NON-LOOSE UNKNOTS, OVERTWISTED DISCS, AND THE CONTACT MAPPING CLASS GROUP OF S^3

T. VOGEL

ABSTRACT. We show that up to isotopy there are exactly two oriented non-loose Legendrian unknots in S^3 with the same classical invariants (only one overtwisted contact structure on S^3 admits an unknot with these properties).

This can be used to prove a result attributed to Y. Chekanov implying that the contact mapping class group of an overtwisted contact structure on S^3 depends on the contact structure. In addition we show that the identity component of the contactomorphism group of an overtwisted contact structure on S^3 does not always act transitively on the set of boundaries of overtwisted discs.

1. INTRODUCTION

In 3-dimensional contact topology there is a fundamental dichotomy between tight and overtwisted contact structures pioneered by D. Bennequin [Be] and developed further by Y. Eliashberg [El92].

Usually, the attention is restricted to tight contact structures. These contact structures are characterized by the absence of so-called overtwisted discs. This is because a theorem of Y. Eliashberg shows that the classification of contact structures on a fixed closed 3-manifold up to diffeomorphism which contain an overtwisted disc up to isotopy coincides with the classification of plane fields up to homotopy.

What is important in Eliashberg's result is that one needs to fix and control an overtwisted disc. In particular, if one considers the isotopy problem for Legendrian or transverse links, Eliashberg's theorem can be applied effectively when the complement of the links is overtwisted.

This is not always the case. As K. Dymara noted in [Dy01] it may happen that a Legendrian or transverse knot has a tight complement. In other words it intersects all overtwisted discs. A knot with this property is called non-loose. Also, it may happen that two Legendrian knots have the same classical invariants (the Thurston-Bennequin invariant and the rotation number), each knot has an overtwisted complement but the complement of the union of the two knots is tight. In these situations one cannot apply Eliashberg's theorem directly to construct for example Legendrian isotopies between two non-loose Legendrian knots with the same classical invariants.

In her preprint [Dy04] K. Dymara gives several examples of such non-loose Legendrian knots in S^3 and J. Etnyre constructed more examples in

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[Et08]. He also found examples of pairs non-loose Legendrian knots whose complements are not diffeomorphic.

Non-loose Legendrian knots are interesting because one can obtain interesting tight contact structures from surgeries on non-loose Legendrian knots. Furthermore, as shown in this paper, non-loose unknots can be used to obtain non-trivial information about overtwisted contact structures and not only about additional structures like for example Legendrian knots.

A coarse classification (i.e. up to diffeomorphism) of non-loose unknots in S^3 is due to Y. Eliashberg and M. Fraser in [ElF09] and independently to J. Etnyre (see [Et13]). It turns out that on S^3 there is a unique contact structure which admits a non-loose unknot, we denote it by ξ_{-1}. In particular, it turns out that a non-loose unknot has positive Thurston-Bennequin invariant. A coarse classification of certain rationally null-homologous knots in lens spaces was obtained by H. Geiges and S. Onaran [GeO].

One main result of this paper is the classification of non-loose Legendrian unknots in S^3 up to Legendrian isotopy.

Theorem. *Let K, K' be two non-loose Legendrian unknots with* $\mathrm{tb}(K) = \mathrm{tb}(K') = 1$. *Then K and K' are isotopic. When K is oriented, then there is no Legendrian isotopy which reverses the orientation of K.*

The first part of this statement follows from Theorem 3.7 while the second part is the content of Theorem 3.9. The argument for the first part relies on Eliashberg's classification theorem and a study of the characteristic foliation on a spheres containing the non-loose unknot as well as deformations of the sphere which result in overtwisted discs. The proof of Theorem 3.7 uses a classic argument from differential topology.

We will also classify non-loose unknots with $\mathrm{tb}(K) = n$ up to isotopy. Then n has to be positive and $\mathrm{rot}(K) = \pm(n-1)$ and K is isotopic to exactly one of two standard unknots with the same classical invariants. Non-loose unknots with Thurston-Bennequin invariant one are called minimal.

An important problem concerning the classification of non-loose unknots in S^3 that is left open in this article is the following problem.

Problem. *Let K be a minimal non-loose oriented unknot in S^3 with* $\mathrm{tb}(K) = 1$. *Decide effectively whether another minimal non-loose oriented unknot S^3, which could be given in terms of a front projection for example, is isotopic to K or $\overline{K}$ (i.e. K with its orientation reversed).*

More problems can be found in [Et13] and [BaO]. Etnyre's article also contains proofs of most of the known results on non-loose knots. In [BaO] K. Baker and S. Onaran propose to quantify the degree of non-looseness of Legendrian/transverse knots and suggest a number of problems related to their invariants. Another reference is [LOSZ] where the authors give an example of a pair of non-loose transverse knots with the same classical invariants which are not isotopic.

We give two applications of the classification of non-loose Legendrian unknots up to isotopy: First, we prove that there is a pair of boundaries of overtwisted discs in (S^3, ξ_{-1}) which are not Legendrian isotopic to each other. More precisely, it is not always possible to find a contact isotopy in a contact manifold which moves an overtwisted ball into another such ball.

The other application is the description of the contact mapping class group $\pi_0\left(\mathrm{Cont}_+(\mathrm{S}^3, \xi)\right)$ of contact structures on S^3 which preserve the orientation of the contact structure. The claim in the following theorem is attributed to Y. Chekanov in [ElF98] without any indication of proof.

Theorem 4.5. *Let ξ be an overtwisted contact structure on S^3 and denote by $\mathrm{Diff}_+(S^3, \xi)$ the group of diffeomorphisms of S^3 which preserve ξ and an orientation of this plane field. Then*

$$\pi_0\left(\mathrm{Diff}_+(S^3,\xi)\right) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \textit{if } \xi \simeq \xi_{-1} \\ \mathbb{Z}_2 & \textit{otherwise.} \end{cases}$$

The additional $\mathbb{Z}_2$-summand in the case $\xi \simeq \xi_{-1}$ is determined by considering the action of contactomorphisms on isotopy classes of oriented minimal non-loose Legendrian unknots. In [Dy01] K. Dymara introduced an invariant which detects the other $\mathbb{Z}_2$-factor for all overtwisted contact structures on S^3. Finally, recall that the group of diffeomorphisms preserving the oriented tight contact structure on S^3 is connected according to [El92].

Theorem 4.5 can hopefully be applied to obtain the classification of non-loose links in overtwisted contact structures once the classification up to contact diffeomorphism is known.

This paper is organized as follows: Section 2 contains the relevant definitions and examples of non-loose knots. In Section 3 we first review the coarse classification using the approach taken in [ElF09]: The normal form for the characteristic foliation on a disc bounding a non-loose unknot is used heavily in the next sections. After the construction of examples of contact diffeomorphisms preserving a particular non-loose unknot we prove that every non-loose unknot is isotopic to a standard example $K \subset (S^3, \xi_{-1})$. Then we show that K and $\overline{K}$ are not isotopic as oriented knots. Finally, in Section 4 we discuss the connected components of the group of orientation preserving contact diffeomorphisms on S^3 for overtwisted contact structures.

Acknowledgments: The first time I heard about non-loose knots was in discussions with J. Etnyre during which he developed a surgery based proof of the coarse classification of non-loose unknots and I thank him for his explanations. Others, in particular H. Geiges, P. Massot and S. Onaran, have spent some time discussing the problem with me and I want to thank all of them. It is a great pleasure for me to thank the Mittag-Leffler Institute (and T. Ekholm and Y. Eliashberg) for organizing a program on symplectic topology in the fall of 2015 where a portion of this manuscript was prepared.

2. PRELIMINARIES

In this section we review standard definitions, we review Giroux's description of contact structures in terms of families of characteristic foliations and we discuss basic facts about non-loose knots and some examples. Finally, in Section 2.6 we discuss an invariant introduced by K. Dymara in [Dy01] which can be used to show that the group of diffeomorphisms preserving an oriented overtwisted contact structure on S^3 is not connected.

2.1. Tight and overtwisted contact structures.

Definition 2.1. A *contact structure* is a smooth plane field ξ on a 3-manifold which is locally defined by a 1-form α such that $\alpha \wedge d\alpha$ never vanishes.

Manifolds admitting contact structures are necessarily oriented and we will only consider cooriented contact structures which are positive (i.e. $\alpha \wedge d\alpha > 0$. Throughout this paper, we assume that ξ is oriented as a plane field.

An important fact about contact structures is Gray's theorem which we will use:

Theorem 2.2. *Let ξ_t be a family of contact structures on a closed 3-manifold. Then there is an isotopy ψ_t such that $\psi_{t*}(\xi_0) = \xi_t$.*

Parametric and relative versions of this theorem also hold. One fact which comes out of the proof of Theorem 2.2 by the Moser method is the following: If K is a Legendrian knot which is tangent to ξ_t for all t, then ψ_t can be chosen to preserve K (but not pointwise, of course). The details can be found in Section 2.2 of [Ge], for example.

Definition 2.3. An embedded disc $D \longrightarrow M$ in a contact manifold (M, ξ) is *overtwisted* if $T_pD = \xi(p)$ for all points $p \in \partial D$. A contact structure is *tight* if there is no overtwisted disc, otherwise it is *overtwisted*.

The following two theorems of Eliashberg highlight the fundamental difference between overtwisted contact structures (which are very flexible) and tight ones (for which rigidity phenomena appear).

Theorem 2.4 (Eliashberg [El92])**.** *Let ξ be the germ of a tight contact structure along ∂B^3. Then the space of tight contact structures on the ball which coincide with ξ near ∂B^3 is weakly contractible.*

In particular, the relative homotopy type of the plane field on $(B^3, \partial B^3)$ is completely determined be the boundary data and the requirement that ξ is a tight contact structure.

Now let ξ_Δ be a contact structure defined near a disc Δ and on a neighborhood of a compact set N so that Δ is an overtwisted disc for ξ_Δ. We will use the following notation:

$$\mathrm{Cont}(M, N, \xi_\Delta) = \{ \text{ contact structures equal to } \xi_\Delta \text{ near } \Delta \cup N\}$$
$$\mathrm{Distr}(M, N, \xi_\Delta) = \{ \text{ plane fields which coincide with } \xi_\Delta \text{ near } \Delta \cup N\}.$$

Both spaces carry the C^k-topology with $k \geq 1$. The proof of the following theorem can be found in [El89]. It is discussed also in [Gi93] and [Ge]. Y. Huang proved a version not covering families of contact structures using bypasses in [Hu13].

Theorem 2.5 (Eliashberg [El89]). *Let ξ_Δ be a contact structure as above and $N \subset M$ a compact set in the complement of Δ such that $M \setminus N$ is connected. Then*

$$\mathrm{Cont}(M, N, \xi_\Delta) \hookrightarrow \mathrm{Distr}(M, N, \xi_\Delta)$$

is a weak homotopy equivalence.

The statement can be enhanced even further: Not only a relative version with respect to compact subsets N of M holds, but also a relative version with respect to compact subsets of the parameter space the parameter space. More precisely, if $\zeta_s, s \in S$, is a family of plane fields in $\mathrm{Distr}(M, N, \xi_\Delta)$ with compact parameter space S and $S' \subset S$ is a compact set such that $\zeta_s \in \mathrm{Cont}(M, N, \xi_\Delta)$ for $s \in S'$ then one does not have to change ζ_s for $s \in S'$ when applying Theorem 2.5.

Combining these two relative versions one can for example obtain a family of contact structure from a family $\zeta_s, s \in [0, 1]$, of plane fields with the following properties:

- ζ_s is a contact structure near discs Δ_1 respectively Δ_2 for all s.
- Δ_1 respectively Δ_2 are overtwisted discs of ζ_s when $s \in [0, 2/3]$ respectively $[1/3, 1]$.
- Δ_1 and Δ_2 are disjoint.

Also, if an overtwisted disc Δ is moved by an isotopy, then Theorem 2.5 can be adapted to this situation.

Theorem 2.5 implies that to a large extent the study of overtwisted contact structures reduces to the study of homotopy classes plane fields. Using trivializations of the tangent bundle of the underlying manifold M, homotopy classes of families of plane fields correspond to families of maps $M \longrightarrow S^2$ up to homotopy. Since all higher homotopy groups of S^2 are non-trivial by [IMW], the algebraic topology questions that arise here are difficult. We will discuss this for $M = S^3$ in Section 2.6.

Theorem 2.5 can be used to produce families of overtwisted contact structures on S^3 with prescribed homotopy type as families of plane fields. The following consequence of Theorem 2.5 which can be found [Dy01].

Lemma 2.6. *Let ξ be a contact structure on a collar N of the boundary of B^3 which is overtwisted. For every homotopy class of families of plane fields $\zeta_s, s \in K$, where K is a compact manifold, which coincides with ξ on N there is a family of contact structures ξ_s in the same homotopy class.*

Proof. This is a direct application of Theorem 2.5: By assumption, there is always a fixed overtwisted disc in N, so every family of plane fields $\zeta_s, s \in K$, as in the lemma on B^3 can be homotoped to a family of contact structures $\xi_s, s \in K$. □

Before we discuss Legendrian and transverse knots we note one consequence of Theorem 2.4.

Proposition 2.7 (V. Colin, [Co99])**.** *Let $B_0, B_1 \subset M$ be two closed balls in a contact manifold such that $B_0 \subset \mathring{B}_1$ such that the contact structure is tight on an open neighborhood V of B_1. Then there is a contact isotopy of M with support in V which moves B_1 into the interior of B_0. If there is a pair of unknotted Legendrian arcs γ_0, γ_1 which are transverse to ∂B_0 and ∂B_1 each arc intersects ∂B_0 and ∂B_1 exactly once then one can choose the isotopy so that it preserves γ_0, γ_1.*

Definition 2.8. A *Legendrian knot* in a contact manifold (M, ξ) is an embedding $S^1 \longrightarrow M$ so that the tangent space of the image is contained in ξ. A knot K in (M, ξ) is *transverse* if its tangent space is transverse to $\xi(p)$ at every point $p \in K$, usually a transverse knot is oriented so that the orientation coincides with the coorientation of ξ.

In addition to their isotopy type as embedded curves, null-homologous Legendrian or transverse knots have the following classical invariants.

Definition 2.9. Let Σ be a connected Seifert surface for a Legendrian knot K. Then the *Thurston-Bennequin invariant* $\mathrm{tb}(K)$ of K is the algebraic intersection number of the push off of K along a vector field transverse to ξ with Σ. The *rotation number* of K is the number of full turns the positive tangent vector of K makes compared to an oriented framing of $\xi\big|_\Sigma$ (Σ is a surface with boundary) as one moves along the oriented curve K.

The *self-linking number* $\mathrm{sl}(K)$ of a null-homologous transverse knot K is the algebraic intersection number of a Seifert-surface Σ with K' where K' is obtained from K by pushing K away from itself using a vectorfield tangent to ξ which vanishes nowhere along Σ.

In general, these invariants depend on the choice of $[\Sigma] \in H_2(M, K; \mathbb{Z})$, but if $M = S^3$ then the Thurston-Bennequin invariant, the rotation number and the self-linking number are independent of this choice. Note that tb is independent of the orientation of the know while the rotation number changes sign when the orientation of K is reversed.

For an oriented Legendrian knot K one obtains a transverse knot by pushing K away from itself using a vector field N tangent to ξ so that along the knot the vectorfield N followed by the tangent direction of K is an oriented basis of ξ. The resulting knot will be denoted by K^+ and according to [Be] the classical invariants of K and K^+ are related as follows

$$\mathrm{sl}(K^+) = \mathrm{tb}(K) - \mathrm{rot}(K). \tag{1}$$

Theorem 2.10 (Eliashberg [El92])**.** *Let (M, ξ) be a contact 3-manifold. Then ξ is tight if and only if*

$$-\mathrm{tb}(K) + |\mathrm{rot}(K)| \leq -\chi(\Sigma) \tag{2}$$

for every null-homologous Legendrian knot K and Seifert surface Σ.

2.2. **Surfaces in contact manifolds.** Let Σ be an oriented surface embedded in a contact manifold (M, ξ). If $\partial\Sigma \neq \emptyset$ we require that the boundary is tangent to ξ. The following terminology will be used also when ξ is a plane field (in this context we will use the notation ζ instead of ξ).

Definition 2.11. The *characteristic foliation* $\xi(\Sigma)$ on Σ is defined by the singular line field $T\Sigma \cap \xi$. It determines the germ of ξ along Σ up to isotopy preserving $\xi(\Sigma)$. A point $p \in \Sigma$ is called *singular* if $T_p\Sigma = \xi(p)$, when ξ and Σ are oriented, a singular point is *positive* if the orientations coincide, otherwise it is *negative*.

The characteristic foliation is oriented so that the orientation of $\xi(\Sigma)$ followed by the coorientation of ξ is the orientation of Σ. For singular points there is a well defined notion of positive/negative divergence. Positive singularities have positive divergence while the divergence at negative singularities is negative. The orientation convention turns positive/negative elliptic singularities into sources/sinks.

Because of the contact condition, the characteristic foliation near an isolated singular point has a particular form. For example, the index of such a critical point can only be ± 1 or 0 (a proof is in [V]).

Generically, the singular line field $\xi(\Sigma)$ on Σ defines a Morse-Smale foliation, i.e.

- all singularities are non-degenerate,
- all closed orbits are hyperbolic, and
- there are no connections between hyperbolic singularities.

Giroux [Gi91] has proved that these properties together imply that Σ is there is a contact vector field which is transverse to Σ.

Definition 2.12. An embedded surface with Legendrian boundary in a contact manifold is *convex* if there is a contact vector field transverse to Σ.

Of course, the characteristic foliation does not have to be Morse-Smale to be convex. For example, connections between hyperbolic singularities of the same sign never prevent convexity.

Giroux has found a topological property of the characteristic foliation on Σ which determines whether or not Σ is convex.

Definition 2.13. A collection Γ of curves and arcs on a surface Σ with $\partial\Gamma \subset \partial\Sigma$ *divides* a singular foliation if

(i) Γ is transverse to $\xi(\Sigma)$ and Γ decomposes Σ into two subsurfaces Σ_+, Σ_- (not necessarily connected), and
(ii) there is a vector field X and an area form ω on Σ such that $L_X\omega > 0$ respectively $L_X\omega < 0$ on the interior of Σ_+ respectively Σ_- pointing out of Σ_+.

Theorem 2.14 (Giroux [Gi91]). *An oriented compact surface $\Sigma \subset (M, \xi)$ with Legendrian boundary is convex if and only if $\xi(\Sigma)$ admits a dividing set. The dividing set is unique up to isotopy an isotopy through multicurves*

transverse to the characteristic foliation. A convex surface in a contact manifold has a neighborhood where ξ is tight if and only if
$\Sigma \simeq S^2$ *and* $\xi(\Sigma)$ *has a connected dividing set or*
$\Sigma \not\simeq S^2$ *and non component of a dividing set of* $\xi(\Sigma)$ *bounds a disc.*

If one of the last two conditions in the theorem are violated for a convex surface, we will say that this surface contains an obvious overtwisted disc.

Because of the Poincaré-Bendixson theorem it is easier to describe the necessary and sufficient conditions on $\xi(\Sigma)$ for Σ to being convex when $\Sigma \simeq S^2$.

Lemma 2.15. *An embedded sphere S^2 in a contact manifold (M, ξ) whose characteristic foliation has only isolated singularities is convex if and only if*

- *each closed leaf of $S^2(\xi)$ is hyperbolic, and*
- *there is no connection from a negative singular point to a positive singular point.*

Proof. If Σ is convex and Γ is a dividing set, then all positive singularities are contained in Σ_+ while the negative singularities lie in Σ_- Also, a closed leaf is hyperbolic if and only if there is a couple (X, ω) as in Definition 2.13 on a neighborhood of the closed leaf. Thus in order to be convex, (i) and (ii) have to be satisfied.

For the opposite direction, recall that by the Poincaré-Bendixson theorem all limit sets of leaves of $\xi(\Sigma)$ are either closed leaves or cycles formed by finitely many leaves of $\xi(\Sigma)$ which connect singular points. Therefore if a characteristic foliation on S^2 satisfies (i) and (ii) then one can construct a triple (X, ω, Γ) as in the proof of Proposition 2.6 of [Gi91]. □

There are effective methods to manipulate the characteristic foliation on surfaces, in particular for convex surfaces [Gi91, Gi00]. The following lemma can be found in [Gi91] and does not require convexity.

Lemma 2.16 (Elimination Lemma)**.** *Let $\Sigma \subset (M, \xi)$ be an embedded surface such that the characteristic foliation has a leaf γ connecting two singular points with the same sign such that the index of one of the singularities is* 1 *while the other has index* -1.

Then Σ can be isotoped on a neighborhood of γ such that the these two singular points have disappeared from the characteristic foliation on the isotoped surface and γ remains Legendrian.

Conversely, one can also create a pair of singularities satisfying the conditions of Lemma 2.16.

There is another possible modification which is particularly relevant when considering surfaces with Legendrian boundary. The modification shown in Figure 1 preserves the boundary of a surface(represented by the thickened horizontal line). A reference for this modification is [ElF09]. Note that if the hyperbolic singular point is positive respectively negative, then its unstable respectively its stable leaves have to be part of the boundary.

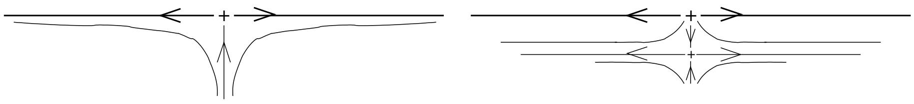

FIGURE 1. Splitting hyperbolic singular points on a Legendrian boundary of a surface

Finally, we note that the rotation number of the a Legendrian knot K with Seifert surface Σ can be computed from $\xi(\Sigma)$ assuming all singular points of $\xi(\Sigma)$ are generic. Let $e_\pm$ respectively $h_\pm$ be the number of elliptic respectively hyperbolic singular points with the sign $\pm$ in the interior of Σ. Then

$$\mathrm{rot}(K) = e_+ - e_- - h_+ + h_- \tag{3}$$

when the singularities on the boundary are all either hyperbolic or all elliptic (this can be deduced from [Et05]).

2.3. **Tomographie.**

2.3.1. *Movies of characteristic foliations.* Let $\Sigma \times (-\delta, \delta), \delta > 0$, be the closure of a tubular neighborhood of an oriented surface in a contact manifold (M, ξ). We assume that S is compact and $\partial\Sigma \times \{t\}$ is Legendrian for all $t \in (-\delta, \delta)$.

Definition 2.17. The *movie* of ξ on $\Sigma\times(-\delta, \delta)$ is the family of characteristic foliations of ξ on $\Sigma \times \{t\} = \Sigma_t, t \in (-\delta, \delta)$.

According to [Gi00] (see also [Ge]) the movie determines the contact structure up to isotopy. If the characteristic foliation on $\Sigma \times \{0\}$ violates one of the conditions (i),(ii) of Lemma 2.15, then this has consequences for the characteristic foliation on Σ_t for t close to 0. The following definitions and lemmas are due to Giroux [Gi00]. As an application of these results, Giroux gave a beautiful proof of Bennequin's theorem:

Theorem 2.18 (Bennequin)**.** *A contact structure on $S^2 \times [-1, 1]$ such that every sphere of the product decomposition is convex and has connected dividing set is tight.*

Definition 2.19. A *retrogradient connection* is a leaf connecting a positive singular point with a negative one such that the leaf is oriented towards the positive singularity. A retrogradient connection is *non-degenerate* if both endpoints are non-degenerate singular points.

The following two lemmas are key for the understanding of non-convex spheres in contact manifolds:

Lemma 2.20 (Lemme de croisement, 2.14 in [Gi00])**.** *Let ξ be an oriented positive contact structure on $\Sigma\times(-1, 1)$ such that there is a non-degenerate retrogradient connection in $\xi(\Sigma_0)$. For $t > 0$ respectively for $t < 0$, the corresponding stable leaf in $\xi(\Sigma_0)$ lies over respectively under the corresponding unstable leaf.*

Here and in the next lemma the words over and under refer to the coorientation of ξ.

Lemma 2.21 (Lemme de naissance-mort, 2.12 in [Gi00])**.** *Let γ be a closed leaf of the characteristic foliation on $\xi(\Sigma_0)$ which is attractive on one side and repelling on the other. If the repelling side is above γ, then for $t > 0$ small enough $\xi(\Sigma_t)$ has two non-degenerate closed orbits in a neighborhood of γ while this neighborhood contains no closed leaves for $t < 0$. If the repelling side is below γ, then it is the other way round.*

The following lemma shows that a single retrogradient connection appearing in a movie of a contact structure on $S^2 \times [-1, 1]$ which is not obviously overtwisted (i.e. an overtwisted disc is present on one of the leaves) is tight.

Lemma 2.22. *Let ξ be a contact structure on $N \simeq S^2 \times [-1, 1]$ (with a generic product decomposition) which is tight near the boundary such that the characteristic foliation on S^2_t is convex except when $t = 0$ and the characteristic foliation on S^2_0 has exactly one retrogradient connection.*

Then ξ is isotopic to a family of contact structures on $S^2 \times [-1, 1]$ such that all spheres S^2_t are convex and ξ is tight.

Proof. For $t = 0$ we eliminate all hyperbolic singularities which do not take part in the retrogradient saddle-saddle connection: Let Γ^-_t be the graph whose vertices are negative singularities and whose edges are unstable leaves of negative singularities. For $t \neq 0$ this is a tree because it follows from the assumptions that ξ is tight on $S^2 \times [-1, 0)$ and $S^2 \times (0, 1]$. When $t = 0$, then Γ^-_t consists of two connected components, one of these components is not closed (the ω-limit set of the edge γ forming the retrogradient saddle-saddle connection is a positive hyperbolic singularity p_+ and does not lie in Γ^-_0). By Lemma 2.20 both unstable leaves of p_+ have their ω-limit set in the closed component of Γ^-_0, otherwise $\xi(S^2_t)$ contains obvious overtwisted discs for $t > 0$ or $t < 0$.

For t close to zero we can therefore eliminate those singular points which belong to the non-closed component of Γ^-_0. □

2.3.2. *Movies on $S^2 \times [-1, 1]$.* Let $\xi_\sigma, \sigma \in [0, 1]$, be a smooth family of contact structures on $S^2 \times [-1, 1]$ where the product decomposition may also vary smoothly and the contact structures and the product decomposition are constant near the boundary of $S^2 \times [-1, 1]$.

We make the following genericity assumptions on the movie of ξ_σ on $S^2 \times [-1, 1]$ with respect to product decomposition associated to the parameter value σ:

- For all parameter values σ and all $t \in [-1, 1]$, the characteristic foliation on $\xi_\sigma(S^2_t), t \in [-1, 1]$ has only finitely many singular points.
- The parameter values where $\xi_\sigma(S^2_t)$ has a simply degenerate closed orbit (i.e. the first derivative of the holonomy along the close leaf

vanishes, the second derivative does not) is a union of finitely many hypersurfaces in $[-1,1] \times [0,1]$ which intersect transversely.
- The parameter values where $\xi_\sigma(S^2_t)$ has a doubly degenerate closed orbit (i.e. the first and the second derivative of the holonomy along the close leaf vanishes, the third derivative does not) is a union of finitely many submanifolds of codimension 2 of $[-1,1] \times [0,1]$.

By Lemma 2.20 and Lemma 2.21 the submanifolds of $[-1,1] \times [0,1]$ associated to connections pointing from negative singularities to positive ones (i.e. retrogradient connections) or to degenerate closed leaves are never tangent to the foliation of $[-1,1] \times [0,1]$ given be the first factor. In particular, if the stable leaf of a positive singularity coincides with the unstable leaf of a negative singularity, then this coincidence is non degenerate.

In Section 2.I of [Gi00] Giroux gives a beautiful proof of Bennequin's theorem based in particular on Lemma 2.20 and Lemma 2.21.

The main step in his proof is the following claim: For a generic family of contact structures on $S^2 \times [-1,1]$ configurations like the one shown in Figure 2 do not occur. In this figure, the t-coordinate corresponds to the horizontal direction, the vertical direction represents the parameter σ of the family. The curved lines intersecting once transversely in the box represent the locus where $\xi_\sigma(S^2_t)$ is not convex (by Lemma 2.20 and Lemma 2.21 these lines are transverse to the horizontal direction). Finally, the numbers on the complement of the non-convexity locus give the number of connected components of the dividing set.

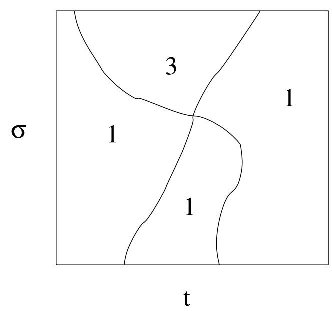

FIGURE 2. Forbidden configuration according to Giroux's proof of Bennequin's theorem

Let ξ_σ be a family of overtwisted contact structures on $S^2 \times [-1,1]$ which is tight near the boundary. We define

$$T_\pm : S \longrightarrow [-1,1]$$

$$T_-(\sigma) = \sup\{t \in [-1,1] \text{ such that } \xi_\sigma \text{ is tight on } S^2 \times [-1,t)\} \tag{4}$$

$$T_+(\sigma) = \inf\{t \in [-1,1] \text{ such that } \xi_\sigma \text{ is tight on } S^2 \times (t,1]\}.$$

Since ξ_σ is overtwisted, these functions take values in $(-1,1)$.

Lemma 2.23. *$T_\pm$ are continuous functions.*

Proof. If the characteristic foliation on $\xi_\sigma(S^2_t)$ was convex for $t = T_\pm(\sigma)$ then the region where ξ_σ is tight could be extended using the contact vector field transverse to S^2_t, so S^2_t is not convex with respect to ξ_σ. The characteristic foliation on this sphere must have a retrogradient connection or a degenerate orbit by Lemma 2.15. Moreover, by Lemma 2.22 (and gluing results from [Co97]) the dividing set on $S^2 \times \{T_+(\sigma) - \varepsilon\}$ cannot be connected for small enough $\varepsilon > 0$, the same holds for $S^2 \times \{T_-(\sigma) + \varepsilon\}$.

By our genericity assumptions $(T_\pm(\sigma), \sigma)$ lies on the closure of finitely many smooth hypersurfaces transverse to the foliation on $[-1, 1] \times [0, 1]$ given by the first factor. Therefore $T_\pm$ are continuous. $\square$

The following lemma follows from Giroux's proof of Bennequin's theorem.

Lemma 2.24. *Let ξ_σ be a family of overtwisted contact structures on $S^2 \times [-1, 1]$ such that ξ_σ is tight on a neighborhood $S^2 \times \{\pm 1\}$ for all σ. Then $T_-(\sigma) \leq T_+(\sigma)$.*

Proof. The proof of Bennequin's theorem in [Gi00] shows that if the movie of a generic contact structure on $S^2 \times [-1, 1]$ contains no obvious Legendrian knot (i.e. a loop formed by leaves of the characteristic foliation on a sphere $S^2 \times \{t\}, t \in (-1, 1)$) which violates the Thurston-Bennequin inequality, then the contact structure is tight.

Then by definition, $S^2 \times \{T_+(\sigma)\}$ contains an obvious Legendrian knot violating the Thurston-Bennequin inequality. Thus for all $t > T_+(\sigma)$, the contact structure ξ_σ on $S^2 \times [-1, t)$ is overtwisted, i.e. $T_-(\sigma) \leq T_+(\sigma)$. $\square$

2.4. **Non-loose knots.** We introduce non-loose knots and some of their properties. This notion is motivated by the following theorem.

Theorem 2.25 (Dymara, [Dy01])**.** *Let ξ be an overtwisted contact structure on S^3 and K_0, K_1 Legendrian knots which are smoothly isotopic,* $\mathrm{tb}(K_0) = \mathrm{tb}(K_1)$ *and* $\mathrm{rot}(K_1) = \mathrm{rot}(K_1)$. *If there is an overtwisted disc D in the complement of $K_0 \cup K_1$, then there is a Legendrian isotopy between K_0 and K_1, i.e. a family of Legendrian knots $K_t, t \in [0, 1]$, interpolating between K_0 and K_1.*

It is therefore more interesting to study knots (or pairs of knots) which do not satisfy the conditions stated in Theorem 2.25.

Definition 2.26. A Legendrian or transverse knot K in an overtwisted contact manifold (M, ξ) is *loose* if ξ is overtwisted on the complement of K. Otherwise it is *non-loose*.

The analogous terminology is used for Legendrian and transverse links. In [ElF98] non-loose knots are called exceptional. It is relatively easy to show that every overtwisted contact structure ξ on a closed manifold M admits a non-loose transverse link. For example, the binding K of an open book of M carrying ξ is a non-loose transverse knot. Also, it is well known that every closed contact manifold (M, ξ) can be obtained by Legendrian surgery on a link in the standard tight contact structure in S^3. The surgery curves induce a non-loose Legendrian link in (M, ξ).

The following modified Thurston-Bennequin inequality is one of the few general results on non-loose knots.

Theorem 2.27 (Swiatkowski [Et13]). *Let $K \subset M$ be a null-homologous Legendrian non-loose knot. Then*

$$-|\mathrm{tb}(K)| + |\mathrm{rot}(K)| \leq -\chi(\Sigma) \tag{5}$$

for every embedded surface Σ with $\partial\Sigma = K$.

Unlike the Thurston-Bennequin inequality (2) for Legendrian knots in tight contact manifolds (5) does not yield an upper bound for the Thurston-Bennequin invariant of a non-loose knot.

Lemma 2.28. *Let $K \subset (M, \xi)$ be a Legendrian knot in an overtwisted contact structure. If a stabilization of K is non-loose, then so is K.*

Proof. We argue by contradiction. Assume that K is loose and let K' be a stabilization of K which is non-loose. The complement of K contains an overtwisted disc D_K. But every stabilization of a Legendrian knot is isotopic to a knot contained in an arbitrarily small neighborhood of K. Thus we can assume that K' is contained in $M \setminus D_K$ contradicting the tightness of $M \setminus K'$. □

Finally, there is a further restriction on the Thurston-Bennequin invariant of non-loose unknots.

Lemma 2.29. *Let K be a non-loose Legendrian unknot in an overtwisted contact manifold (M, ξ). Then $\mathrm{tb}(K) > 0$.*

Proof. Since K is an unknot it bounds a disc D. Obviously $\mathrm{tb}(K) \neq 0$ since otherwise we could choose D to be a convex overtwisted disc and the complement of K would not be tight. Also, if $\mathrm{tb}(K) < 0$ we may choose D to be convex, i.e. there is a tubular neighborhood $D \times [-\varepsilon, \varepsilon]$ such that the contact structure on this neighborhood is tight (because ξ is tight on the complement of K, the dividing set of D does not have a closed component. Since the contact structure on $M \setminus D = M \setminus (D \times \{0\})$ is also tight this tubular neighborhood cannot contain an overtwisted disc. Rounding the edges of $D \times [-\varepsilon, \varepsilon]$ we find a closed ball B such that

- the contact structure on B is tight,
- the contact structure on $M \setminus B$ is tight, and
- the boundary of B is a convex surface.

By [Co97] this implies that ξ is tight contradicting the assumption that ξ is overtwisted. □

Definition 2.30. A non-loose unknot K with $\mathrm{tb}(K) = 1$ is called *minimal*.

2.5. **Examples of non-loose knots.** In this section we discuss examples of non-loose Legendrian knots in S^3. We will focus on the overtwisted contact structure which we will denote by ξ_{-1} later (sometimes we will refer to it as the *standard overtwisted contact structure* on S^3).

According to By work of R. Lutz [Lu] S^1-invariant contact structures on S^1-principal bundles $M \longrightarrow \Sigma$ over oriented closed surfaces are classified up to up to S^1-equivariant isotopy by the projection Γ of

$$\{p \in M \mid \xi(p) \text{ is tangent to the fiber through } p\} \tag{6}$$

to Σ. Because of the contact condition this is an embedded submanifold of dimension 1 in the base surface. When Γ is a simple closed curve, then we obtain ξ_{-1}.

Let K be the fiber over a point of $p \in \Gamma$. A convenient way to establish the tightness of ξ_{-1} on the complement of K relies on techniques developed by E. Giroux in [Gi91, Gi00] (see [Ge] for an exposition in English): One studies the characteristic foliation of ξ_{-1} on $T^2_t = \pi^{-1}(S^1_t)$ where $S^1_t, t \in (0,1)$, is a family of nested circles $S^1_t \subset S^2$ such that as $t \to 0$ respectively $t \to 1$ the circles S^1_t converge to p respectively q where $p \neq q \in \Gamma$. We choose the circles in such a way that each circle is transverse to Γ, the intersection of Γ with S^1_t consists of two points for all $t \in [0,1]$. The fibers above these points form a pair of Legendrian curves. Each curve is either a non-singular leaf of the characteristic foliation $\xi_{-1}(T^2_t)$ or it is a line of singularities of $\xi_{-1}(T^2_t)$. One of these lines is repulsive while the other one is attractive. Therefore the sheet (this is our translation of the french word *feuille* used in [Gi00]) of the movie of characteristic foliations on T^2_t is connected after we add the Legendrian curve $\pi^{-1}(q)$ to $T^2_t, t \in [0,1]$ (thus obtaining a solid torus).

According to [Gi00] (Section 4.B) up to isotopy there is a unique contact structure on the solid torus formed by T^2_t with $t \in [\tau, 1) \cup \pi^{-1}(q)$ which yields a movie with the properties described above. This contact structure is universally tight for all $\tau \in (0,1)$. Hence ξ_{-1} is universally tight on the complement of K.

Alternatively, one can also show that ξ_{-1} is universally tight on $S^3 \setminus K$ by embedding the universal cover of $S^3 \setminus K$ into the $\mathbb{R}^3$ with the standard contact structure. This is the approach taken by K. Dymara in [Dy01, Dy04]

From the above description of ξ and K one can read of $\operatorname{tb}(K) = 1$. The positive push-off of K has self-linking number $+1$ and therefore $\operatorname{rot}(K) = 0$ by (1). Thus K is a minimal non-loose unknot.

In order to describe more examples of non-loose knots we use front projections. Front projections are commonly used to represent Legendrian knots in the standard tight contact structure on $\mathbb{R}^3$, many facts commonly known in that context generalize to the situation considered here with minor modifications. We refer the reader to [Et05] for information about Legendrian knots in the standard contact structure on $\mathbb{R}^3$.

A convenient way to describe the standard overtwisted contact structure ξ_{-1} on S^3 is to identify the complement of the Hopf link with $T^2 \times (0,1)$. The 3-sphere is then the result of the following operation: Compactify $T^2 \times (0,1)$ to obtain $T^2 \times [0,1]$ and collapse the second respectively first factor in $S^1 \times S^1 \times \{i\}$ to a point when $i = 0$ respectively $i = 1$. The image of the boundary components of $T^2 \times [0,1]$ under the quotient map to S^3 is a Hopf link H in S^3.

The restriction of ξ_{-1} to $T^2 \times (0,1)$ is defined by

$$\alpha = \cos\left(\frac{3\pi}{2}z\right)dx - \sin\left(\frac{3\pi}{2}z\right)dy. \tag{7}$$

When one replaces the factor 3 by 1 in this formula, one obtains the tight contact structure. This is the contact structure ξ_{-1} containing the standard non-loose unknot described above. The fibers of the Hopf fibration are linear curves of slope -1 inside the tori $T^2 \times \{t\}, t \in (0,1)$ and the components of H.

Along the Hopf link H the contact planes of ξ_{-1} are tangent to the discs formed by the images of $S^1 \times \{*\} \times [1/3,1]$ and $\{*\} \times S^1 \times [0,2/3]$. All discs in these two families are overtwisted, two such discs are shown in Figure 3. The Hopf link H is transverse to ξ_{-1} and non-loose.

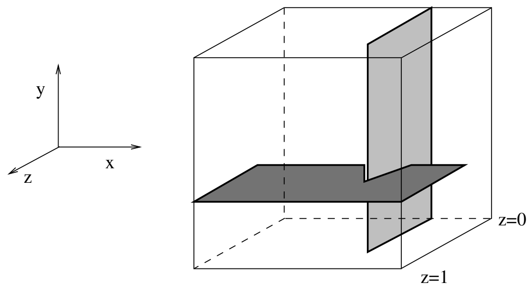

FIGURE 3. Two overtwisted discs in (S^3, ξ_{-1})

For a Legendrian curve in S^3 which is disjoint from the Hopf link H we can consider its image under the front projection $T^2 \times [0,1] \longrightarrow T^2$. Contrary to the case of the standard tight contact structure on S^3, a vector in the tangent space of T^2 does not have a unique preimage which is tangent to ξ_{-1}. More precisely, only lines whose slope is not positive with respect to coordinates x, y on T^2 have a unique preimage.

Let K be a generic Legendrian knot in (S^3, ξ_{-1}) which is disjoint from H. The front projection of K is a closed curve in T^2 whose homotopy class is denoted by $(a,b) \in \mathbb{Z}^2 \simeq \pi_1(T^2)$. The only singularities of this curve are transverse double points and cusps.

Front projections of isotopic Legendrian knots are related via isotopies, Reidemeister moves (as described in [Dy04]) and modifications of the front projection corresponding to the Legendrian knot crossing a component of H. When a segment of a Legendrian knot passing through the image of

$\{z=0\}$ the diagram changes as indicated in Figure 4, when the segment passes through $\{z=1\}$ a similar (horizontal) modification of the front appears.

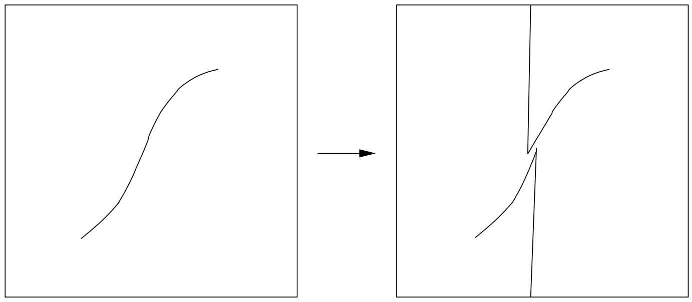

FIGURE 4. A Legendrian segment passing through $\{z=0\}$

According to [Dy04] the Thurston-Bennequin invariant of K is

$$\mathrm{tb}(K) = \mathrm{cross}_+(K) - \mathrm{cross}_-(K) - \frac{1}{2}\mathrm{cusp}(K) - a \cdot b. \tag{8}$$

Here cross_+ respectively cross_- denote the number of positive respectively negative crossings and $\mathrm{cusp}(K)$ denotes the number of cusps of the front projection. Because some lines in a tangent space of T^2 do not have unique Legendrian preimage one has to indicate which strand passes above the other strand at a double point.

By [Dy04] the rotation number of an oriented Legendrian knot can be read of from the front diagram as follows:

$$\mathrm{rot}(K) = \frac{1}{2}\big(\mathrm{cusp}_+(K) - \mathrm{cusp}_-(K)\big) - a - b. \tag{9}$$

A cusp is positive of the tangent space of the knot crosses $\pm\partial_z$ positively with respect to the orientation of ξ given by α. For example, if a cusp has z-coordinate in $(0,\pi/3)$, then the cusp is positive if and only of the knot is oriented downwards, upward cusps are negative in the region $z<\pi/3$.

The non-loose Legendrian knot described at the beginning of this section is diffeomorphic to the knot $K = \{(t,-t,1/2) \,|\, t \in [0,1]\}$. There are more non-loose Legendrian knots in (S^3,ξ_{-1}) which can be easily described in terms of front projections.

For coprime integers m,n consider a linear curve in T^2 representing the homology class $(m,n) \in \mathbb{Z}^2 \simeq H_1(T^2;\mathbb{Z})$. If $mn \leq 0$, then this linear curve has a uniquely determined Legendrian lift $K_{m,n}$ in (S^3,ξ_{-1}) and $K_{m,n}$ is a torus knot with

$$\mathrm{tb}(K_{m,n}) = -mn \qquad\qquad \mathrm{rot}(K_{m,n}) = -m-n.$$

When $n=1$ and $m<-1$ (or vice versa) we obtain non-loose Legendrian unknots with positive Thurston-Bennequin invariant such that the rotation number is $\pm(\mathrm{tb}(K_{m,n})-1)$.

For $m = 0, n = 1$ or $m = 1, n = 0$ the curve $K_{m,n}$ lifts to a Legendrian unknot which bounds overtwisted discs, namely one of the two discs appearing in Figure 3. Of course neither of these unknots is non-loose. According to Proposition 4.9 of [Dy04] complement of their union, a Hopf link, is universally tight.

Finally, we show that a non-loose unknot $K_{m,n}$ can be stabilized to a Legendrian knot isotopic to $K = K_{-1,1}$ (with one of its two possible orientations). In Figure 5 we illustrate this for the case $m = 1, n = -2$, the general case is similar.

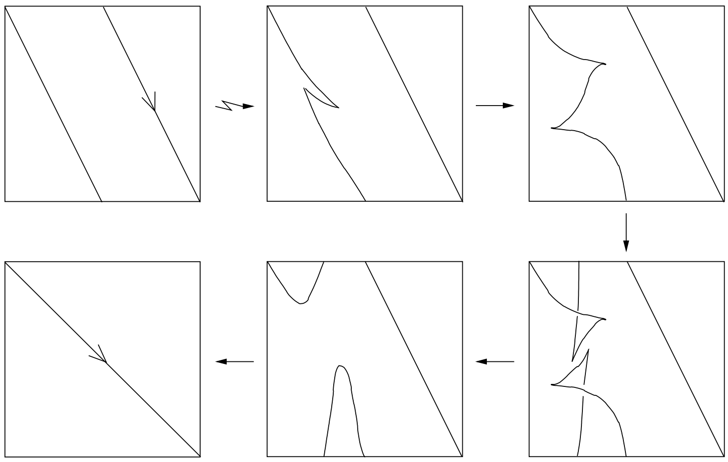

FIGURE 5. Stabilizing $K_{1,-2}$ one obtains $K_{1,-1}$

The first arrow indicates the stabilization, the third downward arrow indicates the move depicted in Figure 4 and the fourth arrow indicates two Reidemeister moves. The last and the second arrow correspond to isotopies.

Example 2.31. Consider the diffeomorphism ψ whose restriction on $T^2 \times (0,1)$ (the complement of the Hopf link H) is

$$\begin{aligned}\psi\big|_{S^3\setminus H} : T^2 \times (0,1) = (\mathbb{R}^2/\mathbb{Z}^2) \times (0,1) &\longrightarrow T^2 \times (0,1)\\ (x,y,z) &\longmapsto (y,x,1-z).\end{aligned}$$

This map extends to a contact diffeomorphism of S^3 which preserves the orientation of ξ and maps $K = K_{1,-1}$ to itself but reverses the orientation of this knot.

2.6. **Homotopy theory of plane fields on S^3.** In this section we discuss homotopical properties of families of contact structures. This leads to the definition of a $\mathbb{Z}_2$-valued invariant d for contactomorphisms on the 3-sphere.

Recall that the tangent bundle of S^3 is trivial (this is true for every oriented 3-manifold). One can use a fixed Riemannian metric and a trivialization to associate to each oriented plane field ξ on S^3 a Gauß map

$$\begin{aligned} g_\xi : S^3 &\longrightarrow S^2 \\ p &\longmapsto \text{unit vector othogonal to } \xi. \end{aligned}$$

By the Thom-Pontrjagin construction homotopy classes of maps from manifolds to m-spheres are in one-to-one correspondence with framed submanifolds of codimension m up to framed cobordisms (cf. [Mi]). Usually, the Thom-Pontrjagin theorem is stated for manifolds without boundary, the case when maps are fixed on the boundary of a manifold was considered e.g. by Y. Huang [Hu14] in the context of plane fields on 3-manifolds.

Homotopy classes of plane fields on an oriented 3-manifold M correspond to homotopy classes of maps to S^2, when $M = S^3$ these homotopy classes correspond to elements of $\pi_3(S^2)$. The homotopy groups of the 2-sphere that matter for us are well known (see for example Chapter 4 of [Ha01] or Chapter 5 of [DFN] for a discussion based on the Thom-Pontrjagin construction)

$$\pi_3(S^2) \simeq \mathbb{Z} \qquad \pi_4(S^3) \simeq \mathbb{Z}_2 \qquad \pi_4(S^2) \simeq \mathbb{Z}_2.$$

A generator of $\pi_3(S^2)$ is the Hopf map

$$\begin{aligned} S^3 \subset \mathbb{C}^2 &\longrightarrow \mathbb{CP}^1 \simeq S^2 \\ (z_0, z_1) &\longrightarrow [z_0 : z_1], \end{aligned}$$

a generator of $\pi_4(S^3)$ is the suspension of the Hopf map and a generator of $\pi_4(S^2)$ is represented by the composition of the Hopf map with its suspension.

Recall that $M = S^3 \simeq \mathrm{SU}(2)$ and consider the $\mathrm{SU}(2)$-invariant framing to define the Gauß map. Homotopy classes of plane fields on S^3 are distinguished by an integer $h(\xi) \in \mathbb{Z}$, the Hopf invariant of its Gauß map defined in [H]. We recall one possible definition of the Hopf invariant.

Given a smooth map $f : S^3 \longrightarrow S^2$, its Hopf invariant can be determined as follows. Let $p \in S^2$ be a regular value of f and fix an oriented basis of T_pS^2. Then $f^{-1}(p)$ is a submanifold of codimension 2 in S^3 with normal bundle $f\big|_{f^{-1}(p)}(T_pS^2)$. This normal bundle is framed using the preimage of the basis of T_pS^2 under f. The Hopf invariant $H(f)$ is the linking number of $f^{-1}(p)$ and a push-off of $f^{-1}(p)$ in the direction of the first vector of the framing. This number is independent of choices other than the homotopy class of the framing and determines f up to homotopy.

Example 2.32. The standard contact structure $TS^3 \cap iTS^3$ is $\mathrm{SU}(2)$-invariant, just like the framing. Therefore the Gauß map is constant in this case and the Hopf invariant vanishes.

A construction which is frequently used to change the homotopy class as plane field of a contact structure is the Lutz-twist. In order to describe it, recall that a transverse knot K in a contact manifold has a tubular neighborhood $N(K) \simeq S^1 \times D^2(\rho)$ such that the contact structure on $N(K)$ is isomorphic to the contact structure defined by $dz + r^2 d\vartheta$, where $D^2(\rho)$ is a disc of radius ρ and (r, ϑ) are polar coordinates on $D^2(\rho)$.

For two smooth functions f, g on $[0, \rho]$ such that

$$\begin{aligned} f(0) &= -1 & \qquad g(r) &= -r^2 \text{ near } r = 0 \\ f(r) &= 1 & \qquad g(r) &= r^2 \text{ near } r = \rho \end{aligned}$$

and $fg' - g'f > 0$. Let ξ' be the contact structure defined by

$$\xi' = \begin{cases} \ker(f(r)dz + g(r)d\vartheta) & \text{on } S^1 \times D^2(\rho) \\ \xi & \text{on the complement of } N(K). \end{cases}$$

Definition 2.33. We say that ξ' is the result of a π-*Lutz twist* along K.

The contact structure ξ' is well defined up to isotopy. When one reverses the orientation of K it is positively transverse to ξ'.

In general, ξ' is not homotopic to ξ even as a plane field. How to determine the difference between the homotopy classes of ξ and ξ' is explained in Chapter 4.3 of [Ge]. What is relevant for us is that applying a π-Lutz twist along a null-homologous transverse knot K in S^3 changes the Hopf invariant by $\mathrm{sl}(K)$. The usual Lutz twist (or 2π-Lutz twist) corresponds two consecutive π-Lutz twists applied to the same knot, it does not change the homotopy type of the plane field.

Remark 2.34. The contact structure obtained by a π-Lutz twist along a transverse knot is always overtwisted since the discs $\{z\} \times D^2(\rho_0)$ are overtwisted for the value $\rho_0 \in (0, \rho)$ for which $(f(\rho_0), g(\rho_0))$ lies on the positive part of the x-axis (ρ_0 is uniquely determined because of $fg' - f'g > 0$).

Example 2.35. Let $\xi = TS^3 \cap iTS^3$ be the standard contact structure on $S^3 \subset \mathbb{C}^2$. This contact structure is $\mathrm{SU}(2)$-invariant. The fibers of the Hopf fibration are transverse curves, each of them has self-linking number -1. After a π-Lutz twist along k distinct fibers one obtains a contact structure with Hopf invariant $-k^2$.

For $k = 1$ we obtain the contact structure which we are interested in most in this article.

When the plane field varies continuously the same is true for the Gauß map. Therefore homotopy classes of 1-parameter families $\xi_t, t \in [0, 1]$, of coorientable plane fields on S^3 are in one-to-one correspondence with the set of homotopy classes of maps $g : S^3 \times [0, 1] \longrightarrow S^2$ (with fixed boundary conditions). This set is isomorphic to $\pi_4(S^2) \simeq \mathbb{Z}_2$. If $M = S^3$, then we choose the framing so that $g(\cdot, 0) = g(\cdot, 1)$ are constant maps.

Following §23.4 in [DFN] we now review how to determine to which element of $\pi_4(S^2)$ a given map $g : S^4 \longrightarrow S^2$ corresponds.

We may assume that g is smooth and pick a regular value p. Then $\Sigma = g^{-1}(p) \subset S^4$ is a framed submanifold of codimension 2. For each oriented simple closed curve γ in Σ consider its normal vector field in Σ. Then γ can be viewed as framed submanifold of S^4 of codimension 3, so it represents an element $\phi(\gamma)$ in $\pi_4(S^3) \simeq \mathbb{Z}_2$. This element depends only on the homology class $\gamma \in H_1(\Sigma; \mathbb{Z}_2)$ and we have defined a map

$$\phi : H_1(\Sigma; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2. \tag{10}$$

Given two simple closed curves γ, γ' in Σ which intersect transversely we can replace $\gamma \cup \gamma'$ by a collection of simple closed curves representing the homology class $\gamma + \gamma'$ by smoothing the intersection points. One then has

$$\phi_g(\gamma + \gamma') = \phi_g(\gamma) + \phi_g(\gamma') + \gamma \cdot \gamma' \qquad \mod 2$$

where $\gamma \cdot \gamma'$ is the intersection pairing.

Thus $\phi_g : H_1(\Sigma; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ is a non-degenerate quadratic form. Recall that the Arf invariant of a $\mathbb{Z}_2$-valued quadratic form on $H_1(\Sigma; \mathbb{Z}_2)$ takes values in $\mathbb{Z}_2$ and is non-trivial if and only if more than one half of the elements $\gamma \in H_1(\Sigma; \mathbb{Z}_2)$ have $\phi_g(\gamma) = 1$. One can show that the Arf invariant of ϕ_g depends only on the homotopy class of $g : S^4 \longrightarrow S^2$.

K. Dymara [Dy01] used the fact $\pi_4(S^2) \simeq \mathbb{Z}_2$ to define a continuous group homomorphism

$$\mathrm{d} : \mathrm{Diff}_+(S^3, \xi) \longrightarrow \mathbb{Z}_2$$

on the group of orientation preserving contactomorphisms. This homomorphism can be used to show that the groups of coorientation preserving contactomorphisms $\mathrm{Diff}_+(S^3, \xi)$ is not connected when ξ is overtwisted.

We recall the definition from [Dy01]. For $\psi \in \mathrm{Diff}_+(S^3, \xi)$ choose a family of diffeomorphisms ψ_t such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$. Such a family exists because ψ is orientation preserving and the group of orientation preserving diffeomorphisms of S^3 is connected according to Cerf's theorem [Ce].

We consider the loop $\psi_{t*}(\xi)$ in the space of oriented plane fields on S^3. (This is even a loop in the space of contact structures on S^3 but we ignore this fact.) Since the Euler class of ξ in $H^2(S^3; \mathbb{Z}) = 0$ vanishes we can pick a trivialization of ξ and extend it to a framing of S^3.

Applying the Gauß map we obtain a loop in the space of maps from S^3 to S^2 based at the constant map, i.e. this loop can be viewed as map $S^4 \longrightarrow S^2$. We define

$$\begin{aligned} \mathrm{d} : \mathrm{Diff}_+(S^3, \xi) &\longrightarrow \pi_4(S^2) \simeq \mathbb{Z}_2 = \{0, 1\} \\ \psi &\longmapsto \left[\left(\psi_{t*}(\xi) \right)_{t \in [0,1]} \right]. \end{aligned}$$

It is proved in [Dy01] that this is a well defined homomorphism on the group of connected components of $\mathrm{Diff}_+(S^3, \xi)$. The proof in [Dy01] that d is well defined contains a minor gap since in the author assumes that every contact structure on S^3 is contactomorphic to a contact structure which is invariant under the standard S^1-action on S^3. However the proof idea still works. In order to see this construct a contact structure with Hopf invariant

k using π-Lutz twists along a collection of knots transverse to the standard contact structure with self linking number -1 which are invariant under the contact diffeomorphism $(w,z) \longmapsto (w,-z)$ of $S^3 \subset \mathbb{C}^2$.

The following diagram shows the various groups involved in the construction of the invariant d.

$$\begin{tikzcd}
\pi_1(\mathrm{Diff}(S^3,\mathrm{or})) = \mathbb{Z}_2 \arrow[r] & \pi_1(\mathrm{Cont}(S^3,\xi)) \arrow[r, "\delta"'] \arrow[d] & \pi_0(\mathrm{Diff}_+(S^3,\xi)) \arrow[l, bend right] \\
& \pi_1(\mathrm{Distr}(S^3,\xi)) \arrow[r, "\text{Gauß-map}", "\text{+ collapse}"'] & \pi_4(S^2) = \mathbb{Z}_2
\end{tikzcd}$$

Here $\mathrm{Distr}(S^3,\xi)$ respectively $\mathrm{Cont}(S^3,\xi)$ denotes the plane fields homotopic to ξ respectively the contact structures isotopic to ξ and $\mathrm{Diff}(S^3,\mathrm{or})$ is the group of orientation preserving diffeomorphisms of S^3. This group is homotopy equivalent to $\mathrm{SO}(4)$ by [Ha83].

The upper line is part of the long exact homotopy sequence of the fibration $\mathrm{Diff}(S^3,\mathrm{or}) \longrightarrow \mathrm{Cont}(S^3,\xi)$ given by $\varphi \longmapsto \varphi_*(\xi)$, the vertical arrow is induced by the inclusion and the map from $\pi_0(\mathrm{Diff}_+(S^3,\xi)) \longrightarrow \pi_1(\mathrm{Cont}(S^3,\xi))$ is the section of δ chosen above.

Remark 2.36. When ξ is overtwisted it follows from Lemma 2.6 that there is a family of contact structures $\xi_t, t \in [0,1]$ with $\xi = \xi_0 = \xi_1$ which represents a non-trivial loop in the space of pane fields on S^3. By Gray's theorem there is a family ψ_t of diffeomorphisms of S^3 such that $\psi_{t*}\xi = \xi_t$. Then $\mathrm{d}(\psi_1) = 1$, i.e. d is surjective.

Later we will use the fact that the resulting contactomophism ψ_1 can be chosen to have support in a ball containing a fixed overtwisted disc Δ.

3. THE CLASSIFICATION OF NON-LOOSE UNKNOTS IN (S^3,ξ)

In this section we prove the main result of this paper, i.e. we classify minimal non-loose Legendrian unknots in S^3 up to Legendrian isotopy. For this, we first review the classification of non-loose unknots up to contact diffeomorphism from [ElF09, Et13] where it is shown that every non-loose unknot K with $\mathrm{tb}(K) = 1$ in S^3 carrying an overtwisted contact structure is diffeomorphic to the example K_{st} discussed at the beginning of Section 2.5. Using

- the coarse classification,
- some particular contact diffeomorphisms constructed in Section 3.2,
- Eliashberg's theorem on overtwisted contact structures

we prove in Section 3.3 that K is isotopic to the standard non-loose unknot. However this isotopy does not respect orientations of these knots. The proof that K_{st} and $\overline{K}_{st}$ are not isotopic as oriented Legendrian can be found in Section 3.4.

Except for the coarse classification we will consider only non-loose Legendrian unknots K with $\mathrm{tb}(K) = 1$ in this section. Non-loose Legendrian unknots with higher Thurston-Bennequin number will be classified in Section 4.2.

3.1. **The coarse classification.** In this section we recall the coarse classification of non-loose unknots in S^3, i.e. the classification up to contactomorphism from [ElF09] and [Et13]. This classification is a corollary of the classification of tight contact structures on the solid torus by E. Giroux [Gi00] and K. Honda [Ho]. The following theorem summarizes the information which is needed to put a Seifert disc of a non-loose Legendrian unknot in a standard form which will then imply that (S^3, ξ, K) is diffeomorphic to the standard model described in Section 2.5.

Proposition 3.1. *Let ξ be a contact structure on S^3 and $K \subset S^3$ a non-loose unknot with* $\mathrm{tb}(K) = n > 0$. *Then* $\mathrm{rot}(K) = \pm(n-1)$.

From this proposition we now deduce the coarse classification of non-loose unknots in S^3, the definition of $K_{m,n}$ can be found in Section 2.5.

Theorem 3.2. *Let ξ be an overtwisted contact structure on S^3 and K a non-loose Legendrian unknot with* $\mathrm{tb}(K) = n > 0$ *and* $\mathrm{rot}(K) = \pm(n-1)$. *Then (S^3, ξ, K) is diffeomorphic to $(S^3, \xi_{-1}, K_{\pm 1, \mp n})$.*

We give the proof of this because it yields a normal form for the characteristic foliation on the Seifert surface of a non-loose unknot which will be used later.

Proof. Let ξ be an overtwisted contact structure on S^3 and K a non-loose oriented Legendrian unknot. We assume $\mathrm{tb}(K) = n > 0$ because of Lemma 2.29. Choose an oriented spanning disc for K. We will simplify the characteristic foliation on D to bring it in a standard form. For this we assume that D is generic so that the singular points of $\xi(D)$ are either elliptic or hyperbolic.

We first consider the singular points of $\xi(D)$ on the boundary of D and simplify the characteristic foliation on the surface as follows.

- Positive hyperbolic singular points on ∂D whose unstable leaves lie on ∂D are replaced by positive elliptic points (cf. Figure 1).
- Negative hyperbolic singular points on ∂D whose stable leaves lie on ∂D are replaced by negative elliptic points.

After this, a negative (positive) elliptic singularity is connected to either a negative (positive) singularity or to a negative hyperbolic (positive) hyperbolic singularity. The second case can be further simplified using the elimination lemma which provides us with a deformation of D which cancels a hyperbolic singular point with a elliptic singularity of the same sign while keeping the boundary of D Legendrian throughout the deformation.

From now on we assume that on ∂D there are no canceling pairs of singularities left. But then, if there an elliptic singularity left at all, then all singular points on ∂D are elliptic and they alternate between negative and positive along ∂D and the Thurston-Bennequin invariant of K is negative.

If ∂D contains no singular points, then $\mathrm{tb}(K) = 0$ which is impossible for a non-loose Legendrian unknot in an overtwisted contact manifold. Therefore $\xi(D)$ has only hyperbolic singular points on the boundary which alternate between positive and negative.

The interior of D can now be simplified further. By genericity we may assume that there are no connections between hyperbolic singular points such the connecting separatrix lies in the interior of D. Then every unstable leaf of a negative hyperbolic singularity has a negative elliptic singularity as its ω-limit set since ξ is tight on $M \setminus K$. Therefore all negative hyperbolic singular points of $\xi(D)$ in the interior of D can be eliminated and the same is true for the positive singular points in the interior of D.

From now on we assume $\mathrm{rot}(K) \geq 0$. Since there are stable and unstable leaves of singular points on the boundary of D, $\xi(D)$ must have positive and negative singular points in the interior. According to (3)

$$\mathrm{rot}(K) = e_+ - e_- = n - 1.$$

Hence there are at least n positive elliptic singularities. But there cannot be more since one the boundary, there are exactly n singular points whose unstable leaves can come from positive elliptic singularities. If no leaf coming from an elliptic singular point ends at a hyperbolic singular point, then the basin of this elliptic singularity is either a sphere (which is impossible since the Seifert surface is a surface with boundary) or there is an overtwisted disc inside of D.

Hence $e_+ = n$ (and $e_- = 1$) and every positive elliptic singular point is connected to a negative hyperbolic point in ∂D. All unstable leaves of positive hyperbolic on ∂D end at the same negative elliptic singular point in the interior. This determines the characteristic foliation on D up to homeomorphism. For $n = 1$ and $n = 3$ it is shown in Figure 6. □

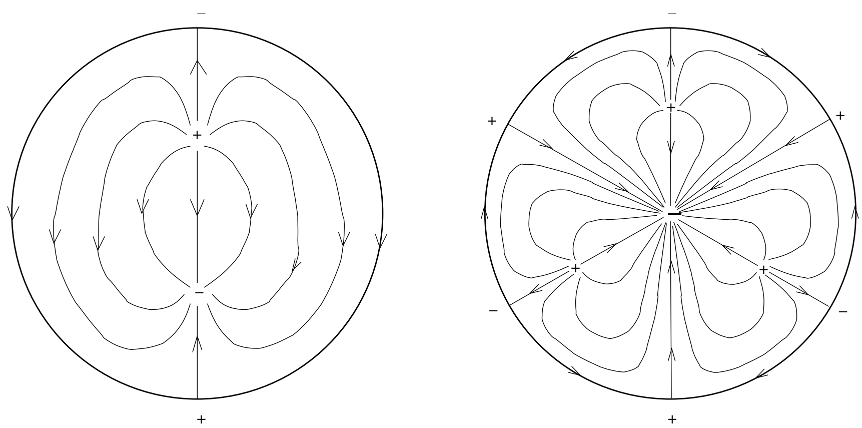

FIGURE 6. Disc bounding a minimal non-loose unknot and an unknot K with $tb(K) = 3$ and $\mathrm{rot}(K) = 2$

Notice that the characteristic foliation shown in Figure 6 is very similar to the one shown in Figure 22 of [ElF98], but there is a crucial difference:

The picture in [ElF98] contains no retrogradient singular points and actually depicts a convex surface while discs bounding non-loose unknots are never convex, there are always two retrogradient connection in K. In view of Lemma 2.20 the appearance of a minimal non-loose unknot as union of leaves of the characteristic foliation on a sphere is a phenomenon of codimension 2.

Remark 3.3. In Section 2.5 we have shown that the positive/negative stabilization of a non-loose Legendrian unknot K with $\mathrm{tb}(K) > 1$ results in a non-loose Legendrian unknot K' with $\mathrm{tb}(K') = \mathrm{tb}(K) - 1$ and $\mathrm{rot}(K') = \mathrm{rot}(K) \pm 1$.

We have shown before that the negative stabilization of $K_{1,-n}$ is non-loose when $n > 1$. By Theorem 3.2 the positive stabilization of $K_{1,-n}$ is loose.

We take the opportunity to introduce a schematic way to depict essential information on the characteristic foliation on sphere in the tubular neighbourhoood containing a (piecewise smooth) non-loose unknot which is minimal. The dashed lines in the following figure represent unstable leaves of negative singularities which take part in a retrogradient connection before and after the retrogradient connection occurs. The solid straight lines represent the remaining part of the graph formed by negative singular points and unstable leaves of negative singularities. This graph is a tree and there are non closed leaves since the knot is supposed to be non-loose.

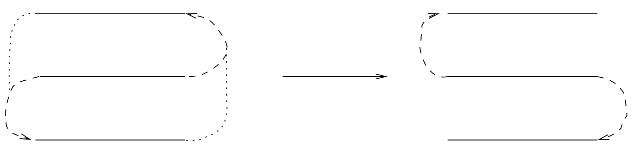

FIGURE 7. Schmeatic representation of the movie on the spheres in a product neighborhood of a sphere carrying a minimal non-loose Legendrian unknot

3.2. **Contact diffeomorphisms preserving non-loose unknots.** For the proof of our classification results we need to construct a contact diffeomorphism

$$\psi : (S^3, \xi_{-1}) \longrightarrow (S^3, \xi_{-1})$$

which preserves K, the orientation of the plane field and has $\mathrm{d}(\psi) = 1$. This will be the content of the second example in this section, the first is a preparation for the second. We will use the consequences of the Thom-Pontrjagin construction which were outlined in Section 2.6.

Before explaining the construction we note that although this section does not refer directly to bypasses some of the material presented here could be rephrased using the methods developed by Y. Huang in [Hu13, Hu14].

First, we consider how a particular operation on a given contact structure affects the Hopf invariant. This construction will yield an alternative construction of a non-loose unknot in S^3.

Example 3.4. Start with the standard contact structure $\xi_{st} = TS^3 \cap iTS^3$. It is $\mathrm{SU}(2)$-invariant and we use it to orient S^3. We choose an $\mathrm{SU}(2)$-invariant framing to define the Gauß maps (the Gauß map of ξ_{st} is constant).

Consider the transverse unknot $H = S^1 \times \{0\}$ together with a tubular neighborhood $N(H) \simeq S^1 \times D^2$ such that the contact structure on this neighborhood is invariant under translations in the S^1-direction and under rotations of the disc.

We fix a round 2-sphere $S_0 \subset S^3$ which is orthogonal to $\mathbb{C} \times \{0\} \subset \mathbb{C}^2$ such that $S_0 \cap N(H) \supset \{1\} \times D^2$ (here $1 = (1,0) \in H \subset \mathbb{C}^2$ is a point on H) and we orient S_0 so that $\xi_{st}(S_0)$ has a positive singular point in the midpoint of the disc $\{1\} \times D^2$. Then S_0 is convex and $\xi_{st}(S_0)$ has exactly two singular points (both elliptic), these are the intersection points of S_0 with H.

We isotope ξ_{st} to a new contact structure ξ with the following properties:

- ξ is S^1-invariant on $N(H) \simeq S^1 \times D^2$, and
- the characteristic foliation of ξ on $\{1\} \times D^2$ is diffeomorphic to the one shown in Figure 8.

S^2 contains a disc D where the characteristic foliation is homeomorphic to the singular foliation shown in the smaller disc D in Figure 8.

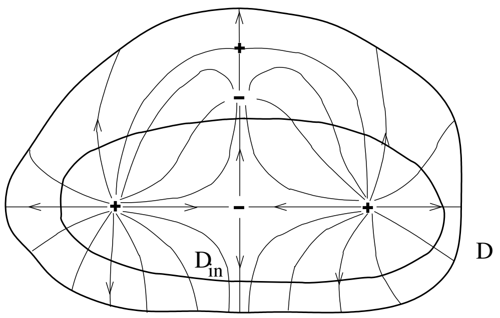

FIGURE 8. Characteristic foliation on D and D_{in}

To obtain a new plane field ζ_1 we consider a product neighborhood $S^2 \times (-\infty, \infty)$ around the convex sphere $S^2 = S^2 \times \{0\}$ oriented so that the product orientation is positive and the contact structure is translation invariant. By assumption, there is a disc $D_{in} \subset D$ such that the characteristic foliation is transverse to D and $\xi(D)$ has three singular points, two of them are elliptic and positive, the remaining one is negative and hyperbolic.

Because the boundary of D_{in} is transverse to the characteristic foliation there is an isotopy $\rho_s, s \in (-1,1)$, of D_{in} such that

- $\rho_s = \mathrm{id}$ for s close to -1,

- ρ_s is constant and preserves the characteristic foliation for s close to 1,
- ρ_s preserves the characteristic foliation near ∂D_{in} for all s,
- the hyperbolic singularity p is a fixed point of ρ_s for all s,
- the unstable leaves of p are interchanged by ρ_s for s close to 1, and
- as s varies from -1 to 1, the unstable leaves of p rotate by a counter clockwise half turn.

A new plane field ζ_1 is determined (up to homotopy) by the requirement that $\zeta_1 = \xi$ outside of $D^2_{in} \times (-1,1)$ and the characteristic foliation of ζ_1 on $D^2_{in} \times \{s\}$ is the image of the characteristic foliation of ξ under ρ_s.

For $k \in \mathbb{N}$ the plane field ζ_k is defined by applying the above operation k times. For negative k one uses $|k|$-times the isotopy σ_s^{-1} instead of σ_s.

When one passes from ξ to ζ_k the homotopy type of the plane field changes. In order to determine how the Hopf invariant changes we choose a framing of the tangent bundle so that TD is tangent to the span of the first two components of the framing and so that the framing is vertically invariant on the ball $D \times (-1,1) \subset D \times (-\infty,\infty)$ (recall that $\zeta_k = \xi$ outside of this ball). We consider the Gauß maps g, g_k of ξ, ζ_k.

Because all singular points of the characteristic foliation on D are non-degenerate we may assume that $g(-TD^2(p,0))$ is a regular value of g and g_k. By construction the Thom-Pontrjagin submanifolds of g and g_k coincide, but their framings are different along the component H. A direct computation in terms of local coordinates near the negative hyperbolic singularity coordinates shows that $H(g_k) = H(g) + k$.

For $k > 0$, the plane field constructed above is not a contact structure because the resulting movie violates Lemma 2.20. If $k \leq 0$, then one obtains a contact structure with Hopf invariant k because the movie one constructs is the not only the movie of a plane field but the movie of a contact structure (see Section 2.3.1).

We now specialize to $k = -1$. In this case one can choose the characteristic foliation and ρ_s so that there is exactly one positive hyperbolic singularity q in D and its stable leaves both come from D_{in}. One has to choose ρ_s so that for each stable leaf of q there is exactly one coincidence with an unstable leaf of q of ρ_s^{-1} for exactly one $s \in (-1,1)$.

When this parameter is the same for both coincidences (say for $s = 0$) the union of the pair of stable/unstable leaves on the sphere $S^2 \times \{s = 0\}$ is a non-loose unknot K: The characteristic foliation on the sphere carrying this knot can be deformed so that on both discs bounding the knot the characteristic foliation has the normal form discussed in Section 3.1. Also, the balls in the complement of a neighborhood of this sphere are tight by construction. Therefore the unknot constructed starting from Example 3.4 is contactomorphic to the standard non-loose unknot in S^3.

Using this description of the non-loose unknot we next construct an example of a family of contact structures on S^3 which is not homotopic to the constant family. From this family of contact structures we obtain a contact diffeomorphism ψ_1 with non-trivial Dymara invariant. In later applications it will be important to ensure that ψ_1 maps K to itself (it will turn out that in this example ψ_1 reverses the orientation of the knot).

Example 3.5. We continue to consider the situation from Example 3.4 for $k = -1$. Our goal to obtain a non-trivial loop of contact structures on S^3 which preserves K as a set.

For $\tau \in [0, 2\pi]$ let $\psi_\tau : S^3 \longrightarrow S^3$ be the composition of two rotations: The first rotation is a rotation around a plane orthogonal to H by the angle τ while the second one is a rotation by $-\tau/2$ around the plane containing H. The family of contact structures $\psi_\tau(\xi_{st})$ is constant because these rotations preserve the complex structure on $\mathbb{C}^2$. As above deform ξ_{st} to a new contact structure so that on the second factor of $N(H) = S^1 \times D^2$ the characteristic foliation of ξ is diffeomorphic to the one shown in Figure 8. This can be done relative to a neighborhood of $\{0\} \times S^1$ and ξ can be chosen so that it is invariant under $\psi_{2\pi}$. Then the loop of contact structures $\xi(\tau) = \psi_{\tau *}(\xi)$ is contractible in the space of plane fields by Theorem 2.4.

Let R be the unit vector tangent to the fibers of the Hopf fibration. This vectorfield is positively transverse to ξ_{st} everywhere. We may assume that R is a component of the framing and that $-R$ is a regular value of the Gauß map of ξ. Since the loop $\xi_\tau, \tau \in [0, 2\pi]$, is null homotopic the associated Thom-Pontrjagin manifold of $S^3 \times S^1$ (carrying the family of plane fields tangent to S^3 induced by ξ_τ) is framed cobordant to the empty manifold. In particular, the sum of the Arf invariants of all components of the Thom-Pontrjagin submanifold is zero.

We now modify the family ξ_τ so that the framing of one component of the Thom-Pontrjagin submanifold changes so the associated Arf invariant changes while we do not modify the framing of other components.

For this we consider the family of thickened spheres $\psi_\tau(S^2 \times (-\delta, \delta)) \subset S^3$ and apply to each such family the modification discussed in Example 3.4. We obtain a closed loop of contact structures $\xi_{-1}(\tau)$. The only part of the Thom-Pontrjagin submanifold affected by this construction is the component containing H. The framing of this submanifold changes so that the framing makes a full turn (compared to a $\mathrm{SU}(2)$-invariant framing) as one moves along H and also as one moves along $p \times [0, 2\pi]$ for $p \in H$.

We identify $S^3 \times \{0\}$ with $S^3 \times \{2\pi\}$ and consider the situation in $S^3 \times S^1$. The Thom-Pontrjagin submanifold associated to the constant family ξ is the same as the Thom-Pontrjagin submanifold of $\psi(\tau)$ but the framing of the component containing H has changed in such a way that the Arf invariant also changes. Hence $\xi_{-1}(\tau)$ is a homotopically non-trivial loop of plane fields on S^3.

Note that by construction, we also obtain a family K_τ of Legendrian knots (with respect to $\xi_{-1}(\tau)$) so that $K_{2\pi} = K_0$ but the orientation is reversed. Thus $\psi_{2\pi}$ is a contact diffeomorphism of ξ_{-1} which reverses the orientation of K, preserves the orientation of ξ_{-1} and has $\mathrm{d}(\psi_{2\pi}) \neq 0 \in \pi_4(S^2)$.

The following lemma summarizes the result of the construction given in Example 3.5 with $\psi = \psi_{2\pi}$.

Lemma 3.6. *Let $K \subset S^3$ be the standard Legendrian unknot in (S^3, ξ_{-1}). Then there is $\psi \in \mathrm{Diff}_+(S^3, \xi_{-1})$ which*

- *preserves the orientation of ξ_{-1} and $\mathrm{d}(\psi) = 1$,*
- *$\psi(K) = K$ and the orientation of K is reversed.*

3.3. **The classification up to isotopy for non-oriented unknots.** In this section we classify non-oriented non-loose unknots in S^3 up to isotopy. This classification makes heavy use of Eliashberg's classification theorem. The overtwisted disc one has to control in the application of that theorem is obtained from the non-loose unknot viewed as subset of a sphere such that the unknot is the union of two retrogradient connections of the characteristic foliation.

In the following we denote the minimal non-loose unknot described at the beginning of Section 2.5 by L.

Theorem 3.7. *Let ξ_{-1} be the overtwisted contact structure with Hopf invariant -1 on S^3. Then every minimal non-loose Legendrian unknot K is isotopic to L or $\overline{L}$.*

The following theorem will prove a slightly stronger statement: A smooth isotopy ψ_t such that ψ_1 preserves the contact structure and has $\mathrm{d}(\psi_1) = 1$ can be deformed to a contact isotopy relative to the endpoints. It should be noted that the existence of such an isotopy does not imply anything about the orientations of K and L.

Proof. Let K be a non-loose unknot. For convenience we remove two open Darboux balls from the complement of $L \cup K$ such that the characteristic foliation on the boundary of each of these balls is convex. We thus view L as non-loose knot in $S^2 \times [-1, 1]$. The contact isotopy moving L to K will have support inside this smaller space.

By the coarse classification of non-loose unknots we already know that there is a contact diffeomorphism ψ with $\psi(L) = K$. In particular, ψ preserves the orientation of S^3 and by Cerf's theorem there is an isotopy $\psi_s, s \in [0, 1]$, of S^3 connecting ψ to the identity. By Lemma 3.6 we may assume that $\mathrm{d}(\psi) = 0$. Let ψ_σ be an isotopy with $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$.

Because the space of 3-balls in $\mathbb{R}^3$ is simply connected (p.5 in [Ce]) we can choose ψ_s so that it preserves the two balls we have removed from S^3.

In order to construct the contact isotopy we will apply Gray's theorem to a family of contact structures on $M = S^2 \times [-1, 1]$. The parameter space will be $I^2 = [0, 1] \times [0, 1]$. We will now describe

- the contact structures $\xi_{\sigma,\tau}$ on M
- and restrictions on the product decomposition $M \simeq S^2 \times [-1, 1]$

for parameter values (σ, τ) contained in a neighborhood of ∂I^2 and near the boundary of M.

t close to ± 1	$\xi_{\sigma,\tau} = \xi_{-1}$ independent from σ and τ The product decomposition near ∂M is independent from σ, τ.
τ close to 0	$\xi_{\sigma,\tau} = \xi_{-1}$ independent from σ and τ The product decomposition of M is the image of the original product decomposition under ψ_σ.
τ close to 1	$\xi_{\sigma,\tau} = \psi_{\sigma *}\xi_{-1}$, but independent from τ The product decomposition of M is the image of the original product decomposition under ψ_σ.
σ close to 0	$\xi_{\sigma,\tau} = \xi_{-1}$ independent from σ and τ $M \simeq S^2 \times [-1,1]$ so that $L \subset S^2 \times \{0\}$ is in normal form.
σ close to 1	$\xi_{\sigma,\tau} = \xi_{-1}$ independent from σ and τ $M \simeq \psi_1(S^2 \times [-1,1])$ so that $K \subset \psi_1(S^2 \times \{0\})$ is in normal form.

Figure 9 illustrates the situation. Each point in the box represents a sphere, horizontal lines represent the manifold $S^2 \times [-1,1]$. Some of the vertices are labeled with coordinates. On the front face $\{\tau = 1\}$, there is an obvious family (parametrized by σ) of Legendrian knots interpolating between K and L but the contact structure is not constant. This is indicated by the curved line on the front face of the box. The straight lines on the top respectively bottom face correspond to the knots L respectively K. On the back face $\{\tau = 0\}$ the contact structure is constant.

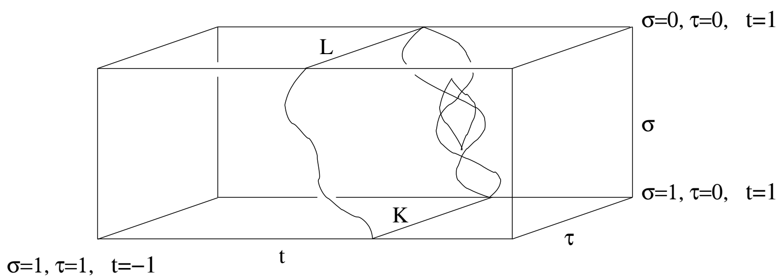

FIGURE 9. The setup for the construction of an isotopy from L to K

For $\tau = 1$, the knot $\psi_\sigma(L)$ is contained in a sphere of the product decomposition. Because $\mathrm{tb}(L) > 0$, this sphere can't be convex. Thus the line on front of the box is also a subset of

$$\left\{(\sigma,\tau,t) \,|\, \xi_{\sigma,\tau}\left(S^2_{\sigma,t}\right) \text{ is not convex}\right\}.$$

The same is true for the curved lines on the back face $\{\tau = 0\}$.

The strategy of the proof is now to extend the family of contact structures near the boundary of $S^2 \times [-1,1]$ to the entire manifold for all parameter values. By Gray's theorem this then induces a family $\varphi_{\sigma,\tau}$ of contact isotopies such that $\varphi_{\sigma,\tau *}(\xi_{\sigma,0}) = \xi_{\sigma,\tau}$. Then

$$\varphi_{\sigma,1}^{-1}(\psi_\sigma(L)) \tag{11}$$

is a family of Legendrian knots (parametrized by σ) in $(S^2 \times [-1,1], \xi)$. Because the contact structure is independent of τ when $\sigma = 0$ or $\sigma = 1$, it follows that $\varphi_{0,\tau} = \mathrm{id}_M = \varphi_{1,\tau}$. Therefore the family of Legendrian knots in (11) interpolates between L and K.

Consider the functions $T_\pm$ defined in (4) on the part of the parameter space I^2 where the contact structure $\xi_{\sigma,\tau}$ is already defined. By Lemma 2.23 these functions are continuous and $\xi_{\sigma,\tau}$ is overtwisted on every neighborhood of $S^2\times[-1, T_-(\sigma,\tau)]$ and $S^2\times[T_+(\sigma,\tau), 1]$. According to Lemma 2.24 $T_-(\sigma,\tau) \leq T_+(\sigma,\tau)$. First, we reduce to the case that $T_-(\sigma,0) < T_+(\sigma,0)$ for all $\sigma \in (0,1)$. The equality $T_-(\sigma,0) = T_+(\sigma,0)$ means that there is a non-unknot of ξ_{-1} on the leaf $S^2\times\{T_\pm(\sigma,0)\}$ of the product decomposition of $N \times \{\sigma, 0\}$ contains a non-loose unknots. Generically, this happens for finitely many $\sigma \in [0,1]$ and all non-loose unknots appearing in this way are minimal.

Assume that for $\sigma_0 \in (0,1)$ the equality $T_+(\sigma_0,0) = T_-(\sigma_0,0)$ holds. (In Figure 9 there is exactly value for σ with these properties.) Then there is an orientation preserving contact diffeomorphism F_{σ_0} such $F_{\sigma_0} \circ \psi_{\sigma_0}$ maps L to the non-loose unknot in $S^2\times\{T_-(\sigma_0,0)\} \subset N\times\{\sigma_0,0\}$. By Lemma 3.6 we may assume that $\mathrm{d}(F_{\sigma_0} \circ \psi_{\sigma_0}) = 0$ so we can use an isotopy connecting ψ_{σ_0} to $F_{\sigma_0} \circ \psi_{\sigma_0}$ define a family of contact structures on $\{\sigma = \sigma_0\} \subset N \times [0,1]^2$ together with a family of non-loose Legendrian unknots interpolating between $\psi_{\sigma_0}(L)$ and the non-loose unknot in $N \times \{\sigma_0, 0\}$.

After finitely many steps we may assume that $T_-(\sigma,0) = T_+(\sigma,0)$ if and only if $\sigma = 0$ or $\sigma = 1$.

The construction of $\xi_{\sigma,\tau}$ is done in several steps. All applications of Eliashberg's theorem (Theorem 2.5) will be relative to neighborhoods of the boundary of $S^2 \times [-1,1]$ (i.e. the subset N appearing in that theorem will contact a neighborhood of the boundary). Also we will no longer mention the dependence of the product decomposition on σ systematically (nor will this be reflected in the notation). In applications of Eliashberg's theorem the overtwisted disc we use are either isotopic and sometime we have two disjoint overtwisted discs.

This will be obvious except for one configuration: Assume that near (σ_0, τ_0) (with τ_0 fixed) the neighborhood of the sphere with $t = T_-(\sigma_0,\tau_0)$ is equivalent to the one shown in Figure 10. The characteristic foliation on the sphere $S^2\times\{T_-(\sigma_0,\tau_0)\}$ contains a knot which is non-loose in a tubular neighborhood of the sphere. However it is not globally non-loose because there is another overtwisted disc in sphere close to $S^2\times\{T_+(\sigma_0,\tau_0)\}$. Hence one can eliminate the intersection point in the left hand part of Figure 10 in Figure 10 as indicated in the right hand side of Figure 10. This makes use of $T_-(\sigma,\tau_0) < T_+(\sigma,\tau_0)$ for $\sigma \in (0,1)$.

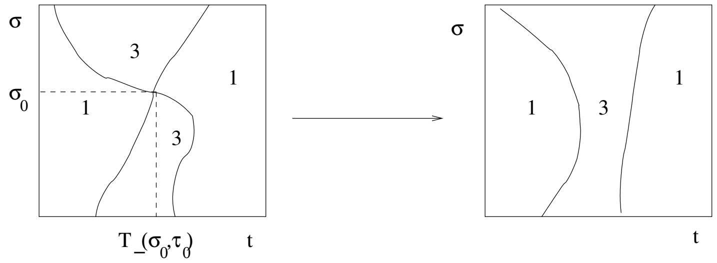

FIGURE 10. Removing a locally (but not globally) non-loose unknot

$\boldsymbol{\tau \in [4/5, 1]}$ **:** We start with $\{\tau = 1\}$. Here, the characteristic foliation on $S^2 \times \{0\}$ is non-generic because for each σ, the characteristic foliation on $S^2 \times \{0\}$ has two simultaneous connections between two saddles (the union of the corresponding pair of stable/unstable leaves is the non-loose unknot $\psi_\sigma(L)$). This is represented by the thickened curve in the right hand part of Figure 11.

We perturb this family of contact structures such that for $\sigma \not\in \{0,1\}$ there no such coincidence of retrogradient saddle-saddle connection by a perturbation of the product structure (this author finds it easier to think in these terms), by Gray's theorem this is equivalent to a perturbation of $\xi_{\sigma,1}$.

In order to carry out this perturbation we on uses a slight generalization of Lemma 2.20: We perturb the spheres $S^2 \times \{0\}$ so that one retrogradient saddle-saddle connections occur for $t = 0$ while the other such connections occurs for $t > 0$ unless $\sigma = 0$ or $\sigma = 1$.

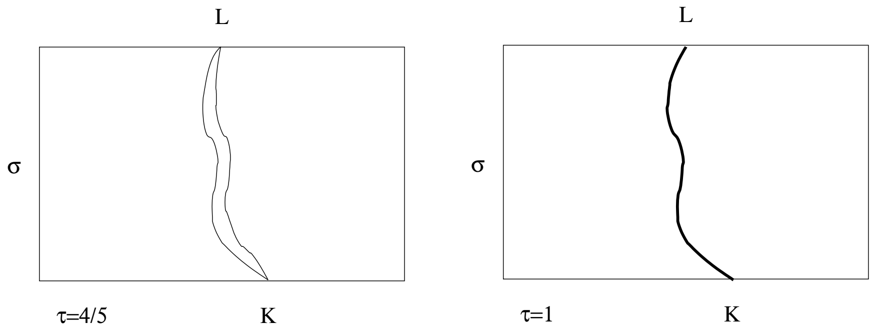

FIGURE 11. The simplification of $\xi_{\sigma,\tau}$ between $\{\tau = 4/5\}$ and $\{\tau = 1\}$

$\boldsymbol{\tau \in [0, 1/5]}$ **:** We apply Eliashberg's theorem to the family of contact structures $\xi_{\sigma,0}$ on $S^2 \times [-1,1]$. As neighborhood of the overtwisted disc we take an open neighborhood of $S^2 \times [-1, T_-(\sigma, 0)]$. The contact structures $\xi_{\sigma,1/5}$ satisfy the following properties:

(i) $\xi_{\sigma,1/5} = \xi_{\sigma,0}$ on a neighborhood of $S^2 \times [-1, T_-(\sigma, 0)]$

(ii) The characteristic foliation of $\xi_{\sigma,1/5}$ on $S^2 \times \{T_+(\sigma, 1/5)\}$ has exactly one retrogradient connection and no other degeneracies. For t slightly smaller respectively bigger than $T_+(\sigma, 1/5)$ the dividing set on $S^2 \times \{t\}$ has one respectively three connected components.

This is shown schematically in Figure 12. The neighborhood of $S^2 \times [-1, T_-(\sigma, 0)]$ is the region to the left from the dashed curve. We ignore what is going on in the shaded area between the dashed and the rightmost almost vertical curve representing the curve $(\sigma, T_+(\sigma, 1/5))$. Using Theorem 2.5 one can construct such a family of contact structures. Moreover, by Lemma 2.6 one can also arrange that $\xi_{\sigma,1/5}$ is homotopic to $\xi_{\sigma,0}$ as a family of plane fields (rel. to $S^2 \times [-1, T_-(\sigma, 0)]$). Then according to Theorem 2.5 there is a family of contact structures $\xi_{\sigma,\tau}, \tau \in [0, 1/5]$, interpolating between $\xi_{\sigma,0}$ and $\xi_{\sigma,\tau}$.

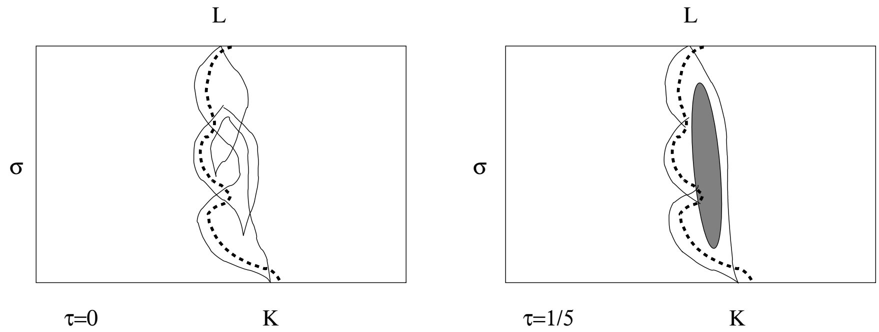

FIGURE 12. The simplification of $\xi_{\sigma,\tau}$ between $\{\tau = 0\}$ and $\{\tau = 1/5\}$

$\boldsymbol{\tau \in [1/5, 2/5]}$ **:** In this step, we merely isotope the contact structures such that

$$T_+(\sigma, 2/5) = T_+(\sigma, 4/5).$$

$\boldsymbol{\tau \in [2/5, 3/5]}$ **:** We apply Eliashberg's theorem to the contact structures $\xi_{\sigma,2/5}$ on $S^2 \times [-1, 1]$. As neighborhood of the overtwisted disc we take and open neighbourhhood of $S^2 \times [T_+(\sigma, 2/5), 1]$. The contact structures $\xi_{\sigma,3/5}$ satisfy the following properties:

(i) $\xi_{\sigma,\tau} = \xi_{\sigma,2/5}$ on a neighborhood of $S^2 \times [T_+(\sigma, 2/5), 1]$ for $\tau \in [2/5, 3/5]$.

(ii) $T_-(\sigma, 3/5) = T_-(\sigma, 4/5)$

(iii) $\xi_{\sigma,3/5} = \xi_{\sigma,4/5}$ on a neighborhood of $S^2 \times [-1, T_-(\sigma, 3/5)]$.

By Theorem 2.5 there is such a family of contact structures. Moreover, by Lemma 2.6 one can also arrange that $\xi_{\sigma,3/5}$ is homotopic to $\xi_{\sigma,2/5}$ as a family of plane fields (rel. to $S^2 \times [T_+(\sigma, 2/5), 1]$, in Figure 13 this corresponds to the region to the right of the thickened dashed curve). Then according to Theorem 2.5 there is a family of contact structures $\xi_{\sigma,\tau}, \tau \in [2/5, 3/5]$, interpolating between $\xi_{\sigma,2/5}$ and $\xi_{\sigma,3/5}$. Again we do control the contact structure in the open set corresponding to the shaded region in Figure 13.

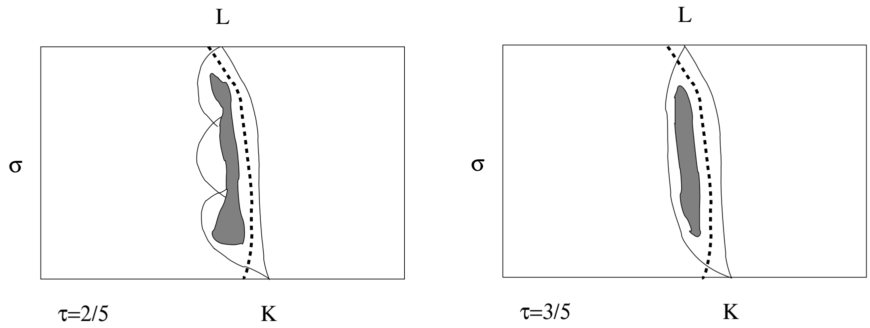

FIGURE 13. The simplification of $\xi_{\sigma,\tau}$ between $\{\tau = 2/5\}$ and $\{\tau = 3/5\}$

$\boldsymbol{\tau \in [3/5, 4/5]}$ **:** We use Eliashberg's theorem (Theorem 2.5) one last time. This time we use an open neighborhood of $S^2 \times [-1, T_-(\sigma, 3/5)]$ where $\xi_{\sigma,3/5} = \xi_{\sigma,4/5}$ already (this has been arranged in the previous step). Because $\mathrm{d}(\psi) = 0$, the family of plane fields $\xi_{\sigma,3/5}$ is homotopic to $\xi_{\sigma,4/5}$ relative to $S^2 \times [-1, T_-(\sigma, 3/5)]$. According to Eliashberg's theorem these two families of contact structures are homotopic through contact structures.

Thus have defined a family of contact structures $\xi_{\sigma,\tau}, \sigma, \tau \in [0, 1]$ on $S^2 \times [-1, 1]$. Gray's theorem yields a family of isotopies $\varphi_{\sigma,\tau}$ which in turn yield the desired family of Legendrian knots. Since we do not keep track of the orientation of the Legendrian knot, this only proves (as claimed) that K is isotopic to L or $\overline{L}$ as Legendrian knot. □

Remark 3.8. For $\tau \in [4/5, 1]$ we perturbed a family of non-loose Legendrian unknots to obtain overtwisted discs. The choice involved is the choice of the retrogradient connection which occurs first in the movie associated to a perturbation of the original family of spheres and the orientation of the knot.

By comparing the perturbation in the proof above with how a stabilization changes the characteristic foliation on a Seifert surface of the knot (see Figure 6) one sees that the overtwisted discs could also have been obtained by stabilizing the non-loose unknot. Here one has to choose an orientation and the sign of the stabilization. The Legendrian unknots obtained in this way are $K_{1,0}, K_{0,1}$ or the same knots with different orientations (cf. Section 2.5).

3.4. **The classification up to isotopy for oriented unknots.** In this section we prove the main result of this article.

Theorem 3.9. *Let K be a minimal non-loose unknot in S^3. Then K and $\overline{K}$ are not isotopic as oriented Legendrian knots.*

Applications including the classification of non-loose unknots with arbitrary (positive) Thurston-Bennequin number can be can be found in Section 4.

In order to prove Theorem 3.9 we will use two results. The first concerns the space of foliations by spheres of $S^2 \times [0,1]$ and is based on Hatcher's theorem [Ha81] on the space of diffeomorphisms of $S^2 \times S^1$. The second result is more technical and uses standard theorems on transversality to establish that certain degenerate configurations of characteristic foliations on leaves foliations by spheres occur on topologically tame subsets of the product of the leaf space with the parameter space.

3.4.1. *Foliations on $S^2 \times S^1$ and spheres containing a non-loose unknot.* We recall Hatcher's theorem on the homotopy type of $\mathrm{Diff}(S^2 \times S^1)$ and apply this result to show that the space of foliations on $S^2 \times [0,1]$ which coincide with the product foliation near the boundary.

Theorem 3.10 (Hatcher, [Ha81])**.** *The map*

$$\begin{aligned} O(2) \times O(3) \times \Omega SO(3) &\longrightarrow \mathrm{Diff}(S^2 \times S^1) \\ (A, B, \gamma_t) &\longmapsto \big((p,\tau) \longmapsto (B \circ \gamma_\tau(p), A\tau)\big) \end{aligned}$$

is a weak homotopy equivalence.

Let $\mathrm{Diff}_\partial(S^2 \times [0,1])$ be the group of diffeomorphisms of $S^2 \times [0,1]$ which coincide with the identity near the boundary and $\mathrm{Diff}_\partial(S^2 \times [0,1], \mathcal{F})$ the subgroup of those diffeomorphisms which preserve the product foliation. It is a corollary of Theorem 3.10 that

$$\mathrm{Diff}(S^2 \times [0,1], \mathcal{F}) \longrightarrow \mathrm{Diff}(S^2 \times [0,1])$$

is a weak homotopy equivalence. Now let $\mathcal{FOL}(S^2 \times [0,1])$ be the space of foliations on $S^2 \times [0,1]$ which coincide with the product foliation near the boundary (we view foliations as plane fields to define the C^k-topology on $\mathcal{FOL}(S^2 \times [0,1])$). The map

$$\begin{aligned} \mathrm{Diff}(S^2 \times [0,1]) &\longrightarrow \mathcal{FOL}(S^2 \times [0,1]) \\ f &\longmapsto \big(f(S^2 \times \{t\})\big)_{t \in [0,1]} \end{aligned}$$

is a Serre fibration. From Theorem 3.10 we therefore obtain the following corollary.

Corollary 3.11. *The space $\mathcal{FOL}(S^2 \times [0,1])$ is weakly contractible.*

3.4.2. *Retrogradient connections on spheres.* The second ingredient is a result on retrogradient connections present in the characteristic foliation on leaves of foliations or families of such foliations by spheres on $(S^2 \times [-1,1], \xi)$. When we consider families of foliations we still can parametrize each leaf space by $[-1,1]$.

Let $\mathcal{F}$ be a foliation from $\mathcal{FOL}(S^2 \times [0,1])$ and assume that the leaf $S_{t_0}(\xi)$ contains a non-degenerate retrogradient connection γ. We parametrize a foliated neighborhood of this leaf by $S^2 \times (t_0 - \varepsilon, t_0 + \varepsilon)$ so that the foliation by the first factor coincides with the given foliation and fix a fiber T of $S^2 \times (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow S^2 = S_{t_0}$.

Next, we will construct 1-parameter families of perturbations $\mathcal{F}_x, x \in (-\delta, \delta)$, with compact support in $S^2 \times (t_0 - \varepsilon, t_0 + \varepsilon) \setminus T$ and $\delta > 0$.

For $p \in \gamma$ choose a compactly supported Legendrian vector field X vanishing near T and outside of $S^2 \times [t_0 - \varepsilon, t_0 + \varepsilon]$ such that X is transverse to the leaves of $\mathcal{F}$ whenever it does not vanish.

Then the flow φ_x of X is well defined and by Lemma 2.20 there is still a retrogradient connection close to γ on nearby spheres $S_{\Gamma(x)}$ intersecting T in $\Gamma(x)$ if $|x| < \delta$ is small enough. Let

$$\begin{aligned} \Gamma : (-\delta, \delta) &\longrightarrow (t_0 - \varepsilon, t_0 + \varepsilon) = T \\ x &\longmapsto \Gamma(x). \end{aligned}$$

The contact property of ξ ensures that t_0 is a regular value of Γ.

We have described the perturbation for a fixed foliation $\mathcal{F}$ but the construction can be carried out in the same way for finite dimensional families of characteristic foliations.

This type of perturbation can be applied simultaneously to different retrogradient connections and because all characteristic foliation have only finitely many singular points there are only finitely many retrogradient connections. Therefore standard transversality theory (e.g. Theorem 2.7 in Chapter 3.2 of [Hi]) implies the following proposition:

Proposition 3.12. *Let $(N = S^2 \times [-1,1], \xi)$ be a contact manifold with convex boundary and $\mathcal{F}_{\sigma,\tau}, \sigma, \tau \in (0,1)$ a 2-parameter family of foliations on N which is constant near ∂N. Assume that the following conditions are satisfied for (t, σ, τ) outside of a compact set in $P = (-1,1) \times (0,1) \times (0,1)$:*

(i) *The set $(t, \sigma, \tau) \in P$ for which there are*
 - *two retrogradient connections and one simply degenerate singular point,*
 - *three retrogradient connections, or*
 - *one retrogradient connection and two degenerate singularities*

 is discrete and there aren't any more degenerate configurations.

(ii) *The submanifolds of P corresponding to non-degenerate retrogradient connections intersect each other transversely.*

(iii) *The collection of points where there is a retrogradient connection and a degenerate singular point intersects the submanifolds of P corresponding to other retrogradient connections transversely.*

After a C^r-small perturbation of $\mathcal{F}_{\sigma,\tau}$ relative to the complement of a small open neighborhood of the compact set one may assume that these conditions are satisfied everywhere in P.

One should note that the fact that there are only finitely many singular points implies only that there are finitely many leaves of the characteristic foliations taking part in retrogradient connections. It does not mean that the space of instances, where a retrogradient connection occurs is compact. For example, assume that $\xi(S_{t_0})$ a closed leaf such that the holonomy is attractive on one side while it is repelling on the other. When a stable leaf of a positive hyperbolic singularity and an unstable leaf of a negative hyperbolic singularity accumulate on the degenerate closed leaf, then the set of leaves of a foliation containing S_{t_0} as a leaf which contain a retrogradient connection is non-compact. By the Poincaré-Bendixson theorem this phenomenon does not occur when the characteristic foliations have no closed leaves, for example when ξ is tight.

We close this section with a preparatory lemma which yields a normalization of the characteristic foliation on a sphere containing a non-loose unknot.

Lemma 3.13. *Let $K_\sigma \subset S^3$ be a family of non-loose Legendrian unknots with $\mathrm{tb}(K_\sigma) = 1$. Then there are two balls $B_0, B_1 \subset S^3$ and a family of foliations $\mathcal{F}_\sigma$ of $S^3 \setminus (B_0 \cup B_1)$ by spheres such that*

- $\mathcal{F}_0 = \mathcal{F}_1$,
- *K_σ is contained in a leaf S_σ of $\mathcal{F}_\sigma$, and*
- *the characteristic foliation on S_σ has exactly two singularities along K_σ.*

Proof. The family of knots $K_\sigma, \sigma \in [0,1]$, misses two small balls B_0, B_1 which we assume to be Darboux balls with convex boundary. We choose a sphere $\mathcal{F}_0$ with the desired properties such that S_0 is a leaf carrying K_0 such that the characteristic foliation is in normal form

By Gray's theorem we can choose a contact isotopy ψ_σ of S^3 with support on the complement of $B_0 \cup B_1$ so that $\psi_\sigma(K_0) = K_\sigma$ and we consider $\mathcal{F}'_\sigma = \psi_\sigma(\mathcal{F}_0)$. This is a family of foliations with all desired properties except that $\mathcal{F}_0 = \mathcal{F}'_0 \neq \mathcal{F}'_1$ in general.

Let $S_1 = \psi_1(S_0)$ and consider one of the discs D_1 in S'_1 which bound K_1. As in the proof of the Roussarie-Thurston normal form [Ro] for surfaces in 3-manifolds carrying a Reebless foliation one can show that this disc is isotopic to one of the discs in S_0 relative to $K_0 = K_1$ in the complement of $B_0 \cup B_1$. This can be done in such a way that the characteristic foliation has two singularities along K_1 throughout the deformation. Doing this for both discs in S'_1 which bound K_1 we apply Corollary 3.11 to obtain the desired family of foliations $\mathcal{F}_\sigma$. $\square$

3.4.3. *Proof that K and $\overline{K}$ are not isotopic.* We will finally prove the main theorem of this paper. It is based on the following idea: We argue by contradiction. From a Legendrian isotopy from K to $\overline{K}$ we construct a 2-parameter family (parametrized by $(\sigma,\tau) \in [0,1]^2$) of foliations $\mathcal{F}_{\sigma,\tau}$ by spheres on $S^2 \times [-1,1]$ (each leaf space is parametrized by $t \in [-1,1]$)

such that the collection $\mathcal{L}$ of parameters $(t, \sigma, \tau) \in [-1,1] \times [0,1]^2$ such that the leaf $S^2 \times \{t\}$ of $\mathcal{F}_{\sigma,\tau}$ contains a minimal non-loose unknot has the following properties:

- $\mathcal{L}$ is a piecewise smooth submanifold of $[-1,1] \times [0,1]^2$ of codimension 2.
- $\mathcal{L}$ is properly embedded in $(-1,1] \times [0,1]^2$ and $\partial([-1,1] \times [0,1]^2)$ contains exactly one boundary point of $\mathcal{L}$.

Since compact 1-manifolds with boundary have an even number of boundary points this is a contradiction.

Proof of Theorem 3.9. Assume that $K_\sigma, \sigma \in [0,1]$, is a family of oriented Legendrian knots in (S^3, ξ) such that $K_0 = K$ and $K_1 = \overline{K}$. This isotopy avoids two points of S^3. In the following we can therefore consider S^3 with two small open balls removed, this space N is of course diffeomorphic to $S^2 \times [-1,1]$.

We will consider the space $(S^2 \times [-1,1]) \times [0,1]^2$ and as in the proof of Theorem 3.7 the product decomposition of N will vary.

- For $\tau = 0$ we choose an identification of N with $S^2 \times [-1,1]$ such that $S = S^2 \times \{0\}$ contains K. We require that the characteristic foliation of ξ on S is in standard form (c.f. Theorem 3.2).
- For $\sigma = 0$ and $\sigma = 1$ we consider the same identification of N with $S^2 \times [-1,1]$.
- When $\tau = 1$ we choose a family of spheres S_σ such that
  - $K_\sigma \subset S_\sigma$, and
  - S_0 and S_1 coincide with the sphere $S^2 \times \{0\}$ from the identifications chosen above.
- Now we extend S_σ to a family of smooth foliations by spheres on N such that S_σ is a leaf on the foliation $N \times \{(\sigma, \tau = 1)\}$. By Lemma 3.13 we may assume that the characteristic foliation of ξ on S_σ has exactly two singular points along K_σ.

This fixes the boundary conditions. The vertical dashed thickened line in Figure 14 corresponds to a constant family of Legendrian knots K(when $\tau = 0$) while the thickened curve on the front face represents the family K_σ.

In the next step we perturb the family of spheres in a particular way from $S^2 \times \partial\left([-1,1] \times [0,1]^2\right)$ to a neighborhood of this set. On each sphere of $S^2 \times [-1,1] \times \{(\sigma,\tau)\}$ with $(\sigma,\tau) \in \partial I^2$ containing the non-loose unknot K_σ, starting with $\tau = 1$ and $\sigma = 1/2$, we consider the retrogradient connection $\gamma_{\sigma,\tau}$ whose orientations coincides with the orientation of the Legendrian knot as one moves along ∂I^2.

Because the isotopy K_σ reverses the orientation of K, the retrogradient connection one obtains after returning to $\tau = 1, \sigma = 1/2$ for the first time is opposite to the one we have started with.

For each point of $\partial[0,1]^2$ we choose a small deformation of the family of spheres as follows:

- For the point $(\sigma = 1/2, t = 1)$ the family of spheres is unchanged.
- For all other points the deformation is constructed as follows. Fix a small disc $D_{\sigma,\tau}$ intersecting the retrogradient connection chosen above (but not the other) such that the characteristic foliation of ξ on $D_{\sigma,\tau}$ has no singular points. To obtain the deformation we push the interior of the small discs slightly into the direction given by the coorientation of the spheres (and extending this deformation to nearby spheres).

  As (σ, τ) approaches $(\sigma = 1/2, \tau = 1)$, the size of the deformation converges to zero (with respect to every C^r-norm, $1 \leq r \in \mathbb{Z}$,) so that we obtain a family of deformations depending smoothly on (σ, τ).

Sufficiently small deformations as above do not introduce new singular points of the characteristic foliation on the deformed spheres and the retrogradient connections which formed the non-loose unknot before the deformation now appear on two different spheres: The retrogradient connection in the part of the spheres which was not deformed occurs later than the retrogradient connection intersecting the small disc used for the construction of the deformation.

We use this family of deformations to extend the family of spheres from ∂I^2 to a neighborhood of ∂I^2 in I^2. By construction the retrogradient connection in the interior of I^2 occur on different spheres except for the constant deformation chosen for $(\sigma = 1/2, \tau = 1)$. These are represented by the thickened dashed lines in Figure 14.

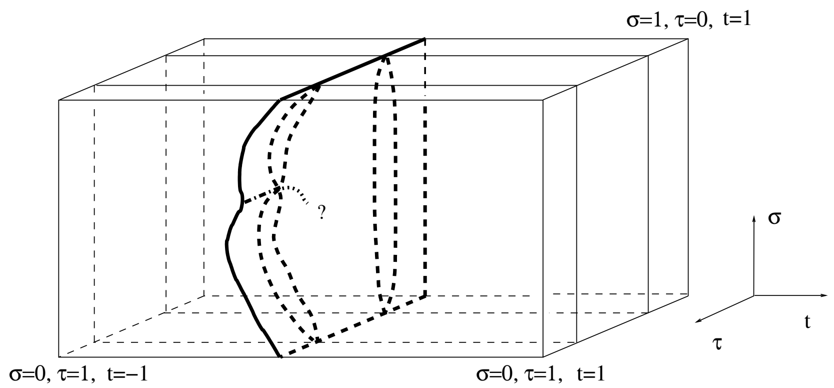

FIGURE 14. Germ of a 1-manifold obtained from an orientation reversing isotopy of K

According to Theorem 3.10 and its consequences we can extend the family of foliations by spheres on N we have constructed for parameter values in a neighborhood of I^2 to the entire parameter space I^2. The spheres are parametrized by three parameters: $t \in [-1, 1]$ and $(\sigma, \tau) \in [0, 1]^2$. Moreover, we can assume that on each of these spheres the characteristic foliation of ξ has only finitely many singular points.

We now consider the space $\mathcal{L}$ of those parameter values (t, σ, τ) for which the corresponding sphere contains a non-loose piecewise smooth Legendrian unknot with minimal Thurston-Bennequin invariant.

Claim: $\mathcal{L}$ is a piecewise smooth properly embedded submanifold of codimension 2 whose boundary is contained in $\partial([-1,1] \times [0,1]^2)$.

Before proving the claim note that it implies the theorem since each submanifold component contributes an even number of boundary points. This contradicts the fact that there is only one boundary point in $[-1,1] \times [0,1]^2$.

For the proof of the claim we may assume by Proposition 3.12 that the points in $(-1,1) \times (0,1)^2$ where the corresponding characteristic foliation has exactly two retrogradient connections is a codimension 2-submanifold. Also, three simultaneous retrogradient connections respectively any configurations which are more degenerate occur at isolated points respectively not at all.

In particular, a generic point in $\mathcal{L}$ has a neighborhood in $(-1,1) \times (0,1)^2$ where $\mathcal{L}$ is a smooth submanifold of codimension 2. Points where $\mathcal{L}$ is not smooth arise at points where three retrogradient connections occur simultaneously. One example of such a configuration is shown schematically (as in Figure 7 on 24) in Figure 15.

The top row depicts schematically a characteristic foliation with three retrogradient connections corresponding to a point $(t_0, \sigma_0, \tau_0) = p \in P$. The union of the solid and the dashed lines in the left part correspond to Γ_- (recall that this is the graph formed by negative singular points and unstable leaves) before the retrogradient connection (before refers to the second factor of $S^2 \times [-1,1]$) while Γ_- after the retrogradient connection is shown in the right part of the top row in Figure 15. The dotted line on the left correspond to the shape of Γ_- after the retrogradient connections occur. In the figure, we denoted the unstable leaves taking part in a retrogradient connection by A, B, C.

All three retrogradient connections are present in all characteristic foliations for parameter values in a small neighborhood U of p and any pair of them corresponds to a submanifold of codimension 2 in U. Some of these configurations correspond to a sphere $S_{t_0} \subset N \times \{(\sigma_0, \tau_0)\}$ such that ξ_{σ_0,τ_0} has a tight neighborhood in N while others don't.

The threefold retrogradient connection in the top part of Figure 15 can be resolved in six different ways to a pair of retrogradient connections occurring simultaneously while the third connection appears before or after the simultaneous connections. In the configuration shown in the top row of Figure 15 exactly two such resolutions correspond to a sphere $S_t \subset N \times \{(\sigma, \tau)\}$ such that $\xi_{\sigma,\tau}$ is tight on a neighborhood of S_t. These configurations are labeled with a filled triangle . The letter and the sign in the six resolutions below indicates which retrogradient connection is fixed in a state it is in before the triple point (indicated by a minus) or after the triple point (indicated by a plus).

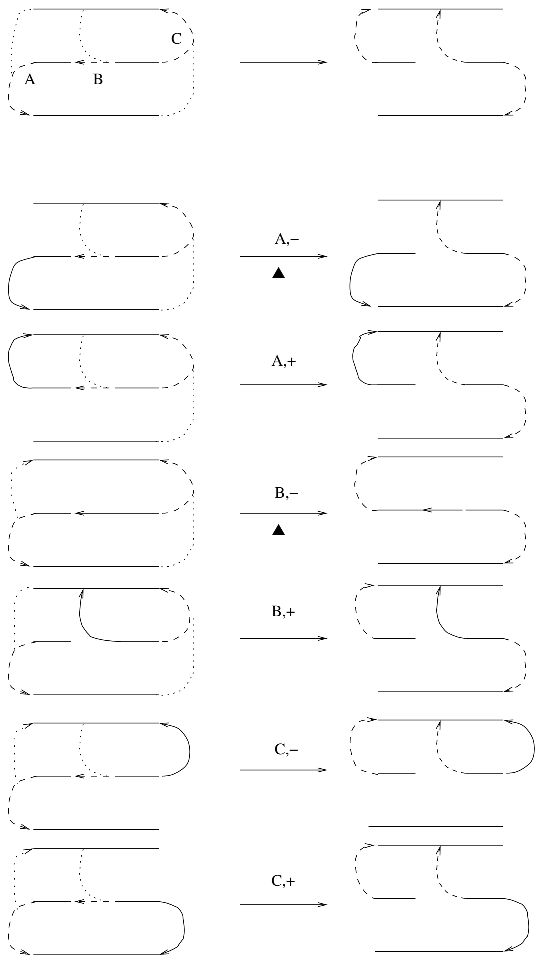

FIGURE 15. Three retrogradient connections - a non-smooth point of $\mathcal{L}$

There are configurations of three retrogradient connections which correspond to smooth points of $\mathcal{L}$, like for example the one shown in Figure 16.

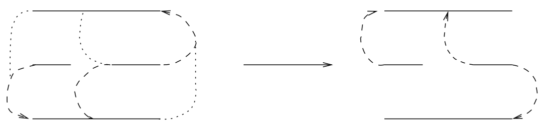

FIGURE 16. Three retrogradient connections - a smooth point of $\mathcal{L}$

All configurations of three retrogradient connections which correspond to spheres in N whose complement is tight lead to a situation as the previous two examples.

In order to finish the proof of the claim it remains to show that $\mathcal{L}$ is a *compact* piecewise smooth submanifold.

Let S_n be a sequence of leaves of $\mathcal{F}_{\sigma_n,\tau_n}$ containing a non-loose Legendrian knot K_n with $\mathrm{tb}(K_n) = 1$. We may assume that

- S_n converges to a sphere S_∞ (we denote the corresponding parameter values by $t_\infty, \sigma_\infty, \tau_\infty$) of $\mathcal{F}_{\sigma_\infty,\tau_\infty}$.
- K_n converges to a union $\mathcal{K}$ of leaves of $\xi(S_\infty)$ which is a closed subset of S_∞.

We require that the limit point $(t_\infty, \sigma_\infty, \tau_\infty)$ lies in the interior of $[-1,1] \times [0,1]^2$.

$\mathcal{K}$ cannot contain a non-degenerate closed leaf since every sphere close to enough to S_∞ contains closed leaf contradicting the fact that K_n is non-loose. For degenerate closed leaves Lemma 2.21 asserts that for $t > t_\infty$ or $t < t_\infty$ the sphere $S(t, \sigma_\infty, \tau_\infty)$ contains an attractive closed leaf. This prevents the existence of non-loose unknots in spheres close to S_∞. Thus we may assume that $\xi(S_\infty)$ has no closed leaf.

If S_∞ contains a cycle (made from stable and unstable leaves of hyperbolic singularities) with one sided holonomy, then after an isotopy of S_∞ which moves S_∞ into its complement the characteristic foliation on the isotoped sphere contains an overtwisted disc. This contradicts the assumption on $(t_\infty, \sigma_\infty, \tau_\infty)$ being the limit set of parameter values corresponding to spheres containing non-loose unknots. Therefore S_∞ does not contain any limit cycles with one sided holonomy. All leaves of $\xi(S_\infty)$ have singular points as limit sets.

By construction, S_∞ contains at least two retrogradient connections and the complement $N \setminus S_\infty$ is tight. Assume that S_∞ contains exactly two retrogradient connections. Then the graph Γ_- consisting of negative singular points and their unstable leaves has three connected components. All these components have to be trees but not all of them are closed subsets of S_∞. If each of these components takes part in at most one retrogradient connection, then the one can eliminate all hyperbolic singularities on S^∞ by an isotopy preserving S_∞.

Then S_∞ has a tight neighborhood and ξ itself would be tight. Therefore there is one component of Γ_- which takes part in both retrogradient connections and the same is true for Γ_+ (consisting of positive singular points and their stable leaves). The union of these two components contains a non-loose Legendrian unknot whose Thurston-Bennequin invariant is one.

Assume now that the limit sphere contains three retrogradient connections. All such configurations correspond to contact structures which are not tight on a neighborhood of S_∞, or the configuration is a smooth/non-smooth point of $\mathcal{L}$ (some examples are depicted in Figure 15 and 16).

Since we have assumed that we are in a generic case, we have discussed all the cases that can occur. Thus $\mathcal{L}$ is a piecewise smooth submanifold of codimension 2 in $[-1,1] \times [0,1]^2$ which is also compact and properly embedded. $\square$

The argument used in this proof also proves the following, very similar, result. Let K be a non-loose unknot in (S^3, ξ_{-1}) with some orientation. Then we denote the unknot obtained as positive stabilization of K by Δ, K with reversed orientation is $\overline{K}$ and the positive stabilization of $\overline{K}$ is $\overline{\Delta}$. Note that $\Delta, \overline{\Delta}$ are unknots with vanishing Thurston-Bennequin invariant and $\mathrm{rot}(\Delta) = \mathrm{rot}(\overline{\Delta}) = 1$.

Theorem 3.14. *$\overline{\Delta}$ is not isotopic to Δ.*

Proof. Assume that there is an isotopy φ_σ of S^3 which moves Δ to $\overline{\Delta}$. As in in the previous proof $\varphi_\sigma(\Delta), \sigma \in [0,1]$, misses two balls which we assume to be so small that they are contained in Darboux domains and have convex boundary. The complement of these balls is $N \simeq S^2 \times [-1,1]$. We reconsider the setting of the proof of Theorem 3.9. On $N \times [0,1]^2$ we consider the restriction of ξ_{-1} to N on all $N \times \{\sigma, \tau\}$. We fix a family of foliations $\mathcal{F}_{\sigma,\tau}, \sigma, \tau \in [0,1]$ by spheres on N so that

- $\mathcal{F}_{\sigma,\tau}$ is constant near ∂N,
- a sphere of $\mathcal{F}_{\sigma=0,\tau=1}$ contains Δ,
- $\mathcal{F}_{\sigma,1} = \varphi_\sigma(\mathcal{F}_{0,1})$,
- $\mathcal{F}_{0,\tau}, \mathcal{F}_{1,\tau}$ is independent of τ,
- on $N \times \{\sigma, \tau = 0\}$ we pick a family of foliations interpolating between $\mathcal{F}_{\sigma=0,0}$ and $\mathcal{F}_{\sigma=1,0}$ (already fixed) as follows: a leaf of $\mathcal{F}_{1/2,0}$ contains a non-loose unknot K. To obtain $\mathcal{F}_{\sigma,0}$ for $\sigma \leq 1/2$ and $\sigma \geq 1/2$ first deform $\mathcal{F}_{1/2,0}$ to make overtwisted discs appear which are isotopic to Δ respectively $\overline{\Delta}$ for $\sigma < 1/2$ respectively $\sigma > 1/2$ and use the use the isotopy to extend the given family of foliations to a family of foliations for $(\sigma,\tau) \in \partial([0,1]^2)$ such that precisely one leaf of one foliation contains a non-loose unknot (namely K on a leaf of $\mathcal{F}_{1/2,0}$).

As in the proof of Theorem 3.9 one shows that such a configuration is not possible contradicting the assumption that Δ is isotopic to $\overline{\Delta}$. $\square$

The above theorem implies that the identity component of $\mathrm{Diff}_+(S^3, \xi_{-1})$ does not act transitively on the set of boundaries of overtwisted discs. We will continue to study related phenomena in the next section.

4. APPLICATIONS

As an application of Theorem 3.9 we give the proof of a statement of Y. Chekanov. The proof of that theorem also requires further results on the action of the contactomorphism group on boundaries of overtwisted discs, these will be discussed first.

4.1. **The action of contactomorphisms on the set of boundaries of overtwisted discs.**

It is an easy corollary of Eliashberg's classification result that the group of all contactomorphisms of an overtwisted contact structure ξ acts transitively on the set $\mathcal{U}_0(\xi)$ of Legendrian unknots with vanishing

Thurston-Bennequin invariant and rotation number one. In this section we will show that the connected component of the identity of $\mathrm{Diff}_+(S^3,\xi_{-1})$ does act transitively on $\mathcal{U}_0(\xi_k), k\neq -1$. We have seen above that this is not the case when $k=-1$.

For this we use a couple of lemmas which will be used in the second application, too.

Lemma 4.1. *Let (M,ξ) be a contact manifold and $\Delta\in\mathcal{U}_0$. Then there is a Lutz tube along a transverse unknot with self-linking number -1 containing Δ such that the Lutz tube is contained in a ball whose boundary is convex and has a tight neighborhood. If the π-Lutz twist is undone, then the contact structure becomes tight on the ball.*

π-Lutz twists as in this lemma will be called simple.

Proof. First of all, note that one can choose a convex disc D bounding Δ such the dividing set on D is connected. In order to see this consider any convex disc D bounding Δ, fix an overtwisted disc D' in the complement of D and construct a contact structure with the desired properties using Theorem 2.5. For this one uses assumption that $\mathrm{rot}(\Delta)=1$.

Then consider a plane field ζ on M which is homotopic to ξ satisfying the following conditions.

- $\zeta=\xi$ near Δ.
- $\zeta=\xi$ near D'.
- ζ is a contact structure on a closed ball B^3 such that
  - B^3 is disjoint from D' and contains D in its interior,
  - the boundary of B^3 is convex with respect to ζ and the dividing set on ∂B^3 is connected, and
  - $\zeta\big|_{B^3}$ is obtained from the tight contact structure determined by $\zeta(\partial B^3)$ by a single π-Lutz twist along a transverse unknot and D is one of the obvious overtwisted discs obtained from a π-Lutz twist.
- ζ is homotopic to ξ as a plane field.

According to Theorem 2.5 ζ is homotopic to a contact structure and ζ is isotopic to ξ relative to a neighborhood of Δ (one uses the overtwisted disc D' for the application of Theorem 2.5), so we can assume that ζ itself is a contact structure. The ball with the desired properties is obtained from B^3 and the contact structure ζ using the fact that ξ and ζ are isotopic contact structures. □

The last lemma allows us to establish a contact topological (rather than homotopy theoretic) criterion, which distinguishes ξ_{-1} from those positive overtwisted contact structures on S^3 which are not diffeomorphic to ξ_{-1}.

Proposition 4.2. *Let ξ be an overtwisted contact structure on S^3. Then ξ is isotopic to ξ_{-1} if and only if there is a closed ball $B^3\subset M$*

- *which contains a simple Lutz tube, and*

- *∂B^3 has a tight neighborhood*

such that the complement $S^3 \setminus B^3$ is tight.

Proof. First, note that according to the description of ξ_{-1} outlined in Example 2.35 this contact structure is obtained from the tight contact structure on S^3 by a single Lutz twist along a transverse unknot with self-lining number -1. Thus there is a ball containing that Lutz twist whose complement is tight.

Conversely, if there is ball with tight complement and convex boundary who contains a simple Lutz twist in its interior, then undoing the Lutz twist on B^3 we obtain a tight contact structure on B^3. By Colin's gluing theorem [Co97] undoing the Lutz twist yields a tight contact structure on S^3. In other words, ξ is obtained from the tight contact structure on S^3 by a single π Lutz twist. Then ξ is homotopic to (and hence isotopic) to ξ_{-1}. $\square$

This has the following consequence.

Theorem 4.3. *Let ξ be an overtwisted contact structure on S^3 which is not isotopic to ξ_{-1}. Then for every pair $\Delta_1, \Delta_2 \in \mathcal{U}_0(\xi)$ there is an overtwisted disc Δ which is disjoint from $\Delta_1 \cup \Delta_2$.*

Proof. According to Lemma 4.1 we may assume that Δ_1 is obtained by a π-Lutz twist along a transverse unknot in a tight ball B^3. By Proposition 4.2, the complement of this ball is overtwisted.

We will attempt to isotope Δ_2 without moving Δ_1 so that the result lies inside a small neighborhood V of B^3 such that $\overline{V} \setminus \mathring{B}^3$ is tight. The process described below either works or we find an overtwisted disc in the complement of $\Delta_1 \cup \Delta_2$. Because the complement of V is overtwisted by [Co97] the proves the desired result.

Let D, D' be discs bounding Δ_2 such that $D \cup D'$ is an embedded sphere. Without loss of generality we assume that Δ_2 and $S = D \cup D'$ are transverse to ∂B^3. Let $x, x' \in \partial B^3 \cap \Delta_2$ be two intersection points such that there is an arc $\delta_1 \subset \Delta_2 \cap (S^3 \setminus B^3)$ whose interior of δ_2 does not meet ∂B^3. Together, δ_2 and an arc from $S \cap \partial B^3$ bound a disc in S whose interior does not contain segments of Δ_2.

Pick a Legendrian arc δ_2' connecting x and x' in a tight neighborhood of S such that $\delta_2 \cap \delta_2'$ is a Legendrian unknot bounded by $\widehat{D}$ in the complement of B^3 which is disjoint from other pieces of Δ_2.

We may assume that $\widehat{D}$ is convex since we may stabilize δ_2'. Either the dividing set on $\widehat{D}$ contains a closed component or not. In the first case one finds an overtwisted disc in the complement of $\Delta_1 \cup \Delta_2$, in the second case $\widehat{D}$ has a tight neighborhood and by Proposition 2.7 we can push the segment δ_2 of Δ_2 into a tight neighborhood of ∂B^3.

After finitely many steps we either found an overtwisted disc in the complement of $\Delta_1 \cup \Delta_2$ or we have isotoped Δ_2 into a neighborhood of B^3. In the latter case we are done. In the first case, note that if the complement of the neighborhood of ∂B^3 containing the isotoped knot Δ_2 is a tight ball, then the contact structure is isotopic to ξ_{-1} contrary to our assumption. $\square$

This theorem has a corollary using Theorem 2.25.

Corollary 4.4. *Let (S^3, ξ) be an overtwisted contact structure which is not isotopic to ξ_{-1}, then for every pair Δ_1, Δ_2 of Legendrian unknots with rotation number one and vanishing Thurston-Bennequin-invariant, there is a contact isotopy moving Δ_1 to Δ_2.*

For oriented knots the same is true provided that the rotation numbers of the two knots coincide.

This is contrasted by Theorem 3.14 for the contact structure ξ_{-1} on S^3.

4.2. **Chekanov's theorem on $\pi_0(\mathrm{Diff}_+(S^3, \xi))$ for overtwisted contact structures ξ.** The following theorem is stated in Remark 4.15 of [ElF98] where it is attributed to Y. Chekanov without an indication of a proof.

Theorem 4.5 (Chekanov)**.** *Let ξ be an overtwisted oriented contact structure on S^3 and $\mathrm{Diff}_+(S^3, \xi)$ the diffeomorphisms of S^3 which preserve ξ and its orientation. Then*

$$\pi_0\left(\mathrm{Diff}_+(S^3, \xi)\right) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \textit{if } \xi \simeq \xi_{-1} \\ \mathbb{Z}_2 & \textit{otherwise.} \end{cases}$$

Proof. According to [Dy01] the homomorphism

$$\mathrm{d} : \mathrm{Diff}_+(S^3, \xi) \longrightarrow \mathbb{Z}_2$$

which was described at the end of Section 2.6 is onto for all overtwisted contact structures on S^3. We first assume $\xi \simeq \xi_{-1}$. Let $K \subset S^3$ be a non-loose Legendrian unknot with $\mathrm{tb}(K) = 1$. According to Theorem 3.7 and Theorem 3.9 there is a well defined group homomorphism

$$\begin{aligned} &\kappa : \mathrm{Diff}_+(S^3, \xi) \longrightarrow \mathbb{Z}_2 \\ &\psi \longmapsto \begin{cases} 0 & \text{if } \psi(K) \text{ is Legendrian isotopic to } K \\ 1 & \text{if } \psi(K) \text{ is Legendrian isotopic to } \overline{K}. \end{cases} \end{aligned} \tag{12}$$

By Lemma 3.6, there is an orientation preserving contact diffeomorphism ψ such that $\mathrm{d}(\psi) \neq 0$ which maps K to itself but reverses its orientation. Moreover, when one applies the proof of surjectivity of d to the ball around one of the overtwisted discs shown in Figure 3 (as in indicated in Remark 2.36), then the resulting contact diffeomorphisms has non-trivial Dymara invariant and it preserves the overtwisted disc up to isotopy. As we have shown above a contact diffeomorphism of ξ' which reverses to orientation of the non-loose unknot interchanges (up to contact isotopy) the two overtwisted discs in Figure 3. This is true because the positive stabilization of the standard non-loose Legendrian unknot yields one of the overtwisted discs in Figure 3 while the negative stabilization (i.e. the positive stabilization when the orientation of K is reversed) yields the other overtwisted disc. Since ψ preserves one of the two overtwisted discs in Figure 3 and these two overtwisted discs are not isotopic $\psi(K)$ is isotopic to K as oriented knot. Thus (d, κ) is a surjective homomorphism

$$(\mathrm{d}, \kappa) : \mathrm{Diff}_+(S^3, \xi_{-1}) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

It remains to show that all orientation preserving contact diffeomorphisms in the kernel of this map are isotopic to the identity inside of $\mathrm{Diff}_+(S^3, \xi)$. In order to show this we apply the proof of Theorem 3.7 with a few modifications.

First of all, we may assume that ψ preserves two Darboux balls pointwise. We denote the complement of their interior by $N \simeq S^2 \times [-1, 1]$. Now the restriction map

$$\pi_1\left(\mathrm{Diff}_+(D^3, \xi_{Darboux})\right) \longrightarrow \pi_1\left(\mathrm{Diff}_+(S^2)\right) \simeq \pi_1\left(\mathrm{SO}(3)\right) = \mathbb{Z}_2$$

induced by the restriction map is surjective. From the Smale conjecture (see item 8 in the Appendix of [Ha83]) it follows that there is family of diffeomorphisms $\psi_\sigma, \sigma \in [0, 1]$, of N (such that $\psi_\sigma = \mathrm{id}$ on a neighborhood of ∂N) with $\psi = \psi_1$. We may assume in addition that $\psi_1(K) = K$ pointwise.

We now use the proof of Theorem 3.7 to find a family of diffeomorphisms φ_σ with $\varphi_0 = \varphi_1 = \mathrm{id}$ so that $\varphi_\sigma \circ \psi_\sigma$ is a contact diffeomorphism. The overall strategy is the same but slightly simpler, and one applies the modification performed for $\tau \in [4/5, 1]$ in the proof above twice: Once to the constant family of Legendrian knots in $\{\tau = 0\}$ and once more to the family $\psi_\sigma(K)$ in $\{\tau = 1\}$.

The hypothesis that $\mathrm{d}(\psi_1) = 0$ ensures that the 2-parameter family of plane fields/contact structures used in the proof of Theorem 3.7 exists, the fact that $\kappa(\psi) = 0$ implies that one does not need to introduce double points as in shown in the front part (close to $\{\tau = 1\}$ in Figure 14) when one separates the simultaneous retrogradient connections by a perturbation of the sphere carrying the knot.

This finishes the proof of the theorem in the case when $\xi \simeq \xi_{-1}$. Finally, we consider the case $\xi \not\simeq \xi_{-1}$.

Let ψ be a contact diffeomorphism of (S^3, ξ) which preserves the orientation of ξ and $\mathrm{d}(\psi) = 0$ and choose an overtwisted disc B. By Theorem 4.3 the complement of $\partial B \cup \psi(\partial B)$ is overtwisted. Therefore we may assume that ψ preserves an overtwisted disc and since $\mathrm{d}(\psi)$ is trivial we can apply Theorem 2.5 to obtain an isotopy connecting ψ to the identity. $\square$

The following consequence is immediate. We denote connected component of the space of contact structure on S^3 which are isotopic to a particular contact structure ξ by $\mathrm{Cont}(S^3, \xi)$.

Corollary 4.6. *Let ξ be an overtwisted contact structure on S^3. Then*

$$\pi_1\left(\mathrm{Cont}(S^3), \xi\right) \simeq \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } \xi \simeq \xi_{-1} \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

Proof. This follows from the long exact sequence of the fibration

$$\mathrm{Diff}_+(\mathrm{S}^3, \xi) \longrightarrow \mathrm{Diff}(S^3, \mathrm{or}) \longrightarrow \mathrm{Cont}(\mathrm{S}^3, \xi)$$

and the fact that one can choose ξ to be invariant under rotations along around one particular complex plane in $\mathbb{C}^2 \supset S^3$. $\square$

As final application of the classification of minimal non-loose unknots and of Theorem 4.5 we complete the classification of non-loose unknots in S^3. We already know the coarse classification: If $K \subset (S^3, \xi)$ is a non-loose unknot, then

$$\mathrm{tb}(K) = n > 0 \qquad\qquad \mathrm{rot}(K) = \pm(n-1)$$

and there is an orientation preserving contact diffeomorphism mapping K to $K_{-n,1}$ respectively $K_{n,-1}$ if $\mathrm{rot}(K) = n-1$ respectively $\mathrm{rot}(K) = -(n-1)$. We will consider the first case. We know two examples of knots with the same classical invariants as K, namely $K_{-n,1}$ and $K_{-1,n}$.

We have furthermore explained (Figure 5) that after $n-1$ negative stabilizations of $K_{1,-n}$ respectively $K_{-n,1}$ we end up with $L_{1,-1}$ respectively $K_{-1,1}$. These two knots are not Legendrian isotopic by Theorem 3.9. Therefore $K_{1,-n}$ is not isotopic to $K_{-1,n}$.

Theorem 4.7. *Let $K \subset S^3$ be a non-loose Legendrian unknot with $\mathrm{tb}(K) = n$ and $\mathrm{rot}(K) = n-1$. Then K is isotopic to either $K_{-n,1}$ or $K_{1,-n}$.*

Proof. By Chekanov's theorem it is sufficient to show that if $\mathrm{d}(\psi) = 1$ and $\kappa(\psi) = 0$, then $\psi(K_{-n,1})$ is isotopic to $K_{-n,1}$.

We may assume that ψ preserves $K_{-1,1}$ together with a tubular neighborhood N_0 of $K_{-1,1}$ pointwise. The claim follows if we show that $K_{-n,1}$ is isotopic to a knot contained in N_0.

Now $K_{-n,1}$ can be obtained from $K_{-1,1}$ by a $(n-1)$-fold band connected sum with $K_{-1,0}$. The $(n-1)$ copies of $K_{-1,0}$ are unlinked copies of the boundary of an overtwisted disc (as in Figure 17).

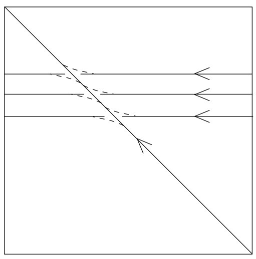

FIGURE 17. Obtaining $K_{-4,1}$ from $K_{-1,1}$ and three copies of $L_{-1,0}$

This collection of $n-1$ unknots is Legendrian isotopic (in the complement of $K_{-1,1}$ and relative to the region where the band connected sum is formed) to a Legendrian link contained in N_0. But this is clear since a negative stabilization of $K_{-1,1}$ (the stabilization is contained N_0) is isotopic to $K_{-1,0}$. We have seen in Figure 5 that this isotopy can be chosen so that it does not move $K_{-1,1}$ nor the region were the band connected sum is performed. $\square$

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MATHEMATISCHES INSTITUT, LUDWIG-MAXIMILIANS-UNIVERSITÄT, THERESIENSTR. 39, 80333 MÜNCHEN, GERMANY

E-mail address: tvogel@math.lmu.de