphic function $f \in \mathscr{O}(U \setminus A)$, which is bounded in the neighbourhood of each point $a \in A$, extends uniquely to a holomorphic function on U (*Riemann's first theorem on removable singularities*).

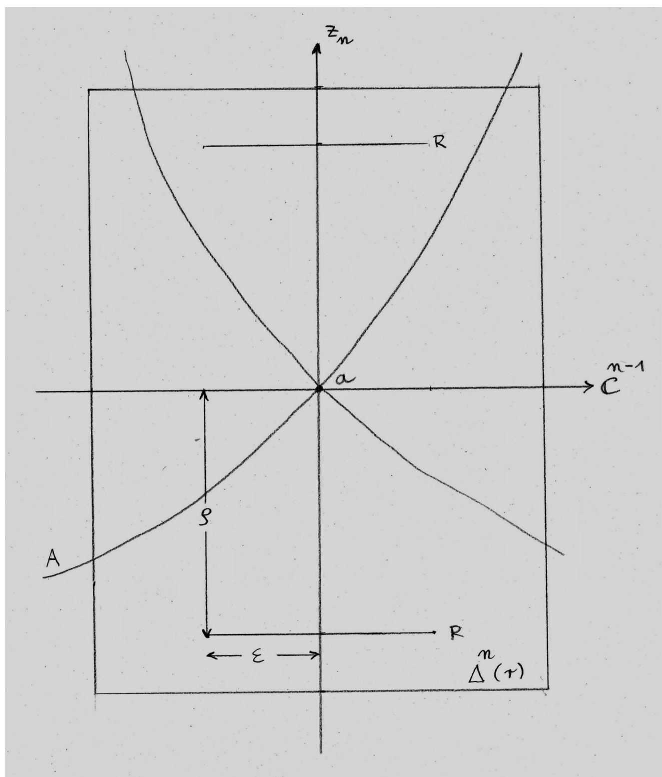

Fig. 0.1 Analytic set A in a neighbourhood of a

Discussion: Thursday, 18.11.2021, 12.15 pm.

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Problems 05

17. For a topological space X and a presheaf $\mathscr{F}$ of Abelian groups on X show the equivalence of the following properties:

- The presheaf $\mathscr{F}$ is a sheaf.

- For each open $U \subset X$ and each open covering $\mathscr{U} = (U_i)_{i \in I}$ of U the following sequence of morphisms of Abelian groups is exact:

$$0 \to \mathscr{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathscr{F}(U_i) \xrightarrow{\beta} \prod_{j,k \in I} \mathscr{F}(U_{jk})$$

  with

$$\alpha(\phi) := (\phi|U_i)_i \text{ and } \beta((\phi_i)_i) := (\phi_j|U_{jk} - \phi_k|U_{jk})_{j,k},\ U_{jk} := U_j \cap U_k.$$

18. A presheaf $\mathscr{F}$ on a topological space X *satisfies the identity theorem* if for each domain $G \subset X$ holds: Two sections $f,\ g \in \mathscr{F}(G)$ are equal if there exists a point $x \in G$ with equal germs

$$f_x = g_x \in \mathscr{F}_x$$

Show: On a complex manifold X both sheaves $\mathscr{O}_X$ and $\mathscr{M}_X$ satisfy the identity theorem.

19. For a presheaf $\mathscr{F}$ of Abelian groups on a topological space X define a topological space $|\mathscr{F}|$, the *étale space* of $\mathscr{F}$, as follows:

- Consider the disjoint union of stalks

$$|\mathscr{F}| := \dot{\bigcup}_{x \in X} \mathscr{F}_x$$

- For each open set $U \subset X$ and for each $f \in \mathscr{F}(U)$ define the set of germs

$$[U,\ f] := \{f_x \in \mathscr{F}_x :\ x \in U\} \subset |\mathscr{F}|.$$

Show: i) The set $\mathscr{B}$ of all sets

$$[U,\ f],\ U \subset X \text{ open},$$

is the base of a topology on $|\mathscr{F}|$.

ii) The projection

$$p : |\mathscr{F}| \to X,\ f_x \in \mathscr{F}_x \mapsto x \in X,$$

is a local homeomorphism.

Hint: Show that p is continuous and open with bijective
restrictions $p|U : [U,f] \to U$.

20. Consider a topological space X and a presheaf $\mathscr{F}$ on X. A continuous map

$$s : U \to |\mathscr{F}| \text{ with } p \circ s = id_U$$

on an open set $U \subset X$ is named a *section* on U against p. Show:

i) The family

$$\mathscr{F}^{sh}(U) := \{s : U \to |\mathscr{F}| : \ s \text{ section}\},\ U \subset X \text{open},$$

with the canonical restriction of maps is a sheaf.

ii) Construct an isomorphism of sheaves on X

$$\mathscr{F}^{sh} \xrightarrow{\simeq} \hat{\mathscr{F}}$$

Discussion: Thursday, 25.11.2021, 12.15 pm.

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Problems 06

21. Consider the exact sequence of presheaf morphisms

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

on a topological space X. For an open covering $\mathscr{U}$ of X show the exactness of the following segment of the induced sequence

$$H^q(\mathscr{U},\mathscr{F}) \xrightarrow{\alpha_q} H^q(\mathscr{U},\mathscr{G}) \xrightarrow{\beta_q} H^q(\mathscr{U},\mathscr{H}),\ q \in \mathbb{N}.$$

22. Consider a complex manifold X and on X the exact sequence of sheaf morphisms

$$0 \to \mathscr{O} \xrightarrow{j} \mathscr{M} \to \mathscr{D} \to 0$$

with the canonical injection j and the sheaf (*divisor sheaf*)

$$\mathscr{D} := coker\ [\mathscr{O} \xrightarrow{j} \mathscr{M}]$$

Show: i) Each additive Cousin distribution c on X defines a section

$$div(c) \in \mathscr{D}(X).$$

ii) An additive Cousin distribution c on X has a solution iff

$$\delta_0^*(div(c)) = 0 \in H^1(X,\mathscr{O})$$

23. Consider the open set $X := \mathbb{C}^2 \setminus \{(0,0)\} \subset \mathbb{C}^2$ and the analytic set

$$A := \mathbb{C}^* \times \{0\} \subset X.$$

Show: The canonical exact sequence of sheaves on X

$$0 \to \mathscr{I}_A \to \mathscr{O}_X \to \mathscr{O}_A \to 0$$

induces a sequence of global sections

$$0 \to \mathscr{I}_A(X) \to \mathscr{O}_X(X) \to \mathscr{O}_A(X) \to 0$$

which is not exact.

24. Consider a continuous map $f : X \to Y$ between topological spaces. Show:

i) The direct image functor

$$f_* : \underline{Sh}_X \to \underline{Sh}_Y$$

between the categories of sheaves on Abelian groups is left-exact.

ii) The direct image functor is not right-exact.

Hint: Consider $A \subset X$ from Problem 23, the sequence $\mathscr{O}_X \to \mathscr{O}_A \to 0$ and

$$f : X \to \mathbb{C},\ (z_1, z_2) \mapsto z_2.$$

Discussion: Thursday, 2.12.2021, 12.15 pm.

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Problems 07

25. Let X be a topological space.

i) Show: A sheaf $\mathscr{F}$ on X restricts for each subset open $U \subset X$ to a sheaf $\mathscr{F}|U$ on U by defining for each $V \subset U$ open

$$(\mathscr{F}|U)(V) := \mathscr{F}(V)$$

and taking the relevant restrictions from $\mathscr{F}$.

ii) For each pair of sheaves $\mathscr{F}, \mathscr{G}$ on X show: The presheaf

$$\mathscr{H}om(\mathscr{F}, \mathscr{G})(U) := Hom(\mathscr{F}|U, \mathscr{G}|U),\ U \subset X \text{ open},$$

with the canonical restrictions is a sheaf. Here $Hom(\mathscr{F}|U, \mathscr{G}|U)$ denotes the Abelian group of sheaf morphisms

$$\mathscr{F}|U \to \mathscr{G}|U$$

between the restricted sheaves.

26. Consider a hypersurface X in the polydisc $\Delta \subset \mathbb{C}^n$, i.e. an analytic submanifold $X \subset \Delta$ with $dim\, X = n-1$.

i) Show: Each point $a \in \Delta$ has an open neighbourhood $U_a \subset \Delta$ and a holomorphic function $f_a \in \mathscr{O}(U_a)$ with

$$X \cap U_a = \{z \in U_a : f_a(z) = 0\}$$

ii) Show: There exists a single holomorphic function $f \in \mathscr{O}(\Delta)$ with

$$X = \{z \in \Delta :\ f(z) = 0\}$$

27. For a sheaf of rings $\mathscr{R}$ on a topological space X show: For each open set $U \subset X$ and each section $u \in \mathscr{R}(U)$ holds:

$$u \in \mathscr{R}(U) \text{ unit} \iff u_x \in \mathscr{R}_x \text{ unit for all } x \in U$$

28. i) Consider a Hausdorff space X and a presheaf $\mathscr{F}$ on X which satisfies the identity theorem, see Problem 18. Show: The étale space $|\mathscr{F}|$ is a Hausdorff space.

ii) Construct a Hausdorff space X and a sheaf $\mathscr{F}$ on X with a non-Hausdorff étale space $|\mathscr{F}|$.

Discussion: Thursday, 9.12.2021, 12.15 pm.

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Problems 08

29. Consider a topological space X, a closed subspace $A \subset X$ with injection

$$i : A \to X,$$

and a sheaf $\mathscr{F}$ on A. The direct image $i_*\mathscr{F}$ is a sheaf on X, named the *extension of* $\mathscr{F}$ to X. Show for the stalks of $i_*\mathscr{F}$:

$$(i_*\mathscr{F})_x = \begin{cases} \mathscr{F}_x & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

30. Consider a topological space X, an open set $U \subset X$ with injection

$$i : U \to X,$$

and a sheaf $\mathscr{F}$ on U. Denote by $i_!\mathscr{F}$ the sheafification of the presheaf on X

$$V \mapsto \begin{cases} \mathscr{F}(V) & \text{if } V \subset U \\ 0 & \text{otherwise} \end{cases}$$

for open $V \subset X$. The sheaf $i_!\mathscr{F}$ is named the *extension of* $\mathscr{F}$ to X. Show:

For each $x \in X$ the stalk of $i_!\mathscr{F}$ satisfies

$$i_!\mathscr{F})_x = \begin{cases} \mathscr{F}_x & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases},$$

and the restriction satisfies

$$(i_!\mathscr{F})|U = \mathscr{F}.$$

31. Consider a complex manifold X and a free $\mathscr{O}$-module $\mathscr{H}$ of finite rank, i.e $\mathscr{H} \simeq \mathscr{O}^k$ for a suitable $k \in \mathbb{N}$. Let $\mathscr{F}, \mathscr{G} \subset \mathscr{H}$ be two coherent submodules.

i) Show: The $\mathscr{O}$-module

$$\mathscr{F} + \mathscr{G}$$

is coherent. Here

$$(\mathscr{F}+\mathscr{G})(U) := (\mathscr{F}(U)+\mathscr{G}(U)) \subset \mathscr{H}(U),\ U \subset X \text{ open.}$$

ii) Show: The $\mathscr{O}$-module

$$\mathscr{F} \cap \mathscr{G}$$

is coherent. Here

$$(\mathscr{F} \cap \mathscr{G})(U) := (\mathscr{F}(U) \cap \mathscr{G}(U)) \subset \mathscr{H}(U),\ U \subset X \text{ open.}$$

32. Consider a complex manifold X and two $\mathscr{O}$-modules $\mathscr{F},\mathscr{G}$ on X. Show:

i) The sheaf $\mathscr{H}om_{\mathscr{O}}(\mathscr{F},\mathscr{G})$, see Problem 25, is an $\mathscr{O}$-module sheaf.

ii) For each $x \in X$ exists a canonical map between $\mathscr{O}_x$-modules

$$\phi : (\mathscr{H}om_{\mathscr{O}}(\mathscr{F},\mathscr{G}))_x \to Hom_{\mathscr{O}_x}(\mathscr{F}_x,\mathscr{G}_x).$$

iii) If $\mathscr{F}$ is coherent then the canonical map ϕ from part ii) is an isomorphism.

Discussion: Thursday, 16.12.2021, 12.15 pm.

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Problems 09

These problems may serve to recall the lecture until now and to become more familiar with the relation between the results.

33. The top ten:

- Choose your ”top ten” from the ”List of results” in the lecture notes.
- Draw a directed graph to visualize the logical structure underlying these top ten: Each result is a vertex, each logical conclusion is a directed edge of the graph.

34. Provide some examples from the lecture which illustrate the following two principles of problem solving:

- *Shrinking*: Obtaining a local solution after shrinking the domain of definition.
- *Extending*: Combining local solutions to obtain a global solution.

Discussion: Thursday, 13.1.2022, 12.15 pm.

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Problems 10

35. Consider a topological space X and a subspace $Z \subset X$ with injection $j : Z \hookrightarrow X$. For a sheaf $\mathscr{F}$ on X with étale space

$$p : |\mathscr{F}| \to X$$

define for each open $V \subset Z$

$$\mathscr{F}(V) = \{s : V \to |\mathscr{F}| : \ s \text{ section against } p\}$$

i) Show: The presheaf

$$V \mapsto \mathscr{F}(V),\ V \subset Z \text{ open },$$

is a sheaf on Z. The sheaf is named $\mathscr{F}|Z$, the *restriction of* $\mathscr{F}$ to Z.

ii) Consider an open $U \subset X$ with injection $j : U \hookrightarrow X$ and set $A := X \setminus U$ with injection $i : A \hookrightarrow X$. For each sheaf $\mathscr{F}$ on X show:

On X exists a short exact sequence of sheaves

$$0 \to j_!(\mathscr{F}|U) \to \mathscr{F} \to i_*(\mathscr{F}|A) \to 0$$

36. Consider a complex manifold X with structure sheaf $\mathscr{O}$. Show:

i) For two coherent $\mathscr{O}$-modules $\mathscr{F}$, $\mathscr{G}$ also the tensor product

$$\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G}$$

is a coherent $\mathscr{O}$-module.

ii) For two coherent ideal sheaves $\mathscr{I}_1,\ \mathscr{I}_2 \subset \mathscr{O}$ also the product

$$\mathscr{I}_1 \cdot \mathscr{I}_2 \subset \mathscr{O}$$

is a coherent ideal sheaf.

37. For two coherent $\mathscr{O}$-modules $\mathscr{F}$, $\mathscr{G}$ on a complex manifold X with structure sheaf $\mathscr{O}$ show: The $\mathscr{O}$-module

$$\mathscr{H}om_{\mathscr{O}}(\mathscr{F},\mathscr{G})$$

is coherent.

38. Give a direct proof that the Hartogs figure from Fig. 1.3 (Lecture notes) is not holomorphically convex.

Discussion: Thursday, 20.1.2022, 12.15 pm.

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Problems 11

39. Consider a complex manifold X and an open, relatively-holomorphically convex subset $Y \subset X$.

Show: For each pair (K, U) with compact $K \subset Y$ and open $U \subset Y$ satisfying

$$\hat{K}_{X,Y} \subset U$$

exists an analytic polyhedron P, defined in an open subset of Y by finitely many holomorphic functions from $\mathscr{O}(X)$, which satisfies

$$\hat{K} \subset P \subset\subset U$$

40. Let X be a complex manifold which is

i) holomorphically separable and

ii) locally uniformizable.

Show: If there exist finitely many global holomorphic functions $f_1, ..., f_k \in \mathscr{O}(X)$ such that the set

$$\{x \in X : \ |f_j(x)| \leq 1 \text{ for all } j = 1, ..., k\}$$

is compact, then the open set

$$D := \{x \in X : \ |f_j(x)| < 1 \text{ for all } j = 1, ..., k\}$$

is a Stein manifold.

41. Consider a Stein manifold X, a holomorphic function $f \in \mathscr{O}(X)$, and denote by

$$V(f) := \{x \in X : \ f(x) = 0\} \subset X$$

the zero set of f.

Show: The complex manifold $X \setminus V(f)$, the complement of a hypersurface in X, is a Stein manifold.

42. Show: Each analytic polyhedron in a complex manifold X is relatively-holomorphically convex with respect to X.

Discussion: Thursday, 27.1.2022, 12.15 pm.

Problems 12

43. For two Stein manifolds $X_1,\ X_2$ show: Also the product $X_1 \times X_2$ is a Stein manifold.

44. Let X be a complex manifold and $Y \subset X$ an analytic submanifold of X. Show:

i) If X is holomorphically spreadable then also Y.

ii) If X is holomorphically convex then also Y.

iii) If X is a Stein manifold then also Y.

Hint: You may use without proof a remark from the lecture notes.

45. Consider a complex manifold X with structure sheaf $\mathscr{O}$ and a coherent $\mathscr{O}$-module $\mathscr{F}$ on X. Show for each point $x \in X$: There exists an open neighbourhood $U \subset X$ of x such that for each $\mathscr{O}_x$-submodule $F \subset \mathscr{F}_x$ the $\mathscr{O}(U)$-submodule

$$F_U := \{s \in H^0(U, \mathscr{F}) :\ s_x \in F\} \subset H^0(U, \mathscr{F})$$

is closed with respect to the canonical Fréchet topology.

Hint: You may assume $U = \Delta$ a polydisc. The argument for the particular case $\mathscr{F} = \mathscr{O}$ is part of a proof from the lecture.You may reduce the general case by showing that the complement $H^0(\Delta, \mathscr{F}) \setminus F_\Delta$ is open.

46. Consider a Stein manifold X with structure sheaf $\mathscr{O}$ and a coherent $\mathscr{O}$-module $\mathscr{F}$ on X. For a given point $x \in X$ denote by

$$F \subset \mathscr{F}_x$$

the $\mathscr{O}_x$-submodule generated by the germs of all sections from $H^0(X, \mathscr{F})$.

i) Show: For each finite system

$$f_{1,x},...,f_{x,k} \in \mathscr{F}_x,\ j = 1,...,k,$$

of generators of the $\mathscr{O}_x$-module $\mathscr{F}_x$ exists a relatively-holomorphically convex neighbourhood $U \subset X$ of x and representatives

$$f_1,...,f_k \in H^0(U,\mathscr{F})$$

of the system of generators.

ii) Show: The submodule

$$F_U := \{f \in H^0(U,\mathscr{F}) :\ f_x \in F\} \subset H^0(U,\mathscr{F})$$

is dense with respect to the canonical Fréchet topology.

iii) Show: The submodule $F_U \subset H^0(U,\mathscr{F})$ from part ii) is closed with respect to the canonical Fréchet structure. Conclude

$$F_U = H^0(U,\mathscr{F}).$$

iv) Conclude Theorem A for X without referring to Theorem B.

Discussion: Thursday, 3.2.2022, 12.15 pm.

Selected Solutions

45 .

W.l.o.g. there exists a polydisc $\Delta \subset \mathbb{C}^n$ with $x = 0 \in \Delta$ and an epimorphism of sections

$$H^0(\Delta, \mathscr{O}^p) \xrightarrow{\pi} H^0(\Delta, \mathscr{F}) \to 0$$

Set

$$F_\Delta := \{f \in \mathscr{F}(\Delta) : \ f_0 \in F\}$$

i) Define the inverse image of germs

$$G := \pi^{-1}(F) \subset \mathscr{O}_x^p$$

and

$$G_\Delta := \{g \in H^0(\Delta, \mathscr{O}^p) : \ g_0 \in G\}$$

Due to the proof of Proposition 5.17 the submodule

$$G_\Delta \subset H^0(\Delta, \mathscr{O}^p)$$

is closed, and its complement

$$H^0(\Delta, \mathscr{O}^p) \setminus G_\Delta$$

is open.

ii) The surjective linear map between Fréchet spaces

$$H^0(\Delta, \mathscr{O}^p) \xrightarrow{\pi} H^0(\Delta, \mathscr{F}) \to 0$$

is open. Because

$$H^0(\Delta, \mathscr{O}^p) \setminus G_\Delta \subset H^0(\Delta, \mathscr{O}^p)$$

is open due to part i), also the image

$$\pi(H^0(\Delta, \mathscr{O}^p) \setminus G_\Delta) \subset H^0(\Delta, \mathscr{F})$$

is open.

iii) Due to

$$G = \pi^{-1}(F) \subset \mathscr{O}_0^p,$$

for each section $s \in H^0(\Delta, \mathscr{O}^p)$ holds

$$s_0 \notin G \implies \pi(s)_0 = \pi(s_0) \notin F$$

As a consequence, the complement

$$\pi(H^0(\Delta,\mathscr{O}^p)\setminus G_\Delta) = \pi(H^0(\Delta,\mathscr{O}^p)\setminus\{\pi(s)\in H^0(\Delta,\mathscr{O}^p:\ \pi(s)_0\notin F\} =$$

$$= H^0(\Delta,\mathscr{F})\setminus\{f\in H^0(\Delta,\mathscr{F}):\ f_0\notin F\} = \{f\in H^0(\Delta,\mathscr{F}):\ f_0\in F\} =$$

$$F_\Delta \subset H^0(\Delta,\mathscr{F})$$

is closed.

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i) Analytic polyhedra in X form a neighbourhood base of x. There exists a common polyhedron U with $x\in U$ where representatives of the generators are defined.

ii) Follows from the density of the restriction

$$H^0(X,\mathscr{F}) \to H^0(U,\mathscr{F})$$

because $U\subset X$ is relatively-homolorphic convex.

iii) Due to Problem 45

$$F_U \subset H^0(U,\mathscr{F})$$

is closed. Together with part ii) follows $F_U = H^0(U,\mathscr{F})$,

iv) Part i) and iii) imply Theorem A.