WiSe 2018/19

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# Symplectic geometry

## Exercise sheet 2

**Exercise 1.** Let $\omega$ be an area form on $S^2$ and consider $S^2 \times S^2$ with the symplectic forms

$$\Omega_0 = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$$
$$\Omega_1 = \frac{1}{2}\mathrm{pr}_1^*\omega + 2\mathrm{pr}_2^*\omega$$

where $\mathrm{pr}_i$, $i = 1, 2$, is the projection on the $i$-th factor of $S^2 \times S^2$. Show that there is no diffeomorphism $\phi$ of $S^2 \times S^2$ such that $\phi^*\Omega_1 = \Omega_0$ although the volumes of $(M, \Omega_0^2)$ and $(M, \Omega_1^2)$ are equal.

Is there a symplectic structure on $S^2 \times S^2$ so that $S^2 \times \{p\}$ is a Lagrangian submanifold for some $p \in S^2$?

**Exercise 2.** Let $(M^{2n+1}, \xi)$ be a contact manifold. Show that if $n$ is odd, there is a canonical orientation of $M$ so that $\alpha \wedge (d\alpha)^n$ is positive for every local defining form $\alpha$ of $\xi$.

Moreover, show that the hyperplane field $\xi$ admits a canonical orientation if $n$ is even.

**Exercise 3.** Consider $S^{2n-1} \subset \mathbb{C}^n$ and show that

$$\alpha = \sum_{i=1}^{n} (x_i dy_i - y_i dx_i)$$

defines a contact structure on $S^{2n-1}$.

**Exercise 4.** a) Prove that around every point $p$ in a contact manifold $(M^{2n+1}, \xi)$ there is a local coordinate system $z, x_1, y_1, \ldots, x_n, y_n$ defined on a neighborhood $U$ of $p$ so that

$$\xi|_U = \ker\left(dz - \sum_i x_i dy_i\right).$$

b) Let $\xi_t$ be a cooriented smooth family of contact structures on the closed manifold $M$. Show that there is a family of diffeomorphisms $\psi_t : M \longrightarrow M$ so that $\psi_{t*}(\xi_0) = \xi_t$. For this, pick a family of contact forms $\alpha_t$ defining $\xi_t$ and find an isotopy of $M$ so that $\psi_t^*\alpha_t = f_t\alpha_0$ for a family of functions $f_t$. (The coorientation assumption is unnecessary, and is made for convenience only.)

Hand in on Thursday October, 31 during the exercise class.