WiSe 2018/19

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Symplectic geometry

Exercise sheet 12

Exercise 1. Consider $S^2 \subset \mathbb{R}^3$ with the symplectic form $\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$ and the function

$$H(x_1, x_2, x_3) = f(x_3)$$

so that $f(x_3)$ is strictly increasing. Show that the Hamiltonian vector field of H has a non-trivial periodic orbit of period 1 if $f(1) - f(-1)$ is sufficiently large.

Exercise 2. Assume that $s > 1/2$ and $x \in H^s(S^1, \mathbb{R})$. Show that the Fourier series (which converges in L^2)

$$x(t) = \sum_{k \in \mathbb{Z}} e^{2\pi i k t} x_k, x_k \in \mathbb{R}$$

converges uniformly and conclude that x is continuous and that the embedding $H^{1/2}(S^1, \mathbb{R}) \hookrightarrow C^0(S^1, \mathbb{R})$ is continuous.

Exercise 3. Let M^m, N^n be disjoint closed oriented submanifolds of $\mathbb{R}^{k+1}$. The linking map is

$$\begin{aligned} \lambda : M \times N &\longrightarrow S^k \\ (x, y) &\longmapsto \frac{y - x}{\|y - x\|}. \end{aligned}$$

If $m + n = k$, then the degree of λ is the linking number $\mathrm{lk}(M, N)$.

a) Prove that $\mathrm{lk}(M, N) = (-1)^{(m+1)(n+1)} \mathrm{lk}(N, M)$.

b) Show that if there is a submanifold with boundary $L \subset \mathbb{R}^{k+1}$ which is compact, oriented, bounds $M = \partial L$ as oriented manifold and is disjoint from N, then

$$\mathrm{lk}(M, N) = 0.$$

Exercise 4. Let $M = S^1 \subset R^2 \subset \mathbb{R}^3$ oriented as the boundary of D^2 and N the closed curve $\partial(\{0\} \times [0, k] \times [-k, k]), k > 1$ oriented so that the orientation of the piece lying on the x_3 axes is oriented upwards.

Compute $\mathrm{lk}(M, N)$.

Hand in on Wednesday January 23 during the exercise class.