DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
WINTER TERM 2019/20

RIEMANN SURFACES

Joachim Wehler

Problems 01

1. Consider the canonical map

$$\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1,\ (z_0, z_1) \mapsto (z_0 : z_1).$$

i) Show that π is open.

ii) Conclude that the topological space $\mathbb{P}^1$ is second countable.

2. Without using the corresponding result from the lecture show by explicit calculation

$$\mathscr{O}(\mathbb{P}^1) = \mathbb{C}$$

i.e. all holomorphic functions on $\mathbb{P}^1$ are constant.

Hint. For a holomorphic function $f \in \mathscr{O}(\mathbb{P}^1)$ consider the Taylor expansions of $f \circ \phi_j^{-1},\ j = 0, 1$, with respect to the standard atlas of $\mathbb{P}^1$.

3. Use the result $\mathscr{O}(X) = \mathbb{C}$ for a compact Riemann surface X to conclude Liouville's theorem: Every bounded entire function is constant.

4. Assume $n \geq 1$ and consider a polynomial

$$f(z) = z^n + a_1 \cdot z^{n-1} + ... + a_{n-1} \cdot z + a_n \in \mathbb{C}[z].$$

i) Represent f as a non-constant holomorphic map

$$\mathbb{P}^1 \to \mathbb{P}^1.$$

ii) Use a result from the lecture to show that f has a zero.

Discussion: Problem session on Monday, 21.10.2019, no submission

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Problems 02

5. i) Show: Any fractional linear transformation

$$f(z) := \frac{az+b}{cz+d}$$

with a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$$

is a meromorphic function on $\mathbb{C}$ and extends uniquely to a holomorphic map

$$f : \mathbb{P}^1 \to \mathbb{P}^1.$$

ii) Determine the value $f(\infty)$ of the holomorphic map from part i).

iii) For which matrices $A \in GL(2,\mathbb{C})$ holds $f = id_{\mathbb{P}^1}$?

6. Show: The group $Aut(\mathbb{C})$ of holomorphic automorphisms of the complex plane is the group of all affine-linear maps

$$\mathbb{C} \to \mathbb{C},\ z \mapsto a \cdot z + b,\ a \in \mathbb{C}^*,\ b \in \mathbb{C}.$$

Hint: You may show first that any holomorphic automorphism f satisfies

$$\lim_{z \to \infty} |f(z)| = \infty.$$

Then conclude that f is a polynomial.

7. Consider an arbitrary Riemann surface X. For each open set $U \subset X$ set

$$\mathscr{B}(U) := \{f : U \to \mathbb{C} |\ \text{f holomorphic and bounded}\}.$$

For the presheaf $\mathscr{B}$ defined as

$$\mathscr{B}(U),\ U \subset X \text{ open},$$

with the canonical restrictions show:

The presheaf $\mathscr{B}$ satisfies the first sheaf axiom, but not the second.

8. Let X be topological space and $\mathscr{F}$ a presheaf of Abelian groups on X. Prove the equivalence of the following two conditions:

i) The presheaf $\mathscr{F}$ is a sheaf.

ii) For each open $U \subset X$ and for each open covering $(U_i)_{i \in I}$ of U the following sequence of Abelian groups is exact:

$$0 \to \mathscr{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathscr{F}(U_i) \xrightarrow{\beta} \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j),$$

i.e. α is injective and *im* $\alpha = $ *ker* β. Here

$$\alpha(f) := (f_i)_i \text{ with } f_i := f|U_i$$

and

$$\beta((f_i)_i) := (f_{ij})_{ij} \text{ with } f_{ij} := (f_j - f_i)|U_i \cap U_j.$$

Discussion: Monday, 28.10.2019

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WINTER TERM 2019/20

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Selected Solutions 02

5 . i) A fractional linear map

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}),$$

is a meromorphic function on $\mathbb{C}$. Accordingly it extends to a holomorphic map

$$f : \mathbb{C} \to \mathbb{P}^1.$$

We have

$$\lim_{z \to \infty} f(z) = \frac{a + (b/z)}{c + (d/z)} = \begin{cases} a/c & c \neq 0 \\ \infty & c = 0 \end{cases}$$

Hence the function further extends to a holomorphic map

$$f : \mathbb{P}^1 \to \mathbb{P}^1$$

ii) According to part i)

$$f(\infty) = \begin{cases} a/c & c \neq 0 \\ \infty & c = 0 \end{cases}$$

iii) Claim:

$$f = id_{\mathbb{P}^1} \iff A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \in \mathbb{C}^*.$$

Apparently, $\dfrac{az+0}{0+a} = z$. Assume for all $z \in \mathbb{C}$

$$f(z) = \frac{az+b}{cz+d} = z$$

Then

- $f(0) = b/d = 0 \implies b = 0$
- $f(\infty) = \infty \implies c = 0$
- $f(1) = 1 \implies a = d$ and $a \neq 0$.

6 . i) We claim: Any biholomorphic map

$$f : \mathbb{C} \to \mathbb{C}$$

satisfies

$$\lim_{z\to\infty} |f(z)| = \infty.$$

We give two different proofs.

- Open neighbourhoods of ∞ are the complements of compact subsets. Assume an open neighbourhood of the form

$$V := \mathbb{C} \setminus \overline{D}_R(0).$$

For any $R > 0$ the inverse image

$$f^{-1}(\overline{D}_R(0)) \subset \mathbb{C}$$

is compact because the inverse map f^{-1} is continuous. Hence

$$f^{-1}(\overline{D}_R(0)) \subset \overline{D}_{R_1}(0)$$

for suitable $R_1 > 0$. Hence

$$f(\mathbb{C} \setminus \overline{D}_{R_1}(0)) \subset \mathbb{C} \setminus \overline{D}_R(0), \text{ i.e. } f(U) \subset V,$$

with

$$U := \mathbb{C} \setminus \overline{D}_{R_1}(0)$$

an open neighbourhood of ∞, which proves the claim.

- (Idea: J. Kruse) We first exclude that the isolated singularity ∞ is an essential singularity of f: Otherwise the Casorati-Weierstrass theorem implies for a neighbourhood of ∞

$$V := \mathbb{C} \setminus K$$

with compact

$$K \subset \mathbb{C},\ K \neq \emptyset,$$

that

$$f(V) \subset \mathbb{C}$$

is dense. After choosing an open neighbourhood $U \subset \mathbb{C}$ of 0 with

$$U \cap V = \emptyset$$

openess of f implies:

$$\emptyset \neq f(U) \subset \mathbb{C}$$

is open. Hence

$$f(U) \cap f(V) \neq \emptyset$$

which contradicts f being bijective. Secondly, Liouville's therorem implies that the function f is not bounded, because f is not constant. As a consequence, the isolated singularity is a pole, which proves the claim.

ii) *f is a linear polynomial*: The substitution $w := 1/z$ implies

$$g : \mathbb{C}^* \to \mathbb{C},\ g(w) := f(1/w) = f(z),$$

satisfies

$$\lim_{w \to 0} g(w) = \infty.$$

Hence $w = 0$ is a pole of g, hence for suitable $k \in \mathbb{N}$

$$g(w) = \sum_{n=-k}^{\infty} c_n \cdot z^n.$$

As a consequence

$$f(z) = \sum_{n=-k}^{\infty} c_n \cdot z^{-n}.$$

Holomorphy of f implies $c_n = 0$ for all $n \geq 1$:

$$f(z) = \sum_{n=0}^{k} c_n \cdot z^n$$

is a polynomial of degree at most $= k$. Biholomorphy of f implies degree $= 1$.

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Problems 03

9. Consider two tripel (z_1, z_2, z_3) and (w_1, w_2, w_3), each with pairwise distinct points from $\mathbb{P}^1$. Then exists a unique fractional linear transformation f satisfying for $j = 1, 2, 3$

$$f(z_j) = w_j.$$

Hint: First show that one may restrict to $(w_1, w_2, w_3) = (0, 1, \infty)$.

10. The group $Aut(\mathbb{P}^1)$ of holomorphic automorphisms of $\mathbb{P}^1$ or *Möbius transformations* is isomorphic to the group

$$SL(2,\mathbb{C})/\{\pm id\}$$

under the isomorphism

$$SL(2,\mathbb{C})/\{\pm id\} \xrightarrow{\simeq} Aut(\mathbb{P}^1)$$

induced from

$$SL(2,\mathbb{C}) \to Aut(\mathbb{P}^1), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a \cdot z + b}{c \cdot z + d}.$$

11. Let X be a topological space.

i) Consider a sheaf $\mathscr{F}$ of Abelian groups on X, an open set $U \subset X$ and a section $f \in \mathscr{F}(U)$. Show the equivalence:

$$f = 0 \in \mathscr{F}(U) \iff \pi_x^U(f) = 0 \in \mathscr{F}_x \text{ for all } x \in U.$$

ii) For two sheaf morphisms

$$\mathscr{F}_1 \xrightarrow{f} \mathscr{F} \text{ and } \mathscr{F} \xrightarrow{g} \mathscr{F}_2$$

show: If for an open set $U \subset X$ and for all $x \in U$ the morphisms of stalks satisfy

$$0 = [g_x \circ f_x : \mathscr{F}_{1,x} \to \mathscr{F}_{2,x}]$$

then the morphisms on the level of sections satisfy

$$0 = [g_U \circ f_U : \mathscr{F}_1(U) \to \mathscr{F}_2(U)].$$

12. i) For a topological space $(X, \mathscr{T})$ and a family $\mathscr{B}$ of open subsets of X prove the equivalence of the following two properties:

- The family $\mathscr{B}$ is a *basis* for $\mathscr{T}$, i.e. each open set $U \subset X$ is the union of elements from $\mathscr{B}$.

- For each open set $U \subset X$ and each point $x \in U$ exists an element B from $\mathscr{B}$ with
$$x \in B \subset U.$$

ii) Let X be a set and $\mathscr{B}$ a family of subsets of X with the following property:

- For each pair $B_1, B_2 \in \mathscr{B}$ and for each $x \in B_1 \cap B_2$ exists an element B from $\mathscr{B}$ with
$$x \in B \subset B_1 \cap B_2.$$

Show: The family
$$\mathscr{B} \cup \{\emptyset\} \cup \{X\}$$
is a basis for a topology on X.

Discussion: Monday, 4.11.2019

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Selected Solutions 03

9 . i) W.l.o.g. we assume

$$(w_1, w_2, w_3) = (0, 1, \infty).$$

Depending on the choice of (z_1, z_2, z_3) we consider the following fractional linear transformation

- $z_1, z_2, z_3 \neq \infty$:

$$f(z) := \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

- $z_1 = \infty$:

$$f(z) := \frac{z_2 - z_3}{z - z_3}$$

- $z_2 = \infty$:

$$f(z) := \frac{z - z_1}{z - z_3}$$

- $z_3 = \infty$:

$$f(z) := \frac{z - z_1}{z_2 - z_1}.$$

In each case

$$(f(z_1), f(z_2), f(z_3)) = (0, 1, \infty).$$

ii) To prove the uniqueness of f it suffices to show: The only fractional transformation f with three pairwise distinct fixed points is the identity. If for $z \neq \infty$

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d} = z$$

then

$$c \cdot z^2 + (d - a) \cdot z - b = 0.$$

The quadratic equation has

- two solutions iff $c \neq 0$
- exactly one solution iff $c = 0$ and $a \neq d$

- infinitely many solutions iff $c = 0$ and $a = d$ and $b = 0$

- no solution iff $c = 0$ and $a = d$ and $b \neq 0$. Then $f = id_{\mathbb{P}^1}$.

If for $z = \infty$

$$f(\infty) = \infty$$

then

$$f(\infty) = \frac{a}{c} = \infty$$

which implies $c = 0$ and

$$f(z) = \frac{a}{d} \cdot z + \frac{b}{d},\ d \neq 0.$$

Hence in any case, f has only one further fixed point besides ∞. As a consequence, any fractional linear transformation $f \neq id$ has at most two fixed points, q.e.d.

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Problems 04

13. On the Riemann surface $\mathbb{P}^1$ let $\mathscr{O}^0$ be the sheaf of holomorphic functions which vanish at $z = 0 \in \mathbb{P}^1$, i.e. for each open set $U \subset \mathbb{P}^1$

$$\mathscr{O}^0(U) := \begin{cases} \{f \in \mathscr{O}(U) : \ f(0) = 0\} & 0 \in U \\ \mathscr{O}(U) & 0 \notin U \end{cases}$$

Analogously let $\mathscr{O}^\infty$ be the sheaf of holomorphic functions on $\mathbb{P}^1$ which vanish at $z = \infty \in \mathbb{P}^1$. Set

$$\mathscr{F} := \mathscr{O}^0 \oplus \mathscr{O}^\infty$$

and consider the sheaf morphism

$$ad : \mathscr{F} \to \mathscr{O}$$

which is defined by the addition of functions

$$ad_U : \mathscr{F}(U) \to \mathscr{O}(U), \ (f_1, f_2) \mapsto f_1 + f_2, \ U \subset \mathbb{P}^1 \text{ open},$$

Show: For each $x \in \mathbb{P}^1$ the induced morphism of stalks

$$ad_x : \mathscr{F}_x \to \mathscr{O}_x$$

is surjective, but for some $U \subset X$ the morphism of groups of sections

$$ad_U : \mathscr{F}(U) \to \mathscr{O}(U)$$

is not surjective.

14. Let X be a topological space. For a morphism

$$f : \mathscr{F} \to \mathscr{G}$$

between two sheaves of Abelian groups on X show:

$$\mathscr{F}(U) \xrightarrow{f_U} \mathscr{G}(U) \text{ bijective for all open } U \subset X \iff \mathscr{F}_x \xrightarrow{f_x} \mathscr{G}_x \text{ bijective for all } x \in X.$$

15. Show: For each pair $(k_1, k_2) \in \mathbb{Z}^2$ the twisted sheaves on $\mathbb{P}^1$

$$\mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2) \text{ and } \mathscr{O}(k_1 + k_2)$$

are isomorphic, i.e. there exists a sheaf morphism

$$f : \mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2) \to \mathscr{O}(k_1 + k_2)$$

such that the induced morphisms f_x on the stalks are isomorphisms for all $x \in \mathbb{P}^1$.

16. Let X be a Riemann surface and $\mathscr{L}$ an invertible sheaf on X.

i) Show: The dual sheaf

$$\mathscr{L}^\vee = \mathscr{H}om_{\mathscr{O}}(\mathscr{L}, \mathscr{O})$$

is invertible.

Hint. You may prove first $\mathscr{H}om_{\mathscr{O}}(\mathscr{O}, \mathscr{O}) \simeq \mathscr{O}$.

ii) For $X = \mathbb{P}^1$ and $k \in \mathbb{Z}$ construct a canonical sheaf morphism

$$\mathscr{O}(-k) \to \mathscr{H}om_{\mathscr{O}}(\mathscr{O}(k), \mathscr{O}).$$

Show:

$$\mathscr{H}om_{\mathscr{O}}(\mathscr{O}(k), \mathscr{O}) \simeq \mathscr{O}(-k).$$

Discussion: Monday, 11.11.2019

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Selected Solutions 04

13 . For $x \in \mathbb{P}^1$ the induced morphism on stalks is

$$ad_x : F \to R$$

with

$$R \simeq \mathbb{C}\{z\}$$

and

$$F \simeq \begin{cases} \mathfrak{m} \oplus R & x = 0 \\ R \oplus R & x \in \mathbb{C}^* \\ R \oplus \mathfrak{m} & x = \infty \end{cases}$$

with

$$\mathfrak{m} = \{f \in R : f(0) = 0\}.$$

Apparently the addition ad_x is surjective.

One has

$$\mathscr{F}(\mathbb{P}^1) = \mathscr{O}^0(\mathbb{P}^1) \oplus \mathscr{O}^\infty(\mathbb{P}^1) = \{0\} \oplus \{0\} = \{0\},$$

but

$$\mathscr{O}(\mathbb{P}^1) = \mathbb{C}.$$

Hence $ad_{\mathbb{P}^1}$ is not surjective.

16 .

- *Commutative algebra*: Consider a commutative ring R with 1. Then the canonical multiplication map

$$\mu_R : R \to Hom_R(R,R),\ a \mapsto \mu_R(a) := [R \to R,\ b \mapsto a \cdot b]$$

  is an isomorphism of R-modules: Elements $\phi \in Hom_R(R,R)$ are determined by their value $\phi(1)$.

- *$\mathscr{O}$-module sheaves*: Let X be a Riemann surface and $\mathscr{F}$, $\mathscr{G}$ two $\mathscr{O}$-module sheaves on X. The $\mathscr{O}$-module structure defines by multiplication a sheaf morphism

$$\mu : \mathscr{O} \to \mathscr{H}om_{\mathscr{O}}(\mathscr{F},\mathscr{G}).$$

For each $x \in X$ we consider its induced morphism of stalks

$$\mu_x : \mathscr{O}_x \to (\mathscr{H}om_{\mathscr{O}}(\mathscr{G},\mathscr{G}))_x$$

It is induced by the following commutative diagrams, which exist for open neighbourhoods $U \subset X$ of x,

$$\begin{array}{ccc} \mathscr{O}(U) & \xrightarrow{\mu_U} & Hom_{\mathscr{O}|U}(\mathscr{F}|U,\mathscr{G}|U) \\ \downarrow \pi^U_x & & \downarrow \pi^U_x \\ \mathscr{O}_x & \overset{\mu_x}{\dashrightarrow} & (\mathscr{H}om_{\mathscr{O}}(\mathscr{G},\mathscr{G}))_x \end{array}$$

- *$\mathscr{H}om$ and stalks*: On the level of stalks we define for each $x \in X$ a morphism

$$\alpha : (\mathscr{H}om_{\mathscr{O}}(\mathscr{F},\mathscr{G}))_x \to Hom_{\mathscr{O}_x}(\mathscr{F}_x,\mathscr{G}_x)$$

such that the following diagram commutes:

$$\begin{array}{ccccc} & & Hom_{\mathscr{O}|U}(\mathscr{F}|U,\mathscr{G}|U) & & \\ & \swarrow \pi^U_x & & \searrow \beta & \\ (\mathscr{H}om_{\mathscr{O}}(\mathscr{F},\mathscr{G}))_x & & \overset{\alpha}{\dashrightarrow} & & Hom_{\mathscr{O}_x}(\mathscr{F}_x,\mathscr{G}_x) \end{array}$$

To define α represent a given element

$$\phi_x \in (\mathscr{H}om_{\mathscr{O}}(\mathscr{F},\mathscr{G}))_x$$

in an open neighbourhood U of x by a sheaf morphism

$$\phi_U : \mathscr{F}|U \to \mathscr{G}|U.$$

The latter induces a morphism of stalks

$$\beta(\phi_U) := (\phi_U)_x : \mathscr{F}_x \to \mathscr{G}_x.$$

Define

$$\alpha(\phi_x) := \beta(\phi_U).$$

One checks that the definition does not depend on the choice of the representative.

- *Hom and direct limit commute for the structure sheaf*: We specialize the result of the previous part to the structure sheaf

$$\mathscr{F} := \mathscr{G} := \mathscr{O}$$

and show that

$$\alpha : (\mathscr{H}om_{\mathscr{O}}(\mathscr{O},\mathscr{O}))_x \to Hom_{\mathscr{O}_x}(\mathscr{O}_x,\mathscr{O}_x)$$

is an isomorphism of stalks.

We claim: Any $\mathscr{O}|U$-linear sheaf morphism

$$\psi : \mathscr{O}|U \to \mathscr{O}|U,\ U \subset X \text{ open neighbourhood of } x,$$

is the multiplication by the holomorphic function

$$f := \psi_U(1) \in \mathscr{O}(U).$$

For the proof note that for any connected open $V \subset U$ the following diagram commutes:

$$\begin{array}{ccc} \mathscr{O}(V) & \stackrel{\psi_V}{\dashrightarrow} & \mathscr{O}(V) \\ \pi_x^V \downarrow & & \downarrow \pi_x^V \\ \mathscr{O}_x & \xrightarrow{\psi_x} & \mathscr{O}_x \end{array}$$

If $g \in \mathscr{O}(V)$ and

$$h := \psi_V(g) \in \mathscr{O}(V)$$

then

$$h_x = \psi_x(g_x) = \psi_x(g_x \cdot 1_x) = g_x \cdot \psi_x(1_x) = g_x \cdot f_x \in \mathscr{O}_x$$

The equality implies

$$h = f|V \cdot g$$

due to the identity theorem and proves the claim.

The identification of each $\mathscr{O}|U$-linear morphism

$$\psi : \mathscr{O}|U \to \mathscr{O}|U$$

with the multiplication by

$$f := \psi_U(1) \in \mathscr{O}(U)$$

shows that the $\mathscr{O}_x$-linear map

$$\alpha : (\mathscr{H}om_{\mathscr{O}}(\mathscr{O},\mathscr{O}))_x \to Hom_{\mathscr{O}_x}(\mathscr{O}_x,\mathscr{O}_x)$$

is an isomorphism.

- *Isomorphism of the multiplication morphism*: With $R := \mathscr{O}_x$ the composition

$$\mu_R = [R \xrightarrow{\mu_x} (\mathscr{H}om_{\mathscr{O}}(\mathscr{O},\mathscr{O}))_x \xrightarrow{\alpha} Hom_R(R,R)]$$

is an isomorphism due to part 1) and part 3). As a consequence also

$$\mathscr{O}_x \xrightarrow{\mu_x} (\mathscr{H}om_{\mathscr{O}}(\mathscr{O},\mathscr{O}))_x$$

is an isomorphism. Exercise 14 implies that the multiplication morphism

$$\mu : \mathscr{O} \to \mathscr{H}om_{\mathscr{O}}(\mathscr{O},\mathscr{O})$$

is an isomorphism.

- *Dual of twisted sheaves*: The multiplication morphism

$$\mu : \mathscr{O}(-k) \to \mathscr{H}om_{\mathscr{O}}(\mathscr{O}(k),\mathscr{O})$$

is defined on the level of sections: For open $U \subset \mathbb{P}^1$ each
element $s \in \mathscr{O}(-k)(U)$ defines by multiplication a morphism

$$\mathscr{O}(k)|U \to \mathscr{O}|U$$

Because the transformation g_{01}^{-k} of the local functions of sections in $\mathscr{O}(-k)$ and the transformation $g_{01}^{k_1}$ of the local functions of sections in $\mathscr{O}(k)$ multiply to

$$g_{01}^{-k} \cdot g_{01}^{k} = 1.$$

The morphism above induces an isomorphism on the level of stalks, because in a neighbourhood where the invertible sheaves restrict to the structure sheaf

$$\mathscr{O} \xrightarrow{\simeq} \mathscr{H}om_{\mathscr{O}}(\mathscr{O},\mathscr{O})$$

Hence the morphism μ is an isomorphism of invertible sheaves on $\mathbb{P}^1$.

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Problems 05

17. Consider a map $p : X \to Y$ between topological spaces. Show the equivalence:

$$p \text{ local homeomorphism } \iff p \text{ unbranched covering projection.}$$

18. i) Show: The exponential map

$$exp : \mathbb{C} \to \mathbb{C}^*$$

is an unbounded, unbranched covering projection.

ii) Conclude: Each holomorphic function

$$f : G \to \mathbb{C}^*$$

with a simply-connected domain $G \subset \mathbb{C}$ has a holomorphic logarithm, i.e. a holomorphic function

$$F : G \to \mathbb{C}$$

with

$$exp(F) = f.$$

19. Consider a presheaf $\mathscr{F}$ on a locally-connected Hausdorff space X which satisfies the identity theorem. Show: The étale space $|\mathscr{F}|$ is a Hausdorff space.

Hint: For two germs $f_x \neq g_y$ you may consider separately the cases $x \neq y$ and $x = y$.

20. Let $X \subset \mathbb{C}$ be open and $x \in X$ a given point. The sheaf $\mathscr{F}$ on

$$Y := X \setminus \{x\}$$

of locally constant integer-valued functions induces a presheaf $\mathscr{F}^X$ on X with

$$\mathscr{F}^X(U) = \begin{cases} \mathscr{F}(U) & x \notin U \\ 0 & x \in U \end{cases}$$

for connected open $U \subset X$, and restrictions derived from the restrictions of $\mathscr{F}$.
Show:

i) The presheaf $\mathscr{F}^X$ is a sheaf on X.

ii) The stalks at x satisfy

$$(\mathscr{H}om(\mathscr{F}^X, \mathscr{F}^X))_x \neq \{0\} \text{ and } Hom(\mathscr{F}^X_x, \mathscr{F}^X_x) = 0.$$

Discussion: Monday, 18.11.2019

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Selected Solutions 05

20 i) For connected open $U \subset X$ and an open covering $\mathscr{U} = (U_i)_{i \in I}$ of open sets, each compatible family $(f_i)_{i \in I}$ of sections $f_i \in \mathscr{F}^X(U_i),\ i \in I$, defines a locally constant function f on U, hence a constant $f \in \mathbb{Z}$.

If $x \in U$ then $x \in U_i$ for at least one $i \in I$ and we have $f_i = 0$. Hence $f = 0$.

ii) On one hand, we have

$$\mathscr{F}^X_x = 0 \text{ and } Hom(\mathscr{F}^X_x, \mathscr{F}^X_x) = 0$$

because $\mathscr{F}^X(U) = 0$ for each connected neighborhood $U \subset X$ of x.

On the other hand

$$(\mathscr{H}om(\mathscr{F}^X, \mathscr{F}^X))_x \neq 0$$

because for any open neighbourhood $U \subset X$ of x the restriction

$$\mathscr{F}^X|U \neq 0.$$

Hence the identity morphisms

$$id : \mathscr{F}^X|U \to \mathscr{F}^X|U,\ U \subset X \text{ open neighbourhood of } x,$$

define an element

$$0 \neq id \in (\mathscr{H}om(\mathscr{F}^X, \mathscr{F}^X))_x \neq 0.$$

Note: As a consequence, the canonical morphism

$$(\mathscr{H}om(\mathscr{F}^X, \mathscr{F}^X))_x \to Hom(\mathscr{F}^X_x, \mathscr{F}^X_x)$$

is not injective.

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RIEMANN SURFACES

Problems 06

21. Consider a compact Riemann surface X and finitely many points $p_1,...,p_n \in X$. Set

$$X' := X \setminus \{p_1,...,p_n\}$$

and consider a non-constant holomorphic function

$$f : X' \to \mathbb{C}.$$

Show: The image of f comes arbitrary close to every $c \in \mathbb{C}$, i.e.

$$\overline{f(X')} = \mathbb{C}.$$

22. Consider an unbranched covering projection

$$p : (Y,y_0) \to (X,x_0)$$

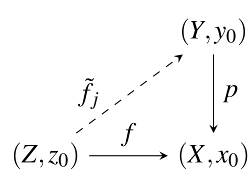
of topological Hausdorff spaces and a continous map

$$f : (Z,z_0) \to (X,x_0)$$

with Z a connected topological space. Assume the existence of two continuous maps

$$\tilde{f}_j,\ j = 1,2,$$

which render commutative the following diagram



Show: $\tilde{f}_1 = \tilde{f}_2$.

23. Consider a holomorphic map

$$f : T_1 \to T_2$$

between two complex tori

$$T_j := \mathbb{C}/\Lambda_j,\ j = 1,2, \text{ with canonical projections } \pi_j : \mathbb{C} \to T_j$$

and assume $f(0) = 0$.

i) Show: There exists a unique holomorphic map

$$F : \mathbb{C} \to \mathbb{C}$$

with $F(0) = 0$ and such that the following diagram commutes

$$\begin{array}{ccc} (\mathbb{C},0) & \overset{F}{\dashrightarrow} & (\mathbb{C},0) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ (T_1,0) & \xrightarrow{f} & (T_2,0) \end{array}$$

ii) Show: There exists a unique $\alpha \in \mathbb{C}$ satisfying

$$\alpha \cdot \Lambda_1 \subset \Lambda_2$$

and for all $z \in \mathbb{C}$

$$F(z) = \alpha \cdot z.$$

24. Consider a Riemann surface X, a point $x \in X$ and a holomorphic germ $f_a \in \mathscr{O}_a$. Show: Two maximal global analytic continuations of f_a

$$(p, f, b) \text{ and } (p', f', b')$$

are biholomorphically equivalent, i.e. there exists a biholomorphic map

$$F : (Y', b') \to (Y, b)$$

such that the following diagram commutes

$$\begin{array}{ccc} (Y',b') & \overset{F}{\dashrightarrow} & (Y,b) \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & (X,a) & \end{array}$$

and $f' = F^*(f)$.

Discussion: Monday, 25.11.2019

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Problems 07

25. Consider a holomorphic unbounded, unbranched covering projection

$$p : Y \to X$$

between two Riemann surfaces and a holomorphic function $f \in \mathscr{O}_Y(Y)$. For a given point $b \in Y$ set

$$a := p(b) \in X \text{ and } f_a := p_*(f_b) \in \mathscr{O}_{X,a}.$$

For the tuple

$$(p, f, b)$$

show the equivalence:

- The tuple (p, f, b) is a maximal global analytic continuation of $f_a \in \mathscr{O}_{X,a}$
- For any two distinct points $b_1,\ b_2 \in Y_a$

$$p_*(f_{b_1}) \neq p_*(f_{b_2}).$$

26. Consider a Riemann surface X. Show: The definition of the exterior derivations

$$d,\ d',\ d'' : \mathscr{E}_X^j \to \mathscr{E}_X^{j+1},\ j = 0, 1,$$

does not depend on the choice of charts of X.

27. For a complex torus T show: Each holomorphic map

$$f : \mathbb{P}^1 \to T$$

is constant.

28. Let

$$R := \{f : U \to \mathbb{C} |\ U \subset \mathbb{C} \text{ open neighbourhood of } 0,\ f \text{ smooth}\}$$

be the ring of smooth functions in a neighbourhood of zero,

$$\mathfrak{m} \subset R$$

its maximal ideal, and

$$T^1 R := \mathfrak{m}/\mathfrak{m}^2$$

the cotangent space of R. A *derivation* of R is a $\mathbb{C}$-linear map

$$D : R \to \mathbb{C}$$

which satisfies the product rule

$$D(f_1 \cdot f_2) = Df_1 \cdot f_2(0) + f_1(0) \cdot Df_2,\ f_1, f_2 \in R.$$

Denote the complex vector space of derivations of R by

$$Der(R, \mathbb{C})$$

Show:

i) Each derivation $D \in Der(R, \mathbb{C})$ restricts to the zero map

$$D|\mathbb{C} = 0$$

on the subspace $\mathbb{C} \subset R$ of constant functions.

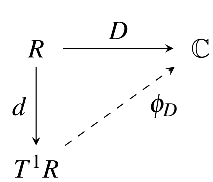
ii) Each derivation

$$D \in Der(R, \mathbb{C})$$

induces a $\mathbb{C}$-linear map

$$\phi_D : T^1 R \to \mathbb{C}$$

such that the following diagram commutes



Here d denotes the differential, defined as

$$df := f - f(0) \quad \mod \mathfrak{m}^2.$$

iii) The map

$$\phi : Der(R, \mathbb{C}) \to Hom_{\mathbb{C}}(T^1 R,\ \mathbb{C}),\ D \mapsto \phi_D,$$

is an isomorphism of complex vector spaces.

Note. The vector space $Der(R, \mathbb{C})$ is named the *tangent space* of R.

Hint ad iii): You may first prove that the differential d satisfies the product rule.

Discussion: Monday, 2.12.2019

DEPARTMENT OF MATHEMATICS — RIEMANN SURFACES
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Selected Solutions 07

25 . i) The maximal global analytic continuation $f_a \in \mathscr{O}_{X,a}$ is uniquely determined. It has been constructed by using $Z \subset |\mathscr{O}|$. The points of Z correspond bijectively to those germs of $\mathscr{O}_{X,a}$ which originate from $f_a \in \mathscr{O}_{X,a}$ by analytic continuation along a path in X.

By definition of the holomorphic function f on Z for $b_1 \in Z_a$ the germ of $f_{b_1} \in \mathscr{O}_{Z,b_1}$ maps via p_* to the germ from the stalk $\mathscr{O}_{X,a}$ which equals $b_1 \in Z$ (tautological definition). Hence

$$b_1 \neq b_2 \implies p_*(f_{b_1}) \neq p_*(f_{b_2}).$$

ii) Assume

$$b_1 \neq b_2 \implies p_*(f_{b_1}) \neq p_*(f_{b_2}).$$

Consider the maximal analytic continuation (q, g, c) of $f_a \in \mathscr{O}_{X,a}$ with

$$q : (Z, c) \to (X, a).$$

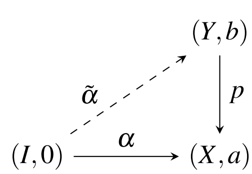
We define

$$F : Z \to Y$$

as follows: A point $\zeta \in Z$ is a germ $f_x \in \mathscr{O}_{X,x}$ which originates from f_a by analytic continuation along a path α in X from a to $x := q(\zeta)$. Because

$$p : Y \to X$$

is an unbounded, unbranched covering projection and I is connected and simply connected, the path α lifts to a unique path $\tilde{\alpha}$ in Y such that the following diagram commutes:



Here $b \in Y$ is the unique point from the fibre Y_a with

$$p_*(f_b) = f_a \in \mathscr{O}_{X,a}.$$

Set

$$F(\zeta) := \tilde{\alpha}(1) \in Y.$$

Then (p, f, b) induces the maximal global analytic continuation via F, and hence any global continuation of f_a.

28 . i) The product rule

$$D(1) = D(1 \cdot 1) = D1 \cdot 1 + 1 \cdot D1 = 2 \cdot D(1)$$

implies $D(1) = 0$ and by $\mathbb{C}$-linearity $D|\mathbb{C} = 0$.

ii) The product rule implies

$$D|\mathfrak{m}^2 = 0.$$

Therefore D induces a unique $\mathbb{C}$-linear map ϕ_D which renders commutative the given diagram.

iii) One checks that the map

$$\phi : Der(R, \mathbb{C}) \to Hom_{\mathbb{C}}(T^1 R, \mathbb{C})$$

is $\mathbb{C}$-linear. We define

$$\psi : Hom_{\mathbb{C}}(T^1 R, \mathbb{C}) \to Der(R, \mathbb{C}),\ \chi \mapsto D := \chi \circ d$$

Note that $d : R \to T^1$ satisfies the product rule, because

$$d(f_1 \cdot f_2) = f_1 \cdot f_2 - f_1(0) \cdot f_2(0) \mod \mathfrak{m}^2 =$$

$$= f_1 \cdot f_2 - f_1(0) \cdot f_2(0) - (f_1 - f_1(0))(f_2 - f_2(0)) \mod \mathfrak{m}^2 =$$

$$= f_1(0) f_2 + f_2(0) f_1 - 2 \cdot f_1(0) \cdot f_2(0) \mod \mathfrak{m}^2 =$$

$$= f_1(0)(f_2 - f_2(0)) \mod \mathfrak{m}^2 + f_2(0)(f_1 - f_1(0)) \mod \mathfrak{m}^2 =$$

$$= f_1(0) \cdot df_2 + f_2(0) \cdot df_1.$$

As a consequence, also the composition

$$D := \chi \circ d : R \to \mathbb{C}$$

is a derivation. One checks that ϕ and ψ are inverse maps, q.e.d.

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Problems 08

29. Consider the holomorphic differential form

$$\omega = \frac{dz}{z} \in \Omega^1(\mathbb{C}^*).$$

i) Show: The form ω extends uniquely to a meromorphic differential form

$$\tilde{\omega} \in \mathscr{M}(\mathbb{P}^1).$$

Determine the residues of $\tilde{\omega}$ at its singularities.

ii) Show: There exists a unique $k \in \mathbb{Z}$ such that $\tilde{\omega}$ defines a global meromorphic section of the twisted sheaf $\mathscr{O}(k)$. Define a sheaf isomorphism

$$\Omega^1 \to \mathscr{O}(k).$$

Note. A global meromorphic section of $\mathscr{O}(k)$ is a pair of meromorphic functions

$$(s_0, s_1) \in \mathscr{M}^1(U_0) \times \mathscr{M}^1(U_1)$$

satisfying $s_0 = g_{01}^k \cdot s_1$.

iii) Does there exist a non-zero holomorphic differential form on $\mathbb{P}^1$?

30. Consider a torus $T = \mathbb{C}/\Lambda$ with a complex atlas

$$\mathscr{A} = (z_i : U_i \to V_i)_{i \in I}$$

such that for all $i, j \in I$ the difference

$$z_i - z_j : U_i \cap U_j \to \mathbb{C}$$

is locally constant with values in Λ.

i) Show: The family $(dz_i)_{i \in I}$ with $dz_i \in \Omega^1(U_i)$ is a global holomorphic form on T, named

$$dz \in \Omega^1(T).$$

ii) Show: There exists an isomorphism of sheaves on T

$$\Omega^1 \to \mathscr{O},\ (f_i\, dz_i)_{i \in I} \mapsto (f_i)_{i \in I},$$

iii) Show

$$\Omega^1(T) \simeq \mathbb{C}$$

and conclude: For any meromorphic function $f \in \mathscr{M}(T)$ holds

$$0 = \sum_{p \in T} res(f;p).$$

31. Consider a holomorphic map

$$p : X \to Y$$

between Riemann surfaces. By means of the sheaf morphism

$$p^* : \mathscr{E}_Y \to p_* \mathscr{E}_X,\ f \mapsto p^* f := f \circ p,$$

define for a chart $z : U \to V$ of Y the pullbacks - using the same notations -

$$p^* : \mathscr{E}_Y^1(U) \to (p_* \mathscr{E}_X^1)(U),\ f \cdot dz + g \cdot d\bar{z} \mapsto p^* f \cdot d(p^* z) + p^* g \cdot d(p^* \bar{z})$$

and

$$p^* : \mathscr{E}_Y^2(U) \to (p_* \mathscr{E}_X^2)(U),\ f \cdot dz \wedge d\bar{z} \mapsto p^* f \cdot d(p^* z) \wedge d(p^* \bar{z})$$

Show:

These local pullbacks glue to global pullbacks independent from the choice of charts, i.e. to sheaf morphisms

$$p^* : \mathscr{E}_Y^1 \to p_* \mathscr{E}_X^1 \text{ and } p^* : \mathscr{E}_Y^2 \to p_* \mathscr{E}_X^2.$$

They respect holomorphy, i. e.

$$p^*(\mathscr{O}_Y) \subset p_* \mathscr{O}_X \text{ and } p^*(\Omega_Y^1) \subset p_* \Omega_X^1.$$

32. Consider

$$\phi := exp : \mathbb{C} \to \mathbb{C}^* \text{ and } \eta := \frac{dz}{z} \in \Omega^1_{\mathbb{C}^*}(\mathbb{C}^*).$$

Determine the pullback

$$\phi^* \eta \in \Omega^1_{\mathbb{C}}(\mathbb{C}).$$

Discussion: Monday, 9.12.2019

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Selected Solutions 08

31 . E.g., consider

$$p^*: \mathscr{E}_Y^{1,0}(U) \to (p_*\mathscr{E}_X^{1,0})(U) = \mathscr{E}_X^{1,0}(p^{-1}(U))$$

and two charts

$$z,\ w: U_{ij} \to \mathbb{C}$$

with $w = \psi(z)$ holomorphic. We have

$$w = \psi(z) \implies dw = \psi'\ dz$$

hence

$$dw = \psi'\ dz$$

If

$$f\ dz = \eta = g\ dw$$

then

$$f\ dz = g \cdot \psi'\ dz \text{ or } f = g \cdot \psi'.$$

As a consequence, there are equivalences

$$p^*\eta \text{ well defined} \iff p^*f\, d(p^*z) = p^*g\, d(p^*w) \iff (f \circ p)\, d(z \circ p) = (g \circ p)\, d(w \circ p) \iff$$

$$(f \circ p)\ d(z \circ p) = (g \circ p) \cdot (\psi' \circ p) \cdot d(z \circ p) \iff f \circ p = (g \circ p) \cdot (\psi' \circ p)$$

which is satisfied.

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Problems 09

33. Show: On a Riemann surface X the sequence of sheaf morphisms with j the canonical injection

$$0 \to \mathbb{C} \xrightarrow{j} \mathscr{O} \xrightarrow{d} \Omega^1 \to 0$$

is exact.

34. Consider a non-constant holomorphic map

$$f : X \to Y$$

between two Riemann surfaces. For two points $b \in Y$ and $a \in X_a$ denote by

$$k := \mathrm{v}(f;\ a) \in \mathbb{N}^*$$

the multiplicity of f at a. For a holomorphic differential form

$$\omega \in \Omega^1_Y(Y \setminus b)$$

show: The pullback

$$f^*\omega \in \Omega^1_X(X \setminus X_b)$$

satisfies

$$res(f^*\omega;\ a) = k \cdot res(\omega;\ b).$$

35. Let X be a Riemann surface. A differential form $\omega \in \mathscr{E}^1_X(X)$ with

$$d\omega = 0$$

has a *primitive* $F \in \mathscr{E}_X(X)$ if

$$dF = \omega.$$

Show: For any differential form $\omega \in \mathscr{E}^1_X(X)$ with $d\omega = 0$ exists a Riemann surface Y and a holomorphic unbounded, unbranched covering projection

$$p : Y \to X$$

such that the pullback $p^*\omega \in \mathscr{E}^1_Y(Y)$ has a primitive.

Hint: Consider the sheaf $\mathscr{F}$ on X of local primitives of ω defined as

$$\mathscr{F}(U) := \{f \in \mathscr{E}_X(U) : \ df = \omega\}$$

and its étale space $p : |\mathscr{F}| \to X$. The exact de Rahm sequence implies that p is an unbounded, unbranched covering projection. The definition of $F : Y \to \mathbb{C}$ is tautological.

36. Show: On a topological space X the covariant functor "global sections"

$$\Gamma(X,-) : \underline{Sheaf}_X \to \underline{Ab},\ \Gamma(X,\mathscr{F}) := \mathscr{F}(X),$$

is *left-exact*, i.e. for any short exact sequence of sheaves of Abelian groups on X

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

the sequence of Abelian groups

$$0 \to \Gamma(X,\mathscr{F}) \xrightarrow{\Gamma(\alpha)} \Gamma(X,\mathscr{G}) \xrightarrow{\Gamma(\beta)} \mathscr{H}(X)$$

is exact.

Here $\underline{Sheaf}_X$ denotes the category of sheaves of Abelian groups on X and $\underline{Ab}$ denotes the category of Abelian groups.

Discussion: Monday, 16.12.2019

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Selected Solutions 09

33 . The question is local. Hence we may assume $X = \mathbb{C}$ and $x = 0 \in \mathbb{C}$.

- *Exactness at* $\mathbb{C}$: Injection
$$\mathbb{C} \hookrightarrow \mathscr{O}_x$$
- *Exactness at* $\mathscr{O}_x$: Apparently
$$d \circ j = 0.$$
Conversely: If
$$df = 0$$
then the holomorphic germ $f \in \mathscr{O}_x$ is locally constant because
$$\partial f = \overline{\partial} f = 0.$$
- *Exactness at* Ω^1_x: Consider
$$\omega = g \cdot dz \in \Omega^1_x.$$
If $f \in \mathscr{O}_x$ then
$$df = d'f = \frac{\partial f}{\partial z} dz = g\ dz \iff \frac{\partial f}{\partial z} = f' = g.$$
One obtains a primitive of g by formal integration of the taylor series: If
$$g(z) = \sum_{n=0}^{\infty} c_n \cdot z^n$$
then define
$$f(z) := \sum_{n=0}^{\infty} \frac{c_n}{n+1} \cdot z^{n+1}$$

34 . The claim is local with respect to $b \in Y$ and $a \in X_b$. We may assume $Y \subset \mathbb{C}$ a disk with $b = 0$, and $X \subset \mathbb{C}$ a disk with $a = 0$, and
$$f(z) = z^k,\ k \neq 0.$$

Consider a holomorphic form

$$\omega(w) = h(w)\ dw \in \Omega^1(Y \setminus \{b\})$$

With

$$w = f(z) = z^k$$

by definition

$$(f^*\omega)(z) = (f^*h)(z)\ d(f^*w) = (h \circ f)(z) \cdot d(z^k) = h(z^k) \cdot k \cdot z^{k-1} dz.$$

The Laurent expansion

$$h(w) = \sum_{n=-\infty}^{\infty} c_n \cdot w^n$$

implies

$$h(z^k) = \sum_{n=-\infty}^{\infty} c_n \cdot z^{kn}$$

and

$$z^{k-1} \cdot h(z^k) = \sum_{n=-\infty}^{\infty} c_n \cdot z^{kn+(k-1)}.$$

From $kn + (k-1) = -1$ follows

$$k(n+1) - 1 = -1 \text{ or } n = -1$$

Hence

$$res_w(h(w);\ 0) = c_{-1} = res_z(z^{k-1} \cdot h(z^k);\ 0)$$

and

$$k \cdot res(\omega;\ b) = k \cdot res_w(h(w);\ 0) = k \cdot res_z(z^{k-1} \cdot h(z^k);\ 0) =$$
$$= res_z(h(z^k) \cdot k \cdot z^{k-1};\ 0) = res(f^*\omega;\ a).$$

36 . Cf. “Otto Forster: Lectures on Riemann Surfaces.” Lemma 15.8.

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Problems 10

37. i) For a vector space V and a semi-norm

$$p : V \to \mathbb{R}_+$$

show:

$$p(0) = 0 \text{ and } p(\mathrm{v}) \geq 0 \text{ for all } \mathrm{v} \in V.$$

ii) Consider a Fréchet space V with its topology defined by the sequence $(p_n)_{n\in\mathbb{N}}$ of semi-norms. Show:

$$V \text{ Hausdorff } \iff \text{ For each } \mathrm{v} \in V,\ \mathrm{v} \neq 0,\ \text{exists } n \in \mathbb{N} \text{ with } p_n(\mathrm{v}) \neq 0.$$

38. Consider a disk

$$D = D_r(0) \subset \mathbb{C},\ 0 < r < \infty,$$

and the space $L^2(D)$ of square-integrable holomorphic functions on D. For the monomials

$$\phi_n(z) := z^n,\ n \in \mathbb{N},$$

compute the Hermitian products

$$< \phi_n, \phi_m >,\ n, m \in \mathbb{N}.$$

39. For a simply connected Riemann surface X show

$$H^1(X, \mathbb{C}) = 0.$$

40. For a simply connected Riemann surface X show

$$H^1(X, \mathbb{Z}) = 0.$$

Hint: Use Exercise 39

Discussion: Monday, 13.1.2020

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Problems 11

41. Show:

$$H^1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}.$$

Hint: Apply Leray's theorem to the covering $\mathscr{U} = (U_1, U_2)$ with

$$U_1 := \mathbb{C}^* \setminus \mathbb{R}_+ \text{ and } U_2 := \mathbb{C}^* \setminus \mathbb{R}_-$$

42. Find a Riemann surface X, an open covering $\mathscr{U}$ of X and a sheaf $\mathscr{F}$ on X with

$$H^1(\mathscr{U}, \mathscr{F}) \neq H^1(X, \mathscr{F}).$$

43. Denote by $D \subset \mathbb{C}$ the unit disk and by $D^* := D \setminus \{0\}$ the punched unit disk.

i) Show: The function

$$f : D^* \to \mathbb{C},\ f(z) := 1/z,$$

does not belong to $L^2(D^*, \mathscr{O})$.

ii) Show: The restriction map

$$L^2(D, \mathscr{O}) \to L^2(D^*, \mathscr{O})$$

is an isomorphism.

44. Consider a Riemann surface X.

i) For a pair of relatively compact open subsets

$$V \subset\subset U \subset X$$

show: There are only finitely many connected components of U which intersect with V.

ii) For two finite coverings of X

$$\mathscr{V} << \mathscr{U}$$

show: The restriction

$$Z^1(\mathscr{U},\mathbb{C}) \to Z^1(\mathscr{V},\mathbb{C})$$

has finite-dimensional image.

iii) For compact X give a direct proof: There exist

- a finite family of charts for X

  $$(\phi_i : U_i \to D_i,\ D_i \subset \mathbb{C} \text{ disk })_{i \in I}$$

  with

  $$\mathscr{U} = (U_i)_{i \in I}$$

  a covering of X,

- and an open covering $\mathscr{V} = (V_i)_{i \in I}$ of X with

  $$\mathscr{V} << \mathscr{U}$$

  and $\phi_i(V) \subset \mathbb{C}$ a disk for all $i \in I$.

iv) Show: For compact X

$$dim_{\mathbb{C}}\ H^1(X,\mathbb{C}) < \infty.$$

Discussion: Monday, 20.1.2020

Selected Solutions 11

44 . i) Connected components of X are open and pairwise disjoint. The compact set $\overline{V}$ is covered by all connected components of U. Hence $\overline{V}$ is already covered by finitely many connected components of U.

ii) For a finite covering $\mathscr{U} = (U_i)_{i\in I}$ of X there are only finitely many pairs $(i,j) \in I^2$. For each pair $(i,j) \in I$ the intersection

$$V_i \cap V_j \subset\subset U_i \cap U_j$$

is contained in finitely many connected components of $U_i \cap U_j$ due to part i). For any cocycle

$$(f_{ij})_{ij} \in Z^1(\mathscr{U}, \mathbb{C})$$

the element $f_{ij} \in \mathbb{C}(U_i \cap U_j)$ is constant on each connected component of $U_i \cap U_j$. Hence the restriction

$$Z^1(\mathscr{U}, \mathbb{C}) \to Z^1(\mathscr{V}, \mathbb{C})$$

has finite-dimensional image.

iii) If X is compact, then we choose a finite open covering $\mathscr{U} = (U_i)_{i\in I}$ of X, such that for each $i \in I$ the set U_i is homeomorphic to a disk $D_i \subset \mathbb{C}$. Any shrinking $\mathscr{W} \subset\subset \mathscr{U}$ extends to a shrinking

$$\mathscr{V} = (V_i)_{i\in I} << \mathscr{U}$$

such that for all $i \in I$ the set $V_i \subset\subset U_i$ is homeomorphic to a disk which is relatively compact in D_i.

iv) Both coverings $\mathscr{U}$ and $\mathscr{V}$ from part iii) are Leray coverings of X for the sheaf $\mathbb{C}$. Hence the identity

$$H^1(X, \mathbb{C}) = H^1(\mathscr{U}, \mathbb{C}) \to H^1(\mathscr{V}, \mathbb{C}) = H^1(X, \mathbb{C})$$

factors over the restriction from part ii). As a consequence

$$dim_{\mathbb{C}}\, H^1(X, \mathbb{C}) < \infty.$$

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Problems 12

45. Show: There is no meromorphic function on a torus with a single pole, and this pole has order $= 1$.

46. For a compact Riemann surface X show:

i) The injection $\mathbb{Z} \hookrightarrow \mathbb{C}$ induces an injection

$$H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathbb{C})$$

ii) The $\mathbb{Z}$-module $H^1(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank.

Hint. Similarly to Exercise 44 show first that $H^1(X, \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module.

47. For a Riemann surface X show:

i) Any open covering $\mathscr{U}$ of X has a locally-finite, countable refinement

$$\mathscr{W} = (W_i)_{i \in \mathbb{Z}} < \mathscr{U}$$

and a subordinate integer-valued partition of unity, i.e. a family

$$(\phi_i : X \to \mathbb{Z})_{i \in Z}$$

with $\phi_i | X \setminus V_i = 0$ for all $i \in \mathbb{Z}$ and

$$\sum_{i \in \mathbb{Z}} \phi_i = 1$$

ii) The divisor sheaf $\mathscr{D}$ on X satisfies

$$H^1(X, \mathscr{D}) = 0.$$

48. Show for the divisor class group of the projective space

$$Cl(\mathbb{P}^1) \simeq \mathbb{Z}$$

Discussion: Monday, 27.1.2020

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Selected Solutions 12

46 . i) The injectivity follows from the proof of Exercise 40: If an integer valued cocycle splits in $H^1(X,\mathbb{C})$ then it splits already in $H^1(X,\mathbb{Z})$.

ii) Similarly to exercise 44, part i) and ii) for two finite coverings

$$\mathscr{V} << \mathscr{U}$$

of X the image of the restriction

$$Z^1(\mathscr{U},\mathbb{Z}) \to Z^1(\mathscr{V},\mathbb{Z})$$

is a free $\mathbb{Z}$-module of finite rank. Due to compactness of X we may assume the existence of two finite coverings of X

$$\mathscr{V} << \mathscr{U}$$

with simply connected covering sets. Hence both coverings are Leray coverings with respect to the sheaf $\mathbb{Z}$. As a consequence the identity

$$H^1(X,\mathbb{Z}) = H^1(\mathscr{U},\mathbb{Z}) \to H^1(\mathscr{V},\mathbb{Z}) = H^1(X,\mathbb{Z})$$

factorizes over the restriction

$$Z^1(\mathscr{U},\mathbb{Z}) \to Z^1(\mathscr{V},\mathbb{Z})$$

and the image of the restriction

$$H^1(\mathscr{U},\mathbb{Z}) \to H^1(\mathscr{V},\mathbb{Z})$$

is finitely generated. Hence

$$H^1(X,\mathbb{Z})$$

is a finitely-generated $\mathbb{Z}$-module. The inclusion

$$H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{C}) \simeq \mathbb{C}^n$$

excludes any torsion elements of $H^1(X,\mathbb{Z})$. Therefore $H^1(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank.

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49. Prove $H^1(\mathbb{P}^1, \mathscr{O}^*) \simeq \mathbb{Z}$.

Hint. You may use without proof $H^2(S^2, \mathbb{Z}) \simeq \mathbb{Z}$.

50. For a twisted sheaf $\mathscr{O}(k),\ k \in \mathbb{Z}$, on $\mathbb{P}^1$ determine a divisor $D \in Div(\mathbb{P}^1)$ with

$\mathscr{O}_D \simeq \mathscr{O}(k)$ and determine $deg\ D$.

51. Consider a Riemann surface X.

i) Show: For any divisor $D \in Div(X)$ the $\mathscr{O}$-module sheaf $\mathscr{O}_D$ is invertible.

ii) For two divisors $D_1, D_2 \in Div(X)$ show:

$$\mathscr{O}_{D_1} \otimes_{\mathscr{O}} \mathscr{O}_{D_2} \simeq \mathscr{O}_{D_1+D_2}.$$

iii) For a divisor $D \in Div(X)$ conclude:

$$(\mathscr{O}_D)^\vee \simeq \mathscr{O}_{-D}.$$

52. Consider a compact Riemann surface X.

i) Show

$$dim\ H^0(X, \Omega^1) = g(X)$$

ii) Consider two non-zero forms $\eta_1,\ \eta_2 \in H^0(X, \mathscr{M}^1)$. Show:

$$div\ \eta_1 - div\ \eta_2 \in Div(X)$$

is a principal divisor.

iii) Show: Any divisor

$$K := div\ \eta \in Div(X)$$

with a non-zero form $\eta \in H^0(X, \mathscr{M}^1)$ satisfies

$$deg\ K = 2g(X) - 2.$$

Discussion: Monday, 3.2.2019

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Selected Solutions 13

50 . If $k \geq 0$ we consider the divisor $D = k \cdot P$ with the point divisor $P \in Div(\mathbb{P}^1)$ belonging to the point

$$p = 0 = (1:0) \in \mathbb{P}^1.$$

Choose the holomorphic section $s \in H^0(X, \mathscr{O}(k))$ which is defined with respect to the standard covering by

$$s = (s_0, s_1) \text{ with } s_0 = (z_1/z_0)^k,\ s_1 = 1.$$

We define a sheaf morphism

$$\mathscr{O}_D \to \mathscr{O}(k)$$

on a given open set $U \subset \mathbb{P}^1$

$$\mathscr{O}_D(U) \to \mathscr{O}(k)(U), f \mapsto f \cdot s|U.$$

Because f has a pole at p of order at most k and s has a zero at p of order k, the function $(f \cdot s|U)$ is holomorphic. On the intersetion U_{01} we have

$$f \cdot s_0 = f \cdot (g_{01}^k \cdot s_1) = g_{01}^k \cdot (f \cdot s_1)$$

Hence the morphism is well-defined. The sheaf morphism is an isomorphism on the stalks, hence an isomorphism of sheaves. We have $deg\ D = k$.

The case for $k < 0$ can be proved analogously, or considered a consequence of Exercise 51.

51 . i) For a given divisor $D \in Div(X)$ exist an open covering $\mathscr{U} = (U_i)_{i \in I}$ and a cochain $(f_i)_{i \in I} \in C^0(\mathscr{U}, \mathscr{M}^*)$ satisfying for all $i \in I$

$$D|U_i = di\ (f_i$$

For each $i \in I$ the sheaf morphism

$$\mathscr{O}_D|U_i \to \mathscr{O}|D,\ g \mapsto g \cdot f_i,$$

is well-defined and an isomorphism on stalks.

ii) Multiplication defines a morphism of sheaves

$$\mathcal{O}_{D_1} \otimes_{\mathcal{O}} \mathcal{O}_{D_2} \to \mathcal{O}_{D_1+D_2}, \text{ induced from } f_1 \otimes f_2 \mapsto f_1 \cdot f_2,$$

which is an isomorphism on stalks. Note that the left hand side $\mathcal{O}_{D_1} \otimes_{\mathcal{O}} \mathcal{O}_{D_2}$ is a sheafification.