Existence of Engel structures

L'existence de structures d'Engel

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Abstract

Every 4–manifold with trivial tangent bundle admits an Engel structure. *To cite this article: Thomas Vogel, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Toute variété de dimension 4 dont le fibré tangent est trivial admet une structure d'Engel. *Pour citer cet article : Thomas Vogel, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Version française abrégée

Une structure d'Engel est un champs des plans différentiable $\mathcal{D}$ sur une variété de dimension 4 qui satisfait les conditions

$$\mathrm{rang}[\mathcal{D},\mathcal{D}] = 3 \qquad\qquad \mathrm{rang}[\mathcal{D},[\mathcal{D},\mathcal{D}]] = 4\ .$$

Ici on note par $[\mathcal{D},\mathcal{D}]$ l'ensemble des vecteurs tangents de M qu'on peut obtenir comme commutateur $[X,Y]$ pour des sections locales X,Y de $\mathcal{D}$. Les structures d'Engel sont stables dans le sens de la théorie des singularités. Il y a peu de types des distributions qui sont stables dans ce sens. Ce sont les champs des lignes, les structures de contact sur les variétés de dimension impaire, les structures de contact paires sur les variétés de dimension paire et les structures d'Engel en dimension 4, [6]. En particulier, les structures d'Engel sont une peculiarité de la dimension 4. Ces faits sont une motivation pour l'étude des structures d'Engel.

A chaque structure d'Engel $\mathcal{D}$ sur une variété M on peut associer des distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM \tag{1}$$

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où $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ est une structure de contact paire et $\mathcal{W}$ est un champs de lignes associé à $\mathcal{E}$. Le feuilletage induit est tangent à $\mathcal{D}$ et il est appelé le *feuilletage caractéristique* de $\mathcal{D}$. Il ne dépend que de $\mathcal{E}$. Sur une hypersurface H qui est transverse à $\mathcal{W}$ on a une structure de contact $TH \cap \mathcal{E}$ et un champs de lignes $TH \cap \mathcal{D}$ tangentes à la structure de contact sur H. Ces lignes sont appelées *lignes d'intersection.* Si M est orientée on obtient une orientation de $\mathcal{W}$ et de $TH \cap \mathcal{E}$. Dans ce cas, une orientation de $\mathcal{D}$ induit une orientation des lignes d'intersection sur H. En utilisant (1) on obtient la proposition suivante.

Proposition 0.1 *Si M est une variété orientable qui admet une structure d'Engel orientable, le fibré tangent de M est trivial.*

Dans [5] cette proposition est attribuée à V. Gershkovich. Le résultat principal de [8] annoncé dans cette note est la réciproque de la Proposition 0.1.

Théorème 0.2 *Une variété de dimension quatre dont le fibré tangent est trivial admet une structure d'Engel orientable.*

La preuve du Théorème 0.2 utilise la décomposition d'une variété en anses rondes. Une anse ronde R_k de dimension n et d'indice $k \in \{0, \ldots, n-1\}$ est definie par $R_k = D^k \times D^{n-k-1} \times S^1$. On attache R_k à une variété M avec bord en utilisant un plongement de $\partial_- R_k = \partial D^k \times D^{n-k-1} \times S^1$ dans ∂M.

Soit M une variété fermée de dimension 4 dont le fibré tangent est trivial. On fixe une décomposition en anses rondes de M. Une telle décomposition existe grâce au théorème suivant.

Théorème 0.3 (Asimov [1]) *Si M est une variété fermée et connexe de dimension $n \neq 3$, il y a une décomposition de M en anses rondes si et seulement si la caractéristique d'Euler de M est nulle.*

Sur les anses rondes on fixe des structures d'Engel orientables qui serviront comme modèles. On choisit les modèles tels que leurs feuilletages caractéristiques soient transverse à $\partial_- R_k$ et $\partial_+ R_k = \partial R_k \setminus \partial_- R_k$. Les feuilletages caractéstiques sont orientées tels qu'ils sortent de R_k par $\partial_+ R_k$ et ils y rentrent par $\partial_- R_k$.

Soit M' une variété munie d'une structure d'Engel dont le feuilletage caractéristique est transverse au bord et sort le long de ∂M. On fixe une des structures d'Engel modèles sur R_k et un plongement $\varphi : \partial_- R_k \longrightarrow \partial M$. Si φ préserve les structures de contact avec leurs orientations et les lignes d'intersection orientées on obtient une structure d'Engel différentiable sur $M' \cup_\varphi R_k$.

Pour la preuve du Théorème 0.2 on attache les anses rondes dans la décomposition de M en anses rondes l'une après l'autre. On montre qu'à chaque étape on peut modifier les fonctions de recollement des anses rondes et qu'il est possible de choisir une structure d'Engel sur l'anse ronde telle que les conditions sur la fonction de recollement soient satisfaites. Une difficulté importante est de construire un ensemble assez grand des structures d'Engel modèles sur les anses rondes.

Une démonstration detaillée du Théorème 0.2 sera publiée ultérieurement.

1. Introduction

An Engel structure is a smooth maximally non–integrable plane field on a 4–manifold. Engel structures are stable under C^2–small perturbations. Around every point of a manifold with an Engel structure $\mathcal{D}$, there are coordinates x, y, z, w such that $\mathcal{D}$ is the intersection of the kernels of $dz - x\,dy$ and $dx - w\,dy$. This normal form for Engel structures is due to F. Engel [3]. Engel structures arise for example as generic germs of distributions of planes at a point in $\mathbb{R}^4$. In particular Engel structures are stable in the sense of singularity theory.

There are only a few types of distributions with this stability property. In dimension n the stable germs of distributions arise for distributions of rank $k = 1$ or $k = n-1$ for arbitrary n or if $k = 2$ and $n = 4$, c.f. [6]. The case $k = 1$ corresponds to foliations of rank 1 while the case $k = n-1$ is realized by contact structures if n is odd and by even contact structures if n is even. If $k = 2$ and $n = 4$, the stable distribution germ is an Engel structure. Even among the distributions with stable germs, Engel structures appear to be special due to their exceptional appearance in dimension 4. This motivates the study of Engel structures.

As we explain in Section 2, an Engel structure $\mathcal{D}$ on a 4–manifold M induces a flag of distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM \qquad (2)$$

where each distribution has corank 1 in the distribution to its right. The distribution $\mathcal{E}$ is an even contact structure and it carries a distinguished orientation. The line field $\mathcal{W}$ depends only on $\mathcal{E}$. An orientation of $\mathcal{W}$ induces an orientation of TM and vice versa. From this one obtains the following proposition.

Proposition 1.1 *An orientable 4–manifold which admits an orientable Engel structure has trivial tangent bundle.*

This proposition can be found in [5], where it is attributed to V. Gershkovich. Up to now one can find only few examples of closed Engel manifolds in the literature (cf. [6,4]). The main result of [8] announced in this note is the converse of Proposition 1.1.

Theorem 1.2 *Every 4–manifold M with trivial tangent bundle admits an orientable Engel structure.*

In Section 3 we discuss Theorem 1.2 and another result from [8]. Detailed proofs will be published elsewhere.

2. Properties of Engel structures

In this section we give the definition of Engel structures and we explain some properties which will be used in the discussion of our results in Section 3. When $\mathcal{D}, \mathcal{D}'$ are distributions on M, then $[\mathcal{D}, \mathcal{D}']$ at $p \in M$ consists of those tangent vectors of M which can be obtained by evaluation of the commutator $[X, X']$ of local sections X of $\mathcal{D}$ and X' of $\mathcal{D}'$ at p.

Definition 2.1 *A plane field $\mathcal{D}$ on a 4–manifold M is an* Engel structure *if*

$$\mathrm{rank}[\mathcal{D}, \mathcal{D}] = 3 \qquad \textit{and} \qquad \mathrm{rank}[\mathcal{D}, [\mathcal{D}, \mathcal{D}]] = 4 \ .$$

In our discussion we shall also encounter contact structures and even contact structures. Let us recall the definitions.

Definition 2.2 *A distribution $\mathcal{C}$ of hyperplanes on a manifold of odd dimension $2k+1$ is a* contact structure *if it is locally defined by a 1–form α with the property $\alpha \wedge d\alpha^k \neq 0$.*

A field $\mathcal{E}$ of hyperplanes on a manifold of even dimension $2k$ is an even contact structure *if it is locally defined by a 1–form α such that the restriction of $d\alpha$ to $\mathcal{E} = \ker(\alpha)$ has maximal rank.*

The definitions of contact structures and of even contact structures are very similar. Still there is a significant difference. If α is a defining form for a contact structure, then the restriction of $d\alpha$ to $\ker(\alpha)$ is non–degenerate by definition. On the other hand if α defines an even contact structure, then the 2–form $d\alpha$ has even rank while the rank of $\mathcal{E}$ is odd. Thus the restriction of $d\alpha$ to $\mathcal{E}$ has a kernel of dimension one since the rank of the restriction of $d\alpha$ to $\mathcal{E}$ is assumed to be maximal. This kernel is independent of the choice of α. The resulting foliation $\mathcal{W}$ of rank one is called the *characteristic foliation* of $\mathcal{E}$. From the definition of $\mathcal{W}$ it follows that the local flow of every vector field tangent to $\mathcal{W}$ preserves $\mathcal{E}$, i.e. $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$.

Notice that a hyperplane distribution $\mathcal{E}$ on a 4–dimensional manifold M is an even contact structure if and only if $[\mathcal{E}, \mathcal{E}] = TM$. In particular if $\mathcal{D}$ is an Engel structure, then $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ is an even contact structure and one can associate the characteristic foliation $\mathcal{W}$ of $\mathcal{E}$ to $\mathcal{D}$. In order to establish the existence of the flag of distributions (2) it remains to show $\mathcal{W} \subset \mathcal{D}$.

Lemma 2.3 *Let $\mathcal{D}$ be an Engel structure. The characteristic foliation $\mathcal{W}$ of $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is tangent to $\mathcal{D}$.*

Proof : Assume that $\mathcal{W}_p$ is not contained in $\mathcal{D}_p$ at $p \in M$. Let α be a local defining form for $\mathcal{E}$ on a neighbourhood of p. If X, Y are two linearly independent local sections of $\mathcal{D}$ around p, then $d\alpha(X_p, Y_p) \neq 0$

by the assumption on $\mathcal{W}$. On the other hand $d\alpha(X,Y) = -\alpha([X,Y]) = 0$ since $[X,Y]$ is a local section of $\mathcal{E} = [\mathcal{D},\mathcal{D}]$. Thus the assumption $\mathcal{W}_p \not\subset \mathcal{D}_p$ leads to a contradiction. $\square$

Let $\mathcal{D}$ be an Engel structure on M and H a hypersurface transversal to the characteristic foliation $\mathcal{W}$. Then $TH \cap \mathcal{E}$ is a distribution of rank 2. Since H is transversal to the characteristic foliation $TH \cap \mathcal{E}$ is a contact structure. It follows from transversality and Lemma 2.3 that $TH \cap \mathcal{D}$ is a Legendrian line field. We refer to $TH \cap \mathcal{D}$ as the *intersection line field* on H. An orientation of $\mathcal{W}$ and $\mathcal{D}$ induces an orientation of the intersection line field.

Contact structures in dimension 3 induce an orientation of the underlying manifold. Therefore an orientation of $\mathcal{W}$ induces an orientation of M using the contact orientation on hypersurfaces transversal to $\mathcal{W}$. If X, Y is a local frame for $\mathcal{D}$, then $X, Y, [X,Y]$ is a local frame for $\mathcal{E}$ and the orientation defined by this frame is independent of the choice of X, Y. Proposition 1.1 follows immediately from these observations and (2).

Let H, H' be hypersurfaces transversal to $\mathcal{W}$ and assume that $p \in H$ lies on the same leaf of $\mathcal{W}$ as $p' \in H'$. Then the holonomy of $\mathcal{W}$ induces a diffeomorphism between neighbourhoods of p in H respectively p' in H' which preserves contact structures since every flow tangent to $\mathcal{W}$ preserves the even contact structure. This allows us to give the following geometric interpretation of the condition $[\mathcal{D},\mathcal{D}] = \mathcal{E}$.

Suppose that the hypersurface H is transversal to $\mathcal{W}$ and let W be a vector field tangent to $\mathcal{W}$ which does not vanish along H. Consider the image H_t of H under the local flow φ_t of W at time t. Then the image of the intersection line field on H_t under φ_{-t} is a Legendrian line field on H which rotates without stopping as t increases.

A classical construction of Engel structures is called prolongation. It is described in the following example.

Example 1 Let N be a 3–dimensional manifold with contact structure $\mathcal{C}$. Consider the projectivization $\mathbb{P}\mathcal{C}$ of $\mathcal{C}$ consisting of 1–dimensional subspaces of $\mathcal{C}$. Thus $\mathbb{P}\mathcal{C}$ is a fiber bundle over M with fiber $\mathbb{RP}^1$. We denote the bundle projection by pr *and Legendrian lines by λ. The distribution on $\mathbb{P}\mathcal{C}$*

$$\mathcal{D}_{\mathcal{C}} = \left\{ v \in T_\lambda \mathbb{P}\mathcal{C} \,\middle|\, \mathrm{pr}_*(v) \in \lambda \text{ for } \lambda \in \mathbb{P}\mathcal{C} \right\}$$

is an Engel structure. The induced even contact structure is $\mathrm{pr}_*^{-1}(\mathcal{C})$ and the characteristic foliation of $\mathcal{D}_{\mathcal{C}}$ corresponds to the fibers of $\mathbb{P}\mathcal{C}$.

Another construction due to H. J. Geiges [4] yields an Engel structure on the mapping torus of a diffeomorphisms of a 3–manifold if the mapping torus has trivial tangent bundle.

3. Discussion of the main results

Here we make only a few remarks concerning the proof of Theorem 1.2. Detailed proofs of the results presented in this note will be published elsewhere.

For open 4–manifolds with trivial tangent bundle one can use Gromov's h–principle for open differential relations which are invariant under the action of the group of diffeomorphisms of M, c.f. [2]. This yields Engel structures on open 4–manifolds with trivial tangent bundle. Therefore we assume that M is 4–dimensional, closed, connected with trivial tangent bundle.

A round handle of dimension n and index $k \in \{0, \dots, n-1\}$ is defined to be $R_k = D^k \times D^{n-k-1} \times S^1$. Round handles are attached to manifolds with boundary using embeddings of $\partial_- R_k = \partial D^k \times D^{n-k-1} \times S^1$ into ∂M. We say that M admits a round handle decomposition if M can be obtained from the disjoint union of several round handles of index 0 by successively attaching round handles.

Theorem 3.1 (Asimov [1]) *A closed connected manifold of dimension $n \neq 3$ admits a decomposition into round handles if and only if its Euler characteristic vanishes.*

The analogous statement in dimension 3 is wrong [7]. One can assume that the round handles are ordered according to their index.

For the proof of Theorem 1.2 we fix model Engel structures on round handles of dimension 4 such that the characteristic foliation is transversal to $\partial_- R_k$ and $\partial_+ R_k = \partial R_k \setminus \partial_- R_k$. The characteristic foliations are oriented such that they point outwards along $\partial_+ R_k$ and inwards along $\partial_- R_k$ for $k = 0, 1, 2, 3$.

Since M has trivial tangent bundle, the Euler characteristic of M vanishes. By Theorem 3.1 we can fix a round handle decomposition of M. We attach the round handles successively. Suppose that we have constructed an Engel structure on $M' \subset M$ with oriented characteristic foliation such that the characteristic foliation is transversal to the boundary of M' and points outwards.

Let φ be an attaching map for R_k and equip R_k with a model Engel structure. If φ preserves oriented contact structures and oriented intersection line fields, then R_k can be attached to M' such that the model Engel structure on R_k extends the Engel structure on M' to a smooth oriented Engel structure on $M' \cup_\varphi R_k$. The characteristic foliation of the resulting Engel structure is oriented and transversal to the boundary of $M' \cup_\varphi R_k$.

Given an arbitrary attaching map φ for R_k we isotope φ such that the condition on the attaching map we have explained above is satisfied for a suitable choice of the model Engel structure on R_k. In this process we also modify the Engel structure on M' slightly. One of the main difficulties of the proof is to construct sufficiently many model Engel structures on round handles.

We finish this note with the discussion of another theorem from [8].

Theorem 3.2 *Let $(M_1, \mathcal{D}_1)$ and $(M_2, \mathcal{D}_2)$ be Engel manifolds such that the characteristic foliations of both Engel structures have closed transversals H_1 respectively H_2. Then $M_1 \# M_2 \# (S^2 \times S^2)$ admits an Engel structure $\mathcal{D}$ which coincides with $\mathcal{D}_1$ and $\mathcal{D}_2$ away from tubular neighbourhoods of H_1 and H_2. The characteristic foliation of $\mathcal{D}$ also admits a closed transversal.*

In order to prove Theorem 3.2 we cut M_1 along H_1 and M_2 along H_2. Then we attach two round handles of index 1 and 2 with model Engel structures such that we obtain a closed manifold with an Engel structure. Then we show that the resulting manifold is diffeomorphic to $M_1 \# M_2 \# (S^2 \times S^2)$.

Notice that in Theorem 3.2 we make no orientability assumption and that the construction can be iterated. The condition that the characteristic foliation of $\mathcal{D}_i, i = 1, 2$ admits a closed transversal can be replaced by an assumption concerning the number of full twists of $\mathcal{D}_i$ around a leaf of $\mathcal{W}_i$ in $\mathcal{E}_i$, c.f. the interpretation of the condition $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ in Section 2.

When M_1 and M_2 are parallelizable, then so is $M_1 \# M_2 \# (S^2 \times S^2)$. In this situation Theorem 1.2 guarantees the existence of an Engel structure on $M_1 \# M_2 \# (S^2 \times S^2)$. The advantage of Theorem 3.2 is that we obtain an Engel structure which is closely related to the original Engel structures on M_1 and M_2.

If $\mathcal{C}$ is a contact structure on a 3–manifold which is trivial as a bundle, then the Engel structure $\mathcal{D}_C$ on $\mathbb{P}\mathcal{C}$ from Example 1 satisfies the hypothesis in Theorem 3.2. For example if $\mathcal{C}$ is the standard contact structure on S^3, then the application of Theorem 3.2 yields an Engel structure on

$$M_k = (k+1)(S^3 \times S^1) \# k(S^2 \times S^2)$$

for all $k \geq 1$. One can prove that M_k is not the total space of a fibration over S^1 and that M_k is not the projectivization of a subbundle of rank two of the tangent bundle of an orientable 3–manifold. In particular it is impossible to apply the construction of Geiges (c.f. [4]) or the prolongation construction (c.f. Example 1) to obtain an Engel structure on M_k.

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