LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

WiSe 2018/19

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Symplectic geometry

Exercise sheet 6

Exercise 1. Let $h : A \longrightarrow A$ be a homeomorphism of a region $A \subset \mathbb{R}^2$. Assume that p is an isolated fixed point of h in the interior of A. The index $\mathrm{ind}_h(p)$ is defined as the index of a simple closed loop around p so that p is the only fixed point lying in of the closed disc bounded by the loop.

a) Let $A = \mathbb{R}^2$ and $h(u, v) = (\lambda u, \mu v)$ with $\lambda\mu \neq 0$. Compute $\mathrm{ind}_h(0)$ in terms of the signs of (λ, μ).

b) Assume that $\widetilde{h} : \widetilde{A} \longrightarrow \widetilde{A}$ is the lift of a homeomorphism h to the universal cover $\widetilde{A}$ of A **where A is an annulus** and assume that all fixed points of $\widetilde{h}$ are isolated and lie in the interior of $\widetilde{A}$. Let $p_1, \ldots, p_k$ be representatives of all classes of fixed points of $\widetilde{h}$ (if there are any). Prove that

$$\sum_{i=1}^{k} \mathrm{ind}_{\widetilde{h}}(p_i) = 0.$$

c) Let h be an area preserving twist map of the annulus A with finitely many fixed points. Prove that h has a fixed point with negative index.

Exercise 2. Let r, φ be polar coordinates on the annulus $A = \{0 < a^2 \leq r \leq b^2\}$ and

$$h(\varphi, r) = (\varphi + r^2, r).$$

Prove that h is an area preserving map and find a generating function.

Exercise 3. a) Let ω be symplectic form and α a closed 1-form. Show that there is a unique vector field X_α so that $\omega(X_\alpha, \cdot) = \alpha$. Prove that the flow of X_α preserves ω and α.

b) Now assume that $\omega = d\lambda$ is exact. Show that there is a unique vector field Y (the Liouville vector field) so that $i_Y \omega = \lambda$. Compare ω and $\phi_t^* \omega$ where ϕ_t is the time-t-flow of Y.

c) Let $I \subset (M^{2n}, \omega = d\lambda)$ be a submanifold which is tangent to the Liouville vector field Y and $p \in I$ a point so that for every compact set $K \subset I$ and $\varepsilon > 0$ there is t_K so that $\phi_{t_K}(K) \subset B_\varepsilon(p) \cap L$. Show that I is isotropic, in particular its dimension is $\leq n$.

Exercise 4. Compute the Liouville vector field on $\mathbb{R}^{2n}$ for the 1-forms

$$\alpha_k = -\sum_{i=1}^{n-k} \left(\frac{1}{2} q_i dp_i - \frac{1}{2} p_i dq_i \right) - \sum_{i=n-k+1}^{n} (+2 q_i dp_i + p_i dq_i)$$

with $k \in \{0, \ldots, n\}$. Compare the Liouville vector fields L_k with the gradient vector fields of the functions (Morse functions)

$$f_k = \frac{1}{4} \sum_{i=1}^{n-k} (q_i^2 + p_i^2) + \sum_{i=n-k+1}^{n} (q_i^2 - p_i^2/2).$$

Try to exhibit I_k with the properties as in exercise 3.

Hand in on Wednesday November, 28 during the exercise class.