MODULAR FORMS
AND ELLIPTIC CURVES
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Problems 01

1. i) Consider a domain $G \subset \mathbb{C}$ and two meromorphic functions f_1, f_2 defined on G. Assume a sequence $(z_\nu)_{\nu \in \mathbb{N}}$ of points from G with an accumulation point in G such that

$$f_1(z_\nu) = f_2(z_\nu) \text{ for all } \nu \in \mathbb{N}.$$

Here $f_i(z) := \infty$ for a pole z of $f_i, i = 1, 2$.

Show: $f_1 = f_2$ (Identity theorem).

Hint: You may presuppose the identity theorem for holomorphic functions.

ii) Show the existence of two holomorphic functions with isolated singularities which do not satisfy a corresponding identity theorem.

2. Compute the singularities and the type of singularity for the following functions, and determine the residues:

i)

$$f(z) = \frac{1 - \cos\ z}{z^3}$$

ii)

$$f(z) = \frac{1}{e^z + 1}$$

3. The complex projective space $\mathbb{P}^1(\mathbb{C})$ (Riemann sphere) is the set of equivalence classes with respect to the equivalence relation on $\mathbb{C}^2 \setminus \{0\}$

$$z \sim w \iff \exists \lambda \in \mathbb{C}^* \text{ with } w = \lambda \cdot z.$$

The standard notation for the equivalence class of $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$ is

$$(z_1 : z_2) \in \mathbb{P}^1(\mathbb{C})(\textit{homogeneous coordinates}).$$

The complex plane $\mathbb{C}$ is considered the subspace

$$\{(z_1 : z_2) \in \mathbb{P}^1(\mathbb{C}) : \; z_2 = 1\}.$$

Define

$$PGL(2,\mathbb{C}) := GL(2,\mathbb{C}) / \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C}^* \right\}.$$

The group $PGL(2,\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ as group of holomorphic automorphisms:

$$PGL(2,\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}), ([\gamma],(z_1 : z_2)) \mapsto (a \cdot z_1 + b \cdot z_2 : c \cdot z_1 + d \cdot z_2)$$

with

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C}).$$

Show for the group of holomorphic automorphisms:

i)

$$Aut(\mathbb{C}) \simeq \left\{ [\gamma] \in PGL(2,\mathbb{C}) : \gamma = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\} \simeq$$

$$\simeq \{f : \mathbb{C} \to \mathbb{C}, z \mapsto a \cdot z + b : a, b \in \mathbb{C}, a \neq 0\} \text{ (Linear transformations)}$$

ii)

$$Aut(\mathbb{P}^1(\mathbb{C})) \simeq PGL(2,\mathbb{C}) \text{ (Moebius transformations)}$$

4. i) From `https://pari.math.u-bordeaux.fr/` install the open-source software PARI.

ii) Consider the two lattices

$$\Lambda_i := \mathbb{Z} + \mathbb{Z}\, i$$

and

$$\Lambda_\rho := \mathbb{Z} + \mathbb{Z}\, \rho, \; \rho := e^{2\pi i/3}.$$

Write a PARI-program to compute for each lattice Λ

- the Weierstrass function $\wp$ and its derivative $\wp'$ - at least up to order $= 12$
- and the values $\wp'(\omega/2), \; \omega \in \Lambda$, at the half-lattice points.

Hint: Use the functions *ellperiods* and *ellwp*. If the source code of the program is contained in a text-file stored in the working direction of PARI, then PARI executes the file by the command $\backslash r$ *filename* (no quotation marks).

Discussion: Tuesday, 10.11.2020, 12.15 pm.

Problems 02

5. In the lecture it was proved: Each holomorphic elliptic function is constant. This result follows also from the maximum principle for compact Riemann surfaces.

Determine in an analogous way two further results about elliptic functions from the lecture which follow from the theory of compact Riemann surfaces.

In the following problems 6) and 7) consider a fixed lattice

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$$

with lattice constants

$$g_2 := 60 \cdot G_{\Lambda,4}, \ g_3 := 140 \cdot G_{\Lambda,6}.$$

6. Consider the cubic polynomial

$$F(T) := 4 \cdot T^3 - g_2 \cdot T - g_3 \in \mathbb{C}[T].$$

Relate the zeros of F to the Weierstrass function $\wp$ of Λ, and show that all zeros are pairwise distinct.

7. i) For any point $w \in \mathbb{C} \setminus (\Lambda/2)$ determine modulo Λ the poles and zeros with their orders of the elliptic function with respect to Λ

$$f(z) := \frac{\wp'(z)}{\wp(z) - \wp(w)}.$$

ii) Determine explicitly an elliptic function with respect to Λ

$$f \in \mathbb{C}(\wp)[\wp']$$

with two distinct first order poles modulo Λ and no other poles mod Λ.

8. Confirm by a PARI-program your result from Exercise 7), part i) for the two lattices Λ_i and Λ_ρ from Exercise 4).

Hint: Use the function *ellwp* to compute or to approximate the relevant values of $\wp$ and $\wp'$.

Discussion: Thursday, 19.11.2020, 12.15 pm.

Problems 03

9. (Schwarz Lemma) On $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ consider a holomorphic function

$$f : \Delta \to \Delta \text{ with } f(0) = 0.$$

i) Show:

$$|f'(0)| \le 1 \text{ and } |f(z)| \le |z| \text{ for all } z \in \Delta.$$

ii) Show: If

$$|f'(0)| = 1 \text{ or } |f(z_0)| = |z_0| \text{ for a point } z_0 \neq 0$$

then exist $\alpha \in \mathbb{R}$, such that for all $z \in \Delta$

$$f(z) = e^{2\pi i \cdot \alpha} \cdot z \text{ (Rotation)}$$

Hint: Consider the function $f(z)/z$, use the maximum principle.

10. On a Riemann surface X denote by $\mathscr{O}$ the holomorphic structure sheaf, i.e. the sheaf of holomorphic functions

$$\mathscr{O}(U) := \{f : U \to \mathbb{C} |\ f \text{ holomorphic}\},\ U \subset X \text{ open},$$

with the canonical restriction of functions. Denote by $\mathscr{O}^*$ the subsheaf of holomorphic functions without zeros. For a holomorphic line bundle L on X denote by $\mathscr{L}$ its sheaf of holomorphic sections.

Show the equivalence:

$$\mathscr{L} \simeq \mathscr{O} \iff \mathscr{L} \text{ has a holomorphic section without zeros}$$

Hint: Use the representation of $\mathscr{L}$ by a cocycle

$$g = (g_{ij})_{i,j \in I} \in Z^1(\mathscr{U}, \mathscr{O}^*)$$

with respect to an open covering $\mathscr{U} = (U_{ij})_{i,j \in I}$ of X.

11. Show: Meromorphic functions on a Riemann surface X correspond bijectively to holomorphic maps

$$f : X \to \mathbb{P}^1,$$

which are not constant $= \infty \in \mathbb{P}^1$.

Hint: For calculations on $\mathbb{P}^1$ use the standard covering $\mathscr{U} = (U_0, U_1)$ with

$$U_j := \{(z_0 : z_1) \in \mathbb{P}^1 : \ z_j \neq 0\}, \ j = 0, 1,$$

and the corresponding charts

$$\phi_j : U_j \to \mathbb{C}, \ j = 0, 1,$$

$$w_0 := \phi_0(z_0 : z_1) := \frac{z_1}{z_0} \ \text{ and } \ w_1 := \phi_1(z_0 : z_1) := \frac{z_0}{z_1}$$

12. A *divisor* on a compact Riemann surface X is a formal sum

$$D := \sum_{x \in X} n_x(x) \cdot x$$

with integers $n_x \in \mathbb{Z}$, and $n_x = 0$ for all but finitely many $x \in X$. The *degree* of D is

$$deg\ D := \sum_{x \in X} n_x$$

For each $x \in X$ set

$$D(x) := n_x$$

The divisors of non-zero meromorphic functions $f \in \mathscr{M}(X)^*$

$$(f) := div\ f := \sum_{x \in X} n_x \cdot x, \ n_x := ord(f;\ x),$$

are named *principal divisors*. Hence

$$n_x = \begin{cases} n & \text{if } f \text{ has a zero at } x \text{ of order } n \\ -n & \text{if } f \text{ has a pole at } x \text{ of order } n \end{cases}$$

Show: i) The divisors on X form an additive group $Div(X)$ in a canonical way. The divisors of degree $= 0$ form a subgroup

$$Div_0(X) := \{D \in Div(X) : deg\ D = 0\}$$

ii) A principal divisor $D \in Div(X)$ satisfies $deg\ D = 0$.

Hint: You may use a general theorem about holomorphic maps between compact Riemann surfaces.

Discussion: Tuesday, 24.11.2020, 12.15 pm.

Problems 04

13. Consider the neighbourhoods of $\infty \in \mathbb{H}^*$

$$U_R := \{x + i \cdot y \in \mathbb{H} : y > R\} \cup \{\infty\},\ R > 0$$

and the lines

$$L_R := \{x + i \cdot y \in \mathbb{H} : y = R\} \cup \{\infty\},\ R > 0.$$

For

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$$

determine $S(\infty) \in \mathbb{H}^*$ and the images

$$S(L_R) \subset \mathbb{H}^* \text{ and } S(U_R) \subset \mathbb{H}^*.$$

Hint: Use that a fractional-linear transformation maps lines and circles to lines or circles.

14. On a Riemann surface X one considers the following sheaves: The sheaf $\mathscr{O}$ of holomorphic functions, the multiplicative subsheaf $\mathscr{O}^*$ of holomorphic functions without zeros, the sheaf $\mathscr{M}$ of meromorphic functions and the subsheaf $\mathscr{M}^*$ of non-zero meromorphic functions.

On a compact Riemann surface X consider a divisor

$$D = \sum_{x \in X} n_x \cdot x \in Div(X).$$

For each open set $U \subset X$ define the $\mathbb{C}$-vector space

$$\mathscr{O}_D(U) := \{f \in \mathscr{M}^*(U) : div\ f \geq -D|U\} \cup \{0\}$$

i.e. for all $x \in U$

$$f \text{ has in } x \begin{cases} \text{a pole of order at most } n_x & if\ n_x \geq 0 \\ \text{a zero of order at least } -n_x & if\ n_x \leq 0 \end{cases}$$

i) Show: The attachment

$$U \mapsto \mathscr{O}_D(U),\ U \subset X \text{ open},$$

with the restriction of functions

$$\mathscr{O}_D(U) \to \mathscr{O}_D(V),\ V \subset U \subset X \text{ open},$$

defines a subsheaf $\mathscr{O}_D \subset \mathscr{M}_X$, named the *sheaf of multiples* of $-D$.

ii) Show: The sheaf $\mathscr{O}_D$ is isomorphic to the sheaf of sections of the holomorphic line bundle L_D on X, with the latter sheaf defined by the following cocycle

$$g := (g_{ij})_{i,j \in I} \in Z^1(\mathfrak{U}, \mathscr{O}_X^*) :$$

Here $\mathfrak{U} = (U_i)_{i \in I}$ is a covering of X by open subsets $U_i \subset X$ with meromorphic functions $g_i \in \mathscr{M}^*(U_i)$ such that

$$div\ g_i = D|U_i,\ i \in I,$$

and

$$g_{ij} := \frac{g_i}{g_j} \in \mathscr{O}^*(U_i \cap U_j).$$

15. Consider a compact Riemann surface X of genus $g(X)$.

i) Show: If $D \in Div(X)$ and $deg\ D < 0$ then

$$H^0(X, \mathscr{O}_D) = 0.$$

ii) The canonical divisor $K \in Div(X)$ is the divisor of a non-zero meromorphic section of the sheaf Ω^1 of differential forms on X. Show:

$$deg\ K = 2g(X) - 2.$$

16. Consider $N \in \mathbb{N}$ and the modular curve $X_0(N)$ of genus $g = g(X_0(N))$.

i) Write a PARI program which outputs the degree of the canonical branched covering

$$X_0(N) \to X(1) \simeq \mathbb{P}^1$$

induced from the injection

$$\Gamma_0(N) \hookrightarrow \Gamma$$

as well as the constants of $X_0(N)$

$$g,\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_\infty.$$

ii) Enhance the program from part i) to output the result for all primes $N < 100$.

Hint: You may outsorce some computations as separate functions to be called from the main program, e.g. https://wstein.org/edu/Fall2001/124/lectures/lecture16/.

Discussion: Thursday, 3.12.2020, 12.15 pm.

Problems 05

17. For the Γ-action on $\mathbb{H}^*$ determine and prove the explicit form of the isotropy group Γ_∞.

18. The Bernoulli numbers $B_n,\ n \in \mathbb{N}$, are defined by the convergent power series

$$\frac{z}{e^z - 1} =: \sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot z^n$$

i) Show: For each odd $n \geq 3$ holds $B_n = 0$.

ii) Show by induction for $N \in \mathbb{N}^*$

$$\sum_{n=0}^{N} \binom{N+1}{n} \cdot B_n = 0 \text{ and } B_0 = 1.$$

19. Consider a divisor $D \in Div(\mathbb{P}^1)$. Show:

i) $dim\ H^0(\mathbb{P}^1, \mathscr{O}_D) = max\{0, 1 + deg\ D\}$

ii) $dim\ H^1(\mathbb{P}^1, \mathscr{O}_D) = max\{0, -1 - deg\ D\}$.

20. For a compact Riemann surface X of genus $g \geq 2$ compute the dimension of its moduli space

$$dim\ H^1(X, \Theta_X).$$

Here Θ_X denotes the tangent bundle of X. It is the dual of the line bundle Ω^1_X.

Discussion: Thursday, 10.12.2020, 12.15 pm.

Problems 06

21. i) For the two Eisenstein series of weight respectively 4 and 6 determine the values

$$G_4(\rho) \text{ and } G_6(i)$$

ii) Derive the following values of the modular j-invariant

$$j[i] = 1728 \text{ and } j([\rho]) = 0,\ [i],[\rho] \in X(1) = \Gamma\backslash\mathbb{H}^*.$$

iii) Validate your results by using appropriate PARI commands.

22. Using the Eisenstein series G_2 set for each $\tau \in \mathbb{H}^*$

$$\tilde{G}(\tau) := G_2(\tau) - \frac{\pi}{Im\ \tau}$$

i) Show: For all $\gamma \in \Gamma$

$$\tilde{G}[\gamma]_2 = \tilde{G}$$

Hint: Use without proof the formula for $G_2[S]_2$ from the lecture.

ii) Argue why $\tilde{G} \notin M_2(\Gamma)$.

23. The lecture proved a theorem on the relation between modular forms and meromorphic differential forms on the modular curve. As a direct application of the theorem compute $dim\ M_k(\Gamma)$ for the weights

$$k \in \{0,2,4,6,8,10,12\}$$

24. Consider the Fourier series

$$\sum_{n=0}^{\infty} \tau_n \cdot q^n$$

of the normalized discriminant form $\dfrac{\Delta}{(2\pi)^{12}}$

i) Write a PARI-progam which outputs for $n \leq 7$ the values

$$\tau(n) := \tau_n$$

ii) Make a conjecture relating the values

$$\tau(p),\ \tau(q) \text{ and } \tau(p \cdot q)$$

for distinct primes p, q and validate your conjecture for $p, q \leq 20$ by a PARI program.

iii) Make a conjecture relating the values

$$\tau(m),\ \tau(n) \text{ and } \tau(m \cdot n)$$

for coprime positive integers $n,\ m$, i.e. $gcd(n, m) = 1$, and validate your conjecture for $n, m \leq 10$ by a PARI program.

iv) Show

$$\tau(2) \cdot \tau(2) \neq \tau(4)$$

Discussion: Thursday, 17.12.2020, 12.15 pm.

Problems 07

The following list of questions and problems may serve for repetition of the topics from the lecture so far. Intentionally, the issues are not arranged in a logical order.

1. Name four different characterizations of modular forms.

2. Name some steps necessary for computing the dimension of modular forms by applying the Riemann-Roch theorem.

3. What about modular forms with odd weight?

4. How to study holomorphic maps between complex tori?

5. Which object is the moduli space of complex tori?

6. Where are the zeros located of the discriminant form Δ?

7. Are there modular forms of the modular group of weight $k = 2$?

8. Explain the definition of the Eisenstein series?

9. Explain the Fourier expansion of a modular form of Γ.

10. When is a line bundle very ample?

11. Name some modular forms of Γ with rational respectively integer Fourier coefficients.

12. What is the twisted line bundle on $\mathbb{P}^n$?

13. Why do meromorphic functions correspond to holomorphic maps to $\mathbb{P}^1$?

14. Explain the role of the line bundle $\mathscr{L}_{N,k}$ on the modular curve $X_0(N)$ for the computation of $M_k(\Gamma_0(N))$.

15. Name the very-ampleness criterion.

16. Explain the Riemann-Roch theorem.

17. State the definition of a Hecke congruence subgroup.

18. Name the distinct types of isotropy groups of the action of Γ on $\mathbb{H}^*$.

19. What about modular forms of Γ with negative weight?

20. State the differential equation of the Weierstrass function $\wp$, and give some examples illustrating the importance of the differential equation.

21. Explain Serre's theorem on duality.

22. Describe the neighbourhoods of $\infty \in \mathbb{H}^*$ and of rational points $q \in \mathbb{Q} \subset \mathbb{H}^*$.

23. What is a cusp? How many cusps do exist for Γ?

24. How does the general modular curve $X_0(N),\ N \in \mathbb{N}$, relate to the modular curve $X := \Gamma\backslash\mathbb{H}^*$?

25. Explain the difference between the level and the weight of a modular form.

26. Name some important ideas in the proof that complex tori embed as cubics into $\mathbb{P}^2$.

Discussion: Thursday, 7.1.2021, 12.15 pm.

Problems 08

25. For $j = 1,2$ find a compact Riemann surface X_j and a line bundle $\mathscr{L}_j$ on X_j such that

$$H^0(X_1, \mathscr{L}_1) = H^1(X_1, \mathscr{L}_1) = 0$$

and

$$H^0(X_2, \mathscr{L}_2) = H^1(X_2, \mathscr{L}_2) \neq 0.$$

26. i) Express the normalized Eisenstein series E_8 as a polynomial in the generators $E_4,\ E_6$ of the algebra $M_*(\Gamma)$.

ii) Show for all $n \geq 1$:

$$\sigma_7(n) = \sigma_3(n) + 120 \cdot \sum_{m=1}^{n-1} \sigma_3(m) \cdot \sigma_3(n-m)$$

27. For $N \in \mathbb{N}^*$ set

$$\mu_N := \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Q})^+$$

i) Denote by

$$\Gamma' := \mu_N^{-1} \cdot \Gamma \cdot \mu_N := \{\mu_N^{-1} \cdot \gamma \cdot \mu_N :\ \gamma \in \Gamma\} \subset GL(2, \mathbb{Q})^+$$

the μ_N-conjugate of Γ. Show:

$$\Gamma_0(N) = \Gamma' \cap \Gamma.$$

ii) Show: For any $\alpha \in \Gamma$ exists a matrix $\alpha_{left} \in \Gamma$ such that

$$\mu_N \cdot \alpha = \alpha_{left} \cdot \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in M(2 \times 2, \mathbb{Z}) \cap GL(2, \mathbb{Q})^+$$

Hint: Set $M := \mu_N \cdot \alpha$ and prove the existence of $B \in \Gamma$ with $B \cdot M$ triangular.

28. Consider a level $N \in \mathbb{N}^*$ and set

$$\mu_N := \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \in GL(2,\mathbb{Q})^+$$

Show:

i) For each even weight $k \geq 0$ the map

$$M_k(\Gamma) \to M_k(\Gamma_0(N)),\ f \mapsto f[\mu_N]_k,$$

is a well-defined, injective morphism of complex vector spaces.

ii) The morphism from part i) restricts to a morphism of cusps

$$S_k(\Gamma) \to S_k(\Gamma_0(N)).$$

Discussion: Thursday, 14.1.2021, 12.15 pm.

Problems 09

29. For an elliptic curve (E,O) defined over k denote by $K \in Div(E)$ the canonical divisor. Show the isomorphy:

$$\mathscr{O} \simeq \mathscr{O}_K.$$

30. Consider the plane elliptic curve $E/\mathbb{Q}$ defined by the Weierstrass equation

$$y^2 = x^3 - 43x + 166$$

i) Write a PARI script which computes a torsion subgroup $G \subset E(\mathbb{Q})_{tors}$ of order $= 7$.

ii) Determine a generator $g \in G$ and add to the PARI-script from part i) a routine, which outputs the coordinates

$$(x,y) \in \mathbb{Q}^2 \cup \{\infty\}$$

of all multiples $n * g \in G$ with $n \in 0, 1, ..., 6$.

31. i) Determine the dimension of the cusp spaces

$$S_{12}(\Gamma),\ S_{12}(\Gamma_0(2)) \text{ and } \ S_{12}(\Gamma_0(4)).$$

ii) Denote by

$$\frac{\Delta}{(2\pi)^{12}} \in S_{12}(\Gamma)$$

the normalized discriminant form. Determine a basis of $S_{12}(\Gamma_0(2))$.

iii) Determine three linearly independent elements of $S_{12}(\Gamma_0(4))$ and the Fourier expansion of its elements with respect to $\infty \in \mathbb{H}^*$.

32. Write a Pari script which outputs a basis of $S_{12}(\Gamma_0(4))$ and the Fourier expansion of its elements with respect to $\infty \in \mathbb{H}^*$.

Discussion: Thursday, 21.1.2021, 12.15 pm.

Problems 10

33. i) For $k,\ n \in \mathbb{N}^*$ show

$$\sum_{d|n} \left(\frac{d}{n}\right)^{k-1} = \sum_{d|n} \left(\frac{1}{d}\right)^{k-1}$$

ii) Show: The Fourier coefficients $(a_n)_{n\in\mathbb{N}}$ of the Eisenstein series

$$G_k \in M_k(\Gamma),\ k \geq 4 \text{ even},$$

grow like $(n^{k-1})_{n\in\mathbb{N}}$, i.e. there exist two constants $m_k, M_k > 0$ such that for all $n \in \mathbb{N}$

$$m_k \cdot n^{k-1} \leq |a_n| \leq M_k \cdot n^{k-1}$$

34. Consider the elliptic curves $E/\mathbb{Q}$ defined by the Weierstrass equation

$$Y^2 = X^3 - p \text{ with } p \in \{17, 19, -55, 73\}$$

Write a PARI program with the following features:

i) The program searches for a point $P = (x, y) \in E(\mathbb{Q})$ with integer coordinates satisfying both conditions

$$|x| \leq 500 \text{ and } |y| \leq 500.$$

ii) For a point $P \in E(\mathbb{Q})$ which satisfies the condition from part i) the program computes the points $2P,\ 4P$ and the order of P in the group $E(\mathbb{Q})$.

iii) The program outputs the analytic rank of E.

35. i) Consider the homogeneous cubic polynomial

$$F_{Ferm}(U, V, W) := U^3 + V^3 - W^3 \in \mathbb{Z}[U, V, W]$$

which defines the Fermat variety $Ferm/\mathbb{Q}$ with

$$Ferm(\mathbb{C}) := \{(u : \mathrm{v} : w) \in \mathbb{P}^2(\mathbb{C}) : F_{Ferm}(u, \mathrm{v}, w) = 0\} \subset \mathbb{P}^2(\mathbb{C})$$

Determine the intersection with the line at "infinity"

$$Ferm(\mathbb{C}) \cap \{(u : \mathrm{v} : w) \in \mathbb{P}^2(\mathbb{C}) :\ w = 0\}$$

ii) Prove that the Fermat variety is smooth.

iii) Consider the elliptic curve $E/\mathbb{Q}$ with Weierstrasss equation

$$0 = F(X,Y) := Y^2 - (4 \cdot X^3 - 1728)$$

Show: The map

$$E(\mathbb{C}) \setminus \{\pm(0, 2^3 \cdot 3 \cdot \sqrt{-3})\} \to Ferm(\mathbb{C}),\ (x,y) \mapsto \left(\frac{72+y}{12x}, \frac{72-y}{12x}\right)$$

extends to an automorphism of $\mathbb{P}^2(\mathbb{C})$, defined over $\mathbb{Q}$, which restricts to an isomorphism

$$E/\mathbb{Q} \xrightarrow{\simeq} Ferm/\mathbb{Q}$$

Hint: The inverse of the map is

$$(u, \mathrm{v}) \mapsto \left(\frac{12}{u+\mathrm{v}}, 72 \cdot \frac{u-\mathrm{v}}{u+\mathrm{v}}\right)$$

36. Determine the Mordell-Weil group $E(\mathbb{Q})$ of the elliptic curve $E/\mathbb{Q}$ from Problem 35.

Hint: You may use without proof that Fermat's conjecture holds for the exponent $p = 3$.

Discussion: Thursday, 28.1.2021, 12.15 pm.

Problems 11

37. For $\tau \in \mathbb{H}$ define the product

$$\eta(\tau) := q^{1/24} \cdot \prod_{n=1}^{\infty} (1-q^n),\ q := e^{2\pi i \cdot \tau},\ \tau \in \mathbb{H}.$$

i) Show: The product is normally convergent and defines a holomorphic function

$$\eta : \mathbb{H} \to \mathbb{C}$$

without zeros.

ii) Show for the logarithmic derivation

$$\frac{d}{d\tau}(log\ \eta) = \frac{\eta'}{\eta} = \frac{2\pi i}{24} \cdot \left(1 - 24 \cdot \sum_{n=1}^{\infty} \frac{n \cdot q^n}{1-q^n}\right) = \frac{2\pi i}{24} \cdot E_2$$

with $E_2 := \dfrac{G_2}{2\zeta(2)}$ the normalized Eisenstein series of weight $k = 2$.

iii) Show: The η-function satisfies for $\tau \in \mathbb{H}$ and for the two generators $S, T \in \Gamma$ the transformation law

$$\eta(S(\tau)) = \sqrt{\tau/i} \cdot \eta(\tau) \text{ and } \eta(T(\tau)) = \eta(\tau)$$

with the positive branch of the square root.

Hint: To compute $\eta(S(\tau))$ use part ii). Then you may apply the transformation formula for G_2 from the lecture.

38. For the η-function from Problem 37 prove Jacobi's product representation:

$$\frac{\Delta(\tau)}{(2\pi)^{12}} = \eta^{24}(\tau) = q \cdot \prod_{n=1}^{\infty} (1-q^n)^{24},\ q = e^{2\pi i \cdot \tau},\ \tau \in \mathbb{H}.$$

Hint: Use $S_{12}(\Gamma) = \mathbb{C} \cdot \Delta$.

39. Consider the two modular forms from $S_{28}(\Gamma)$

$$f_1 := \Delta \cdot E_4^4 \text{ and } f_2 := \Delta^2 \cdot E_4$$

Write a PARI-program which outputs the following information:

- The dimension $dim\ S_{28}(\Gamma)$
- the Fourier series of $f_j,\ j = 1, 2,$
- the Fourier series of the Hecke transforms $T_2 f_j,\ j = 1, 2,$
- the matrix of the Hecke operator $T_2 \in End(S_{28}(\Gamma))$ with respect to the basis (f_1, f_2)
- the eigenvalues of T_2
- the Fourier coefficients of two corresponding eigenvectors $(\mathrm{v}_1, \mathrm{v}_2)$ of T_2, normalized with linear Fourier coefficient $= 1$,
- the representation of the eigenvectors with respect to the basis (f_1, f_2).

All Fourier series should be explicitly given up to order $= 4$.

40. Denote by

$$r(n,k,e) := card\ \left\{ v = (v_1, ..., v_k) \in \mathbb{Z}^k : n = v_1^e + ... + v_k^e \right\}$$

the number of representations of $n \in \mathbb{N}$ as a sum of k summands, each the e-power of an integer.

Note. The case of squares, $e = 2$, will be investigated in the lecture in class.

i) Determine by explicit counting the numbers $r(n,4,2),\ n = 0, 1, 2$.

ii) Write a PARI-program which outputs the numbers $r(n,4,2),\ n = 0, .., 9$.

iii) Extend the PARI-program from part ii) as follows:

- The program outputs the function $r(-,k,e)$ for the testcases

  $$(k,e) \in \{(8,2), (4,2), (3,2), (2,2), (1,2)\}$$

  up to order 9.
- For each testcase (e,k) the program outputs a description of the modular space and the ϑ-function underlying the computation.

iv) Determine the smallest $n \in \mathbb{N}$ which cannot be represented as the sum of 3 squares. Determine a representation of n as the sum of 4 squares.

Discussion: Thursday, 4.2.2021, 12.15 pm.

Selected Solutions

4 . See PARI files `Weierstrass_p_function_08`.

16 . See PARI files
`Congruence_subgroup_19, epsilon_2, epsilon_3`

19 . Denote by $K \in Div(\mathbb{P}^1)$ the canonical divisor. We have $deg\ K = -2$.
According to the Theorem of Riemann-Roch and Serre duality we have

$$dim\ H^0(\mathbb{P}^1, \mathscr{O}_D) - dim\ H^0(\mathbb{P}^1, \mathscr{O}_{-D+K}) = 1 + deg\ D.$$

i) If $deg\ D \leq -1$ then

$$dim\ H^0(\mathbb{P}^1, \mathscr{O}_D) = 0.$$

If $deg\ D \geq 0$ then $deg(-D+K) \leq -2$, hence $dim\ H^0(\mathbb{P}^1, \mathscr{O}_{-D+K}) = 0$ and

$$dim\ H^0(\mathbb{P}^1, \mathscr{O}_D) = 1 + deg\ D.$$

Overall

$$dim\ H^0(\mathbb{P}^1, \mathscr{O}_D) = max\{0, 1 + deg\ D\}.$$

ii) We have

$$dim\ H^1(\mathbb{P}^1, \mathscr{O}_D) = H^0(\mathbb{P}^1, \mathscr{O}_{-D+K})$$

and $deg(-D+K) = (deg\ D) - 2$.

Part i) implies

$$dim\ H^1(\mathbb{P}^1, \mathscr{O}_D) = max\{0, -1 - deg\ D\}.$$

24 . See PARI file `Ramanujan_01`.