LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

WiSe 2018/19

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Symplectic geometry

Exercise sheet 5

Exercise 1. a) Let f be a symplectomorphism of (M, ω) and H a Hamiltonian function. Compute the Hamiltonian vector field of f^*H in terms of f and the Hamiltonian vector field of H.

b) For ϕ_t, ψ_t be Hamiltonian diffeomorphism generated by the Hamiltonian functions F_t, G_t. Show that

$$\frac{d}{dt}(\phi_t \circ \psi_t) = X_{F_t} + \phi_{t*} X_{G_t}.$$

c) Show that if f is Hamiltonian, then the same is true for f^{-1}.

Exercise 2. Let f_t, g_t be Hamiltonian flows generated by time independent normalized Hamiltonian functions. Show that $f_t \circ g_t = g_t \circ f_t$ if and only if $\{F, G\} = 0$.

Exercise 3. Let $h : A \longrightarrow A = \{a^2 \leq u^2 + v^2 \leq b^2\}$ be an area preserving homeomorphism which satisfies the *monotone twist condition*, i.e. if a lift $\widetilde{h} = (f, g)$ of h to the universal cover $\widetilde{A} = \{a \leq y \leq b\}$ (as in the lecture) satisfies

$$f(x, a) < x \text{ and } f(x, b) > x \text{ for all } x, \text{ and}$$
$$y < y' \Rightarrow f(x, y) < f(x, y')$$

a) Show that there is a 2π-periodic function $w : \mathbb{R} \longrightarrow \mathbb{R}$ so that $f(x, w(x)) = x$.

b) Conclude that the curve $\widetilde{\Gamma} = \{(x, w(x)) \,|\, x \in \mathbb{R}\}$ projects to a closed curve Γ in A.

c) Consider $\Gamma \cap h(\Gamma)$ to show that $\widetilde{\Gamma}$ and $\widetilde{h}(\Gamma)$ intersect in at least two points which project to different points in A.

d) Prove that h has at least two fixed points.

Exercise 4. Prove that the Maslov index of a smooth closed embedded loop $\gamma : [0, L] \longrightarrow \mathbb{R}^2$ is ± 2 (i.e. $\gamma(0) = \gamma(L)$ and $\dot{\gamma}(0) = \dot{\gamma}(L)$).

Proceed as follows: Pick a straight line in $\mathbb{R}^2$ tangent to γ so that the image of γ lies in one half space bounded by the straight line. Assume that $\gamma(0) = \gamma(L)$ is on the line, that the straight line is the x-axis and that $\dot{\gamma}(0)/ = \|\dot{\gamma}(0)\|\partial_x$.

Consider the following map $d : [0, L] \times [0, L] \longrightarrow \Lambda_1$ to compute the Maslov index of γ

$$d(s,t) = \begin{cases} \dot{\gamma}(t) & \text{if } s = t \notin \{0, L\} \\ \partial_x & \text{if } (s,t) \in \{(0,0), (L,L), (L,0), (0,L)\} \\ \gamma(s) - \gamma(t) & \text{otherwise.} \end{cases}$$

Hand in on Wednesday November, 21 during the exercise class.