WiSe 2018/19

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Symplectic geometry

Exercise sheet 1

Exercise 1. Let (V,ω) be a symplectic vector space and $U \subset V$ a subspace. Show that $\left(U^{\perp_\omega}\right)^{\perp_\omega} = U$ and that there is a symplectic form $\hat{\omega}$ on $U/\left(U \cap U^{\perp_\omega}\right)$ such that $\hat{\omega}(\mathrm{pr}(X),\mathrm{pr}(Y)) = \omega(X,Y)$ for all $X,Y \in U$ where $\mathrm{pr} : U \longrightarrow U/\left(U \cap U^{\perp_\omega}\right)$ is the quotient map.

Exercise 2.

a) Which de Rham-cohomology classes of $S^2 \times S^2$, T^4 are represented by symplectic forms?

b) Let $H^2(M;\mathbb{R})$ be the second de Rham cohomology group of the compact Riemannian manifold M. This is equipped with the norm

$$\|[\omega]\| = \inf\{\|\eta\|_{C^1} \mid \eta \in [\omega]\}.$$

Show that the set of classes in $H^2(M;\mathbb{R})$ which are represented by symplectic forms is open. (Recall that every cohomology class has a harmonic representative.)

Exercise 3. The subject of this exercise is an example of a symplectic manifold which does not admit a Kähler structure. It was found by W. Thurston in 1976.

Consider $G = \mathbb{Z}^4$ (as a set) with the following, non-commutative group structure:

$$a * b = a + L_a b \text{ with } L_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) Show that $L_a L_b = L_{a+b} = L_{a*b}$ and show that $*$ is associative. Verify, that $a^{-1} = -L_{-a}a$.

b) Prove that $\rho_a(x) = a + L_a x$ defines a group action of G on $\mathbb{R}^4$ and verify, that $\|\rho_a(x) - x\| \geq 1$ for $a \neq 0$. Therefore, $N = \mathbb{R}^4/G$ is a smooth manifold.

c) Verify $\rho_a^*\omega = \omega$ for the symplectic structure $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Thus N admits a symplectic structure ω_N such that $\mathrm{pr}^*\omega_N = \omega$ (here $\mathrm{pr} : \mathbb{R}^4 \longrightarrow N$ is the quotient map.)

d) Show that $G/[G,G]$ is isomorphic to $(\mathbb{Z}^3,+)$. This implies that N does not admit a Kähler structure. Recall that $[G,G]$ denotes the smallest subgroup of G containing all commutators $\{ghg^{-1}h^{-1} \mid g,h \in G\}$.

Exercise 4. In this exercise, we study $\mathrm{Sl}(2,\mathbb{R}) = \mathrm{Sp}(2)$ (following a note bayJoa Weber).

a) Show that a symmetric 2×2-matrix A is positive definite if and only if $\det(A) > 0$ and $\mathrm{trace}(A) > 0$.

b) Let $B \in \mathrm{Sl}(2,\mathbb{R})$. Show that B has real eigenvalues if and only if $\mathrm{trace}(B) \geq 2$.

c) Consider the polar decomposition $B = SR$ with

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \qquad R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

where S is symmetric and positive definite while R is orthogonal. Express s_{22} in terms of the other matrix entries and show that

$$\mathcal{S} : (0,\infty) \times \mathbb{R}/(2\pi\mathbb{Z}) \cup (0,0) \longrightarrow \mathcal{M} = \mathbb{R}^+ \times \mathbb{R}$$
$$(\tau,\sigma) \longmapsto (s_{11}(\tau,\sigma) = \cosh(\tau) + \sinh(\tau)\cos(\sigma), s_{12}(\tau,\sigma) = \sinh(\tau)\sin(\sigma))$$

is a homeomorphism, and the restriction to $(0,\infty) \times \mathbb{R}/(2\pi\mathbb{Z})$ is a diffeomorphism onto $\mathcal{M} \setminus \{(1,0)\}$.

Reparametrizing the $\mathbb{R}^+$-factor of $\mathcal{M}$ we obtain an identification of $\mathrm{Sl}(2,\mathbb{R})$ and the open solid torus

$$\mathrm{Sl}(2,\mathbb{R}) \longrightarrow S^1 \times D^2$$
$$(B = SR) \longmapsto (\alpha, r = \tanh^2(\tau(s_{11}, s_{12})), \sigma(s_{11}, s_{12})).$$

d) Determine which points in $S^1 \times D^2$ correspond to matrices B with $|\mathrm{trace}(B)| = 2$. Show that away from $\pm E$ this set is a smooth surface and that

$$C_\pm = \{B \in \mathrm{Sl}(2,\mathbb{R}) \mid \pm\det(B - E) > 0\}$$

are simply connected. Make a sketch.

Hand in on Thursday October, 25 during the lecture.