MAXIMALLY NON–INTEGRABLE PLANE FIELDS ON THURSTON GEOMETRIES

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Abstract. We study Thurston geometries (X, G) with contact structures and Engel structures which are compatible with the action of the isometry group G. We classify geometric contact structures and geometric Engel structures up to equivalence and we compare the geometric Engel structures with other constructions of Engel manifolds.

1. Introduction

In this article we investigate Thurston geometries with adapted Engel structures and contact structures. In [Wa1, Wa2] Wall carries out a similar study for complex structures in dimension 4. The motivation for the consideration of geometric Engel structures compatible with Thurston geometries comes from the fact that Engel manifolds obtained from Thurston geometries have particular properties not present in general.

For example the geometric Engel structure on the Thurston geometry Nil^4 leads to closed Engel manifolds such that all leaves of the characteristic foliation are rigid in the sense of [BrH]. Moreover the contact structures compatible with Thurston geometries lead to tight contact manifolds. It is unknown whether or not there is an analogue of the tight/overtwisted dichotomy for Engel structures, but if there is, then one may hope that geometric Engel structures lead to examples belonging to the analogue of tight contact structures.

Let us discuss the relevant definitions. We start with Thurston geometries and their classification in low dimensions.

Definition 1. Let X be a simply connected complete Riemannian manifold and G a Lie group acting on X. The pair (X, G) is a *Thurston geometry* if

(i) G acts transitively on X,
(ii) the stabilizer $\mathrm{Stab}(x) = \{g \in G \mid gx = x\}$ of a point $x \in X$ is compact,
(iii) G contains a discrete subgroup Γ acting freely on X such that $\Gamma\backslash X$ has finite volume (with respect to a G–invariant metric).

Note that condition (ii) ensures that there is a G–invariant Riemannian metric. Moreover it implies that whenever $\Gamma\backslash X$ is a compact manifold for $\Gamma \subset G$, then $\Gamma\backslash G$ is also compact.

The quotient manifolds $\Gamma\backslash X$ are said to have X*–geometry*. Two Thurston geometries $(X, G), (X', G')$ are *equivalent* if there is a diffeomorphism $\psi : X \to X'$ such that $\psi \circ g \circ \psi^{-1} \in G'$ for all g in G. Diffeomorphisms with this property are called *automorphisms* of the Thurston geometry. If there is a diffeomorphism ψ of X such that $\psi \circ G \circ \psi^{-1} \subset G'$, then (X, G) is a *subgeometry* of (X', G').

We deviate from the notion of equivalence of Thurston geometries used for example in [Thu] where it coincides with our notion of subgeometry (which is not an equivalence relation). Notice that if (X,H) is a subgeometry of (X,G), there is no a priori relation between the groups of automorphisms. The most obvious examples of automorphisms of a Thurston geometry (X,G) are the elements of G itself. In the following example we give another class of automorphisms.

Example 2. Let G be a Lie group acting freely and transitively on X such that (X,G) is a Thurston geometry and ψ is a group automorphism of G. After choosing a base point $x_0 \in X$ we obtain a diffeomorphism $\widetilde{\psi}$ with $\widetilde{\psi}(gx_0) = \psi(g)x_0$. Conjugating elements of G with $\widetilde{\psi}$ we get ψ again. Hence ψ induces an automorphism of the Thurston geometry (X,G).

Automorphisms of G are easy to construct: Since $X \simeq G$ is simply connected there is a one–to–one correspondence between automorphism of G and automorphisms of the Lie algebra $\mathfrak{g}$.

Usually one considers *maximal* Thurston geometries (X,G) where G is maximal among those groups having the properties in Definition 1. When additional geometric structures like contact structures, Engel structures, complex structures or Kähler structures (as in [Wa1, Wa2]) are present, then it is natural to consider non–maximal Thurston geometries.

A Thurston geometry in dimension 2 is equivalent to a subgeometry of $S^2, \mathbb{H}^2$ or $\mathbb{R}^2$. For $\dim(X) = 3$ Thurston obtained a classification discussed e.g. in [Thu]. In the following table the 3–dimensional Thurston geometries are grouped according to the isomorphism type of point stabilizers.

Isomorphism type of $\mathrm{Stab}_0(x)$	Isometry class of X
SO(3)	$S^3, \mathbb{H}^3, \mathbb{R}^3$
SO(2)	$S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ $\mathrm{Nil}^3, \widetilde{\mathrm{Sl}}(2)$
$\{1\}$	Sol^3

Filipkiewicz (cf. [Fil]) obtained the analogous result in dimension 4. The following list can be found in [Wa1].

Isomorphism type of $\mathrm{Stab}_0(x)$	Isometry class of X
SO(4)	$S^4, \mathbb{H}^4, \mathbb{R}^4$
U(2)	$\mathbb{CP}^2, \mathbb{H}^2(\mathbb{C})$
$\mathrm{SO}(2) \times \mathrm{SO}(2)$	$S^2 \times S^2, S^2 \times \mathbb{R}^2, S^2 \times \mathbb{H}^2$ $\mathbb{H}^2 \times \mathbb{R}^2, \mathbb{H}^2 \times \mathbb{H}^2$
SO(3)	$S^3 \times \mathbb{R}, \mathbb{H}^3 \times \mathbb{R}$
SO(2)	$\mathrm{Nil}^3 \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2) \times \mathbb{R}, \mathrm{Sol}^4_0, F^4$
1	$\mathrm{Nil}^4, \mathrm{Sol}^4(m,n), \mathrm{Sol}^4_1$

We will give more details about many of the geometries later. More information can be found in [Fil, Hil, Sc, Thu, Wa1, Wa2]. We will use the classification of non–maximal Thurston geometries up to equivalence from [Wa2].

Theorem 3 (Wall). *The non–maximal Thurston geometries with connected isometry group in dimension ≤ 4 are*

dim	*non–maximal geometries*	
2	$(\mathbb{R}^2, \mathbb{R}^2)$	
3	$(S^3, \mathrm{U}(2))$	$(S^3, \mathrm{SU}(2))$
	$(\mathbb{R}^3, \mathbb{R}^3 \ltimes \mathrm{SO}(2))$	$(\mathbb{R}^3, \mathbb{R}^3)$
	$(\mathrm{Nil}^3, \mathrm{Nil}^3)$	$(\widetilde{\mathrm{Sl}}(2,\mathbb{R}), \widetilde{\mathrm{Sl}}(2,\mathbb{R}))$
4	$(S^2 \times \mathbb{R}^2, \mathrm{SO}(3) \times \mathbb{R}^2)$	$(\mathbb{H}^2 \times \mathbb{R}^2, \mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}^2)$
	$(\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R})$	$(\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^3 \ltimes \mathbb{R})$
	$(S^3 \times \mathbb{R}, \mathrm{U}(2) \times \mathbb{R})$	$(S^3 \times \mathbb{R}, \mathrm{SU}(2) \times \mathbb{R})$
	$(\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R})$	$(\mathrm{Sol}^4_0, \mathrm{Sol}^4(\lambda))$
	$(\mathbb{R}^4, K)$ *with*	$K \in \left\{ \begin{array}{l} \mathrm{U}(2), \mathrm{SU}(2), \mathrm{SO}(3), \\ \mathrm{SO}(2) \times \mathrm{SO}(2), S^1, S^1_{p,q}, \{1\} \end{array} \right\}$

Here $S^1_{p,q}$ denotes the image of S^1 in $\mathrm{SO}(2) \times \mathrm{SO}(2)$ under the map $z \longmapsto (z^p, z^q)$ with coprime integers (p, q). The geometries $(\mathrm{Sol}^4_0, \mathrm{Sol}^4(\lambda))$ will be discussed below. The geometry $(\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^3 \ltimes \mathbb{R})$ does appear in [Wa1], but in [Wa2], Wall claims that this geometry does not admit a cocompact lattice. We shall exhibit such a lattice in Section 3.2.

Next we recall the definitions of contact structures and Engel structures and after that we finally give the definition of geometric contact structures and geometric Engel structures.

Definition 4. A *contact structure* on a manifold of dimension $2k + 1$ is a smooth hyperplane field $\mathcal{C}$ which can be defined locally by a 1–form α with the property $\alpha \wedge d\alpha^k \neq 0$.

We will only consider contact structures in dimension 3. In this case the sign of the form $\alpha \wedge d\alpha$ appearing in Definition 4 is independent from the choice of α. Thus a contact structure induces an orientation of the underlying 3–manifold. Martinet showed that a 3–manifold admits a contact structure if and only if it is orientable [Mar].

Definition 5. An *Engel structure* is a smooth subbundle $\mathcal{D}$ of rank 2 of the tangent bundle of a 4–manifold such that

$$\mathrm{rank}([\mathcal{D}, \mathcal{D}]) = 3$$
$$\mathrm{rank}([\mathcal{D}, [\mathcal{D}, \mathcal{D}]]) = 4 \ .$$

The hyperplane field $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is an *even contact structure* (in dimension four a hyperplane field $\mathcal{E}$ is an even contact structure if and only if $[\mathcal{E}, \mathcal{E}] = TM$) and $\mathcal{W}$ is defined to be the unique line field tangent to $\mathcal{E}$ with the property $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. The foliation induced by $\mathcal{W}$ is the *characteristic foliation* of $\mathcal{E}$. By definition every flow tangent to $\mathcal{W}$ preserves $\mathcal{E}$ and it is easy to show that $\mathcal{W} \subset \mathcal{D}$ if $\mathcal{E}$ is induced by an Engel structure [Mo]. Thus an Engel structure $\mathcal{D}$ on a 4–manifold M induces a flag of distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} = [\mathcal{D}, \mathcal{D}] \subset TM \ . \tag{1}$$

The existence of an Engel structure on a closed manifold M leads to strong restrictions on the topology of the underlying manifold: If both M and $\mathcal{D}$ are are orientable, then TM is trivial. In [Vo] it is shown that the converse is also true. Two other constructions of Engel manifolds can be found in the literature. Since geometric Engel structures will often turn out to be similar to Engel structures obtained by these constructions, we explain them briefly.

Example 6. The first of the two constructions is called *prolongation* and was introduced by E. Cartan. The starting point is a contact structure $\mathcal{C}$ on a 3–manifold N. We consider the projectivization $\mathbb{P}\mathcal{C}$ of $\mathcal{C}$ with the projection $\pi : \mathbb{P}\mathcal{C} \longrightarrow N$. The plane field

$$\mathcal{D} = \{v \in T_l\mathbb{P}\mathcal{C} \mid \pi_*(v) \in l\}$$

is an Engel structure on $\mathbb{P}\mathcal{C}$. The distributions in (1) can be described explicitly: The leaves of $\mathcal{W}$ correspond to the fibers of π and $\mathcal{E} = \pi_*^{-1}\mathcal{C}$.

Example 7. The second construction is due to H.-J. Geiges, [Gei]. Let φ be a diffeomorphism of a closed 3–manifold N such that the mapping torus

$$M_\varphi = N \times [0, 2\pi]/\sim \text{ with } (\varphi(p), 0) \sim (p, 2\pi)$$

has trivial tangent bundle. Let t be the coordinate on the second factor of $N \times [0, 2\pi]$ and ∂_t the corresponding vector field on M_φ. Since TM_φ is trivial we can choose an almost quaternionic structure (I, J, K) on M_φ. In this way we obtain a framing $\partial_t, V_1 = I\partial_t, V_2 = J\partial_t, V_3 = K\partial_t$ of TM_φ whose first component is ∂_t. If $k \in \mathbb{N}$ is large enough, then the plane field spanned by ∂_t and

$$X = \frac{1}{k}\left(\cos(k^2t)V_1 + \sin(k^2t)V_2\right) + V_3 \tag{2}$$

is an Engel structure. This construction has the disadvantage that one can not determine the characteristic foliation or the even contact structure associated to $\mathcal{D}$.

Definition 8. A *geometric contact structure* is a triple $(X, \mathcal{C}, H)$ where (X, H) is a Thurston geometry and $\mathcal{C}$ is an H-invariant contact structure on X. Two geometric contact structures $(X, \mathcal{C}, H)$ and $(X, \mathcal{C}', H')$ are *equivalent* if there is a diffeomorphism $\psi : X \longrightarrow X'$ such that $\psi \circ H \circ \psi^{-1} = H'$ and $\psi_*\mathcal{C} = \mathcal{C}'$.

A *geometric Engel structure* is a triple $(X, \mathcal{D}, H)$ such that the isometries of the Thurston geometry (X, H) preserve the Engel structure $\mathcal{D}$ on X. Equivalence of geometric Engel structures is defined as for geometric contact structures.

We shall classify *maximal* geometric contact structures respectively Engel structure where H is maximal among the groups having the properties in Definition 8.

If $(X, \mathcal{C}, G)$ is a geometric contact structure and $\Gamma \subset G$ is a cocompact lattice acting freely on X, then we obtain a contact structure on $\Gamma\backslash X$ and the analogous statement for Engel structures is also true. By the Gray stability theorem [Gr] the diffeomorphism class of the induced contact structure on $\Gamma\backslash X$ depends only on the isotopy class of the embedding $\Gamma \hookrightarrow H$. The induced Engel structure on the quotient $\Gamma\backslash X$ of a geometric Engel structure depends on the realization of Γ as subgroup of H since Gray's theorem does not generalize to Engel structures [Gol]. As mentioned above these Engel manifolds have special geometric properties.

This article is organized as follows. In Section 2 we discuss geometric contact structures in dimension 3. We shall see that only the Thurston geometries $X = S^3, \mathrm{Nil}^3, \widetilde{\mathrm{Sl}}(2), \mathrm{Sol}^3$ admit geometric contact structures and for these Thurston geometries all geometric contact structures are equivalent.

We continue with geometric Engel structures in Section 3. With the exception of $\mathbb{H}^3 \times \mathbb{R}$ all Thurston geometries with $\mathrm{Stab}_0(x) \simeq \mathrm{SO}(3), \mathrm{SO}(2), \{1\}$ admit a geometric Engel structure and again the geometric Engel structures supported by a particular Thurston geometry are all equivalent. We also compare geometric Engel structures with the Engel structures obtained in Example 6 and Example 7. In the last section we discuss the rigidity of curves tangent to Engel structures and discuss the geometric Engel structures.

Finally, let us fix some notations. If G is a Lie group, then we write G_0 for the connected component of the identity. The Lie algebra of a Lie group $G, H, \mathrm{Nil}^3, \dots$ is denoted by $\mathfrak{g}, \mathfrak{h}, \mathfrak{nil}^3, \dots$.

2. Geometric contact structures

In this section we classify geometric contact structures in dimension 3. First recall that a contact structure on a 3-manifold induces an orientation. Therefore it makes sense to refine the notion of equivalence and ask for the existence of an *orientation preserving* diffeomorphism ψ with the properties in Definition 8.

Theorem 9. *The Thurston geometries $\mathbb{H}^3, \mathbb{R}^3, S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ do not admit geometric contact structures. The geometric contact structures on the remaining Thurston geometries are all equivalent:*

Thurston geometry	*# of equivalence classes*	*with oriented equivalence*
S^3	1	2
Nil^3	1	1
$\widetilde{\mathrm{Sl}}(2)$	1	2
Sol^3	1	2

In order to prove this we consider each Thurston geometry individually. Before doing so, we first make two observations which we will also use in the classification of geometric Engel structures.

Remark 10. If $(X, \mathcal{C}, H)$ is a geometric contact structure, then (X, H) is a subgeometry of a maximal Thurston geometry and X is connected and diffeomorphic to $H/\mathrm{Stab}_H(x)$ for $x \in X$. Now $\mathrm{Stab}_H(x)$ is compact by definition and therefore H has only finitely many connected components. If Γ is a lattice in H such that $\Gamma \backslash X$ is compact, then the same is true for $\Gamma \cap H_0 \subset H_0$. Hence if $(X, \mathcal{C}, H)$ is a geometric contact structure, then the same is true for $(X, \mathcal{C}, H_0)$.

The second observation is that if $(X, \mathcal{C}, H)$ is a geometric contact structure, then H acts transitively on X. Therefore the stabilizer $\mathrm{Stab}_H(x) \subset H$ of $x \in X$ is trivial or acts by rotations/reflections on the contact plane $\mathcal{C}(x)$. Thus H has either dimension 3 or 4.

Whenever a geometric contact structure exists on the Thurston geometry (X, H) it turns out to be tight. Essentially this can be shown by relating geometric contact structures with the contact structure $\ker(dz - x\,dy)$ on $\mathbb{R}^3$ which is tight according to [Be]. We will give only brief arguments and refer the reader to [Be, El, Gi] for definitions and background.

2.1. $X = S^3$. The Riemannian manifold is $S^3 \subset \mathbb{C}^2$ with its round metric. The full isometry group is $G = \mathrm{O}(4)$. Let $\mathcal{C}_{st}$ be the plane field consisting of the complex subspaces of $TS^3 \subset T\mathbb{C}^2$. A possible defining form is $\alpha = x_1\,dy_1 - y_1\,dx_1 + x_2\,dy_2 - y_2\,dx_2$ where x_1, y_1, x_2, y_2 are the usual coordinates on $\mathbb{R}^2 \oplus \mathbb{R}^2 \simeq \mathbb{C}^2$. A short calculation shows $\alpha \wedge d\alpha \neq 0$ on S^3.

By definition, $\mathcal{C}_{st}$ is preserved by $\mathbb{C}$–linear isometries, i.e. by all elements of $\mathrm{U}(2)$ and by complex conjugation κ of $\mathbb{C}^2$. Conversely, let $\varphi \in \mathrm{O}(4)$ satisfy $\varphi_* \mathcal{C}_{st} = \mathcal{C}_{st}$.

Then φ preserves the orthogonal complement of $\mathcal{C}$, i.e. the fibers of the Hopf fibration. Hence φ is an isometry of $\mathbb{C}^2$ which preserves complex subspaces and the complex orientation. Composing φ with suitable elements of

$$H = \mathrm{U}(2) \cup \kappa\mathrm{U}(2)$$

we can achieve that the resulting map fixes both $\mathbb{C}\oplus 0$ and $0\oplus\mathbb{C}$ pointwise. Because the map is linear we obtained the identity. This shows $\varphi \in H$ and that $(S^3, \mathcal{C}_{st}, H)$ is a maximal geometric contact structure.

Now let $(S^3, \mathcal{C}', H')$ be a geometric contact structure. By Remark 10 and Theorem 3 we may assume that H' is connected and contains $\mathrm{SU}(2)$. Moreover the action of $\mathrm{SU}(2)$ by conjugation on the tangent planes in T_eS^3 is transitive (here we identify $\mathrm{SU}(2)$ with S^3). Hence we can assume that $\mathcal{C}_{st}(e) = \mathcal{C}'(e)$. But this implies $\mathcal{C}_{st} = \mathcal{C}'$ because these contact structures are both $\mathrm{SU}(2)$–invariant.

Thus all maximal geometric contact structures on S^3 are equivalent to subgeometries of $(S^3, \mathcal{C}_{st}, \mathrm{U}(2) \cup \kappa\mathrm{U}(2))$. If one allows only orientation preserving diffeomorphisms of S^3, then there are two equivalence classes of geometric contact structures. Note that according to [Sc] all spherical 3–manifolds admit a geometric contact structures since every discrete subgroup of $\mathrm{SO}(4)$ acting freely on S^3 is conjugate to a subgroup of $\mathrm{U}(2)$. Furthermore, it is well know that $\mathcal{C}_{st}$ is tight.

2.2. $X = \mathrm{Nil}^3$**.** The nilpotent Lie group Nil^3 has the matrix representation

$$\mathrm{Nil}^3 = \left\{ [x,y,z] := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle|\, x,y,z \in \mathbb{R} \right\} .$$

The metric on Nil^3 is defined such that the left–invariant vector fields $X, Y, Z \in \mathfrak{nil}^3$ form an orthonormal frame. For the usual flat metric on $\mathbb{R}^2$, the fibration

$$\begin{aligned} \mathrm{pr} : \mathrm{Nil}^3 &\longrightarrow \mathbb{R}^2 \\ [x,y,z] &\longmapsto (x,y) \end{aligned}$$

is a Riemannian submersion. The maximal isometry group G of Nil^3 is generated by left–translations and by lifts of isometries of the plane with respect to the horizontal distribution $\ker(\alpha_{st})$ for $\alpha_{st} = dz - x\,dy$. Hence $G \simeq \mathrm{Nil}^3 \ltimes \mathrm{O}(2)$. If κ denotes the reflection of $\mathbb{R}^2$ along the y-axis, then $\mathrm{O}(2)$ acts on $\mathfrak{nil}^3$ by rotations around Z while κ_* is a reflection of the X, Y plane and maps Z to $-Z$. Thus $\mathfrak{g}$ is generated by $X, Y, Z \in \mathfrak{nil}^3$ and T satisfying

$$[X,Y] = Z \qquad [X,Z] = 0 \qquad [Y,Z] = 0 \tag{3}$$

$$[T,X] = Y \qquad [T,Y] = -X \qquad [T,Z] = 0\,. \tag{4}$$

The lane left–invariant plane field spanned by X, Y is a contact structure $\mathcal{C}_{st}$ which preserved by G. Obviously this is the only plane field with this property and the corresponding contact structure is tight since according to [Be].

Now let $(\mathrm{Nil}^3, \mathcal{C}, H)$ be a geometric contact structure with $H \subset G$ connected and $\dim(H) = 3$. From Theorem 3 it would follow that (Nil^3, H) is equivalent to $(\mathrm{Nil}^3, \mathrm{Nil}^3)$. We now show that actually $(\mathrm{Nil}^3, H') = (\mathrm{Nil}^3, \mathrm{Nil}^3)$: Either $\mathfrak{h} = \mathfrak{nil}^3$ or $\mathfrak{h}$ is transverse to $\mathfrak{nil}^3$. In the second case it follows from (3) and $\dim(H) = 3$ that $Z \in \mathfrak{h}$ and $\mathfrak{h}$ intersects $\mathcal{C}_{st}$ in a line. Because the adjoint action of T does not preserve any one dimensional subspace of the X, Y-plane, only the case $\mathfrak{h} = \mathfrak{nil}^3$ can occur. Therefore $\mathcal{C}$ is invariant under Nil^3. Note that we did not use any orientation reversing diffeomorphism of the manifold Nil^3.

Now since $\mathcal{C}$ is a contact structure it has to be transverse to Z. After conjugation with a suitable element of Nil^3, the plane $\mathcal{C}(e)$ is identified with $\mathcal{C}_{st}(e)$. Since both $\mathcal{C}$ and $\mathcal{C}_{st}$ are Nil^3-invariant these contact structure coincide everywhere on Nil^3. Thus every geometric contact structures $(\mathrm{Nil}^3, \mathcal{C}, H)$ is equivalent to a subgeometry of $(\mathrm{Nil}^3, \mathcal{C}_{st}, G)$. Since we used only orientation preserving automorphisms, only one orientation of the Thurston geometry $(\mathrm{Nil}^3, \mathrm{Nil}^3)$ is represented by a geometric contact structure.

2.3. $X = \widetilde{\mathrm{Sl}}(2)$. Recall that $\mathrm{PSl}(2,\mathbb{R})$ acts freely and transitively on the manifold $S^1T\mathbb{H}^2$ of unit tangent vectors of the hyperbolic plane $\mathbb{H}^2$. We identify $\mathrm{PSl}(2,\mathbb{R})$ with $S^1T\mathbb{H}^2$ and we use the Riemannian metric induced by the hyperbolic metric on $\mathbb{H}^2$. Then the projection of $S^1T\mathbb{H}^2$ onto $\mathbb{H}^2$ is a Riemannian submersion. The Thurston geometry $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is obtained by taking the universal covering of $S^1T\mathbb{H}^2$.

The plane field $\mathcal{C}$ orthogonal to the fibers is the horizontal plane field of the connection on $\mathbb{H}^2$. Because the curvature of $\mathbb{H}^2$ is vanishing nowhere, $\mathcal{C}$ is a contact structure.

The maximal isometry group G of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is generated by the group $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ acting by left–multiplication together with horizontal lifts of isometries such that the resulting map fixes the fiber of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ over a base point in $\mathbb{H}^2$. The latter isometries form the stabilizer of points in that fiber. The plane field $\mathcal{C}$ is preserved by all isometries of $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}), G)$. Hence $\mathcal{C}$ is a maximal geometric contact structure.

Let A, B, C be a basis of $\mathfrak{sl}(2,\mathbb{R})$ such that

$$[A,B] = C \qquad [C,A] = -B \qquad [C,B] = A \tag{5}$$

The plane field spanned by A, B is a geometric contact structure on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ which is invariant under the SO(2)–action given by

$$[W,A] = -B \qquad [W,B] = A \qquad [W,C] = 0 \tag{6}$$

where W is the generator of the Lie algebra of SO(2). Since $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is simply connected, this defines an action of SO(2) on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Thus the identity component of the maximal isometry group of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is the semidirect product $G_0 = \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \ltimes \mathrm{SO}(2)$ (cf. the geometries S^3 and Nil^3).

Next we want to show that every geometric contact structure $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}), \mathcal{C}, H)$ with connected isometry group is equivalent to a subgeometry of $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}), \mathcal{C}_{st}, G_0)$. If H is four–dimensional, then $H = G_0$ and the only G_0–invariant plane field is $\mathcal{C}_{st}$.

If $\dim(H) = 3$, then by Theorem 3 we may assume $H = \widetilde{\mathrm{Sl}}(2,\mathbb{R})$. In order to classify left–invariant contact structures up to equivalence, we consider the coadjoint action on $\mathfrak{sl}^*(2)$. A defining form $\alpha \in \mathfrak{sl}^*(2,\mathbb{R})$ for a left–invariant contact structure is well defined up to multiplication with a real number. According to the description of the coadjoint orbits of $\mathfrak{sl}^*(2,\mathbb{R})$ in [MaRa] the multiple of the contact form α belongs to one of two equivalence classes. Forms in these two conjugacy classes induce different contact orientations of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Thus up to orientation preserving equivalence $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}), \widetilde{\mathrm{Sl}}(2,\mathbb{R}))$ admits two geometric contact structures $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}), \mathcal{C}, \widetilde{\mathrm{Sl}}(2,\mathbb{R}))$ and $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}), \mathcal{C}', \widetilde{\mathrm{Sl}}(2,\mathbb{R}))$.

It remains to show that $(\widetilde{\mathrm{Sl}}(2), \mathcal{C}_{st}, \widetilde{\mathrm{Sl}}(2))$ and $(\widetilde{\mathrm{Sl}}(2), \mathcal{C}', \widetilde{\mathrm{Sl}}(2))$ are equivalent. For this consider the diffeomorphism ψ of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ which is obtained by lifting the map

$$\begin{aligned} \mathrm{Sl}(2,\mathbb{R}) &\longrightarrow \mathrm{Sl}(2,\mathbb{R}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} -a & b \\ c & -d \end{pmatrix} \end{aligned}$$

to the universal covering. Obviously, ψ is orientation reversing. Moreover, conjugation with ψ maps elements of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ to $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Hence ψ is really an automorphism of the Thurston geometry $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}), \widetilde{\mathrm{Sl}}(2,\mathbb{R}))$. The contact structures $\psi_*(\mathcal{C}_{st})$ and $\mathcal{C}_{st}$ induce different orientations. Thus all geometric contact structures on $\widetilde{\mathrm{Sl}}(2)$ are equivalent to subgeometries of $(\widetilde{\mathrm{Sl}}(2), \mathcal{C}_{st}, G)$.

In order to prove that the geometric contact structure $\mathcal{C}_{st}$ on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is equivalent to the tight (cf. [Be]) contact structure $\mathcal{C}' = \ker(dz' - x'\,dy')$ on $\mathbb{R}^3$ we first exhibit a convex surface $\widetilde{\Sigma}$ in $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ such that the line field $T\widetilde{\Sigma} \cap \mathcal{C}_{st}$ is equivalent to the line field on $\Sigma' = \{x' > 0, z' = 0\} \subset \mathbb{R}^3$ induced by $\mathcal{C}'$. In addition we find a Reeb vector field R transverse to $\widetilde{\Sigma}$ whose flow is conjugate to the flow of the Reeb vector field $\partial_{z'}$ of α' such that each flow line of R intersects $\widetilde{\Sigma}$ exactly once.

Now consider the contact vector field induced by horizontal lifts of rotations of $\mathbb{H}^2$ around a point in the interior of $\mathbb{H}^2$. By definition the lifts of these rotations preserve the contact structure and the corresponding vector field are transverse to $\mathcal{C}_{st}$. Each orbit of the flow of R intersects $\widetilde{\Sigma}$ exactly once (note that the rotation of $\mathbb{H}^2$ by 2π yields a deck transformation of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$).

For $\widetilde{\Sigma}$ we take the lift of the surface Σ whose points in $\mathrm{Sl}(2,\mathbb{R})$ are represented by the speed vector field of geodesics starting at a chosen point on the ideal boundary of $\mathbb{H}^2$. It is easy to show that the leaves of the foliation on Σ induced by the contact structure correspond to these geodesics. This foliation is hence equivalent to the foliation on Σ' induced by $\mathcal{C}'$.

According to [Gi] the foliation on $\widetilde{\Sigma}$ determines the contact structure on a neighbourhood of $\widetilde{\Sigma}$. Because one can cover $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ using translates of neighbourhoods of $\widetilde{\Sigma}$ this determines the contact structure on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ up to isotopy. Since the flows of $\partial_{z'}$ and R are conjugate this implies that the contact structure on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is equivalent to the standard tight contact structure on $\mathbb{R}^3$.

2.4. $X = \mathrm{Sol}^3$. The group Sol^3 is the semidirect product $\mathbb{R}^2 \ltimes \mathbb{R}$: We write x, y for the coordinates on $\mathbb{R}^2$ and t for the coordinate on $\mathbb{R}$. The action of $\mathbb{R}$ on $\mathbb{R}^2$ is given by $t \cdot (x,y) = (e^t x, e^{-t} y)$. The isometry group of Sol^3 is generated by Sol^3 and the isometries

$$\begin{aligned} r_1 &: (x,y,t) \longmapsto (-x,y,t) \qquad\qquad r_2 : (x,y,t) \longmapsto (x,-y,t) \\ \tau &: (x,y,t) \longmapsto (y,x,-t)\ . \end{aligned}$$

The maximal isometry group of Sol^3 has eight connected components. Four of them contain orientation preserving isometries. The Lie algebra $\mathfrak{sol}^3$ is generated by the left-invariant vector fields X, Y, T with the relations

$$[X,Y] = 0 \qquad [T,X] = X \qquad [T,Y] = -Y\ . \tag{7}$$

¿From this one can easily check that $\mathcal{C}_\pm = \mathrm{span}\{R, X \pm Y\}$ are geometric contact structure inducing different orientations. Both are preserved by Sol^3 and $\tau, r_1 \circ r_2$. Moreover $r_{1*}\mathcal{C}_+ = \mathcal{C}_-$ implies that $\mathcal{C}_+$ and $\mathcal{C}_-$ are equivalent.

Now we want to show that all geometric contact structures on Sol^3 are equivalent. Let $(\mathrm{Sol}^3, \mathcal{C}, H)$ be a geometric contact structure with connected isometry group. Since G is 3–dimensional we may assume that $H = \mathrm{Sol}^3$. Because of (7), the plane field spanned by X, Y must be transverse to $\mathcal{C}$. Hence there are $x_1, y_1, x_2, y_2 \in \mathbb{R}$ such that $\mathcal{C} = \mathrm{span}\{x_1 X + y_1 Y, T + x_2 X + y_2 Y\}$. Without loss of generality assume $x_1 \geq 0$.

Since $\mathcal{C}$ is a contact structure $x_1, y_1 \neq 0$ and we may assume in addition that $x_2 = 0$. Conjugating with a suitable element of $\exp(\mathbb{R}Y)$ we can achieve that y_2 also vanishes and using elements of $\{0\} \ltimes \mathbb{R} \subset \mathrm{Sol}^3$ we can achieve that $1 = x_1 = \pm y_1$. Thus all geometric contact structures on Sol^3 are equivalent and both orientations of Sol^3 are induced by geometric contact structures.

In order to show that $\mathcal{C}_+$ is equivalent to the standard tight contact structure on $\mathbb{R}^3$, we consider the Sol^3–invariant defining form $\alpha = 1/2\,(e^{-t}dx - e^t dy)$. The Reeb vector field of α (characterized by $i_R d\alpha = 0$ and $\alpha(R) = 1$) is $R = X - Y$. The corresponding flow has a transversal $\{x = 0\}$ intersecting every flow line of R exactly once and the induced foliation on $\{x = 0\}$ consists of horizontal lines parallel to the t–axis.

The standard contact form $dz' - x'\,dy'$ on $\mathbb{R}^3$ (with coordinates x', y', z') has analogous features on $\{x' > 0, z' = 0\}$. This shows that $\mathcal{C}_+$ is isomorphic to the usual tight contact structure on $\mathbb{R}^3$ (cf. Section 2.3).

2.5. $X = \mathbb{R}^3$**.** The metric is the flat metric and the maximal group of isometries is $G = \mathbb{R}^3 \ltimes \mathrm{O}(3)$ acting in the obvious way on $\mathbb{R}^3$. In particular, every subgroup $H \subset G$ acting transitively on $\mathbb{R}^3$ must contain the translations $\mathbb{R}^3$, cf. Theorem 3. But every plane field which is invariant under translations is integrable. Thus there is no geometric contact structure on $\mathbb{R}^3$.

2.6. $X = \mathbb{H}^3$**.** The metric on $X = \mathbb{H}^3$ has constant sectional curvature and the maximal isometry group is $G \simeq \mathrm{PSl}(2, \mathbb{C}) \ltimes \mathbb{Z}_2$ where the generator of $\mathbb{Z}_2$ is the reflection of $\mathbb{H}^3$ along the plane $\mathbb{H}^2 \subset \mathbb{H}^3$.

By Theorem 3, $(\mathbb{H}^3, \mathrm{PSL}(2, \mathbb{C}))$ does not have any proper subgeometries. Since the stabilizer of a point in $\mathbb{H}^3$ is isomorphic to $\mathrm{SO}(3)$, there is no plane field invariant under all elements of $\mathrm{PSL}(2, \mathbb{R})$. Thus there is no geometric contact structure on $\mathbb{H}^3$.

2.7. $X = S^2 \times \mathbb{R}$**.** This is the obvious product geometry. The full isometry group is the product of the isometry groups of S^2 and $\mathbb{R}$.

Suppose that $\mathcal{C}$ is a geometric contact structure on X. Since G acts transitively, $\mathcal{C}$ is either everywhere tangent to the fibers of the projection $S^2 \times \mathbb{R}$ onto the second factor or everywhere transverse to the fibers. On the one hand contact structures have no integral surfaces and on the other hand S^2 does not admit a non–singular line field. Thus there is no geometric contact structure on $S^2 \times R$.

2.8. $X = \mathbb{H}^2 \times \mathbb{R}$**.** Again, this is the obvious product geometry. The isometry group G is the product of the isometry groups of the factors. The dimension of G is four and $G_0 = \mathrm{PSl}(2, \mathbb{R}) \times \mathbb{R}$.

By Theorem 3, this geometry has no proper subgeometry of dimension 3. The only plane field which is invariant under the action of $\mathrm{PSl}(2,\mathbb{R})$ is tangent to the first factor of $\mathbb{H}^2\times\mathbb{R}$. Thus there is no geometric contact structure on this geometry.

3. Geometric Engel structures

Now we come to the main part of this article, the classification of geometric Engel structures up to equivalence. Let us first make some general remarks about the isometry group of a maximal geometric Engel structure $(X,\mathcal{D},G)$.

Every isometry preserving an Engel structure $\mathcal{D}$ has to preserve the subbundles

$$\mathcal{W}\subset\mathcal{D}\subset\mathcal{E}=[\mathcal{D},\mathcal{D}]\subset TM\ . \tag{8}$$

Hence the identity component of the stabilizer of a point $x\in X$ in the group of isometries H preserving an Engel structure $\mathcal{D}$ acts trivially on T_xX. This implies that H has dimension four. Since X is simply connected, G acts freely on X. Thus we can find automorphisms of Thurston geometries as in Example 2.

The other connected components of the stabilizer of x act by isometries on T_xX and they preserve (8). For every local framing X,Y of $\mathcal{D}$ we obtain the local framing $X,Y,[X,Y]$ of $\mathcal{E}$. Note that the orientation of $\mathcal{E}=[\mathcal{D},\mathcal{D}]$ defined by $X,Y,[X,Y]$ is actually independent from X,Y. Thus $\mathcal{E}$ is canonically oriented and every isometry preserving $\mathcal{D}$ must also preserve this orientation of $\mathcal{E}$. Therefore the stabilizer of x in G can have at most four components. The following theorem summarizes the classification of geometric Engel structures.

Theorem 11. *The following Thurston geometries admit a geometric Engel structure which is unique up to equivalence:*

$$\begin{array}{lll}
(S^3\times\mathbb{R},\mathrm{SU}(2)\times\mathbb{R}) & (\mathrm{Nil}^3\times\mathbb{R},\mathrm{Nil}^3\ltimes\mathbb{R}) & (\widetilde{\mathrm{Sl}}(2)\times\mathbb{R},\widetilde{\mathrm{Sl}}(2)\times\mathbb{R})\\
(\mathrm{Sol}^4(m,n),\mathrm{Sol}^4(m,n)) & (\mathrm{Sol}^4_0,\mathrm{Sol}^4(\lambda)) & (\mathrm{Sol}^4_1,\mathrm{Sol}^4_1)\\
(\mathrm{Nil}^4,\mathrm{Nil}^4) & &
\end{array}$$

No subgeometry of the following Thurston geometries admits a geometric Engel structure:

$$\begin{array}{lllll}
S^4 & \mathbb{R}^4 & \mathbb{H}^4 & \mathbb{CP}^2 & \mathbb{HC}^2\\
S^2\times S^2 & S^2\times\mathbb{R}^2 & S^2\times\mathbb{H}^2 & \mathbb{H}^2\times\mathbb{H}^2 & \mathbb{H}^2\times\mathbb{R}^2\\
F^4 & \mathbb{H}^3\times\mathbb{R} & (\mathrm{Nil}^3\times\mathbb{R},\mathrm{Nil}^3\times\mathbb{R}) & &
\end{array}$$

For the proof we consider the individual cases where we also discuss the non–maximal Thurston geometries like for example $(\mathrm{Sol}^4_0,\mathrm{Sol}^4(\lambda))$. We start with those geometries which do not admit geometric Engel structures for topological reasons. Then we discuss $S^3\times\mathbb{R},\mathrm{Nil}^3\times\mathbb{R},\widetilde{\mathrm{Sl}}(2,\mathbb{R})\times\mathbb{R}$ using prolongation. Finally we treat the remaining geometries.

3.1. **Topological obstructions.** Every Engel manifold has vanishing Euler characteristic and signature. Since the underlying space of each Thurston geometry X is simply connected it follows also that the tangent bundle of X is trivial if there is a geometric Engel structure on X.

Therefore $S^4,\mathbb{CP}^2,S^2\times S^2,S^2\times\mathbb{R}^2,S^2\times\mathbb{H}^2$ do not carry geometric Engel structures. Moreover, compact quotients of $\mathbb{H}^4,\mathbb{H}^2\times\mathbb{H}^2,\mathbb{H}^2(\mathbb{C})$ have positive Euler characteristic, cf. [Wa2, Ko]. Therefore these geometries do not admit geometric Engel structures either.

3.2. **Prolongation of geometric contact structures.** Three Thurston geometries of dimension 4 can be treated using a construction similar to prolongation, cf. Example 6. Recall that the 3–dimensional Thurston geometries $X = S^3, \mathrm{Nil}^3$ and $\widetilde{\mathrm{Sl}}(2)$ admit geometric contact structures $(X, \mathcal{C}, H)$ such that H is 4–dimensional, connected and the stabilizer of points in X acts by rotations on the contact plane at that point. The action of H is free and transitive on the manifold $S^1\mathcal{C}$ of unit vectors of the contact structure. The action is isometric for the induced Riemannian metric on $S^1\mathcal{C}$.

Lifting all data to the universal coverings we obtain an isometric group action of $\widetilde{H}$ on $\widetilde{S^1\mathcal{C}}$ which is free and transitive. In particular, the preimage in $\widetilde{H}$ of a cocompact lattice $\Gamma \subset H$ is a cocompact lattice $\widetilde{\Gamma}$ in $\widetilde{H}$. Thus $(S^1\mathcal{C}, \widetilde{H})$ is a Thurston geometry. Similarly to Example 6, there is a natural geometric contact structure on $(\widetilde{S^1\mathcal{C}}, \widetilde{H})$.

Let $\mathrm{pr} : S^1\mathcal{C} \longrightarrow X$ be the projection and $\Lambda : \widetilde{S^1\mathcal{C}} \longrightarrow S^1\mathcal{C}$ be the universal covering map. The plane field

$$\mathcal{D} = \left\{ v \in T_l\widetilde{S^1\mathcal{C}} \;\middle|\; (\Lambda \circ \mathrm{pr})_*(v) \text{ a multiple of } \Lambda(l) \in \mathcal{C} \right\}$$

is $\widetilde{H}$–invariant by definition. Moreover $\mathcal{D}$ is an Engel structure. It is simply the lift of the prolonged Engel structure to the universal covering. Thus we have shown the following proposition.

Proposition 12. *Let $(X, \mathcal{C}, H)$ be a geometric contact structure with connected isometry group H such that the stabilizer in H of a point $x \in X$ acts by rotations of the contact plane $\mathcal{C}(x)$. Using the notation from above, the plane field $\mathcal{D}$ defined by*

$$\mathcal{D} = \left\{ v \in T_l\widetilde{S^1\mathcal{C}} \;\middle|\; (\Lambda \circ \mathrm{pr})_*(v) \textit{ a multiple of } \Lambda(l) \in \mathcal{C} \right\}$$

is a geometric Engel structure on $(\widetilde{S^1\mathcal{C}}, \widetilde{H})$.

It remains to classify all geometric Engel structures on $S^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2, \mathbb{R}) \times \mathbb{R}$. It will turn out that for these Thurston geometries there is a unique geometric Engel structure up to equivalence. Observe that by Theorem 3, all subgeometries of $S^3 \times \mathbb{R}$ and $\widetilde{Sl}(2, \mathbb{R}) \times \mathbb{R}$ with connected four–dimensional isometry group are equivalent. In the case of $\mathrm{Nil}^3 \times \mathbb{R}$ there are two non–equivalent subgeometries with these properties. As we will see in Section 3.2.2 only one class of these subgeometries caries a geometric Engel structure.

In all three cases, the isometry group $\widetilde{H}$ used in Proposition 12 can be decomposed into a semidirect product where $\mathbb{R}$ acts on one of the groups $\mathrm{SU}(2), \widetilde{\mathrm{Sl}}(2, \mathbb{R})$ and Nil^3 by automorphisms. Because $\mathrm{SU}(2)$ and $\widetilde{\mathrm{Sl}}(2, R)$ are semisimple Lie groups, all automorphisms are inner, thus group elements can be used to obtain an action similar to the action of $\mathbb{R}$ if $X = S^3$ or $X = \widetilde{\mathrm{Sl}}(2, R)$. In contrast to this, if $X = \mathrm{Nil}^3$ not all automorphisms are inner. This explains why $\mathrm{Nil}^3 \times \mathbb{R}$ has not a unique subgeometry with four–dimensional connected isometry group.

The argument in [Wa2] that $(\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R})$ is the only minimal subgeometry of $(\mathrm{Nil}^3 \times \mathbb{R}, G \times \mathbb{R})$ appears incomplete because it does not justify why the fact that $\Gamma \cap (\mathrm{Nil}^3 \times \mathbb{R})$ is a lattice for all lattices $\Gamma \subset H$ implies that $\mathrm{Nil}^3 \times \mathbb{R}$ is contained in H where H is the isometry group of a minimal Thurston geometry.

In the next three section we consider the case $X = S^3, \mathrm{Nil}^3, \widetilde{Sl}(2,\mathbb{R})$ and show that all geometric Engel structures on $X \times \mathbb{R}$ are equivalent to $\mathcal{D}$ from Proposition 12.

3.2.1. *$X = S^3$.* Let $(S^3 \times \mathbb{R}, \mathcal{D}, H)$ be a geometric Engel structure such that $H \subset \mathrm{SO}(4) \times \mathbb{R}$ is connected. By Theorem 3 every subgeometry of $S^3 \times S^1$ with connected isometry group of dimension 4 is equivalent to $(S^3 \times S^1, \mathrm{SU}(2) \times \mathbb{R})$. It remains to show that all geometric Engel structures on $(S^3 \times \mathbb{R}, \mathcal{D}, \mathrm{SU}(2) \times \mathbb{R})$ are equivalent to each other.

We identify $S^3 \times \mathbb{R}$ with $\mathrm{SU}(2) \times \mathbb{R}$. The Lie algebra $\mathfrak{su}(2)$ has a basis A, B, C satisfying the relations

$$[A,B] = 2C \quad [B,C] = 2A \quad [C,A] = 2B \tag{9}$$

Assume first that the characteristic foliation $\mathcal{W}$ of $\mathcal{D}$ is tangent to $\mathrm{SU}(2)$. Then $\mathcal{E}(e) \cap \mathfrak{su}(2)$ induces a left–invariant plane field which is invariant under $\mathcal{W}$. This would mean that $\mathcal{E}(e) \cap \mathfrak{su}(2)$ is a left–invariant foliation on $\mathrm{SU}(2)$. It follows from (9) that such a foliation does not exist. Hence $\mathcal{W}$ is transverse to $\mathrm{SU}(2)$. Then $T\,\mathrm{SU}(2) \cap \mathcal{E}$ is a geometric contact structure on S^3 and by Section 2.1 we may assume that it is spanned by $A, B \in \mathfrak{su}(2)$.

Using suitable isometries from $\mathrm{SU}(2)$ we can achieve that $\mathcal{D} \cap T(S^3 \times 0)$ is mapped to the left–invariant line field $\mathbb{R}A$. Then $\mathcal{D}$ is spanned by A and $\partial_t + \lambda B + \mu C$. If both λ and μ are zero, then $\mathcal{D}$ is not an Engel structure. Conjugating with elements of the 1–parameter subgroup corresponding to A we can achieve that $\lambda > 0$ and $\mu = 0$. Finally,

$$\begin{aligned} S^3 \times \mathbb{R} &\longrightarrow S^3 \times \mathbb{R} \\ (x,t) &\longmapsto (x, \lambda \cdot t) \end{aligned} \tag{10}$$

is an equivalence of the geometric Engel structures $\mathcal{D} = \mathrm{span}\{A, \partial_t + \lambda B\}$ and $\mathcal{D}_{st} = \mathrm{span}\{A, \partial_t + B\}$. The maximal isometry group preserving $\mathcal{D}_{st}$ is generated by $\mathrm{SU}(2) \times \mathbb{R}$ and two reflections of $S^3 \times \mathbb{R}$. Thus the maximal isometry group of a geometric Engel structure on $S^3 \times \mathbb{R}$ has four connected components.

3.2.2. *$X = \mathrm{Nil}^3$.* We use the notation from Section 2.2. We want to show that all geometric Engel structures $(\mathrm{Nil}^3 \times \mathbb{R}, \mathcal{D}, H)$ with connected isometry group are equivalent to the one exhibited in Proposition 12. The Lie algebra of the maximal connected isometry group G of Nil^3 is generated by X, Y, Z, W satisfying the relations

$$\begin{array}{llllll} [X,Y] & = Z & [X,Z] & = 0 & [Y,Z] & = 0 \\ [W,X] & = -Y & [W,Y] & = X & [W,Z] & = 0 \end{array} \tag{11}$$

and the identity component of the maximal isometry group of the geometry $\mathrm{Nil}^3 \times \mathbb{R}$ is $G \times \mathbb{R}$. As in Section 3.2.1 one can show that the even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ induces a geometric contact structure on $(\mathrm{Nil}^3, H \cap (G \times 0))$. By Section 2.2 we may assume that this is the standard contact structure $\mathrm{span}\{X, Y\}$ on Nil^3 and that $\mathrm{Nil}^3 = H \cap (G \times 0)$.

Thus $\mathfrak{h}$ is spanned by $\mathfrak{nil}^3$ and $\partial_t + \lambda Z$ for $\lambda \in \mathbb{R}$. If $\lambda = 0$, then $H = \mathrm{Nil}^3 \times \mathbb{R}$. It is clear from (11) that there is no geometric Engel structure on $(\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R})$. Hence we may assume that $\lambda \neq 0$. Applying an equivalence analogous to (10) we can achieve $\mathfrak{h} = \mathfrak{nil}^3 + \mathbb{R}(\partial_t + W)$. The Thurston geometries $(\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R})$ and

($\mathrm{Nil}^3 \times \mathbb{R}, H = \mathrm{Nil}^3 \ltimes \mathbb{R}$) are not equivalent because $\mathrm{Nil}^3 \times \mathbb{R}$ is nilpotent while H is only solvable.

We identify the manifolds $\mathrm{Nil}^3 \times \mathbb{R}$ and H. Then $\mathcal{D}$ is spanned by $\lambda X + \mu Y \neq 0$ and $\partial_t + W + S$ with $S \in \mathfrak{nil}^3$. After conjugation with a suitable element of $\exp(\mathbb{R}(\partial_t + W))$ we may assume $\lambda = 1, \mu = 0$ and that S is a linear combination of Y and Z (recall that conjugation with $\exp(\mathbb{R}W)$ corresponds to rotations of the contact plane at the unit element of Nil^3). But since adding a multiple of Z to $\partial_t + W + S$ induces an isomorphism of H we can even assume $S = \nu Y$ with $\nu \in \mathbb{R}$. The following equivalence of the Thurston geometry (Nil^3, H)

$$\begin{aligned} \partial_t + W &\longmapsto \partial_t + W + \nu Y & X &\longmapsto X \\ Y &\longmapsto Y - \nu Z & Z &\longmapsto Z \end{aligned}$$

finally shows that all Engel structures $\mathcal{D}_\nu = \mathrm{span}\{\partial_t + W + \nu Y, X\}$ are equivalent.

The maximal group of isometries preserving $\mathcal{D}_{st} = \mathcal{D}_0$ is generated by H and the isometries acting on $\mathfrak{h}$ as follows

$$\begin{aligned} \psi_1 : X &\longmapsto -X & Y &\longmapsto -Y & Z &\longmapsto Z & W &\longmapsto -W \\ \psi_2 : X &\longmapsto -X & Y &\longmapsto Y & Z &\longmapsto -Z & W &\longmapsto W\,. \end{aligned}$$

3.2.3. $X = \widetilde{\mathrm{Sl}}(2)$. For a description of the Thurston geometry $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ as well as for the classification of geometric contact structures on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ we refer to Section 2.3. In that section we also described the Lie algebra $\mathfrak{g}$ of the maximal isometry group of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$–geometry by generators and relations, cf. (5) and (6).

We want to show that all geometric Engel structures ($\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}, \mathcal{D}, H$) are equivalent to the geometric Engel structure from Proposition 12. By Theorem 3 we may assume that $H = \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}$. The generator of the Lie algebra of the second factor will be denoted by ∂_t.

Now assume that $\mathcal{D}'$ is a geometric geometric Engel structures on ($\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}$) such that the characteristic foliation $\mathcal{W}'$ of $\mathcal{E}' = [\mathcal{D}', \mathcal{D}']$ is transverse to the first factor. Then $\mathcal{E}'$ induces a geometric contact structure $\mathcal{C}'$ on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ and by Section 2.3 we may assume that $\mathcal{C}'$ is spanned by A, B. The characteristic foliation of $\mathcal{E}'$ is left invariant. But the only left invariant vector field W which preserves $\mathcal{E}$ is ∂_t. This vector field is contained in the center of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ and this contradicts that $\mathcal{D}' \supset \mathcal{W}'$ is an Engel structure.

The characteristic foliation of each geometric Engel structure $\mathcal{D}'$ on $\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}$ is tangent to the first factor. This implies that $\mathcal{E}' = [\mathcal{D}', \mathcal{D}]$ induces a left–invariant foliation on the first factor. From the description of the coadjoint orbits in $\mathfrak{sl}^*(2,\mathbb{R})$ in Section 2.3 one can read of that all left–invariant foliations on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ are conjugate to each other.

Hence we may assume that $\mathcal{E}' \cap \widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is spanned by A, C and that $\partial_t + \mu B$ extends this to a framing of $\mathcal{E}'$ with $\mu \in \mathbb{R}$. If $\mu = 0$, then $\mathcal{E}'$ is a foliation and no even contact structure. After applying an automorphism of the Thurston geometry ($\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}$) similar to (10) we may assume $\mu = 1$. In this case, the characteristic foliation of $\mathcal{E}'$ is spanned by A.

So there is $\nu \in \mathbb{R}$ such that the Engel structure is spanned by A and $\partial_t + B + \nu C$. Finally, conjugation with elements of $\exp(\mathbb{R}A)$ shows that all geometric Engel structures on $(\widetilde{\mathrm{Sl}}(2,\mathbb{R})\times\mathbb{R},\widetilde{\mathrm{Sl}}(2,\mathbb{R})\times\mathbb{R})$ are equivalent to $\mathcal{D}_0$ with $\nu=0$.The maps

$$\begin{aligned}\widetilde{\mathrm{Sl}}(2)\times\mathbb{R} &\longrightarrow \widetilde{\mathrm{Sl}}(2)\times\mathbb{R}\\ \left(\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},t\right) &\longmapsto \left(\begin{pmatrix} a_{11} & -a_{12}\\ -a_{21} & a_{22}\end{pmatrix},-t\right)\\ (A,t) &\longmapsto \left((A^T)^{-1},t\right)\end{aligned}$$

are isomorphisms of $\widetilde{\mathrm{Sl}}(2)\times\mathbb{R}$. Thus the maximal group of Engel preserving isometries has four components, we have seen before that there cannot be more of them.

3.3. **Solvable Geometries.** In this section we show that each of the solvable geometries $\mathrm{Sol}^4(m,n),\mathrm{Sol}^4_1$ and all subgeometries of Sol^4_0 admit a geometric Engel structure which is unique up to equivalence. Then we compare theses Engel structures with the construction of Geiges described in Example 7.

3.3.1. $X=\mathrm{Sol}^4(m,n)$. Let m,n be positive integers such that the zeroes of

$$P(m,n)=-\lambda^3+m\lambda^2-n\lambda+1 \tag{12}$$

are pairwise different real numbers $e^\alpha,e^\beta,e^\gamma$ with $\alpha+\beta+\gamma=0$ and $\alpha>\beta>\gamma$. Other possible configurations of the zeroes of $P(m,n)$ will be discussed below. The Lie group $\mathrm{Sol}^4(m,n)$ is the semidirect product $\mathbb{R}^3\ltimes\mathbb{R}$ with $\mathbb{R}$ acting on $\mathbb{R}^3$ by

$$t\longmapsto\psi(t)=\exp\begin{pmatrix}\alpha t & 0 & 0\\ 0 & \beta t & 0\\ 0 & 0 & \gamma t\end{pmatrix} \tag{13}$$

of $\mathbb{R}$ on $\mathbb{R}^3$. In order to find a cocompact lattice in $\mathrm{Sol}^4(m,n)$ consider $A\in\mathfrak{sl}(3,\mathbb{R})$ such that

$$\exp(A)=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -n & m\end{pmatrix}. \tag{14}$$

The group obtained by replacing the $\mathbb{R}$–action from (13) by $\exp(tA)$ is isomorphic to $\mathrm{Sol}^4(m,n)$ and since all coefficients of $\exp(A)$ are integers, $\mathrm{Sol}^4(m,n)$ contains a cocompact lattice isomorphic to $\mathbb{Z}^3\ltimes\mathbb{Z}$ where $1\in\mathbb{Z}$ acts on $\mathbb{Z}^3$ by $\exp(A)$.

Two couples (m,n) and (m',n') of integers as above yield isomorphic Lie groups if and only if the corresponding triples (α,β,γ) and (α',β',γ') are proportional. A particular instance is $m=n\geq 4$. Because of $\beta=0$ and $\alpha=-\gamma\in\mathbb{R}$ this corresponds to the Thurston geometry $\mathrm{Sol}^3\times\mathbb{R}$.

The Lie algebra $\mathfrak{sol}^4(m,n)$ is generated by X_1,X_2,X_3,T with the relations

$$[T,X_1]=\alpha X_1 \qquad [T,X_2]=\beta X_2 \qquad [T,X_3]=\gamma X_3 \tag{15}$$

while the remaining commutators vanish. If $m\neq n$, the isometry group of $\mathrm{Sol}^4(m,n)$ is generated by $\mathrm{Sol}^4(m,n)$ and

$$\begin{aligned} X_1&\longmapsto\lambda_1X_1 \qquad X_2\longmapsto\lambda_2X_2 \qquad X_3\longmapsto\lambda_3X_3\\ T&\longmapsto T\end{aligned} \tag{16}$$

where $\lambda_1,\lambda_2,\lambda_3=\pm1$. If $m=n\geq 4$, then there is an additional generator $\psi_=$ interchanging X_1 with X_3 and reverses T.

The left–invariant plane field $\mathcal{D}_0=\mathrm{span}(T,X_1+X_2+X_3)$ is a geometric Engel structure since α,β,γ are pairwise different. The characteristic foliation is spanned by $X_1+X_2+X_3$.

If $m \neq n$, the only two components of the maximal isometry group H preserving $\mathcal{D}_{st}$ contain the maps from (16) for $\lambda_1 = \lambda_2 = \lambda_3 = \pm 1$. In the case $m = n \geq 4$, the isometry $\psi_=$ also preserves $\mathcal{D}_{st}$. So in this case the maximal group of isometries preserving the Engel structure has four connected components.

Next we show that all geometric Engel structures on $\mathrm{Sol}^4(m,n)$ are equivalent. Assume that $\mathcal{D}$ is a geometric Engel structure on $\mathrm{Sol}^4(m,n)$. Then $\mathcal{D}$ has to be left–invariant and transverse to the first factor of $\mathbb{R}^3 \ltimes \mathbb{R} \simeq \mathrm{Sol}^4(m,n)$. Since $[X_i, X_j] = 0, i,j = 1,2,3$, there is an automorphism of $\mathrm{Sol}^4(m,n)$ mapping a given $V \notin \mathbb{R}^3 \ltimes \{0\}$ to a non zero multiple of T. Thus we may assume that $\mathcal{D}$ is spanned by T and $x_1X_1 + x_2X_2 + x_3X_3$ for suitable $x_1, x_2, x_3 \in \mathbb{R}$.

By (15) this plane field is and Engel structure if and only if $x_i \neq 0$ for all $i = 1,2,3$. Thus the automorphism of $\mathrm{Sol}^4(m,n)$ defined in (16) with $\lambda_i = x_i^{-1}, i = 1,2,3$ (instead of $\lambda_i = \pm 1$) and fixing T maps $\mathcal{D}$ to $\mathcal{D}_{st}$. This shows that all geometric Engel structures on $\mathrm{Sol}^4(m,n)$ are equivalent.

3.3.2. $X = \mathrm{Sol}^4_0$. Now we consider the case when m,n are such that (12) has two different complex solutions $e^{\lambda}, e^{\overline{\lambda}}$ and a real solution $e^{-2\Re(\lambda)}$. The Lie group $\mathrm{Sol}^4(\lambda)$ is $\mathbb{R}^3 \ltimes \mathbb{R} = (\mathbb{C} \oplus \mathbb{R}) \ltimes \mathbb{R}$ with the $\mathbb{R}$–action

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathrm{Gl}(\mathbb{C} \oplus \mathbb{R}) \\ t &\longmapsto \Big((u,x) \longmapsto \Big(e^{t\lambda}u, e^{-2t\Re(\lambda)}x \Big) \Big) \ . \end{aligned} \tag{17}$$

In order to obtain a Thurston geometry we equip $\mathrm{Sol}^4(\lambda)$ with a left invariant metric and cocompact lattices are obtained using the matrix $A(m,n)$ from (14) in the previous section. This shows that $\mathrm{Sol}^4(\lambda)$ acting on itself by left translations is a Thurston geometry. However it is not maximal: for all possible values of $\lambda \in \mathbb{C}$ the Thurston geometry $(\mathrm{Sol}^4(\lambda), \mathrm{Sol}^4(\lambda))$ is a subgeometry of the same maximal Thurston geometry (Sol^4_0, G) which we describe next.

The Lie group Sol^4_0 is the semidirect product $(\mathbb{C} \oplus \mathbb{R}) \ltimes \mathbb{R}$ where $\mathbb{R}$ acts on $\mathbb{C} \oplus \mathbb{R} = \mathbb{R}^2 \oplus \mathbb{R}$ by

$$t \longmapsto \exp \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -2t \end{pmatrix} . \tag{18}$$

This Lie group does not contain a lattice, cf. [Hil] on p. 137, so $(\mathrm{Sol}^4_0, \mathrm{Sol}^4_0)$ is not a Thurston geometry. However for the left–invariant Riemannian metric

$$g = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2$$

rotations of the complex plane in $(\mathbb{C} \oplus \mathbb{R}) \ltimes \mathbb{R}$ are also isometries. Therefore the identity component of the maximal isometry group G of Sol^4_0 is the semidirect product

$$G_0 = \mathrm{Sol}^4_0 \ltimes \mathrm{SO}(2) \simeq (\mathbb{C} \oplus \mathbb{R}) \ltimes (\mathbb{R} \times S^1) \tag{19}$$

where S^1 acts on Sol^4_0 by rotations of the complex plane in $(\mathbb{C} \oplus \mathbb{R}) \ltimes \mathbb{R}$. In order to realize $\mathrm{Sol}^4(\lambda)$ as a subgeometry of (Sol^4_0, G) we use the embedding

$$\begin{aligned} \mathrm{Sol}^4(\lambda) = (\mathbb{C} \oplus \mathbb{R}) \ltimes \mathbb{R} &\longrightarrow \mathrm{Isom}(\mathrm{Sol}^4_0) \\ ((u,x),t) &\longmapsto \big((u,x), \big(\Re(\lambda)t, \exp(i\Im(\lambda)t) \big) \big) \ . \end{aligned} \tag{20}$$

This embedding is compatible with the action on the manifold $\mathbb{R}^3 \ltimes \mathbb{R} \simeq \mathrm{Sol}^4_0$. This also implies that (Sol^4_0, G) is a Thurston geometry because we obtained cocompact lattices in G from the subgroups $\mathrm{Sol}^4(\lambda)$ whose action on the manifold Sol^4_0 is free and transitive.

¿From the realization of $\mathrm{Sol}^4(\lambda)$ as subgroup of the maximal isometry group of Sol^4_0 one can find a basis U_1, U_2, V, T of $\mathfrak{sol}^4(\lambda)$ satisfying the commutator relations

$$\begin{aligned} [T, U_1] &= \Re(\lambda)U_1 + \Im(\lambda)U_2 \qquad\qquad [T, U_2] = -\Im(\lambda)U_1 + \Re(\lambda)U_2 \\ [T, V] &= -2\Re(\lambda)V \end{aligned}$$

while the remaining commutators vanish. Here T corresponds the rotations of the complex plane in $\mathrm{Sol}^4_0 = (\mathbb{C} \oplus \mathbb{R}) \ltimes \mathbb{R}$. We consider the $\mathrm{Sol}^4(\lambda)$–invariant plane field $\mathcal{D}(\lambda)$ spanned by $T, U_1 + V$. From the commutator relations of $\mathfrak{sol}^4(\lambda)$ it follows

$$\begin{aligned} \mathcal{D}^2 &= \mathcal{D} \oplus \mathbb{R}\left(\Re(\lambda)U_1 + \Im(\lambda)U_2 - 2\Re(\lambda)V\right) \\ \mathcal{D}^3 &= \mathcal{D}^2 \oplus \mathbb{R}\left(\left(\Re^2(\lambda) - \Im^2(\lambda)\right)U_1 + 2\Re(\lambda)\cdot\Im(\lambda)U_2 + 4\Re^2(\lambda)V\right) \ . \end{aligned} \tag{21}$$

Since U_1+V and the two vectors appearing in (21) are linearly independent $\mathcal{D}(\lambda)$ is a geometric Engel structure on $(\mathrm{Sol}^4_0, \mathrm{Sol}^4(\lambda))$ which depends on λ. The characteristic foliation is spanned by $U_1 + V$.

Next we want to show that every geometric Engel structure $(\mathrm{Sol}^4_0, \mathcal{D}, H)$ with connected isometry group is equivalent to one of the geometric Engel structures we have described in this section. By Theorem 3 it is enough to prove that every geometric Engel structure $\mathcal{D}$ on $(\mathrm{Sol}^4_0, \mathrm{Sol}^4(\lambda))$ is equivalent to $\mathcal{D}(\lambda)$.

Obviously, $\mathcal{D}$ must be transverse to $\mathbb{R}^3 \subset \mathrm{Sol}^4(\lambda)$ in order to be an Engel structure. Furthermore since adding an element of $\mathbb{R}^3$ to T induces an inner automorphism of $\mathrm{Sol}^4(\lambda)$, we may assume that T is tangent to $\mathcal{D}$.

It follows from the commutator relations of $\mathrm{Sol}^4(\lambda)$ that $\mathcal{D} \cap \mathbb{R}^3$ is transverse to the plane $\mathbb{C} \oplus 0 \subset \mathbb{R}^3$ and that $\mathcal{D}$ cannot be tangent to V. Now we obtain more automorphisms of the Thurston geometry $(\mathrm{Sol}^4_0, \mathrm{Sol}^4(\lambda))$ by

- rotations of the plane $\mathbb{C} \oplus \{0\} \subset \mathbb{R}^3$,
- multiplying U_1, U_2 by the same real constant,
- multiplying V with a number.

Using these automorphisms we obtain an equivalence between $(\mathrm{Sol}^4_0, \mathcal{D}, \mathrm{Sol}^4(\lambda))$ and $(\mathrm{Sol}^4_0, \mathcal{D}(\lambda), \mathrm{Sol}^4(\lambda))$. The maximal isometry group of the geometric Engel structure is generated by $\mathrm{Sol}^4(\lambda)$ and the automorphism ψ of $\mathrm{Sol}^4(\lambda)$ with $\psi((u, x), t) = ((-u, -x), t)$.

3.3.3. $X = \mathrm{Sol}^4_1$. In the last two sections we considered semidirect products of $\mathbb{R}$ with $\mathbb{R}^3$. Now we turn to Sol^4_1 which is the semidirect product $\mathrm{Nil}^3 \ltimes \mathbb{R}$ with the $\mathbb{R}$–action

$$t \cdot [x, y, z] \longmapsto \left[e^{-t}x, e^t y, z\right] \ . \tag{22}$$

We write $T \in \mathfrak{sol}^4_1$ for the generator of the Lie algebra of $\mathbb{R}$ and X, Y, Z for the usual basis of $\mathfrak{nil}^3$. Then the non vanishing commutators are

$$[T, X] = -X \qquad\qquad [T, Y] = Y \qquad\qquad [X, Y] = Z \ . \tag{23}$$

On Sol^4_1 we choose a left–invariant metric such that $\|X\| = \|Y\|$. According to [Wa2] the maximal isometry group is generated by Sol^4_1 together with the following isometries of $\mathfrak{sol}_1^4$

$$X \longmapsto \lambda_1 X, \quad Y \longmapsto \lambda_2 Y, \quad Z \longmapsto \lambda_1\lambda_2 Z, \quad T \longmapsto T \tag{24}$$

where $\lambda_1, \lambda_2 \in \{\pm 1\}$ together with ψ such that $\psi(X) = Y, \psi(Y) = X, \psi(Z) = -Z$ and $\psi(T) = -T$. The maximal isometry group of Sol_1^4 has eight connected components.

It follows from (23) that the span of T and $X+Y$ is a geometric Engel structure on $(\mathrm{Sol}_1^4, \mathrm{Sol}_1^4)$. We now show that every geometric Engel structure is equivalent to this one. Let $(\mathrm{Sol}_1^4, \mathcal{D}, \mathrm{Sol}_1^4)$ be a geometric Engel structure. By (23) $\mathcal{D}$ has to be transverse to $\mathfrak{nil}^3 \subset \mathfrak{sol}_1^4$ since it is an Engel structure.

After conjugation with suitable elements of $\mathrm{Nil}^3 \subset \mathrm{Sol}_1^3$ we may assume that $\mathcal{D}$ contains $T + \alpha Z$ and some vector in $\mathfrak{nil}^3$. Adding a multiple of Z to T induces an isomorphism of the Lie algebra $\mathfrak{sol}_1^4$ and therefore there is an automorphism of Sol_1^4 such that T is tangent to the image of $\mathcal{D}$ under this automorphism.

Similarly we can arrange that $\mathcal{D} \cap \mathfrak{nil}^3$ is tangent to the X, Y plane. By (23) neither X or Y can be tangent to $\mathcal{D} \cap \mathfrak{nil}^3$. Conjugating with $(0, t) \in \mathbb{R}^3 \ltimes \mathbb{R}$ we can finally achieve that either $X + Y$ or $X - Y$ spans $\mathcal{D} \cap \mathfrak{nil}^3$. The geometric Engel structures $\mathcal{D}_\pm = \mathrm{span}\{T, X \pm Y\}$ induce the same even contact structure with different orientations. Both $\mathcal{D}_+$ and $\mathcal{D}_-$ are preserved by ψ and the isometries from (24) with $\lambda_1 = \lambda_2$. For $\lambda_1 = -\lambda_2$ the isometries from (24) interchange $\mathcal{D}_-$ and $\mathcal{D}_+$. This shows that all geometric Engel structures on $(\mathrm{Sol}_0^4, \mathrm{Sol}_0^4)$ are equivalent to each other.

3.3.4. *Relation to Geiges's construction.* The Engel structures we have obtained in this section are similar to those arising from Geiges's construction. More precisely, for certain lattices Γ in the isometry groups of the solvable geometries $X = \mathrm{Sol}^4(m, n), \mathrm{Sol}_1^4$ and $(X, G) = (\mathrm{Sol}_0^4, \mathrm{Sol}^4(\lambda))$ the Engel structures on the manifold $\Gamma\backslash X$ are slight modifications of the Engel structures obtained in Example 7.

First, we consider the cases $X = \mathrm{Sol}^4(m, n)$ or $X = \mathrm{Sol}_0^4$. Let Γ be a lattice obtained as described in Section 3.3.1. Then the manifold $\Gamma\backslash X$ is the suspension of a diffeomorphism of the 3–torus.

For $X = \mathrm{Sol}^4(m, n) = \mathbb{R}^3 \ltimes \mathbb{R}$ (the action of $\mathbb{R}$ depends on m, n) let X_1, X_2, X_3 be the standard framing of $\mathbb{R}^3$ and X_0 be tangent to the second factor. Consider the span of

$$X_0 \quad \text{and} \quad Y_k = (X_1 + X_2) + X_3 \ . \tag{25}$$

This is the expression arising in (2) when one removes the factor $1/k$ from (2) in Example 7 and applies the resulting formula to the framing $X_0, X_1 + X_2, X_1 - X_2, X_3$. The characteristic foliation is spanned by $X_1 + X_2 + X^3$ and it is tangent to the leaves of the fibers of the mapping torus.

The case $X = \mathrm{Sol}_0^4$ is similar. Let $\Gamma \subset \mathrm{Sol}^4(\lambda)$ be a lattice described in Section 3.3.1 and consider the framing U_1, U_2, V, T of $\mathrm{Sol}^4(\lambda)$. Then the span of T and

$$Y_k = \cos(kt)U_1 + \sin(kt)U_2 + V$$

is the geometric Engel structure $\mathcal{D}_{st}$ from Section 3.3.2. This is again a small modification of (2) for $k = \Im(\lambda)$.

Finally consider Sol_1^4. Choosing a lattice Γ' in Nil^3 which is invariant under the action of $(e,1) \in \mathrm{Nil}^3 \ltimes \mathbb{R}$ one obtains a lattice $\Gamma = \Gamma' \ltimes \mathbb{Z}$ in Sol_1^4. The resulting manifolds $\Gamma\backslash \mathrm{Sol}_1^4$ are mapping tori of diffeomorphisms a Nil^3–manifold, i.e. the fibers are non–trivial circle bundles over T^2. These manifolds carry geometric contact structures, cf. Section 2.2.

The geometric Engel structures on $\Gamma\backslash \mathrm{Sol}_0^4$ are obtained by suspending a diffeomorphism preserving the geometric contact structure on $\Gamma'\backslash \mathrm{Nil}^3$. As can be read of from (22) this diffeomorphism acts on the contact planes expanding one direction and contracting its orthogonal complement. In contrast to the other to solvable geometries, the characteristic foliation corresponds to the vector field induced by the suspension of an diffeomorphism of the fiber $\Gamma'\backslash \mathrm{Nil}^3$.

3.4. $X = \mathrm{Nil}^4$**.** The group Nil^4 can be written as semidirect product $\mathbb{R}^3 \ltimes \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^3$ by

$$\varphi(t) = \exp\begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \tag{26}$$

and T is tangent to the second factor of $\mathbb{R}^3 \ltimes \mathbb{R}$ while Z, Y, W (in this order) correspond to the standard basis of $\mathbb{R}^3$. The Lie algebra $\mathfrak{nil}^4$ is generated by W, T, Y, Z with

$$[T, W] = Y \qquad\qquad [T, Y] = Z \tag{27}$$

and the remaining commutators vanish. The maximal isometry group G is generated by Nil^4 acting by left translations on itself and by those automorphisms of Nil^4 which act on $\mathfrak{nil}^4$ by

$$\begin{aligned} W &\longmapsto \lambda_1 W & \qquad T &\longmapsto \lambda_2 T \\ Y &\longmapsto \lambda_1\lambda_2 Y & \qquad Z &\longmapsto \lambda_1 Z \end{aligned}$$

for $\lambda_1, \lambda_2 = \pm 1$. The left invariant plane field $\mathcal{D}_{st}$ spanned by W, T is a geometric Engel structure whose characteristic foliation is spanned by W. Since $\mathcal{D}_{st}$ is preserved by all automorphisms from (26), $\mathcal{D}_{st}$ is invariant under the maximal isometry group of Nil^4.

Now let $\mathcal{D}$ be a geometric Engel structure on Nil^4. By (27) $\mathcal{D}$ has to be transverse to foliation $[\mathfrak{nil}^4, \mathfrak{nil}^4] = \mathrm{span}\{Y, Z\}$. Thus for suitable constants $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

$$W + \alpha_1 Y + \beta_1 Z \text{ and } T + \alpha_2 Y + \beta_2 Z$$

span $\mathcal{D}$. Adding a multiple of Z to T or W is induces an automorphism of Nil^4. Thus we may assume $\beta_1 = \beta_2 = 0$. Using the adjoint action of $((0, 0, \mathbb{R}), 0) \subset \mathrm{Nil}^4$ one can achieve $\alpha_1 = 0$ and $\alpha_2 = 0$. This implies that all geometric Engel structures on Nil^4 are equivalent to $\mathcal{D}_{st}$.

Before we proceed with the remaining geometries, let us compare the Engel manifolds obtained as quotients of Nil^4 by certain lattices $\Gamma \subset \mathrm{Nil}^4$: For given $\alpha, \gamma \in \mathbb{N}$ and $\beta \in \mathbb{Z}$ the lattice $\Gamma \subset \mathrm{Nil}^4$

$$\begin{aligned} a &= ((0,0,0),1) & \qquad b &= ((0, \alpha\gamma/2 - \beta, \alpha\gamma), 0) \\ c &= ((0, -\gamma, 0), 0) & \qquad d &= ((1,0,0),0) \end{aligned}$$

acts freely on Nil^4 such that the quotient manifold is compact. With these generators the group Γ has the following presentation

$$\Gamma = \big\langle a,b,c,d \big| [b,a] = c^{\alpha}d^{\beta}, [c,a] = d^{\gamma}, [c,b] = [a,d] = [b,d] = [c,d] = 1 \big\rangle \ . \tag{28}$$

According to [Dek] every lattice in Nil^4 has such a presentation. The manifold $\Gamma\backslash\mathrm{Nil}^4$ is a suspension of the diffeomorphism

$$\begin{aligned} T^3 \simeq \mathbb{R}^3/\langle b,c,d\rangle &\longrightarrow \mathbb{R}^3/\langle b,c,d\rangle \simeq T^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\longmapsto \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ . \end{aligned} \tag{29}$$

For this lattice the Engel structure is similar to Engel structures obtained by the construction in Example 7. The Engel structure on $\Gamma\backslash\mathrm{Nil}^4$ is transverse the fibers of

$$\begin{aligned} \mathrm{pr} : \Gamma\backslash\mathrm{Nil}^4 &\longrightarrow T^2 \\ ((a_1,a_2,a_3),t) &\longmapsto (a_3,t) \ . \end{aligned}$$

and tangent to the suspension vector field T. The characteristic foliation is tangent to the fibers of the suspension described in (29). Thus the geometric Engel structure on $\Gamma\backslash\mathrm{Nil}^4$ a modification of the construction of Geiges with $k=0$.

3.5. $X = \mathbb{R}^4$**.** All subgroups of the isometry group of $\mathbb{R}^4$ which act transitively on $\mathbb{R}^4$ must contain the translations of $\mathbb{R}^4$. The only translation invariant plane fields on $\mathbb{R}^4$ are foliations. Thus there is no geometric Engel structure on $\mathbb{R}^4$.

3.6. $X = \mathbb{H}^3\times\mathbb{R}$**.** The maximal isometry group G is the product of the maximal isometry groups of each factor. It is 7–dimensional and has four connected components. According to Theorem 3 there is no subgeometry with 4–dimensional isometry group. Therefore $\mathbb{H}^3\times\mathbb{R}$ does not admit a geometric Engel structure.

3.7. $X = \mathbb{R}^2\times\mathbb{H}^2$**.** The maximal isometry group G is the product of the maximal isometry groups of the factors, it has dimension 6 and four connected component. Like in the case of $\mathbb{H}^3\times\mathbb{R}$ there is no geometric Engel structure because there is no Thurston geometry with 4–dimensional isometry group which is equivalent to $\mathbb{H}^2\times\mathbb{R}^2$, cf. Theorem 3

3.8. $X = F^4$**.** This is the only Thurston geometry in dimension 4 who does not admit compact models, cf. [Wa2]. The manifold underlying this geometry is again $\mathbb{R}^2\times\mathbb{H}^2$ where we use the upper half plane model for $\mathbb{H}^2$. The group is $\mathbb{R}^2\ltimes\mathrm{Sl}(2,\mathbb{R})$ with the standard action of $\mathrm{Sl}(2,\mathbb{R})$ on $\mathbb{R}^2$. This group acts on $\mathbb{R}^2\times\mathbb{H}^2$ by

$$\left((u,v),\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\cdot((x,y),z) = \left((ax+by,cx+dy),\frac{az+b}{cz+d}\right) \ .$$

This group has no 4–dimensional subgroup acting transitively on X. Therefore there is no geometric Engel structure on F^4.

4. Rigidity of integral curves

In this final section we consider integral curves of an Engel structure $\mathcal{D}$ on M, i.e. we consider curves $\gamma : [a,b] \longrightarrow M$ such that $\dot\gamma(t) \in \mathcal{D}$. We denote the space of all integral curves of $\mathcal{D}$ starting at p and ending at q by $\Omega_{\mathcal{D}}(p,q)$ and we use the C^1–topology on $\Omega_{\mathcal{D}}(p,q)$.

Definition 13. An integral curve $\gamma : [a,b] \longrightarrow M$ of $\mathcal{D}$ is *rigid* if there is a neighbourhood U of γ in $\Omega_{\mathcal{D}}(\gamma(a),\gamma(b))$ such that every curve in U is a reparameterization of γ.

In [BrH] R. Bryant and L. Hsu have obtained a complete characterization of rigid integral curves. In order to state it we need to define the development map of a leaf of the characteristic foliation of $\mathcal{D}$. Let p in M and $\mathcal{W}(p)$ the leaf of $\mathcal{W}$ through p. We fix a plane $\mathcal{C} \in \mathcal{E}(p)$ such that $\mathcal{E}(p) = \mathcal{C} \oplus \mathcal{W}(p)$. For q in $\mathcal{W}(p)$ let φ_q be germ of the holonomy of $\mathcal{W}$ mapping a neighbourhood of q to a neighbourhood of p. Recall that the holonomy of $\mathcal{W}$ preserves both $\mathcal{W}$ and $\mathcal{E}$. Therefore the following definition of the *development map* δ_p makes sense

$$\begin{aligned} \delta_p : \mathcal{W}(p) &\longrightarrow \mathbb{P}\mathcal{C} \simeq \mathbb{RP}^1 \\ q &\longmapsto (\varphi_{q*}\mathcal{D}(q)) \cap \mathcal{C}\ . \end{aligned}$$

The condition that $[\mathcal{D},\mathcal{D}] = \mathcal{E}$ has rank 3 everywhere ensures that the development map is an immersion. Rigid integral curves can be characterized as follows.

Theorem 14 ([BrH])**.** *A curve $\gamma : [a,b] \longrightarrow M$ in $\Omega_{\mathcal{D}}(\gamma(a),\gamma(b))$ is rigid if and only if*

(i) *the image of γ is contained in a leaf of $\mathcal{W}$ and*
(ii) *the map $\delta_p \circ \gamma : [a,b] \longrightarrow \mathbb{RP}^1$ is one–to–one except possibly at the endpoints.*

Using this theorem we can show that some geometric Engel structures have the property that every path whose image is tangent to any leaf of the characteristic foliation is rigid. Compact quotients of these geometric Engel structures are probably the first known examples of closed Engel manifolds with this rigidity property. Note that the leaves of the characteristic foliation of Engel structures obtained by prolongation never have this rigidity property since one can choose paths covering the leaves of the characteristic foliation very often. These paths then violate the second condition inTheorem 14. This means that the leaves of the characteristic foliation of the geometric Engel structures on $S^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2,\mathbb{R})$ contain non–rigid segments. Next we consider the solvable geometries and Nil^4.

Proposition 15. *All segments of the characteristic foliation of the geometric Engel structures on $\mathrm{Sol}^4(m,n), \mathrm{Sol}^4(\lambda), \mathrm{Sol}^4_1$ and Nil^4 are rigid.*

Proof. We consider first the Thurston geometries $\mathrm{Sol}^4(m,n) \simeq \mathbb{R}^3 \ltimes \mathbb{R}$ where the action of $\mathbb{R}$ on $\mathbb{R}^3$ is determined by a matrix depending on m,n, cf. Section 3.3.1 for the description of $\mathrm{Sol}^4(m,n)$. As we have shown every geometric Engel structure on $\mathrm{Sol}^4(m,n)$ is equivalent to

$$\mathcal{D}_{m,n} = \mathrm{span}\{X_1 + X_2 + X_3, T\}$$

and the characteristic foliation is spanned by $X_1 + X_2 + X_3$. In particular the holonomy of the characteristic foliation preserves the foliation of $\mathrm{Sol}^4(m,n) \simeq \mathbb{R}^3 \ltimes$

$\mathbb{R}$ given by the first factor. Because $\mathcal{D}$ is never tangent to this foliation all segments of leaves of the characteristic foliation satisfy the second condition in Theorem 14.

The other cases $\mathrm{Sol}^4(\lambda), \mathrm{Sol}^4_1$ and Nil^4 can be treated in the same way. □

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