LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN | MATHEMATISCHES INSTITUT

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Symplectic geometry

Exercise sheet 3

Exercise 1. Let $V = \mathbb{C}^n \simeq \mathbb{R}^{2n}$ be the standard symplectic vector space (with standard complex /symplectic/Hermitian/Euclidean structure). Recall that $\langle\cdot,\cdot\rangle = g(\cdot,\cdot) - i\omega(\cdot,\cdot)$ where $\langle\cdot,\cdot\rangle$ is the standard Hermitian product.

Show that $L \subset \mathbb{C}^n$ is Lagrangian if and only if $L^{\perp_g} = iL$. Assume that $e_1, \ldots, e_n$ is an orthonormal basis of the real subspace $L \subset \mathbb{C}^n$. Prove that $(e_i)_i$ is a unitary basis of $\mathbb{C}^n$. Conversely, show that the real space spanned by a unitary basis of $\mathbb{C}^n$ is Lagrangian.

Let $i\varphi : \mathbb{R}^n \longrightarrow i\mathbb{R}^n$ be linear. Prove that the graph $\{x + i\varphi(x) \mid x \in \mathbb{R}^n\}$ of φ is Lagrangian if and only if φ is symmetric.

Exercise 2. Use the results of the previous exercise to show that the graph of a function $F : \mathbb{R}^n \longrightarrow i\mathbb{R}^n$ is Lagrangian in $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ if and only if there is a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ so that $F = i\nabla f$.

Exercise 3. We use the statements of exercise 1 to study the space/set of Lagrangian subspaces of $\mathbb{C}^n$

$$\Lambda_n = \{L \subset \mathbb{C}^n \mid P \text{ is a Lagrangian subspace}\}.$$

We will need to understand a few properties of Λ_n later.

a) Prove that $\mathrm{U}(n)$ acts transitively on Λ_n and that the stabilizer of $L = \mathbb{R}^n$ in $\mathrm{U}(n)$ is $\mathrm{O}(n)$.

This implies that the map

$$\begin{aligned} \mathrm{U}(n)/\mathrm{O}(n) &\longrightarrow \Lambda_n \\ [A] &\longmapsto A \cdot \mathbb{R}^n \end{aligned}$$

is a well defined bijection which we use to equip the set Λ_n with the structure of a topological space/smooth manifold.

b) Compute the dimension of Λ_n.

Exercise 4. Let $P_x(X) = X^{n+1} + x_1 X^{n-1} + \ldots x_{n-1} X$ be a polynomial with real coefficients $x = (x_1, \ldots, x_{n-1})$. Consider the map

$$\begin{aligned} f : \mathbb{R}^{n-1} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, a) &\longmapsto P_x(a). \end{aligned}$$

a) Show that $V = \left\{(x, a) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid \frac{\partial f}{\partial a} = 0\right\}$ is a submanifold of $\mathbb{R}^{n-1} \times \mathbb{R}$.

b) Give the explicit form of the Lagrangian immersion obtained by reduction from f.

c) For $P_x(X) = X^4 + x_1 X^2 + x_2 X$ one gets

$$V = \{(x_1, x_2, a) \in \mathbb{R}^2 \times \mathbb{R} \mid f_x(a) = 4a^3 + 2x_1 a + x_2 = 0\}$$

Recall that the polynomial $f_x(X)$ has degenerate real solutions if and only if the discriminant

$$D(x_1, x_2) = \frac{x_1^3}{27} + \frac{x_2^2}{8}$$

vanishes, and that f_x has exactly one real solution if and only if $D(x_1, x_2)$ is positive. Use these facts to sketch V in a neighborhood of the origin.

d) Find a function $H : V \longrightarrow \mathbb{R}$ so that $dH = \lambda|_V$. Try to sketch the wave front of V, i.e. the projection of the set

$$\big\{\big(x_1, x_2, a^2, a, z = H(x_1, x_2, a)\big) \mid (x_1, x_2, a) \in V\big\}$$

to the (x_1, x_2, z)-space (may be for one fixed value $x_1 < 0$ and one fixed value $x_1 > 0$.

Hand in on Wednesday November, 7 during the exercise class.