WiSe 2018/19

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Symplectic geometry

Exercise sheet 11

Exercise 1. Let U be bounded, open, nonempty and $W \subset \mathbb{R}^{2n}$ a subspace of codimension 2 and $\omega = \sum dy_i \wedge dx_i$ the standard symplectic form. Then

$$c(U+W) = \infty \text{ if } W^{\perp_\omega} \text{ is isotropic}$$
$$0 < c(U+W) < \infty \text{ if } W^{\perp_\omega} \text{ is not isotropic}$$

Exercise 2. For $n > 1$ let $0 < r_1 \leq \ldots \leq r_n$ and

$$E(r_1, \ldots, r_n) = \left\{ \frac{x_1^2 + y_1^2}{r_1^2} + \ldots + \frac{x_n^2 + y_n^2}{r_n^2} \leq 1 \right\} \subset (\mathbb{R}^{2n}, \omega_0)$$
$$E'(r_1, \ldots, r_n) = \left\{ \frac{x_1^2 + x_2^2}{r_1^2} + \ldots + \frac{y_{n-1}^2 + y_n^2}{r_n^2} \leq 1 \right\} \subset (\mathbb{R}^{2n}, \omega_0).$$

Determine $c(E(r_1, \ldots, r_n))$ and $c(E'(r_1, \ldots, r_n))$ where c is a symplectic capacity.

Exercise 3. Let $Q(r) = (0, r)^{2n} \subset (\mathbb{R}^{2n}, \omega_0)$ with $r > 0$ and define

$$\gamma(M, \omega) := \sup \left\{ r^2 \mid \text{there is a sympl. embedding } \varphi : Q(r) \longrightarrow M. \right\}$$

Show that γ is monotone, conformal and non-trivial in the sense that

$$\gamma(Z(1), \omega_0) < \infty \text{ and } 0 < \gamma(B(1), \omega_0).$$

Exercise 4. Assume that $h : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is a homeomorphism so that $\gamma(h(U)) = \gamma(U)$ for all open sets U. We want to show that h preserves the Lebesgue measure μ on $\mathbb{R}^{2n}$.

a) Justify $\mu(Q(r)) = \gamma(Q(r))^n$ and $\mu(U) \geq (\gamma(U))^n$ for all open set $U \subset \mathbb{R}^{2n}$.

b) Let Q be an open cube whose edges are parallel to the coordinates axes of $\mathbb{R}^{2n}$. Show that $\mu(Q) \leq \mu(h(Q))$.

c) Apply the previous result to prove that $\mu(U) \leq \mu(h(U))$ for all open sets U (use that the Lebesgue measure is inner regular).

d) Use the fact that h is a homeomorphism to finish the proof.

Hand in on Wednesday January 16 during the exercise class.