DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
SUMMER TERM 2019

COMPLEX ANALYSIS

Joachim Wehler, Pascal Stucky

Problems 01

1. Expand the function

$$f : \mathbb{C} \setminus \{1\} \to \mathbb{C},\ f(z) := \frac{1}{1-z},$$

into a power series with center $a = -1$.

Determine the radius of convergence of the resulting power series. In the complex plane scetch the domain of convergence.

2. Prove for a power series

$$\sum_{n=0}^{\infty} c_n \cdot z^n$$

with radius of convergence R:

i) If for suitable $r > 0$ and for all but finitely many indices $n \in \mathbb{N}$

$$\sqrt[n]{|c_n|} < \frac{1}{r}$$

then $R \geq r$.

ii) If for suitable $r > 0$ and for infinitely many indices $n \in \mathbb{N}$

$$\sqrt[n]{|c_n|} > \frac{1}{r}$$

then $R \leq r$.

3. For each of the two power series

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \text{ and } g(z) = \sum_{n=0}^{\infty} z^{n!}$$

determine:

i) The radius of convergence,

ii) All points on the boundary of the disk of convergence where the series is convergent.

4. For each of the two power series

$$f(z) = \sum_{n=0}^{\infty} 2^n \cdot z^{2n} \text{ and } g(z) = \sum_{n=0}^{\infty} cos(n) \cdot z^n$$

determine the radius of convergence.

Hint for $g(z)$: For suitable $\alpha > 0$ holds $|cos(n)| > \alpha$ for infinitely many $n \in \mathbb{N}$.

Deadline: Monday, 29.4.2019, 10.15 a.m., box near room A109

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COMPLEX ANALYSIS

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Problems 02

5. Generalizing the well-known binomial coefficents $\binom{N}{n}$ one defines for arbitrary $\sigma \in \mathbb{C}$ and $n \in \mathbb{N}$

$$\binom{\sigma}{0} := 1, \ \binom{\sigma}{n} := \frac{\sigma \cdot (\sigma - 1) \cdot \ldots \cdot (\sigma - n + 1)}{n!}$$

The *binomial series* with parameter $\sigma \in \mathbb{C}$ is the power series

$$f_\sigma(z) := \sum_{n=0}^{\infty} \binom{\sigma}{n} z^n, \ z \in \mathbb{C}.$$

Show:

i) For any $\sigma \in \mathbb{C}$ and $n \in \mathbb{N}$ holds

$$\binom{\sigma}{n+1} = \frac{\sigma - n}{n+1} \cdot \binom{\sigma}{n}$$

ii) If $\sigma \in \mathbb{N}$ then $f_\sigma(z)$ has radius of convergence $R = \infty$, and for all $z \in \mathbb{C}$

$$f_\sigma(z) = (1+z)^\sigma$$ - give a proof, not a reference :-)

iii) If $\sigma \in \mathbb{C} \setminus \mathbb{N}$ then $f_\sigma(z)$ has radius of convergence $R = 1$.

6. Consider two power series

$$f_1(z) := \sum_{n=0}^{\infty} a_n \cdot z^n, \ f_2(z) := \sum_{n=0}^{\infty} b_n \cdot z^n$$

and the formal sum

$$f_3(z) := \sum_{n=0}^{\infty} (a_n + b_n) \cdot z^n$$

Let $R_i, \ i = 1, 2, 3$, be the respective radius of convergence. Show:

$$R_3 \geq min\{R_1, \ R_2\}, \text{ and } R_3 = min\{R_1, \ R_2\} \text{ if } R_1 \neq R_2$$

7. Consider $a \neq b \in \mathbb{C}$ and

$$G := \mathbb{C} \setminus \{1/a,\ 1/b\}$$

i) Expand the function

$$f : G \to \mathbb{C},\ f(z) := \frac{z}{(1-az)(1-bz)}\ ,$$

into a power series with center $= 0$.

ii) Determine the radius of convergence of the power series of part i).

Hint ad i): Represent f as

$$\frac{f_1(z)}{1-az} + \frac{f_2(z)}{1-bz}$$

and expand each summand into a convergent geometric series.

8. i) Consider a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n,\ c_n \in \mathbb{C}.$$

Show: The coefficients of f satisfy the recursive equations

$$c_0 = 0,\ c_1 = 1,\ c_{n+2} = \alpha c_{n+1} + \beta c_n,\ \alpha, \beta \in \mathbb{C},\ n \geq 0,$$

if and only if f satisfies the equation

$$(1 - \alpha z - \beta z^2) \cdot f(z) = z$$

ii) The *Fibonacci numbers* $c_n \in \mathbb{R}_+$ are recursively defined as

$$c_0 = 0,\ c_1 = 1,\ c_{n+2} = c_{n+1} + c_n,\ n \geq 0.$$

Determine the generator of the Fibonacci numbers, i.e. show:

$$\sum_{n=0}^{\infty} c_n \cdot z^n \text{ is convergent and } \sum_{n=0}^{\infty} c_n \cdot z^n = \frac{z}{1-z-z^2}.$$

Derive a closed form of the Fibonacci numbers.

Deadline: Monday, 6.5.2019, 10.15 a.m., box near room A109

DEPARTMENT OF MATHEMATICS
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COMPLEX ANALYSIS

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Problems 03

9. i) Prove the addition theorem for $z_1,\ z_2 \in \mathbb{C}$

$$exp(z_1 + z_2) = exp(z_1) \cdot exp(z_2)$$

ii) Derive the addition theorems for

$$sin(x+y) \text{ and } cos(x+y)$$

with real arguments $x,\ y \in \mathbb{R}$.

10. i) Determine the radius of convergence R of the power series

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot z^n$$

Which value is $f(1)$?

ii) Show: The series

$$\tilde{f}(1) := 1 + \sum_{n=1}^{\infty} \left(\frac{1}{4n-1} - \frac{1}{2n} + \frac{1}{4n+1} \right)$$

is a convergent rearrangement of the series $f(1)$ with value

$$\tilde{f}(1) = (3/2) \cdot f(1)$$

11. Consider a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n$$

with $c_0 = 1$. Define recursively the sequence $(d_n)_{n \in \mathbb{N}}$ with

$$d_0 := 1,\ d_1 := c_1,\ d_n := -c_1 \cdot d_{n-1} - c_2 \cdot d_{n-2} - ... - c_{n-1} \cdot d_1 - c_n,\ n \geq 2$$

Show:

i) If for suitable $M > 0$ and for all $n \in \mathbb{N}$

$$|c_n| \leq M^n$$

then for all $n \in \mathbb{N},\ n \geq 1$,

$$|d_n| \leq (1/2) \cdot (2M)^n$$

ii) The series

$$g(z) := \sum_{n=0}^{\infty} d_n \cdot z^n$$

is convergent.

iii) For any analytic function

$$f : U \to \mathbb{C}$$

defined on an open set $U \subset \mathbb{C}$ and without zeros, i.e. $f(z) \neq 0$ for all $z \in U$, also the reciprocal function

$$1/f : U \to \mathbb{C}$$

is analytic.

12. Denote by $\mathbb{C}\{z\}$ the set of complex convergent power series with center $= 0$.

Show:

i) With respect to addition and multiplication the set $\mathbb{C}\{z\}$ is a ring with unit.

ii) The subset

$$\mathfrak{m} := \{f \in \mathbb{C}\{z\} : f(0) = 0\} \subset \mathbb{C}\{z\}$$

is an ideal.

iii) The ideal $\mathfrak{m}$ is the unique maximal ideal of $\mathbb{C}\{z\}$. Determine the residue field $\mathbb{C}\{z\}/\mathfrak{m}$?

Note: An *ideal* I of a ring R is a subset $I \subset R$ which is closed with respect to addition and multiplication. A proper ideal $I \subsetneq R$ is *maximal* if there is no ideal J with

$$I \subsetneq J \subsetneq R.$$

Deadline: Monday, 13.5.2019, 10.15 a.m., box near room A109

DEPARTMENT OF MATHEMATICS
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COMPLEX ANALYSIS

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Problems 04

13. Consider the function

$$f:\mathbb{R}\to\mathbb{R},\ f(x):=\begin{cases} x\cdot sin(1/x) & x\in\mathbb{R}^* \\ 0 & x=0\end{cases}$$

i) Show: The function f is continuous.

ii) Does there exist a domain G with $\mathbb{R}\subset G\subset\mathbb{C}$ and an analytic function

$$F:G\to\mathbb{C} \text{ with } F|\mathbb{R}=f?$$

Give a justification of your answer.

14. Determine the power series expansion with center $a=0$ up to terms of order $=4$ of the following analytic functions:

$$f_k:\mathbb{C}\to\mathbb{C},\ f_k(z):=sin^k(z),\ k=1,2,3,4$$

and

$$g:\mathbb{C}\to\mathbb{C},\ g(z):=exp(sin(z)).$$

15. Prove: Any open connected set $U\subset\mathbb{C}$ is path-connected. Hint: Any disk is path-connected. (The problem proves a claim from the lecture.)

16. Determine a domain $G\subset\mathbb{C}$, a sequence of points $(a_n)_{n\in\mathbb{N}}$ with $a_n\in G$ for all $n\in\mathbb{N}$ and

$$\lim_{n\to\infty} a_n\in\partial G,$$

and an analytic function

$$f:G\to\mathbb{C}$$

such that: For all $n\in\mathbb{N}$ holds $f(a_n)=0$, but f does not vanish identically in G.

Deadline: Monday, 20.5.2019, 10.15 a.m., box near room A109

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COMPLEX ANALYSIS

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Problems 05

17. For a domain $G \subset \mathbb{C}$ prove: The ring

$$\mathscr{A}(G) := \{f : G \to \mathbb{C} \mid f \text{ analytic}\}$$

of analytic functions on G is an integral domain, i.e. for $f,\ g \in \mathscr{A}(G)$

$$f \cdot g = 0 \implies f = 0 \text{ or } g = 0 \text{ (no zero divisors).}$$

18. i) Determine all zeros of the two analytic functions

$$sin,\ cos : \mathbb{C} \to \mathbb{C}$$

ii) The period set of a function $f : \mathbb{C} \to \mathbb{C}$ is defined as the set

$$\{\omega \in \mathbb{C} : \text{For all } z \in \mathbb{C} \text{ holds } f(z+\omega) = f(z)\}.$$

Show: The period set of each of the two functions *sin* and *cos* is the set

$$\{k \cdot 2\pi :\ k \in \mathbb{Z}\}$$

19. Consider $\alpha \in \mathbb{C}$ and a domain $G \subset \mathbb{C}$ where an analytic branch of the logarithm function

$$log : G \to \mathbb{C}$$

exists. Define the corresponding branch of the power function z^α as the function

$$f : G \to \mathbb{C},\ f(z) := e^{\alpha \cdot log(z)}\ .$$

Show:

i) Two branches of the power function differ by a factor $e^{k \cdot 2\pi i \alpha}$, $k \in \mathbb{Z}$.

ii) Determine the principal value, i.e. the value computed by using the principal value of the logarithm, of

$$i^i,\ i^\pi,\ i^{-1}.$$

20. For $x, y \in \mathbb{R}$ set

$$G_x := \{z \in \mathbb{C} : Re\, z = x\} \text{ and } H_y := \{z \in \mathbb{C} : Im\, z = y\}$$

i) Prove: For each

$$x \in [0, 2\pi] \setminus \{0, \pi/2, \pi, (3/2)\pi, 2\pi\}$$

the set $sin(G_x)$ is one branch of a hyperbola, and for each $y \in \mathbb{R}^*$ the set $sin(H_y)$ is an ellipse.

Hint: The equation of a hyperbola/ellipse in the (u/v)-plane is

$$\frac{u^2}{a^2} \mp \frac{\mathrm{v}^2}{b^2} = 1$$

ii) Sketch - or visualize by a short clip - the sets

$$sin(G_x) \text{ and } sin(H_y)$$

for

$$x \in \{0, \pi/4, \pi/2, (3/4)\pi, (5/4)\pi, (3/2)\pi, (7/4)\pi\}$$

and

$$y \in \{ln(1/3),\ 0,\ ln(2)\}.$$

Deadline: Monday, 27.5.2019, 10.15 a.m., box near room A109

DEPARTMENT OF MATHEMATICS
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SUMMER TERM 2019

COMPLEX ANALYSIS

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Problems 06

21. Compute the path integral

$$\int_{\gamma_j} \frac{dz}{z},\ j = 1,2,$$

for the two paths connecting the points 1 and -1 in $\mathbb{C}$

$$\gamma_1, \gamma_2 : [0,\pi] \to \mathbb{C},\ \gamma_1(t) := e^{it} \text{and } \gamma_2(t) := e^{-it}.$$

22. Consider the function

$$f : \mathbb{R} \to \mathbb{R},\ f(x) := \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

defined on the real axis. Show:

i) All derivatives $f^{(n)}(x),\ x \in \mathbb{R},\ n \in \mathbb{N}^*$, exist and satisfy $f^{(n)}(0) = 0$.

ii) Does f extend to a differentiable function

$$U \to \mathbb{C}$$

defined on a neighbourhood $U \subset \mathbb{C}$ of $\mathbb{R}$? Give an argument for your answer.

Note. Distinguish between differentiability with respect to one real argument and differentiability with respect to one complex argument.

23. Consider a domain $G \subset \mathbb{C}$ and two differentiable functions

$$f, g : G \to \mathbb{C}$$

with $Re(f) = Re(g)$. Then

$$Im(f) - Im(g) = c$$

for a suitable constant $c \in \mathbb{R}$.

24. The oriented angle between two non-zero complex numbers $z_1,\ z_2 \in \mathbb{C}$ is the argument of their quotient, i.e.

$$\sphericalangle(z_1, z_2) := arg\ \frac{z_2}{z_1} \in [0, 2\pi[.$$

Consider a holomorphic function

$$f : U \to \mathbb{C}$$

defined on an open set $U \subset \mathbb{C}$, with f' having no zeros. For any $z \in U$ consider pairs of continuously differentiable paths

$$\gamma_j : I \to U,\ j = 1, 2,\ I \subset \mathbb{R} \text{ interval},$$

with a point $t_0 \in I$ satisfying

$$\gamma_j(t_0) = z \in U \text{ and } \gamma_j'(t_0) \neq 0,\ j = 1, 2.$$

Show:

$$\sphericalangle\left(\gamma_1'(t_0), \gamma_2'(t_0)\right) = \sphericalangle\left((f \circ \gamma_1)'(t_0), (f \circ \gamma_2)'(t_0)\right) \text{ (Locally conformal at } z\text{)}$$

Hint: You may use the Wirtinger calculus.

Deadline: Monday, 3.6.2019, 10.15 a.m., box near room A109

DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
SUMMER TERM 2019

COMPLEX ANALYSIS

Joachim Wehler, Pascal Stucky

Problems 07

25. Show for a continuous function

$$f : U \to \mathbb{C},$$

defined in an open neighbourhood $\overline{D}_1(0) \subset U$ of the closed unit circle:

$$\overline{\int_{|\zeta|=1} f(\zeta)\, d\zeta} = -\int_{|\zeta|=1} \frac{\overline{f}(\zeta)}{\zeta^2}\, d\zeta.$$

26. Consider a domain $G \subset \mathbb{C}$ and a holomorphic function

$$f : G \to \mathbb{C}.$$

Show that f is constant in each of the following cases:

i)

$$|f| = \mathit{constant}$$

ii) At a point $a \in G$ the modulus $|f|$ assumes a local minimum $|f(a)| \neq 0$, i.e. for all z in an open neighbourhood $U \subset G$ of a holds

$$|f(z)| \geq |f(a)|.$$

iii)

$$f' = 0.$$

27. Consider a map

$$f : U \to \mathbb{C},\ U \subset \mathbb{C} \text{ open},$$

with partial derivatives. For the Wirtinger operators prove:

$$\frac{\partial f}{\partial z} = \overline{\frac{\partial \overline{f}}{\partial \overline{z}}} \text{ and } \frac{\partial f}{\partial \overline{z}} = \overline{\frac{\partial \overline{f}}{\partial z}}$$

28. Consider a map

$$f : U \to \mathbb{C},\ U \subset \mathbb{C} \text{ open},$$

which has partial derivatives. For $z = x + iy \in U$ denote by

$$Jac(f)(x,y) := \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ \mathrm{v}_x(x,y) & \mathrm{v}_y(x,y) \end{pmatrix}$$

its Jacobi matrix at $(x,y) \in \mathbb{R}^2$ and by

$$T_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$$

the induced $\mathbb{R}$-linear map.

Show: Under the identification

$$j : \mathbb{C} \xrightarrow{\simeq} \mathbb{R}^2,\ z = x + iy \mapsto (x,y),$$

the map $T_{\mathbb{R}^2}$ identifies with the $\mathbb{R}$-linear map

$$T_{\mathbb{C}} : \mathbb{C} \to \mathbb{C},\ h \mapsto f_z(z) \cdot h + f_{\bar{z}}(z) \cdot \overline{h},$$

i.e. the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{T_{\mathbb{C}}} & \mathbb{C} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbb{R}^2 & \xrightarrow[T_{\mathbb{R}^2}]{} & \mathbb{R}^2 \end{array}$$

which means

$$j \circ T_{\mathbb{C}} = T_{\mathbb{R}^2} \circ j.$$

Hint: It suffices to check the last equality for the two special arguments $1,\ i \in \mathbb{C}$.

Deadline: Wednesday, 12.6.2019, 10.15 a.m., box near room A109

Department of Mathematics
LMU München
Summer term 2019

Complex Analysis

Joachim Wehler, Pascal Stucky

Problems 08

29. Determine all entire functions f with $f \circ f = f$.

30. Consider a map

$$f : U \to \mathbb{C},\ U \subset \mathbb{C} \text{ open},$$

with continuous partial derivatives. Assume: For any $z \in U$

$$\det\,(Jac(f)(z)) \neq 0,$$

and for any pair of paths $\gamma_j,\ j = 1,2,$ with the properties from Exercise 24 holds

$$\sphericalangle\left({\gamma_1}'(t_0), {\gamma_2}'(t_0)\right) = \sphericalangle\left((f \circ \gamma_1)'(t_0), (f \circ \gamma_2)'(t_0)\right) \text{(Locally conformal at } z)$$

Show: The function f is holomorphic and f' has no zeros.

Hint: Consider the family of paths

$$\gamma_s : I \to \mathbb{C}, \gamma_s(t) := z_0 + e^{is} \cdot t,\ s \in [0, 2\pi],$$

and the pairs (γ_s, γ_0),

31. Consider the function

$$f : \mathbb{C} \setminus \{1,2\} \to \mathbb{C},\ f(z) := \frac{1}{(z-1)\cdot(z-2)}$$

Determine the Laurent series of f with center $= 0$ in each of the following domains $G_j,\ j = 1,2,3$:

i) $G_1 := \{z \in \mathbb{C} : |z| < 1\}$

ii) $G_2 := \{z \in \mathbb{C} : 1 < |z| < 2\}$

iii) $G_3 := \{z \in \mathbb{C} : 2 < |z|\}$.

32. Show: A non-constant entire function

$$f : \mathbb{C} \to \mathbb{C}$$

has dense image, i.e. $f(\mathbb{C}) \subset \mathbb{C}$ is a dense subset.

Deadline: Monday, 17.6.2019, 10.15 a.m., box near room A109

DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
SUMMER TERM 2019

COMPLEX ANALYSIS

Joachim Wehler, Pascal Stucky

Problems 09

33. Consider an entire function

$$f : \mathbb{C} \to \mathbb{C} \text{ with } \lim_{|z|\to\infty} |f(z)| = \infty.$$

Show: The function f is a polynomial.

34. Consider a holomorphic function

$$f : D_r(0) \to \mathbb{C}$$

with a radius $r > 0$. Denote by

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n$$

the Taylor series of f with center $= 0$. Assume the existence of a constant $M > 0$ such that for all $z \in D_r(0)$

$$|f(z)| \le M.$$

Show: If for an index $n \in \mathbb{N}$

$$|c_n| = \frac{M}{r^n}$$

then for all $z \in D_r(0)$

$$f(z) = c_n \cdot z^n.$$

Hint. For $0 < \rho < r$ prove the integral representation

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} |f(\rho \cdot e^{i\phi})|^2 \, d\phi = \sum_{m=0}^{\infty} |c_m|^2 \cdot \rho^{2m}$$

35. For a holomorphic function

$$f : U \to \mathbb{C},\ U \subset \mathbb{C} \text{ open}$$

define

$$M_f(r) := sup\{|f(z)| : z \in U \text{ and } |z| = r\}.$$

i) Consider an entire function

$$f : \mathbb{C} \to \mathbb{C}$$

and assume the existence of a radius $r_0 > 0$ such that for all $r \geq r_0$

$$M_f(r) \leq \sqrt{r} \cdot ln\ r.$$

Show: The function f is constant.

ii) Consider a holomorphic function

$$D^*_\rho(0) \to \mathbb{C}$$

with a radius $\rho > 0$. Assume for all $0 < r < \rho$

$$M_f(r) \leq \frac{|ln\ r|}{\sqrt{r}}.$$

Show: The function f has a removable singularity at $0 \in \mathbb{C}$.

36. Consider a periodic entire function

$$f : \mathbb{C} \to \mathbb{C}$$

satisfying for all $z \in \mathbb{C}$

$$f(z+1) = f(z).$$

i) Show: For all $z \in \mathbb{C}$

$$f(z) = g(e^{2\pi i \cdot z})$$

with a holomorphic function

$$g : \mathbb{C}^* \to \mathbb{C}$$

satisfying

$$g(w) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \cdot \int_0^{2\pi} g(e^{i\phi}) \cdot e^{-in\cdot\phi}\ d\phi \right) \cdot w^n$$

ii) Conclude: For all $z \in \mathbb{C}$ holds the Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \cdot \int_0^{2\pi} f\left(\frac{\phi}{2\pi}\right) \cdot e^{-in\phi}\ d\phi \right) \cdot e^{2\pi in\cdot z}$$

Hint. Find g as commutative completion of the following diagram

$$
\begin{array}{ccc}
\mathbb{C} & & \\
{\scriptstyle exp(2\pi i\cdot(-))}\downarrow & \searrow{\scriptstyle f} & \\
\mathbb{C}^* & \overset{}{\underset{g}{\dashrightarrow}} & \mathbb{C}
\end{array}
$$

Deadline: Monday, 24.6.2019, 10.15 a.m., box near room A109

DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
SUMMER TERM 2019

COMPLEX ANALYSIS

Joachim Wehler, Pascal Stucky

Problems 10

37. For the Bernoulli numbers $(B_n)_{n\in\mathbb{N}}$ prove the recursion formula: If $N \in \mathbb{N}^*$ then

$$\sum_{n=0}^{N} \binom{N+1}{n} \cdot B_n = 0.$$

38. i) Prove the formula

$$tan\ z = cot\ z - 2 \cdot cot\ 2z$$

ii) Prove the Taylor expansion with center $a = 0$

$$tan\ z = \sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{2^{2k} \cdot (2^{2k}-1) \cdot B_{2k}}{(2k)!} \cdot z^{2k-1}$$

and determine the radius of convergence.

39. i) For $2 > |z|$ and $n \in \mathbb{N},\ n \geq 2$, show

$$\frac{1}{z-n} + \frac{1}{z+n} = -2 \cdot \sum_{k=1}^{\infty} \frac{z^{2k-1}}{n^{2k}}$$

ii) For $|z| > 1$ show

$$\frac{1}{z-1} + \frac{1}{z+1} = 2 \cdot \sum_{k=1}^{\infty} \frac{1}{z^{2k-1}}$$

iii) For z in the open annulus

$$A := \{z \in \mathbb{C} :\ 1 < |z| < 2\}$$

prove the Laurent expansion with center $= 0$

$$\pi \cdot \cot(\pi z) = \frac{1}{z} + 2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{z}\right)^{2k-1} - 2 \cdot \sum_{k=1}^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{n^{2k}}\right) \cdot z^{2k-1}.$$

40. i) Show

$$\frac{1}{sin(\pi z)} = \frac{1}{2} \cdot \left(cot\left(\frac{\pi z}{2}\right) + tan\left(\frac{\pi z}{2}\right)\right).$$

ii) Show

$$\frac{1}{sin(\pi z)} = \frac{1}{2} \cdot \left(cot\left(\frac{\pi z}{2}\right) + cot\left(\frac{\pi \cdot (1-z)}{2}\right)\right).$$

iii) Determine the pole set, the pole orders and the principal parts of

$$\frac{\pi}{sin(\pi z)}$$

considered as a meromorphic function in $\mathbb{C}$. Conclude from part i) and ii)

$$\frac{\pi}{sin(\pi z)} = \frac{1}{z} + 2z \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2}.$$

Deadline: Monday, 1.7.2019, 10.15 a.m., box near room A109

DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
SUMMER TERM 2019

COMPLEX ANALYSIS

Joachim Wehler, Pascal Stucky

Problems 11

41. Consider the Weierstrass elementary factor E_p of order $p \in \mathbb{N}^*$.

i) Show: The Taylor series of E_p with center $= 0$ has the form

$$E_p(z) = 1 - \sum_{n=p+1}^{\infty} a_n \cdot z^n$$

with non-negative real coefficients $a_n,\ n \geq p+1$.

Hint: You may compare coefficients of two suitable representations of the derivative $E_p'(z)$.

ii) Conclude from part i): For $|z| \leq 1$ holds the estimate

$$|E_p(z) - 1| \leq |z|^{p+1}$$

Hint: Use $E_p(1) = 0$.

42. Which well-known function equals the canonical product

$$\prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} E_1\left(\frac{z}{n}\right)$$

43. A *divisor* on a non-empty set $U \subset \mathbb{C}$ is a map

$$D : U \to \mathbb{Z}$$

with support

$$supp\ D := \{z \in U : D(z) \neq 0\}$$

a discrete set, closed in U. A divisor D is *non-negative*, denoted $D \geq 0$, if $D(z) \geq 0$ for all $z \in U$. A non-negative divisor is *positive*, denoted $D > 0$, if $D(z) > 0$ for at least one $z \in U$.

Any meromorphic function $f \in \mathscr{M}(U)$ defines on U the divisor

$$(f) := D$$

named a *principal divisor* (= Hauptdivisor), with

$$D : U \to \mathbb{Z},\ D(a) := ord(f;\ a).$$

Show: i) Any divisor D on U decomposes as

$$D = D_1 - D_2$$

with two divisors on U

$$D_1, D_2 \geq 0 \text{ and } supp\ D_1 \cap supp\ D_2 = \emptyset.$$

ii) Any divisor D on $\mathbb{C}$ is a principal divisor, i.e. the divisor of a meromorphic function on $\mathbb{C}$.

44. Consider the Γ-function

$$\Gamma : RH(0) \to \mathbb{C}.$$

i) Show: For any $z \in RH(0)$ and any $n \in \mathbb{N}$ holds

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1) \cdot \ldots \cdot (z+n-1)}$$

ii) Conclude: The Γ-function extends uniquely to a meromorphic function on $\mathbb{C}$, also named Γ.

iii) Show: The meromorphic Γ-function on $\mathbb{C}$ has the pole set

$$P = \{-n :\ n \in \mathbb{N}\}.$$

Each pole has order $= 1$. The principal parts are

$$H_{-n}(z) = \frac{(-1)^n}{n!} \cdot \frac{1}{z+n},\ n \in \mathbb{N}.$$

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COMPLEX ANALYSIS

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Problems 12

45. Show: The compact set

$$A := \overline{D}_1(0) \subset \mathbb{C} \simeq \mathbb{R}^2$$

has a smooth boundary ∂A.

46. i) Consider two functions f, g which are holomorphic in an open neighbourhood of a point $a \in \mathbb{C}$ and assume

$$ord(g;\ a) = 1.$$

Show:

$$res\left(\frac{f}{g};\ a\right) = \frac{f(a)}{g'(a)}.$$

ii) Show

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

47. i) Using Fubini's theorem and polar coordinates compute

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\ dy,$$

and derive

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

ii) Show

$$\int_0^{\infty} cos(x^2)\ dx = \int_0^{\infty} sin(x^2)\ dx = \sqrt{\frac{\pi}{8}}$$

Hint: Integrate e^{-z^2} along the closed path from Figure 0.1, and relate the result to

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx.$$

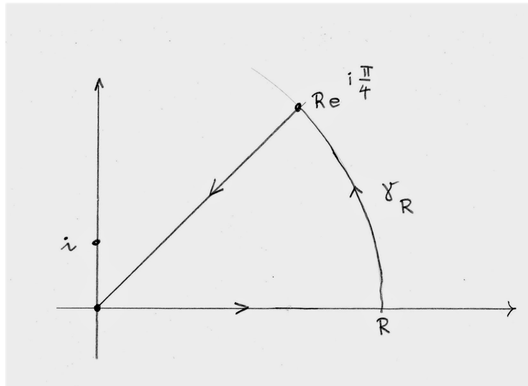

Fig. 0.1 Closed path of integration

48. Derive the product representation of the Γ-function

$$\Gamma(z) = \frac{e^{-C\cdot z}}{z}\cdot\prod_{n=1}^{\infty}\frac{e^{z/n}}{1+(z/n)}$$

with the Euler-Mascheroni constant

$$C := \lim_{N\to\infty}\left[\left(\sum_{n=1}^{N}\frac{1}{n}\right) - ln\, N\right]$$

along the following steps:

i) Prove: The function

$$\gamma(z) := \frac{e^{-C\cdot z}}{z}\cdot\prod_{n=1}^{\infty}\frac{e^{z/n}}{1+(z/n)}$$

is meromorphic on $\mathbb{C}$ with the same pole set as Γ. It satisfies

$$\gamma(z) = \lim_{N\to\infty}\frac{N^z\cdot N!}{z\cdot(z+1)\cdot\ldots\cdot(z+N)}$$

ii) Derive a functional equation for γ, and conclude that γ has the same principal parts as Γ.

iii)(*) Consider the entire functions $g := \Gamma - \gamma$ and

$$S : \mathbb{C}\to\mathbb{C},\ S(z) := g(z)\cdot g(1-z).$$

Show that g is bounded in the strip

$$B_{1,2} := \{z\in\mathbb{C} :\ 1\le Re\, z\le 2\}$$

and conclude it's boundedness in the strip

$$B_{0,1} := \{z \in \mathbb{C} :\ 0 \leq Re\, z \leq 1\}$$

Show for all $z \in \mathbb{C}$

$$S(z+1) = -S(z),$$

and conclude that S is bounded in the strips $B_{0,1}$, $B_{1,2}$, and even in $\mathbb{C}$.

iv) Conclude $S = 0$ and $g = 0$.

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Problems 13

49. Determine the number of zeros of the two polynomials

i) $p(z) := z^8 - 3 \cdot z^2 + 1$ for $|z| > 1$

ii) $q(z) := 3 \cdot z^4 - 7 \cdot z + 2$ for $1 < |z| < 3/2$.

50. Prove: All domains $G \subset \mathbb{C}$, which are star-like with respect to a point $a \in G$, are simply connected.

51. Consider a domain $G \subset \mathbb{C}$ and a fixed point $z_0 \in G$.

i) Consider a holomorphic function

$$f : G \to \mathbb{C}$$

satisfying: For each continuously differentiable path γ in G with $\gamma(0) = z_0$ holds

$$\int_\gamma f(\zeta)\, d\zeta = 0.$$

Show $f = 0$.

ii) On the set of all closed paths γ in G with $\gamma(0) = \gamma(1) = z_0$ the property *being homotopic as closed path in G* defines an equivalence relation. Let $\pi_1(G, z_0)$ be the set of all equivalence classes $[\gamma]$.

Show: For all holomorphic functions $f \in \mathscr{O}(G)$ the map

$$T_f : \pi_1(G, z_0) \to \mathbb{C}, T_f([\gamma]) := \int_\gamma f(z)\, dz,$$

is well-defined.

iii) For $G := \mathbb{C}^*$ and $z_0 := 1$ the winding number defines the bijective map

$$\pi_1(G,z_0) = \{[\gamma] : \gamma(t) = e^{n\cdot 2\pi i\cdot t},\ t\in[0,1],\ n\in\mathbb{Z}\} \simeq \mathbb{Z}.$$

Prove

$$\mathbb{C} = \bigcup_{f\in\mathscr{O}(G)} T_f(\pi_1(G;z_0)),$$

and determine the set

$$\{f\in\mathscr{O}(G) :\ T_f = 0\}.$$

52. Prove Morera's Theorem: Consider a domain

$$G \subset \mathbb{C} \simeq \mathbb{R}^2$$

and a continuous function

$$f : G \to \mathbb{C}.$$

If for all rectangles $R \subset G$ parallel to the coordinate axes of $\mathbb{R}^2$

$$\int_{\partial R} f(z)\, dz = 0,$$

then f is holomorphic.

Hint: Reduce the statement to an open disk $G = D_r(0)$ and define

$$F(z) := \int_\gamma f(\zeta)\, d\zeta$$

by integrating from 0 to z along two adjacent boundary lines of a suitable rectangle. Show: The function F has continuous partial derivatives satisfying the Cauchy-Riemann differential equations, and $F' = f$.

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COMPLEX ANALYSIS

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Problems 14 (*for repetition*)

53. Let f, g be entire functions such that for all $z \in \mathbb{C}$

$$|f(z)| \leq |g(z)|.$$

Show: There exists $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ such that for all $z \in \mathbb{C}$

$$f(z) = \lambda \cdot g(z).$$

54. Show: A non-constant entire function

$$f : \mathbb{C} \to \mathbb{C}$$

has dense image, i.e. $f(\mathbb{C}) \subset \mathbb{C}$ is a dense subset.

55. For each pair of mathematical domains

- (Analysis, Algebra) and
- (Analysis, Topology)

state a result from the lecture *Complex Analysis*, which combines concepts or results from both domains of the given pair.

Justify your answer by writing down one or two arguments.

56. Examine whether the function

$$f(z) := \frac{1}{cos(1/z)}$$

has an isolated singularity at $a = 0$. If yes, determine the type of singularity and the residue of f at a.

57. Consider the function

$$f(z) := \frac{1}{\cosh z - \cos z}$$

i) Determine the type of the singularity of f at $a = 0$.

ii) For the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \cdot z^n$$

determine all coefficients c_n with $n \leq 2$.

No submission!