Ludwig-Maximilians-Universität München
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Lecture in the winter term 2018/19

Symplectic Geometry

Please note: These notes summarize the content of the lecture. Many details and examples are omitted. Sometimes, but not always, I provide a reference for proofs, examples or further reading. I will not attempt to give the first reference where a theorem appeared. Some proofs might take two lectures although they appear in a single lecture in these notes. Changes to this script are made without further notice at unpredictable times. If you find any typos or errors, please let me know.

1. Lecture on October 15 – Definition, basics, linear algebra

- **Definition:** Let M^{2n} be a smooth $2n$-manifold. A *symplectic form* on M is a smooth 2-form ω which is closed ($d\omega = 0$) and non-degenerate, i.e. for all $p \in M$ and $0 \neq X \in T_pM$ there is $Y \in T_pM$ so that $\omega(X, Y) \neq 0$.
- **Examples:** Area forms on surfaces are symplectic forms. If (M_1, ω_1) and (M, ω_2) are symplectic, then $\mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$ is symplectic.
- **Example:** The standard symplectic structure on $\mathbb{R}^{2n}$ is
$$\omega = \sum_i dx_i \wedge dy_i.$$
- **Fact:** ω represents a deRham cohomology class. If you want to know what that means see [Jä] or [BT].
- **Definition:** Let V be a real vector space of dimension $2n$. A *symplectic form* on V is a non-degenerate 2-form $\omega \in \Lambda^2 V^*$.
- If (M, ω) is a symplectic manifold, then (T_pM, ω_p) is a symplectic vector space for all $p \in M$.
- **Example:** Let U a real vector space. Then $U \oplus U^*$ carries a natural symplectic structure:
$$\omega((v, \alpha), (w, \beta)) = \alpha(w) - \beta(v).$$
- **Definition:** Let (V, ω) be a symplectic vector space and U a subspace. Then
$$U^{\perp_\omega} = \{X \in V \mid \omega(X, Y) = 0 \text{ for all } Y \in U\}.$$
- **Definition:** Let (V, ω) be a symplectic vector space and $U \subset V$ a subspace. Then U is
$$\begin{aligned} \text{isotropic} &\iff \omega|_U = 0, \text{ i.e. } U \subset U^{\perp_\omega}. \\ \text{coisotropic} &\iff U^{\perp_\omega} \subset U. \\ \text{symplectic} &\iff U \cap U^{\perp_\omega} = \{0\}, \text{ i.e. } \omega_U \text{ is symplectic} \\ \text{Lagrangian} &\iff U = U^{\perp_\omega}. \end{aligned}$$
- If (V, ω) is a symplectic vector space, then
$$\begin{aligned} V &\longrightarrow V^* \\ X &\longmapsto (Y \longmapsto \omega(X, Y)) \end{aligned}$$
is an isomorphism. Hence, $\dim(U) + \dim(U^{\perp_\omega}) = \dim(V)$ for $U \subset V$ a subspace. In particular, $\dim(L) = \dim(V)/2$ if $L \subset V$ is Lagrangian.

- **Definition:** Let (V,ω) be a symplectic vector space. A symplectic basis is a basis $e_1,\dots,e_n,f_1,\dots,f_n$ such that
$$\omega(e_i,e_j)=\omega(f_i,f_j)=0 \text{ and } \omega(e_i,f_j)=\delta_{ij}.$$
- **Lemma:** Every symplectic vector space admits a symplectic basis.
- Relative versions of this statement hold: for example: Let $e_1,\dots,e_k$ be a basis of an isotropic subspace of V, then this basis extends to a symplectic basis of V.
- **Corollary:** A 2-form on a $2n$-vector space is non-degenerate if and only if $\omega\wedge\ldots\wedge\omega=\omega^n$ is non-vanishing.
- **Corollary:** If (M,ω) is a closed symplectic manifold, then $\omega^k\neq 0\in H^{2k}_{dR}(M)$ for all $0\le k\le n$.
- This is an immediate consequence of Stokes theorem. In particular, symplectic manifolds have a canonical volume form and orientation.
- **Example:** Symplectic structure on cotangent bundles:
  Let M be a smooth manifold. Then define the tautological 1-form λ_{st} on T^*M using the projection $\mathrm{pr}:T^*M\longrightarrow M$ via
$$\lambda_{st}(v)=\alpha(\mathrm{pr}_*v) \text{ for } v\in T_\alpha T^*M.$$
  Then $\omega_{st}:=d\lambda_{st}$ is symplectic: $d\omega_{st}=0$ follows from $d^2=0$, non-degeneracy follows from a computation in local coordinates induced by coordinates on M.

2. Lecture on October, 18 – Compatible (almost) complex structures

- **Definition:** Let V be a real vector space. A linear map $J:V\longrightarrow V$ is a *complex structure* if $J^2=-\mathrm{Id}_V$. A complex structure is compatible with a symplectic form ω if and only if $d_J(X,Y)=\omega(X,JY)$ is a (positive definite) Euclidean metric.
- **Lemma:** Let (V,ω) be symplectic. There is a continuous map
$$\{g \text{ Euclidian sturcture on } V\}\longrightarrow\{J \text{ complex structure compatible with } \omega\}$$
  so that g_J maps to J for every compatible complex structure J.
- **Proof:** For a Euclidean metric g choose an endomorphism A_g of V so that $\omega(X,Y)=g(A_gX,Y)$. Then A_g is antisymmetric. Let P be the unique positive definite matrix so that $P^2=AA^T$ and set $J_g=P^{-1}A_g$.
- **Corollary:** The space of complex structures compatible with ω is contractible.
- **Remark:** If J is compatible with ω, then $\langle X,Y\rangle=g_J(X,Y)-i\omega(X,Y)$ is a Hermitian structure on V.
- **Definition:** Let $(\mathbb{R}^{2n},\omega_{st})$ be the symplectic vector space. The *symplectic group* $\mathrm{Sp}(2n)$ is
$$\mathrm{Sp}(2n)=\{\psi\in \mathrm{Gl}(2n)\,|\,\psi^*\omega=\omega\}.$$
- **Example:** $\mathrm{Sp}(2)=\mathrm{Sl}(2,\mathbb{R})$ is non-compact.
- **Warning:** In the theory of Lie groups, there is a family of compact Lie groups which are also called symplectic (cf. [BtD] p.8). They are quite different from the symplectic groups we consider.
- **Fact:** $\mathrm{Sp}(2n)=\{\psi\in\mathrm{Gl}(2n,\mathbb{R})\,|\,\psi^TJ\psi=J\}$ where J is the standard complex structure on $\mathbb{R}^{2n}\simeq\mathbb{C}^n$.
- **Lemma:**
  1. $\mathrm{Sp}(2n)\subset\mathrm{Gl}(2n,\mathbb{R})$ is a closed subgroups, hence a Lie subgroup. It is closed under transposition, i.e. $\psi^T\in\mathrm{Sp}(2n)$ for $\psi\in\mathrm{Sp}(2n)$.

2. $\det(\psi) = 1$ for $\psi \in \mathrm{Sp}(2n)$.
3. If λ is an eigenvalue of $\psi \in \mathrm{Sp}(2n)$ then so is λ^{-1}. If λ is a zero of $\det(\psi - \lambda\mathrm{Id})$, then so are $\lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$.
4. If $\lambda\mu \neq 1$, then the λ-eigenspace is ω-orthogonal to the μ-eigenspace.

- On $\mathbb{R}^{2n}$ let $J = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$. This is compatible with the standard symplectic structure.

  A complex matrix $Z = X + iY$ acts on $v = x + iy \in \mathbb{R}^n \oplus i\mathbb{R}^n$ via $Zv = (Xx - Yy) + i(Yx - Xy)$. In this way one identifies $\mathrm{Gl}(n, \mathbb{C})$ with a subgroup of $\mathrm{Gl}(2n, \mathbb{R})$.
- **Lemma:** $\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{Gl}(n, \mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{Gl}(n, \mathbb{C}) = \mathrm{U}(n)$.
- **Proposition:** ψ symplectic has a unique decomposition $\psi = PQ$ with P symplectic, symmetric and positive definite and unitary Q.
- **Remark:** One has to check that $P = (\psi\psi^T)^{1/2}$ is symplectic.
- **Remark:** Considering $P_s = (\psi\psi^T)^s$ with $s \in [0, 1]$ one obtains that $\mathrm{U}(n)$ is a deformation retract of $\mathrm{Sp}(2n)$. In particular, $\pi_1(\mathrm{Sp}(2n)) \simeq \mathbb{Z}$.
- **Definition:** An *almost complex structure* on a manifold M is a base point preserving smooth map $J : TM \longrightarrow TM$ so that $J|_{T_pM}$ is a complex structure for all $p \in M$.
- **Definition:** A *complex structure* on a manifold M is a smooth atlas $(\varphi_i : U_i \subset M \longrightarrow \varphi_i(U_i) \subset \mathbb{C}^n)_i$ so that transition functions are holomorphic.
- **Remark:** A complex structure induces an almost complex structure, but not every almost complex structure is obtained in this way [McDS], p.123 ff.
- **Definition:** Let (M, ω) be a symplectic manifold. An almost complex structure is adapted to ω if $g(X, Y) = \omega(X, JY)$ is a Riemannian metric.
- **Definition:** A Kähler manifold (M, J, ω) is a symplectic manifold with a complex structure so that the induced almost complex structure is compatible with ω.
- **Theorem:** Every symplectic manifold admits an adapted almost complex structure.
- **Proof:** Choose a Riemannian metric and for each T_pM choose a complex structure as above.
- **Observation:** If (M, ω, J) is a manifold with almost complex structure compatible with the symplectic structure ω and $N \subset M$ is a submanifold so that $J(TN) = TN$, then $\omega|_N$ is symplectic. Hence, complex submanifolds of Kähler manifolds are symplectic. This is a rich supply of closed symplectic manifolds with interesting topologies.

  Using the fact that $b_{2k+1}(M)$ has to be even when M is Kähler, Thurston [Th] provided an example of a closed manifold which is symplectic but does not admit a Kähler structure.

3. Lecture on October, 21 – Moser method

- **Reference:** Section 3.2. of [McDS], see also Chapter 2 in [Ge]
- **Reminder:** Let X be a complete vector field on M, ϕ_t an isotopy (obtained by integrating the time dependent vector fields X_t) and $\alpha \in \Omega^*(M)$. Then

$L_X\alpha = i_X d\alpha + d i_X \alpha$ and for α_t a smooth family of forms

$$\left.\frac{d}{dt}\right|_{t=t_0} (\phi_t^*\alpha_t) = \phi_{t_0}^* \left(\left.\frac{d}{dt}\right|_{t=t_0} \alpha_t + L_{X_{t_0}}\alpha_{t_0} \right). \tag{1}$$

Let α_t be a smooth family of exact forms. Then there is a family of primitives β_t (i.e. $d\beta_t = \alpha_t$). There are several ways to do this, see for example [BT] for an explicit construction or use Hodge theory [Jä].

- **General Problem/Moser method:** Let α_t be a family of k-forms. Is there a family (an isotopy) of diffeomorphism ϕ_t so that

$$\phi_t^*\alpha_t = \alpha_0$$

and $\phi_0 = \mathrm{id}$. Differentiating this we get $\dot{\alpha}_t = -L_{X_t}\alpha_t$. Conversely, if this equation is satisfied for a smooth family of vector fields X_t, then the induced isotopy ϕ_t has the desired property.
- **Theorem (Moser):** Let Ω_0, Ω_1 be two volume forms on the closed manifold M with the same total volume. Then there is a diffeomorphism ϕ_1 of M so that $\phi_1^*\Omega_1 = \Omega_0$.
- **Proof:** Apply the Moser method to the family $\Omega_t = t\Omega_1 + (1-t)\Omega_0, t \in [0,1]$ of volume forms. Since the total volume of Ω_t is constant, $\dot{\Omega}_t$ is exact (it is obviously closed as form of top degree). Let β_t be a smooth family of primitives. We look for a family of vector fields X_t so that

$$d\beta_t = \dot{\Omega}_t = -d i_{X_t}\Omega_t.$$

If X_t solves $i_{X_t}\Omega_t = -\beta_t$. This equation has a unique solution since Ω_t is a volume form.
- **Theorem (Moser stability):** Let M be closed and ω_t a family of symplectic forms such that $[\omega_t] \in H^2_{dR}(M)$ is constant. Then there is an isotopy ϕ_t so that $\phi_t^*\omega_t = \omega_0$.
- **Proof:** Same as above.
- **Remark:** It is hard to determine whether or not two symplectic forms are connected by a path of symplectic forms. Therefore, the scope of the previous theorem is limited. However, the Moser method can be applied to obtain normal forms/coordinates in which a differential form has a nice (or standard) representation. One then defines a local flow/isotopy of a neighborhood of a subset of the manifold. completeness of vector fields is no longer needed.
- **Theorem (Darboux):** Let (M,ω) be symplectic and $p \in M$ then there are coordinates $(x_1, y_1, \ldots, x_n, y_n)$ around p so that $\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$.
- **Proof:** Pick any coordinate system around p so that

$$\frac{\partial}{\partial x_i} = e_i \text{ and } \frac{\partial}{\partial y_i} = f_i$$

is a symplectic basis for T_pM, ω_p. On a neighborhood of p the family $\omega_t = t\omega + (1-t)\omega_0, t \in [0,1]$ is symplectic. Using the Moser method one finds a family of vector fields X_t with the additional property that $X_t(p) = 0$ for all t. Then one can define a local flow defined on a neighborhood of p which deforms the coordinate system we have into the coordinate system that we want.
- **Definition:** Let (M,ω) be symplectic. Then a submanifold $N \subset M$ is isotropic, coisotropic, symplectic, Lagrangian if $T_pN \subset (T_pM, \omega_p)$ has the corresponding property for all $p \in N$.

- **Definition:** Let $(M,\omega),(M',\omega')$ be symplectic. A diffeomorphism $\phi : M \longrightarrow M'$ is a symplectomorphism if and only if $\phi^*\omega' = \omega$.
- **Definition:** Let M be a manifold of dimension $2n+1$. A 1-form α is a contact form if $\alpha \wedge (d\alpha)$ is a volume form. A *contact structure* is a hyperplane field ξ in TM such that around every point there is a contact form α so that $\ker(\alpha) = \xi$.
- **Theorem (Gray):** Let ξ_t be a family of contact structures on a closed manifold M. Then there is ϕ_t and isotopy so that $\phi_{t*}\xi_0 = \xi_t$.

4. Lecture on October, 25 – Neighborhoods of Lagrangians, Reduction for symplectic vector spaces

- **Reminder:** We will make frequent/implicit use of the *Tubular neighborhood theorem*, see for example Kapitel 12 in [BJ] or Section 4.5 in [Hi].
- **Theorem (Weinstein):** Let (M,ω) be symplectic and $N \subset M$ be a closed Lagrangian submanifold. Then N has a tubular neighborhood which is symplectopmorphic to a neighborhood of the zero section in $(T^*N, d\lambda_{st})$.
- **Proof:** This is yet another application of the Moser method similar to the proof of the Darboux theorem. One shows that the map

$$\begin{aligned} TM/TN &\longrightarrow T^*N \\ v &\longmapsto (w \longmapsto \omega(v,w)) \end{aligned}$$

  preserves the symplectic structure on a neighborhood of the zero section.
- **Lemma:** Let (V,ω) be symplectic, $L \subset V$ Lagrangian and $F \subset V$ coisotropic so that L and F are transverse, i.e. $L + F = V$. Then the map

$$F \cap L \longrightarrow F/F^{\perp_\omega}$$

  is injective and its image is Lagrangian.
- **Proof:** Recall that $F^{\perp_\omega} \subset F$ and $L = L^{\perp_\omega}$. ω induces a symplectic structure on $F/F^{\perp_\omega}$. The kernel of the map in the Lemma is

$$\begin{aligned} L \cap F \cap F^{\perp_\omega} &= L \cap F^{\perp_\omega} = L^{\perp_\omega} \cap F^{\perp_\omega} \\ &= (L+F)^{\perp_\omega} = \{0\}. \end{aligned}$$

  The image of the map is obviously isotropic. Finally,

$$\begin{aligned} \dim(L \cap F) &= \dim(L) + \dim(F) - \dim(L+V) \\ &= \dim(L) + \dim(F) - 2n \\ &= \dim(F) - n. \\ \dim(F/F^{\perp_\omega}) &= \dim(F) - \dim(F^{\perp_\omega}) \\ &= 2(\dim(F) - n). \end{aligned}$$

  Hence, the image of $L \cap F$ is an isotropic subspace of maximal (=half) dimension, i.e. it is Lagrangian.

5. Lecture on October, 29 – Construction of Lagrangians using generating functions

- Consider a 1-form α on N. This can be viewed as a map $\alpha : N \longrightarrow T^*N$ such that $\mathrm{pr} \circ \alpha = \mathrm{id}$ where $\mathrm{pr} : T^*N \longrightarrow N$ is the projection. In particular, α is an embedding of N into T^*N. Then

$$\alpha^*\lambda_{st} = \alpha.$$

Hence, α has Lagrangian image if and only if $d\alpha \equiv 0$. In this way one obtains very special Lagrangian submanifolds. We want to generalize this.

- We will identify $T^*\mathbb{R}^k$ with $\mathbb{C}^k = \mathbb{R}^k \oplus i\mathbb{R}^k$, the imaginary part corresponds to the forms. The coordinates on $\mathbb{R}^k$ are $a_1, \ldots, a_k$.
- Let $f : M \times \mathbb{R}^k \longrightarrow \mathbb{R}$ be smooth. This induces a Lagrangian section $df : M \times \mathbb{R}^k \longrightarrow T^*M \times \mathbb{C}^k$ of the bundle $T^*M \times \mathbb{C}^k \longrightarrow M \times \mathbb{R}^k$, we call the image V_f and we assume that df is transverse to $T^*M \times (\mathbb{R}^k \oplus i0)$. Then

$$V_f \cap (T^*M \times \{0\}) = \left\{ \frac{\partial f}{\partial a_1} = \ldots = \frac{\partial f}{\partial a_k} \right\}.$$

The transversality condition using local coordinates $x_1, \ldots, x_n$ near $p \in M$ is

$$\operatorname{rank} \left(\frac{\partial^2 f}{\partial x_i \partial a_j} \middle| \frac{\partial^2 f}{\partial a_i \partial a_j} \right) = k$$

for all points in $V_f \cap (T^*M \times (\mathbb{R}^k \oplus i0))$. (This makes sure that the condition $L + F = V$ will be satisfied when we apply the linear algebra lemma.) Then $V_f \cap T^*M \times (\mathbb{R}^k \times \{0\})$ is a submanifold in $M \times \mathbb{R}^k$ of codimension k. Now apply the linear algebra lemma to the Lagrangian subspace T_pV_f as Lagrangian, and $F_p = T^*M \times \mathbb{R}^k \times \{0\} \subset T^*M \times \mathbb{C}^k$ as coisotropic subspace. Then

$$V_f \cap T^*M \times (\mathbb{R}^k \times \{0\}) \longrightarrow T^*M = (T^*M \times \mathbb{C}^k)/(T^*M \times (\mathbb{R}^k \times \{0\}))^{\perp_\omega}$$

is a Lagrangian immersion (the quotient map turns embeddings into immersions, in general).

- **Example:** Take $M = \mathbb{R}^n$ and $k = 1$ and $f(x, a) = a\|x\|^2 + a^3/3 - a$. Then

$$\frac{\partial f}{\partial a} = \|x\|^2 + a^2 - 1 = 0,$$

Then $V_f \cap (T^*M \times \mathbb{R} \times \{0\}) \simeq S^n$ with the transversality condition satisfied. Now

$$df(x_1, \ldots, x_n, a) = 2a(x_1dx_1 + \ldots + x_ndx_n) + (\|x\|^2 + a^2 - 1)da.$$

Thus, we get

$$\begin{aligned} S^n \subset (\mathbb{R}^n \times \mathbb{R}) &\longrightarrow \mathbb{C}^n = T^*\mathbb{R}^n \\ (x, a) &\longmapsto (x, 2ax) \text{ or } (1 + 2ia)x \text{ under the identification } T^*\mathbb{R}^n \simeq \mathbb{C}^n. \end{aligned}$$

This map is called the Whitney immersion. It is not an embedding since $(0, \pm 1)$ map both to $0 \in T_0^*\mathbb{R}^n$.

- **Definition:** Let $f : L \longrightarrow (M, d\lambda)$ be a Lagrangian immersion into an exact symplectic manifold with a fixed primitive λ of the symplectic form $d\lambda = \omega$. f is *exact* if $f^*\lambda$ is exact.
- **Example:** If $H^1_{dR}(L) = 0$, then every Lagrangian immersion into an exact symplectic manifold is exact.
- **Example:** If $n = 1$. then the Whitney immersion is parametrized by $f : \mathbb{R} \longrightarrow S^1 \longrightarrow \mathbb{C}$ as

$$f : \alpha \longmapsto (\cos(\alpha), \sin(2\alpha)).$$

If $\lambda = ydx$, then $f^*\lambda = d\left(-\frac{2}{3}(\sin(\alpha))^3\right)$. The function in the bracket descends from $\mathbb{R}$ to S^1 showing that the Whitney immersion is exact when $n = 1$.

6. Lecture on November, 5 – Contactisation of exact symplectic manifolds, wave fronts

- **Definition:** Let $(M, \omega = d\lambda)$ be an exact symplectic manifold. Then its contactisation is the manifold $\mathbb{R} \times M$ with the contact structure ξ defined by $dz - \mathrm{pr}^*\lambda$.
- **Lemma:** $dz - \mathrm{pr}^*\lambda$ is a contact form.
- Let $f : L \longrightarrow M$ be a Lagrangian immersion. If $f^*\lambda = dH$ is exact, then

$$\begin{aligned} F : L &\longrightarrow \mathbb{R} \times M \\ p &\longmapsto (H(p), f(p)) \end{aligned}$$

is a *Legendrian immersion*, i.e. an immersion such that the tangent space of the image is tangent to the contact structure, $F_*(T_pL) \subset \xi(F(p))$. This is equivalent to $F^*(dz - \lambda) = 0$. F is a lift of the $f : L \longrightarrow M$ for the projection $\mathrm{pr}_M : \mathbb{R} \times M \longrightarrow M$.

Thus, a Lagrangian immersion into an exact symplectic manifold lifts to the contactisation if and only if it is exact. While f is not always exact, there is a cover $\pi : \widetilde{L} \longrightarrow L$ so that $f \circ \pi$ is exact.
- **Remark:** Let T^*M and $\lambda = \lambda_{st}$ be a cotangent bundle and $\pi : T^*M \longrightarrow M$ the projection.
- **Definition:** The projection $\pi \circ (\mathrm{id} \times F)$ to $\mathbb{R} \times M$ of the image of the lift F of an exact Lagrangian immersion f to the contactisation is called a *wave front*.
- **Fact:** The wave front $L \longrightarrow \mathbb{R} \times M$ is not an immersion, in general. If $\pi \circ f$ is an immersion at some point, then the Lagrangian immersion (parametrized) can be reconstructed from the (parametrized) wave front: dH can be read off from the wave front since it contains H entirely. Since $F^*(dz - \lambda_{st}) = 0$ one can read of $f^*\lambda_{st}$ from the (parametrized) wave front:

$$\begin{aligned} dH(Y) &= f^*\lambda_{st}(Y) = \lambda_{st}(f_*Y) \\ &= \underbrace{(f(p))}_{\in T^*_{\pi(f(p))}M} (\pi_* f_*(Y)) \end{aligned}$$

Note that $\pi_* f_*(Y)$ can be determined from the wave front.

This determines the coordinate projected away by $\pi : T^*M \longrightarrow M$.
- One can constuct/draw wave fronts of Lagrangian immersions of an orientable closed surface Σ into $\mathbb{C}^2$. Pictures can be found on p. 279 of [AL] or in [Giv] (the later reference allows singularities).
- It would be interesting to know which manifolds admit Lagrangian *embeddings* into a symplectic manifold, like $\mathbb{C}^n \simeq T^*\mathbb{R}^n$. However, a full answer is out of reach.
- **Lemma:** If $f : L \longrightarrow \mathbb{C}^n$ is a Lagrangian immersion, then $TL \otimes \mathbb{C}$ is trivial (as complex vector bundle).
- **Reference:** If you want to know more about vector bundles, I strongly recommend [Mi-C], Chapter 2,3 and 13.
- **Proof:** By the Weinstein neighborhood theorem the normal bundle of the immersion is isomorphic to $J \cdot f_*(TL)$. Thus, at each point of $p \in L$

$$f_*T_pL \otimes J \cdot (f_*T_pL) = T_{f(p)}\mathbb{C}^n \simeq \mathbb{C}^n.$$

- **Remark:** For all oriented closed manifolds of dimension $1, 2, 3$, the bundle $TL \otimes \mathbb{C}$ is trivial. In higher dimensions, there are manifolds which do not admit Lagrangian immersions.
- **Remark:** The necessary condition for the existence of a Lagrangian immersion into $\mathbb{C}^n$ is also sufficient! The story for Lagrangian embeddings is far more complicated.

7. Lecture on November, 8 – Lagrangian embeddings, rigidity, Maslov class

- **Theorem:** If the closed oriented manifold L admits a Lagrangian embedding into $\mathbb{C}^n$, then $\chi(L) = 0$.
- **Corollary:** No even dimensional sphere of positive dimension admits a Lagrangian embedding into M. (cf. Whitney immersion).
- **Proof of Theorem:** Assume $f : L \longrightarrow \mathbb{C}^n$ is a Lagrangian embedding. We use a bunch of facts:

  (1) By Weinsteins Lagrangian neighborhood theorem, the normal bundle of L in $\mathbb{C}^n$ is isomorphic to T^*L (which is isomorphic to TL via a choice of a Riemannian metric on L).

  (2) Let α be a generic section of T^*L (i.e. transverse to the zero section). Then $\alpha(L)$ and L have finitely many intersection points which can be equipped with signs according to whether or not $T_p\alpha(L) \otimes T_pL \subset T_p(T^*L)$ coincide as *oriented* vector spaces. On the one hand, the sum of these signs computes the self intersection number $[L] \cdot [L] \in H_0(\mathbb{C}^n) = \mathbb{Z}$ of the (image of) the fundamental/orientation class $[L] \in H_n(\mathbb{C}^n)$ which has to vanish since $[L] = H_n(\mathbb{C}^n) = 0$.

  (3) On the other hand, the sum of the signs computes also $\chi(L) = \langle \chi(T^*L), [L] \rangle$ by standard theorems from algebraic topology (see Cor. 12.5 on p. 380 in [Br], and the section it is contained in, or Chapter 11 in [Mi-C]).

  Also, note that $\chi(L) = 0$ for all oriented manifolds of odd dimension.
- **Reminder:** (1) The Gauß-map associates to each immersion of a manifold into $\mathbb{R}^n$ the orthogonal complement of the image of the tangent space.

  (2) The space of Lagrangian submanifolds Λ_n in $\mathbb{C}^n$ is homeomorphic to $\mathrm{U}(n)/\mathrm{O}(n)$. The map

  $$(\det)^2 : \Lambda_n = \mathrm{U}(n)/\mathrm{O}(n) \longrightarrow S^1 \subset \mathbb{C}$$

  induces an isomorphism on fundamental groups.

  (3) If (M, ω) is symplectic, and $\gamma : S^1 \longrightarrow M$ is a loop, then γ^*TM is trivial. The same is true when TM is viewed as a symplectic vector bundle (i.e. each fibre is equipped with a symplectic structure). Therefore, also the bundle of Lagrangian Grassmannians along γ is trivial. To each section of the Lagrangian Grassmannian one can associate an element $\mu(\gamma)$ of $\pi_1(\Lambda_n) = \mathbb{Z}$.
- **Definition:** Let $f : L \longrightarrow (M, \omega)$ be symplectic. Then

  $$\begin{aligned} \mu : \pi_1(L) &\longrightarrow \mathbb{Z} \\ [\gamma] &\longmapsto \mu(\gamma) \end{aligned}$$

  is the *Maslov class* of f.
- The following question is interesting: Let $f : L \longrightarrow (M, \omega)$ be a Lagrangian embedding. What can be said about the Maslov class? Unfortunately, we can discuss only a trivial case.

- **Example:** Let $n = 1$ and consider $S^1 \longrightarrow \mathbb{C}$. Then $\Lambda_1 = \mathrm{U}(1)/\mathrm{O}(1) = \mathbb{RP}^1$ and the Maslov index is the winding number. In particular, if γ is a Lagrangian embedding (i.e. a simple closed curve), then $\mu(\gamma) = \pm 2$. Note that every even number appears of $\mu(g)$ for some immersion g.
- **Definition:** Let μ be the Maslov class of a Lagrangian immersion. Then $\|\mu\|$ is the non-negative generator of $\mu(\pi_1(L))$.
- **Remark:** Unlike μ, $\|\mu\|$ does not refer to a particular Lagrangian immersion but only to the image.
- **Theorem (Viterbo):** For a Lagrangian embedding $f : T^n \longrightarrow \mathbb{C}^n$
$$2 \leq \|\mu\| \leq (n+1).$$
- The Maslov cycle, or $\|\mu\|$ can be used to show that two Lagrangian immersions are not homotopic through Lagrangian immersions.

8. Lecture on November, 12 – Hamiltonian vector fields, Poisson bracket, $\mathrm{Ham}(M, \omega)$

- **Definition:** Let (M, ω) be symplectic and $H : M \longrightarrow \mathbb{R}$ smooth. Then there is a unique vector field X_H on M so that
$$i_{X_H}\omega = -dH.$$
X_H is the Hamiltonian vector field of the Hamiltonian function H.
- **Example:** $M = \mathbb{R}^{2n}$ and $\omega = \sum_i dp_i \wedge dq_i$. Then
$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right).$$
Hence,
$$X_H = \sum_i \left(-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} \right).$$
Thus, $\dot{p}_i = -\partial_{q_i} H$ and $\dot{q}_i = \partial_{p_i} H$. These are the Hamiltonian equations from classical mechanics.
- **Example:** In the previous example, let $H = p_i$. Then $X_H = \frac{\partial}{\partial q_i}$. The Hamiltonian diffeomorphisms associated to this function is a family of translations. Compactly supported symplectomorphisms can be obtained by multiplication of H with a bump function.
- **Example:** Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $H(x_1, x_2, x_3) = x_3$. The symplectic form is $\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_1$.
  The Hamiltonian vector field of H is
$$X_H = -\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right)$$
- **Lemma:** X_H preserves ω as well as H.
- **Lemma:** If X_H and X_G are Hamiltonian vector fields of a smooth function H, G, then the same is true for $[X_H, X_G]$. The vector space of Hamiltonian vector fields is a Lie algebra.
- **Definition:** Let $\mathrm{Ham}(M, \omega)$ be the group generated by Hamiltonian vector fields with (maybe time dependent) Hamiltonian function.
- **Remark:** The time dependence is assumed to be piecewise smooth to allow for composition of paths. This turns out to be not necessary, see (2) below.

- **Remark:** This definition requires some caution when M is not closed. Usually, one assumes that H has compact support or some other fixed behavior outside of a compact set which ensures completeness of the Hamiltonian vector field.
- We will not use the following notion much. However, it is important in various contexts (e.g. integrable dynamical systems, including those of infinite dimension).
- **Definition:** Let (M, ω) be symplectic. The Poisson bracket of two smooth functions H, G is
$$\{H, G\} = \omega(X_H, X_G).$$
- **Remark:** $\{H, G\} = \omega(X_H, X_G) = (i_{X_H}\omega)(X_G) = -L_{X_G}H$. Hence, $\{\cdot, G\}$ is a derivation. Obviously, $\{H, G\} = -\{G, H\}$. Moreover, since $L_X i_Y \alpha - i_Y L_X \alpha = i_{[X,Y]}\alpha$ for all forms α one obtains
$$\begin{aligned} i_{[X_H,X_G]}\omega &= L_{X_H} i_{X_G}\omega = d i_{X_H}(\omega(X_G, \cdot)) \\ &= -d\omega(X_H, X_G) = -d\{H, G\}. \end{aligned}$$
Thus, $[X_H, X_G]$ is the Hamiltonian vector field of the function $\{H, G\}$.
- **Proposition:** $\{\cdot, \cdot\}$ is a Lie algebra structure on $C^\infty(M)$.
- **Proof:** $\mathbb{R}$-bilinear and antisymmetry are obvious. The Jacobi identity follows from a little computation using the fact that ω is closed!
$$\begin{aligned} 0 &= d\omega(X_1, X_2, X_3) \\ &= L_{X_1}(\omega(X_2, X_3)) + L_{X_2}(\omega(X_3, X_1)) + L_{X_3}(\omega(X_1, X_2)) \\ &\quad - \omega([X_1, X_2], X_3) - \omega([X_2, X_3], X_1) - \omega([X_3, X_1], X_2) \end{aligned}$$
Assume that X_i is the Hamiltonian vector field associated to f_i. Then
$$\begin{aligned} L_{X_1}(\omega(X_2, X_3)) &= L_{X_1}\{f_2, f_3\} = -\{\{f_2, f_3\}, f_1\} \\ &= -\omega(X_{\{f_2,f_3\}}, X_1) = -\omega([X_2, X_3], x_1). \end{aligned}$$
Combining this with the previous computation one obtains
$$0 = -2(\{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}\{\{f_1, f_2\}, f_3\}).$$
This is the Jacobi identity.
- **Remark:** Therefore, the linear map
$$\begin{aligned} (C^\infty(M), \{\cdot, \cdot\}) &\longrightarrow (\Gamma(M), [\cdot, \cdot]) \\ H &\longmapsto X_H \end{aligned}$$
is a homomorphism of Lie algebras.
- **Definition:** A *Poisson structure* on a manifold is a Lie-bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$.
- **Remark:** Symplectic structures induce Poisson structures, but there are others.
- **Example:** Let $(\mathfrak{g}, [\cdot, \cdot])$ be a finite dimensional Lie algebra. Then $\mathfrak{g}^*$ (viewed as manifold) has a Poisson structure defined as follows: Let f, g be smooth functions on $\mathfrak{g}^*$. Then $df_\alpha \in (T_\alpha \mathfrak{g}^*)^* = (\mathfrak{g}^*)^* = \mathfrak{g}$. Thus, we can define
$$\{f, g\}(\alpha) = \langle \alpha, [df, dg] \rangle$$
The Jacobi identity follows from the analogous property of $[\cdot, \cdot]$. $\mathfrak{g}^*$ can have odd dimension, so the Poisson structure on $\mathfrak{g}^*$ does not come from a symplectic structure in general.
- **Remark:** By a theorem of Weinstein [We], one can obtain symplectic structures on well organized immersed submanifolds from a Poisson structure.

- **Remark:** Let (M, ω) be symplectic such that $H^1_{dR}(M) = 0$. For a family of symplectomorphism ϕ_t one can consider the vector field they generate, i.e. $X_\tau(x) = \frac{d}{dt}\big|_{t=\tau} \phi_t(\phi_\tau^{-1}(x))$.

  This is a symplectic vector field by (1), i.e. $L_{X_t}\omega = 0 = d(i_{X_t}\omega)$. By assumption, there is a function H_t so that $i_{X_t}\omega = -dH_t$. Thus,
$$\mathrm{Symp}_0(M, \omega) = \mathrm{Ham}(M, \omega)$$
  if $H^1(M) = 0$. We will describe the difference between $\mathrm{Symp}_0(M, \omega)$ and $\mathrm{Ham}(M, \omega)$ later.
- **Proposition:** Let f_t, g_t be Hamiltonian isotopies associated to the families of Hamiltonian functions F_t, G_t. Then the product path $h_t = f_t g_t$ is a Hamiltonian path generated by the Hamiltonian function
$$H(x, t) = F(x, t) + G(f_t^{-1}(x), t). \tag{2}$$
- **Proof:** By the chain rule
$$\frac{d}{dt}(f_t \circ g_t)(x) = X_{f_t}(f_t(g_t(x))) + f_{t*}\left(X_{g_t}(g_t(x))\right)$$
  The first summand is the symplectic gradient of f_t, X_{g_t} is the symplectic gradient of g_t. It is an exercise to show that the second summand is $G(f_t^{-1}(x), t)$.
- **Remark:** If X_H is a Hamiltonian vector field of H, then it is also a Hamiltonian vector field for $H + c$ for all constants $c \in \mathbb{R}$. A Hamiltonian function is said to be *normalized*, if the average value is zero on closed manifolds, or when the support is compact on open manifolds.

9. Lecture on November, 15 – Poincaré-Birkhoff fixed point theorem

- Let $A = \{(u, v) \in \mathbb{R}^2 \,|\, a^2 \leq u^2 + v^2 \leq b^2\}$ for $0 < a < b$. The universal cover is $\widetilde{A} = \{a \leq y \leq b\}$ with covering projection $\phi(x, y) = (\sqrt{y}\cos(x), \sqrt{y}\sin(x))$. This map satisfies $\phi^*(du \wedge dv) = -dx \wedge dy/\pi$.
- **Definition:** A map $h : A \longrightarrow A$ is a **twist map** if it preserves the boundary components of A individually and there is a lift $\widetilde{h} = (f, g)$ to the universal cover so that either
$$f(x, a) < x \text{ and } f(x, b) > x \text{ for all } x \text{ or}$$
$$f(x, a) > x \text{ and } f(x, b) < x \text{ for all } x$$
- **Theorem (Poincaré-Birkhoff)** Let $h : A \longrightarrow A$ be an area preserving twist map. Then h has at least two fixed points.
- **Remark:** A collection of simple examples relevant to the condition/conclusion of the theorem can be found on p. 270 in [McDS].
- What follows is essentially the proof from [BN]. It uses several steps/observations.
  1. Let $\widetilde{h} = (f, g)$ be a lift of h certifying that h is a twist map. $\widetilde{h}$ preserves the area form $dx \wedge dy$.
  2. $\widetilde{h} = (f, g)$ satisfies $f(x + 2\pi k, y) = f(x, y) + 2\pi k$ and $g(x + 2\pi k, y) = g(x, y)$ because it is a lift.
  3. We will show that $\widetilde{h}$ has two fixed points which are geometrically distinct, i.e. do not become equal after a translation in x direction by integer multiples of 2π. This is somewhat stronger than what we have to show.

4. Extend $\widetilde{h}$ to $\mathbb{R}^2$ by
$$\widetilde{h} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto \begin{cases} (f(x,y), g(x,y)) & a \le y \le b \\ (f(x,a), y) & y \le a \\ (f(x,b), y) & b \le y. \end{cases}$$
This is not smooth and it is area preserving with respect to $dx \wedge dy$ only on $\widetilde{A}$ in general. It is area preserving everywhere only when $f(x,a) = x \pm c$ and $f(x,b) = x \mp c'$ for positive constants c, c'. Moreover, the extended version of $\widetilde{h}$ has the same periodicity properties as the original version.
5. Let $c : [r,s] \longrightarrow \mathbb{R}^2$ be a curve avoiding fixed points of $\widetilde{h}$. We put $d(t) = \frac{\widetilde{h}(c(t)) - c(t)}{\|\widetilde{h}(c(t)) - c(t)\|} \in S^1$ and choose a lift $\widetilde{d} : [a,b] \longrightarrow \mathbb{R}$ of d to the universal covering $\mathbb{R} \longrightarrow S^1, \tau \longrightarrow e^{i\tau}$. Define
$$\mathrm{ind}_{\widetilde{h}}(c) = \frac{\widetilde{d}(s) - \widetilde{d}(r)}{2\pi}.$$
This measures the total number of turns the vector pointing from $c(t)$ to $\widetilde{h}(c(t))$ makes as one moves along c. The index has some obvious properties (continuity in $\widetilde{h}, c$, orientation reversal of c, concatination of curves,...) we will use. One of them is
$$\mathrm{ind}_{\widetilde{h}}(c) = \mathrm{ind}_{\widetilde{h}^{-1}}(\widetilde{h} \circ c). \tag{3}$$
Since c is not closed, the index is not an integer, in general. Finally, the index of a curve is a homotopy invariant as long as the homotopy avoids fixed points of $\widetilde{h}$ and endpoints do not move.
6. We will consider curves c so that $c(r) \in \{y \le a\}$ and $c(s) \in \{b \le y\}$ which avoid fixed points of $\widetilde{h}$.
7. **Lemma:** Assume that $\widetilde{h}$ has at most one class of fixed points $(x_0 + 2k\pi, y_0), k \in \mathbb{Z}$. Let c, c' be two curves avoiding fixed points of $\widetilde{h}$ going from $\{y \le a\}$ to $\{b \le y\}$. Then
$$\mathrm{ind}_{\widetilde{h}}(c) = \mathrm{ind}_{\widetilde{h}}(c').$$
After reparameterization of the plane we may assume that $x_0 = 0$ if $\widetilde{h}$ has any fixed point.
8. For the proof of the Lemma, let c, c' as above, connect to endpoint of c with the starting point of $-c'$ by a straight line in $\{b \le y\}$ connect the endpoint of $-c'$ with the starting point of c by a straight line in $\{y \le a\}$. We obtain a loop whose index does not change if the loop is homotoped in the complement of the fixed points of $\widetilde{h}$.
This loop is freely homotopic to a collection of rectangles connected to a base point in one of its vertices so that the sides of the rectangle are parallel to the x- or y-axis and every horizontal vertical segment is contained in $\{x = (2k+1)\pi, k \in \mathbb{Z}\}$. Since the number of vertical sides pointing down equals the number of sides pointing up, the index of the loop is zero. By construction
$$\mathrm{ind}_{\widetilde{h}}(\mathrm{loop}) = \mathrm{ind}_{\widetilde{h}}(c) - \mathrm{ind}_{\widetilde{h}}(c') = 0.$$

9. We will now show that there are paths c, c' going from $\{y \leq a\}$ to $\{b \leq y\}$ so that $\operatorname{ind}_{\widetilde{h}}(c) = 1/2$ and $\operatorname{ind}_{\widetilde{h}}(c') = -1/2$. This contradicts the assumption that $\widetilde{h}$ has at most one class of fixed points. The difficulty is now to construct a path c as above and to compute its index with respect to $\widetilde{h}$. For this we consider a perturbation of $\widetilde{h}$.
10. There is $\varepsilon > 0$ so that $\|\widetilde{h}(x,y) - (x,y)\| > 2\varepsilon$ when $x \in (2\pi k + \pi/2, 2\pi k + 3\pi/2)$ with $k \in \mathbb{Z}$. Let
$$\begin{aligned} T_\lambda : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x, y + \lambda\varepsilon(|\cos(x)| - \cos(x))) \end{aligned}$$
with $\lambda \in [0,1]$ and write $T_1 = T$. T_λ is an area preserving homeomorphism of the plane with the same periodicity properties as $\widetilde{h}$. Moreover, $\widetilde{h}$ and $T \circ \widetilde{h}$ have the same fixed point set. Instead of $\widetilde{h}$ we will consider $(T \circ \widetilde{h})$.
11. Let $D_0 = \Big((T \circ \widetilde{h})^{-1}(\widetilde{A})\Big) \cap \{y \leq a\}$ and $D_i = (T \circ \widetilde{h})^i(D_0)$ for all $i \in \mathbb{Z}$. D_0 has non-empty interior, the interiors of D_i, D_k are disjoint if $i \neq k$. By choice of T, D_0 is convex.
12. Construction of c: Since $T \circ \widetilde{h}$ is area preserving on $\widetilde{A}$, there is N so that $D_i \cap \{b \leq y\} \neq \emptyset$. Pick the smallest such N and $p_N \in D_N$ with maximal y-coordinate and let $p_i = (T \circ \widetilde{h})^{i-N}(p_N)$. Note that $p_{n-1} \neq p_n$ so we can pick c_0 to be the straight line from p_{-1} to $p_0 \in D_0$ and $c_i = (T \circ \widetilde{h})^i \subset D_i$. Then the concatination $c_0 c_1 c_2 \dots c_{N+1}$ is embedded as the image under $(T \circ \widetilde{h})^N$ of $c_{-N} \dots c_{-1} c_0$. We use $[0,1]$ as domain of c. The last curve can be easily seen to be embedded because of the form of $T \circ \widetilde{h}$ on $\{y \leq a\}$. Now define
$$c = c_0 c_1 \dots c_N$$
This curve starts at p_{-1} below $\widetilde{A}$ and ends at p_N above this strip.
13. For the computation of the index of c we assume that $f(x,a) > x$ (hence $f(x,b) < x$).
14. By the choice of p_N, no point along $c_0 c_1 c_2 \dots c_N$ has a bigger y-coordinate than p_{N+1} since p_{N+1} lies above p_N. Moreover, no point of $(T \circ \widetilde{h})(c)$ lies below p_{-1}.
15. c has some kind of *positivity property with respect to* $T \circ \widetilde{h}$: for all t, t' with $t' \geq t$
$$c(t) \neq (T \circ \widetilde{h})(c(t')).$$
This allows us to determine $\operatorname{ind}_{T \circ \widetilde{h}}$. By using a homotopy provided by the simplex $\{(t,t') \in [0,1] \,|\, t' > t\}$. Moving along the diagonal with $t(\sigma) = t'(\sigma) = \sigma$ and considering
$$\frac{(T \circ \widetilde{h})(c(t'(\sigma))) - c(t(\sigma))}{\|(T \circ \widetilde{h})(c(t'(\sigma))) - c(t(\sigma))\|}$$
we compute $\operatorname{ind}_{T \circ \widetilde{h}}(c)$. This map from an interval parametrizing the diagonal in $[0,1] \times [0,1]$to S^1 is homotopic relative of to the endpoints to a map corresponding to a path $(t'(\sigma), t(\sigma)), \sigma \in [0,1]$ following the other two sides of the simplex$\{t' \geq t\}$. The direction considered now never points vertically downwards. It therefore makes not a single full turn.

Hence, the index is $+1/2$ up to an error which goes to zero when λ goes to zero. Assuming that there is only one class of fixed points one can homotope the curve relative to its endpoints so that as long as it is in $\widetilde{A}$, its x-coordinate is contained in $[2k\pi + \pi/2, 2k\pi + 3\pi/2]$. The index of the result is close to $1/2$, it remains an admissible curve as λ goes to zero (i.e. it does no meet any fixed point of $T_\lambda \circ \widetilde{h}$) as λ goes from 1 to 0 (recall $T_0 = \mathrm{id}$).

16. Now apply the same construction to $\widetilde{h}^{-1}$ keeping the same T. Since left/right are now reversed, this yields a curve c'' of index $-1/2$ with respect to $\widetilde{h}^{-1}$. By (3) this yields the contradiction we were looking for.

10. Lecture on November, 19 – Periodic points from Poincaré-Birkhoff

- One can apply the theorem under weaker conditions than being a twist map.
- **Corollary:** Let $h : A \longrightarrow A$ be an area preserving map of the annulus so that a lift $\widetilde{h}$ to $\widetilde{A}$ satisfies

$$m = \max_x (f(x,a) - x) < \min_x (f(x,b) - x) = M.$$

  Then h has infinitely many distinct periodic points.
- **Proof:** Let $\widetilde{h}^q = (f^q, g^q)$ for $q \in \{1, 2, 3, \ldots\}$. Then

$$f^q(x,a) - x = \sum_{j=0}^{q-1} (f^{j+1}(x,a) - f^j(x,a)) \leq 2\pi q m$$

$$f^q(x,b) - x = \sum_{j=0}^{q-1} (f^{j+1}(x,b) - f^j(x,a)) \geq 2\pi q M$$

  If q is chosen sufficiently large we can choose a lift $\widetilde{h}_q$ of h^q which certifies the twist condition for h^q. If $q(M - m) > 1$, then there is an integer p so that $mq < p < Mq$. Then

$$f^q(x,a) - 2\pi p < f^q(x,a) - 2\pi q m \leq x \leq f^q(x,b) - 2\pi q M < f^q(x,b) - 2\pi p.$$

  This shows that $(f^q - 2\pi p, g^q)$ is a lift of h^q showing that h^q is a twist map. By the Poincaré-Birkhoff Theorem, h^q has two fixed points which yield periodic points of h whose period is divides q. Moreover, periodic points corresponding to different ratios p/q are geometrically distinct.
- **Reference:** The following is basically Section 8.2/8.3 adapted to our notation.
- **Generating functions:** Let $h : A \longrightarrow A$ be area preserving and $\widetilde{h} = (f, g) : (x_0, y_0) \longmapsto (x_1, y_1)$ a lift to the univ. cover. We assume that $\widetilde{h}$ certifies that h is a twist map and $f(x,a) < x$. In addition, assume

$$\frac{\partial f}{\partial y_0} > 0 \tag{4}$$

  This is a version of the monotone twist condition and allows solving $x_1 = f(x_0, y_0)$ for y_0. Let

$$U = \{(x_0, x_1) \in \mathbb{R}^2 \mid f(x_0, a) \leq x_1 \leq f(x_0, b)\}.$$

- **Lemma:** There is a function $S : U \longrightarrow \mathbb{R}$ so that for $(x_0, y_0), (x_1, y_1) \in \widetilde{A}$ and $(x_0, x_1) \in U$

$$\widetilde{h}(x_0, y_0) = (x_1, y_1) \text{ iff } y_0 = -\frac{\partial S}{\partial x_0}, y_1 = \frac{\partial S}{\partial x_1}$$

- **Proof:**

$$\begin{aligned} \widetilde{A} &\longrightarrow U \\ (x_0, y_0) &\longmapsto (x_0, f(x_0, y_0)) \end{aligned}$$

has an inverse of the form

$$\begin{aligned} U &\longrightarrow \widetilde{A} \\ (x_0, x_1) &\longmapsto (x_0, -u(x_0, x_1)) \end{aligned}$$

Using u one can express $y_1 = g(x_0, y_0) = g(x_0, -u(x_0, x_1)) = v(x_0, x_1)$. Then

$$\begin{aligned} \widetilde{h}^*(y_0 dx_0) - y_0 dx_0 &= y_1 dx_1 - y_0 dx_0 \\ &= v(x_0, x_1) dx_1 + u(x_0, x_1) dx_0 \end{aligned}$$

defines a closed form on U. Since $\pi_1(U) = \{1\}$ there is a function $S(x_0, x_1)$ on U so that $dS = v(x_0, x_1)dx_1 + u(x_0, x_1)dx_0$, i.e.

$$\frac{\partial S}{\partial x_1} = v = y_1 \text{ and } \frac{\partial S}{\partial x_0} = -y_0. \tag{5}$$

- This function is unique up to addition of a constant. It *generates* a symplectomorphism from a Lagrangian section (i.e. a closed form) of T^*U: Above (x_0, x_1) the one form dS is $-y_0 dx_0 + y_1 dx_1$ so that $y_1 = g(x_0, y_0)$ and $x_1 = f(x_0, y_0)$.
- **Properties of S:** Differentiating

$$\frac{\partial S}{\partial x_0}(x_0, f(x_0, y_0) = x_1) = -y_0$$

we get

$$\frac{\partial^2 S}{\partial x_0 \partial x_1}(x_0, x_1) \underbrace{\frac{\partial f}{\partial y_0}(x_0, y_0)}_{>0 \text{ by } (4)} = -1 \Rightarrow \frac{\partial^2 S}{\partial x_0 \partial x_1}(x_0, x_1) < 0$$

The fact that boundary components of $\widetilde{A}$ are preserved implies

$$\frac{\partial S}{\partial x_0}(x_0, x_1) = -a \text{ and } \frac{\partial S}{\partial x_0}(x_0, x_1) = a$$

when $x_1 = f(x_0, a)$. Moreover, recall U is invariant under translation by $2\pi(1, 1)$. S is also periodic

$$\frac{\partial}{\partial x_i}(S(x_0 + 2\pi, x_1 + 2\pi) - S(x_0, y_0)) = 0,$$

so the difference which we differentiate is constant on $\widetilde{a}$. This difference can be computed along segments $\gamma(t) = (x_0, +t, f(x_0 + t, a))$ of the boundary of $\widetilde{A}$

$$\begin{aligned}
S(x_0 + 2\pi, f(x_0 + 2\pi, a)) - S(x_0, f(x_0, a)) &= \int_\gamma dS \\
&= \int_\gamma \left(\frac{\partial S}{\partial x_0} dx_0 + \frac{\partial S}{\partial x_1} dx_1 \right) \\
&= \int_\gamma (-y_0 dx_0 + y_1 dx_1) \\
&= \int_\gamma (-a dx_0 + a dx_1) \\
&= 0.
\end{aligned}$$

- **Remark:** The existence of S allows giving yet another proof of the Poincaré-Birkhoff fixed point theorem with monotone twist condition: Since S is invariant under $(2\pi, 2\pi)$ translation, it attains a minimum and a maximum along the diagonal. At such critical points, the gradient of S is perpendicular to the diagonal (i.e. $(\lambda, -\lambda)$ for $\lambda \in \mathbb{R}$), and by (5) this yields two fixed points.

11. Lecture on November, 22 – Discrete Hamiltonian mechanics, convex billiard, Generating functions for Hamiltonian diffeos

- S allows to study the dynamical system $(x_0, y_0) \longrightarrow (x_1, y_1) = \widetilde{h}(x_0, y_0)$ from a variational point of view: For given $x_0, x_1, \ldots, x_l$ so that $(x_{i+1}, x_i) \in U$ for all $i = 1, \ldots, l$. We look for $y_0, y_1, \ldots, y_l$ so that $(x_{i+1}, y_{i+1}) = \widetilde{h}(x_i, y_i)$ for all i. If there is such a sequence, then

$$\frac{\partial S}{\partial x_1}(x_{i-1}, x_i) + \frac{\partial S}{\partial x_1}(x_i, x_{i+1}) = y_i + (-y_i) = 0$$

  for $i = 1, \ldots, i-1$. Thus, if there are such $y_0, \ldots, y_l$, then $(x_0, x_1, \ldots, x_{l-1}, x_l)$ is a critical point of

$$I_l(x_0, x_1, \ldots, x_l) = \sum_{j=0}^{l-1} S(x_j, x_{j+1})$$

  with respect to variations with fixed endpoints (x_0, x_l).
- **Example (Convex billiards)**: Let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ a smooth curve (whose image is an embedded circle), 2π-periodic so that the image bounds a strictly convex disc D in $\mathbb{R}^2$ (i.e. tangents to the curve meet γ in exactly one point). We assume that γ is parametrized by arclength. Let

$$\begin{aligned}
S : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\
(s_0, s_1) &\longmapsto -\|\gamma(s_1) - \gamma(s_0)\|.
\end{aligned}$$

  The dynamical system associated to this function is the motion of a particle in a convex billiards table with the usual reflection at the boundary.

A computation shows that for $t_0 = \text{angle}(\dot\gamma(s_0), \gamma(s_1) - \gamma(s_0)) \in [0, \pi]$ and $t_1 = \text{angle}(\dot\gamma(s_1), \gamma(s_1) - \gamma(s_0)) \in [0, \pi]$

$$\begin{aligned} \frac{\partial S}{\partial s_0} &= \cos(t_0) \\ \frac{\partial S}{\partial s_1} &= -\cos(t_1) \end{aligned} \tag{6}$$

This allows to verify the monotone twist condition for S. Differentiating the second equation above with respect to s_0 one gets

$$\frac{\partial^2 S}{\partial s_0 \partial s_1} = \sin(t_1)\frac{\partial t_1}{\partial s_0} < 0$$

(as s_0 moves towards s_1, the angle t_1 is decreasing). Thus, S defines an area preserving map of the annulus $\mathbb{R} \times [-1, 1]$ where the second variable is $y = -\cos(t)$ (the $-$-sign accounts for the sign difference between the Lemma and (6)). The boundary is preserved since by continuity $t_0 = 0 \Leftrightarrow t_1 = \pi$ and $t_1 = 0 \Leftrightarrow t_0 = \pi$.

Now s_1 is well-defined only up to addition of multiples of $2\pi k$. We may assume $f(x, -1) = x$, this implies that $f(x, 1) = x - 2\pi$. The corollary of the Poincaré-Birkhoff fixed point theorem implies that billiards in a convex domain has infinitely many periodic orbits.

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- The notion of generating function can be generalized to higher dimension. We consider open subsets of $\mathbb{R}^{2n}$ with the usual symplectic structure, let $(x_0, y_0) \in \mathbb{R}^{2n}$ with $x_0 = (x_{0,1}, \ldots, x_{0,n}), dx_0 = \ldots$ etc. Moreover, $\frac{\partial S}{\partial x_0}$ contains n-components of the gradient of S.
- Let $\psi = (f, g) : \Omega \subset \mathbb{R}^{2n} \longrightarrow \Omega' \subset \mathbb{R}^{2n}$ be a symplectomorphism so that

$$\begin{aligned} \Omega &\longrightarrow U \\ (x_0, y_0) &\longmapsto (x_0, f(x_0, y_0)) \end{aligned} \tag{7}$$

is a diffeomorphism. For $d\psi(z_0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then this is true on a neighborhood of z_0 if $\det(B) \neq 0$. (7) implies, that $\psi(\{x_0\} \times \mathbb{R}^n)$ is a submanifold of Ω' which is every transverse to $\{z_1\} \times \mathbb{R}^n$.
- **Lemma:** If Ω (hence U) is simply connected, then there is a function $S : U \longrightarrow \mathbb{R}$ so that for $(x_0, y_0) \in \Omega, (x_1, y_1) \in \Omega'$ with $(x_0, x_1) \in U$

$$(x_1, y_1) = \psi(x_0, y_0) \Leftrightarrow y_0 = -\frac{\partial S}{\partial x_0}(x_0, x_1), y_0 = -\frac{\partial S}{\partial x_1}(x_0, x_1).$$

- Same as the lemma in the context of area preserving twist maps of the annulus. Conversely:
- **Lemma:** Let $U \subset \mathbb{R}^{2n}$ be open and $S : U \longrightarrow \mathbb{R}$ so that

$$\begin{aligned} F : U &\longrightarrow \Omega \\ (x_0, x_1) &\longmapsto \left(x_0, -\frac{\partial S}{\partial x_0}\right) \end{aligned}$$

is a diffeomorphism. Then

$$\begin{aligned} \psi : \Omega &\longrightarrow \psi(\Omega) = \Omega' \subset \mathbb{R}^{2n} \\ (x_0, -\partial_{x_0} S(x_0, x_1)) &\longmapsto (x_1, \partial_{x_1} S(x_0, x_1)) \end{aligned}$$

is a symplectomorphism.

- **Proof:** We will show that $(\psi \circ F)^*(y_1 dx_1) - F^*(y_0 dx_0)$ is exact. This implies that ψ is a symplectomorphism.

$$\begin{aligned}(\psi \circ F)^*(y_1 dx_1) - F^*(y_0 dx_0) &= F^*(\psi)^*(y_1 dx_1) - F^*(y_0 dx_0)\\ &= \partial_{x_1} S dx_1 + (\partial_{x_0} S) dx_0\\ &= dS.\end{aligned}$$

- Our main source for symplectomorphisms are obtained by integrating time-dependent Hamiltonian vector fields. It is therefore important to know the generating functions in this case (assuming that they exist!) The fact that we did not really do any classical mechanics will haunt us now.
- **Example:** Let $H_t; \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a smooth family of Hamiltonian functions and $\psi = \phi_H^{t_0,t_1}$ the associated symplectomorphism. Let $z = (x,y) : [t_0, t_1] \longrightarrow \mathbb{R}^{2n}$ be a C^2-path and define the action functional

(8)
$$\Phi_H(z) = \int_{t_0}^{t_1} \left(\langle y(t), \dot{x}(t)\rangle - H(t, x(t), y(t))\right) dt$$

The domain of this *action functional* are C^∞-curves connecting given pairs of points z_0, z_1. Consider the unique solution $z(t)$ of the boundary value problem $x(0) = x_0, x(1) = x_1$

$$\dot{x}(t) = \frac{\partial H_t}{\partial y}(x(t), y(t)) \qquad \dot{y}(t) = -\frac{\partial H_t}{\partial x}(x(t), y(t)).$$

Solutions of these equations are critical points of the action functional with fixed boundary conditions:

Let $z_s : [t_0, t_1] \longrightarrow \mathbb{R}^{2n}$ be a smooth 1-parameter family of smooth curves and set

$$\xi(t) = \frac{\partial x_s}{\partial s}(t)\bigg|_{s=0} \qquad \eta(t) = \frac{\partial y_s}{\partial s}(t)\bigg|_{s=0}.$$

Differentiating (8) with respect to s and integrating by parts we get $((x(t), y(t)) = z_{s=0}(t))$

(9)
$$\begin{aligned}\frac{\partial \Phi(z_s)}{\partial s}\bigg|_{s=0} &= \int_{t_0}^{t_1} \left(\langle \eta(t), \dot{x}(t)\rangle - \langle \partial_y H(t, x(t), y(t))), \eta(t)\rangle\right) dt\\ &\quad + \int_{t_0}^{t_1} \Big(\langle y(t), \dot{\xi}(t) - \langle \partial_x H(t, x(t), y(t)), \xi(t)\rangle\Big) dt\\ &= \int_{t_0}^{t_1} \left(\langle \eta(t), \dot{x}(t)\rangle - \langle \partial_y H(t, x(t), y(t))), \eta(t)\rangle\right) dt\\ &\quad - \int_{t_0}^{t_1} \left(\langle \dot{y}(t), \xi(t)\rangle - \langle \partial_x H(t, x(t), y(t)), \xi(t)\rangle\right) dt\\ &\quad + \langle y(t_1), \xi(t_1)\rangle - \langle y(t_0), \xi(t_0)\rangle.\end{aligned}$$

If $\xi(t_0) = \xi(t_1) = 0$, i.e. we consider variation with partially fixed endpoints, then critical points of the action functional are solutions of the Hamiltonian equations.

We require $(x_0, x_1) \in U$, i.e. there is a unique solution $z(t) = (x(t), y(t))$ with $x_0 = x(t_0)$ and $x_1 = x(t_1)$ (this is again a consequence of the monotone twist condition/existence of S). Thus, we can consider

$$F : U \longrightarrow C^\infty([t_0,t_1],\mathbb{R}^{2n})$$
$$(x_0,x_1) \longmapsto (z : [t_0,t_1] \longrightarrow \mathbb{R}^{2n})$$

- **Lemma:** If $\psi = \phi_H^{t_0,t_1}$ admits a generating function, then $S_H(x_0,x_1) = \Phi_H(F(x_0,x_1))$ is a generating function
- **Proof:** We vary the boundary condition by $(x_0 + s\xi_0, x_1 + s\xi_1)$ and consider the unique solutions $z_s(t)$ of the corresponding boundary value problems. Differentiating $\Phi(z_s)$ with respect to s one obtains by (9) and knowing that $z(z)$ satisfies the Hamiltonian equations
$$\frac{\partial S_H}{\partial x_0}\cdot\xi_0 + \frac{\partial S_H}{\partial x_1}\cdot\xi_1 = \langle y_1,\xi_1\rangle - \langle y_0,\xi_0\rangle.$$
This is true for all ξ_0,ξ_1, so this implies the claim.

12. Lecture on November 26, – Hamilton-Jacobi equation and consequences

- Since we are free to choose t_1 we let t_1 vary in $[t_0,t_1]$. We get a smooth family of functions $S(t_1, x(t_0), x(t_1))$.
- **Lemma:** Using awful notation, this function satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}(t,x,x(t)) + H\big(t, x(t), \partial_{x_1}S(t,x,x(t))\big) = 0. \tag{10}$$

- **Proof:** Let $z(t) = (x(t), y(t))$ be a solution of the Hamiltonian equations with $x(t_0) = x$. Then by the previous Lemma
$$S(t,x,x(t)) = \int_{t_0}^{t} \left(\langle y(\tau),\dot{x}(\tau)\rangle - H(\tau, x(\tau), y(\tau))\right) d\tau.$$
Differentiating with respect to t and using the fact that $S(t,\cdot)$ generates $\phi_H^{t_0,t}$ we get
$$\partial_t S(t,x_0,x(t)) + \langle \underbrace{\partial_{x_1}S(t,x,x(t))}_{=y(t)}, \dot{x}(t)\rangle = \langle y(t),\dot{x}(t)\rangle - H(t,x,\underbrace{x(t)}_{=\partial_{x_1}S(t,x,x(t))}).$$
- This has applications to the action of Hamiltonian flows on $T^*\mathbb{R}^n$ on certain Lagrangian submanifolds.
- **Lemma:** Let $S : \mathbb{R}^n \longrightarrow \mathbb{R}$ be smooth and consider the connected Lagrangian submanifold L given by the graph of dS in $T^*\mathbb{R}^n$. The Hamiltonian flow of the time independent function $H : T^*\mathbb{R}^n \longrightarrow \mathbb{R}$ preserves L if and only if

$$H(x,\partial_x S) = c. \tag{11}$$

Thus, the graph of dS is invariant under the Hamiltonian flow if S solves the Hamilton Jacobi equation.
- **Proof:** Assume that H is not constant on L, so $dH(Y) \neq 0$ for some $Y \in T_lL$ with $l \in L$. Then $0 = \omega(X_H, L) = -dH(L)$ cannot be satisfied for $X_H \in T_lL$.
  If $H|_S$ is constant, but $X_H \notin T_lL$ for some l, then since $T_lL^{\perp_\omega} = T_lL$ there is $Y \in T_lL$ so that $0 \neq \omega(X_H, L) = -dH(L) = 0$.
- Thus, solutions of the Hamilton Jacobi equation $H(x,\partial_x S) = c$ give rise to invariant submanifolds.

- **Lemma:** For $U \subset \mathbb{R}^n$ open, $S : [t_0, t_1] \times U \longrightarrow \mathbb{R}$ smooth and $H : [t_0, t_1] \times U \times \mathbb{R}^n \longrightarrow R$ we have

$$\partial_t S_t + H(t, x, d_x S_t) = c \text{ for some constant } c \text{ iff}$$
$$\phi_H^{t_0,t} L_{t_0} = L_t$$

where $L_t = \mathrm{graph}(dS_t)$.

- **Proof:** By the Weinstein neighborhood theorem it is enough to consider the case $t = 0$ when dS_0 describes the zero section of T^*R^n. We are considering a local statement which can be proved using local coordinates. The section L_t moves in direction $\partial_t(dS(x))$. In terms of local coordinates this is the vector $\sum \frac{\partial^2 S_t}{\partial q_i \partial t} \frac{\partial}{\partial p_i}$.
  The vector generating the Hamiltonian flow is

$$X_H = \sum_i \left(-\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right)$$

These two vectors have the same component in the fiber direction if and only if

$$d\partial_t S_t + d(dS)^* H = 0 \Leftrightarrow \partial_t S_t + H(t, x, \partial_x S) = c.$$

for some constant c.

- **Remark:** As above, generating functions can be used to describe dynamical systems with discrete time.
  Assume that $\psi : \Omega \longrightarrow \mathbb{R}^{2n}$ is a symplectomorphism and consider sequences of the form $(x_{i+1}, y_{i+1}) = \psi(x_i, y_i)$ for $i = 0, \dots, l-1$. If ψ admits a generating function, then this sequence is completely determined by $x = (x_0, x_1, x_2, \dots, x_l)$ with $(x_i, x_{i+1}) \in U$ and this sequence is a critical point of the functional

$$I_l(x) = \sum_{i=0}^{l-1} S(x_i, x_{i+1})$$

with respect to variations with fixed endpoints.

- In the following we will see that solutions of the Hamiltonian equation which lie on an invariant Lagrangian give are not only critical points of the action functional, but also minimizers.
- **Lagrangian/Hamiltonian mechanics:** So far, we have considered classical mechanics from the Hamiltonian view point. The Lagrangian formalism is equivalent: $L(t, x, v) : ([t_0, t_1] \times U) \subset \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}$ is a smooth function. On the space of paths $C^1([t_0, t_1], \mathbb{R}^n)$ the *action* is defined as

$$I(x) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t)) dt.$$

Critical points of this functional with respect to variations with fixed endpoints $x(t_0), x(t_1)$ are solutions of the Euler-Lagrange equation

$$\frac{d}{dt} \underbrace{\frac{\partial L}{\partial v}(t, x(t), \dot{x}(t))}_{\in \mathbb{R}^n} - \frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) = 0 \tag{12}$$

The variational problem described above can be transformed if the Legendre condition

$$\det\left(\frac{\partial^2 L}{\partial v_i \partial v_j}\right) \neq 0 \tag{13}$$

is satisfied. The *Legendre transformation* transforms the second order system (12) into a first order system. One introduces new variables (*momenta*)

$$y_i = \frac{\partial L}{\partial v_i}(x, v), i = 1, \ldots, n.$$

(13) ensures by the implicit function theorem that one can recover v_i from (x, y) and L at least locally, i.e. there are local functions

$$v_i = G_i(t, x, y).$$

The *Hamiltonian function* $H : V \longrightarrow \mathbb{R}, V \subset \mathbb{R} \times \mathbb{R}^{2n}$ associated to L (Legendre transformation) is

$$H(t, x, y) = \left(\sum_i y_i v_i\right) - L(t, x, v) = \left(\sum_i y_i G_i\right) - L(t, x, G). \tag{14}$$

Differentiating H we get (using (12))

$$\begin{aligned}
\frac{\partial H}{\partial x_k} &= -\frac{\partial L}{\partial x_k} + \sum_i \left(\sum_j y_j \frac{\partial G_j}{\partial x_k} + \sum_j \underbrace{\frac{\partial L}{\partial v_j}}_{=y_j} \frac{\partial G_j}{\partial x_k}\right) \\
&= -\frac{d}{dt}\frac{\partial L}{\partial v_k} = -\dot{y}_k \\
\frac{\partial H}{\partial y_k} &= \left(v_k + \sum_i y_i \frac{\partial G_i}{\partial y_k}\right) - \sum_i \frac{\partial L}{\partial v_i}\frac{\partial G_i}{\partial y_k} \\
&= v_k = \dot{x}_k
\end{aligned}$$

- The following Lemma provides a sufficient condition ensuring that solutions of the Euler-Lagrange equation are *minimizers*, not just critical points.
- time independent setting: Let $S : \Omega \longrightarrow \mathbb{R}$ be a solution of the Hamilton-Jacobi equation (11) with $H : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ so that

$$\left(\frac{\partial^2 H}{\partial y_i \partial y_j}\right) \text{ is positive definite.}$$

This condition implies the Legendre condition for the Legendre transform to be possible and it yields:

$$L : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

Define $f : \Omega \longrightarrow \mathbb{R}^n$ by

$$f(x) = \partial_y H(x, \partial_x S(x)) \Leftrightarrow \partial_x S(x) = \partial_v L(x, f(x)). \tag{15}$$

Solutions of

$$\begin{aligned}
\dot{x}(t) &= f(x) \\
y(t) &= \partial_x(S(x(t))
\end{aligned}$$

satisfy the Hamiltonian equations: The first one is automatic, the other follows from (11)

$$\begin{aligned} 0 &= \frac{d}{dt} H(x(t), \overbrace{\partial_x S(x(t))}^{=y(t)}) \\ &= \frac{\partial H}{\partial x}(x(t), y(t))\dot{x}(t) + \underbrace{\frac{\partial H}{\partial y}(x(t), y(t))}_{=f(x)=\dot{x}} \dot{y}(t) \end{aligned}$$

- **Lemma:** Let $L : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a time-independent Lagrangian function so that $\left(\frac{\partial^2 L}{\partial v_i \partial v_j}\right)$ is positive definite. Assume that $x : [t_0, t_1] \longrightarrow \Omega$ satisfies $\dot{x}(t) = f(x)$ for $f : \Omega \longrightarrow \mathbb{R}^n$ defined above and $\xi : [t_0, t_1] \longrightarrow \Omega$ is another path in Ω with the same endpoints. Then

$$\int_{t_0}^{t_1} L(x, \dot{x}) dt \leq \int_{t_0}^{t_1} L(\xi, \dot{\xi}) dt$$

- **Proof:** Since $\left(\frac{\partial^2 L}{\partial v_i \partial v_j}\right)$ is positive definite, the function lies above the tangent line of graph(L) which projects to a line in direction $v - f(\xi)$, i.e.

$$L(\xi, f(\xi)) + \langle \partial_v L(\xi, f(\xi)), v - f(\xi) \rangle \leq L(\xi, v) \tag{16}$$

with equality if and only if $\xi = v$. We have seen that

$$(x(t), \partial_x S(x(t)) = y(t))$$

satisies the Hamilton equations. By (14) and the Hamilton-Jacobi equation (10)

$$L(x, f(x)) = \langle \underbrace{\partial_x S(x)}_{=y}, f(x) \rangle - \underbrace{H}_{=c}.$$

for all points x (including $\xi(t)$). Integrating the above inequality, we get

$$\begin{aligned} \int_{t_0}^{t_1} L(x, \dot{x}) dt &= \int_{t_0}^{t_1} \left(\langle \partial_x S(x(t)), f(x(t)) \rangle - c \right) dt \\ &= S(x(t_1)) - S(x(t_0)) - c(t_1 - t_0) \\ &= \int_{t_0}^{t_1} \Big(\langle \partial_x S(\xi), \dot{\xi} \rangle - c \Big) dt \\ &= \int_{t_0}^{t_1} \Big(\langle \partial_x S(\xi), \dot{\xi} - f(\xi) \rangle + \langle \partial_x S(\xi), f(\xi) \rangle - c \Big) dt \\ &= \int_{t_0}^{t_1} \Big(\langle \partial_x S(\xi), \dot{\xi} - f(\xi) \rangle + L(\xi, f(\xi)) \Big) dt \\ &\leq \int_{t_0}^{t_1} L(\xi, \dot{\xi}) dt. \end{aligned}$$

The inequality follows from (16) with $v = \dot{\xi}$ and the second version of the definition (15) of f.

13. Lecture on November, 29 – Other generating functions, Discussion of Arnol'd conjecture for (T^{2n}, ω_{st})

- Here is the theorem we want to prove before Christmas.
- **Theorem (Arnol'd conjecture for T^{2n}):** Let ψ be a Hamiltonian diffeomorphism of the $2n$-torus with its standard symplectic structure given by $(\mathbb{R}^{2n}, \omega_{st})/\mathbb{Z}^{2n}$. Then ψ has at least as many fixed points as a function has critical points (i.e. $2n+1$). If all fixed points are non-degenerate, than ψ has as many fixed points as a Morse function (i.e. a function such that all critical points are non-degenerate) has critical points (i.e. 2^{2n}).
- **Remark:** Of course we will consider the lift of ψ to $\mathbb{R}^{2n}$. Then we will decompose the periodic symplectomorphism ψ into $\psi_{N-1} \circ \ldots \circ \psi_0$ so that ψ_j has a generating function V_j as in (18).

  Another fact that we want to use is that these generating functions V_j are invariant under decktransformations of the universal covering $\mathbb{R}^{2n} \longrightarrow T^{2n}$. So far, we only know that V satisfies a condition $V(x+e_j, y) = V(x,y) + \alpha_j$ and $V(x, y+e_j) = V(x) + \beta_j$ for constants α_j, β_j for all standard generators e_j of $\mathbb{Z}^{2n}$.

  That V_j is the lift of a function on T^{2n} is important since we want to use properties of functions on T^{2n}. In order to show this we need to better understand the difference between symplectomorphisms (isotopic to the identity through symplectomorphisms) and Hamiltonian diffeomorphisms. Note that any translation of T^{2n} is symplectic and isotopic to the identity through translations. However, many translations have no fixed points at all.

  The final part of the proof will then be a study of the relationship between the topology of T^{2n} and the action functional $\Phi : \mathbb{R}^{2nN} \longrightarrow \mathbb{R}$ which is translation invariant (under integral translations). This is the least symplectic part of this discussion.
- **Remark:** The Lefschetz fixed point theorem gives no guarantee for the existence of fixed points of diffeomorphisms of T^{2n} isotopic to the identity since $\chi(T^{2n}) = 0$.

  The example of translation of T^{2n} shows that symplectic/volume preserving maps which are isotopic to the identity through symplectic/volume preserving maps do not have fixed point in general. Thus, the Arnol'd conjecture theorem shows that Hamiltonian diffeo's are essentially less flexible than volume preserving maps (the difference between Hamiltonian and symplectic can be understood). More interestingly, it shows that

$$\overline{\mathrm{Ham}(T^{2n}, \omega_{st)})}^{C^0} \cap \mathrm{Diff}_0(T^{2n}, \mu_{vol})$$

  is a proper subgroup of the measure preserving diffeomorphisms. This is one of the first results implying that the symplectic category is more rigid, that the topological/smooth/measure preserving category.
- We will need a version generating function which is adapted to global symplectomorphism $\psi = (u, v) : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$. If the condition

$$\|d\psi - \mathrm{id}\| \leq 1/2 \tag{17}$$

  is satisfied (operator norm), then the B-submatrix of $d\psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ maybe degenerate, but the A-part is non-degenerate.

- Assume that the map $(x_0, y_0) \longmapsto (u(x_0, y_0), y_0)$ is a diffeomorphism then we can replace the independent variable (x_0, y_0) by $(u(x_0, y_0) = x_1, y_0)$. Under the condition (17) this is always the case!
- **Fact:** Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth map so that $\|Df - \mathrm{id}\| \leq 1/2$. Then f is a diffeomorphism.
- **Proof of Fact:** Let $y \in \mathbb{R}^n$. We try to find $f^{-1}(y)$. For this consider $\Phi(x) = x - f(x)$. Then the map

$$x \longmapsto y + \Phi(x)$$

is contracting and has a therefore a unique fixed point by the Banach fixed point theorem which depends continuously on the input variables y and Φ. Therefore, f is surjective and injective. By the assumption $\|Df - \mathrm{id}\| \leq 1/2$ it is a local diffeomorphism everywhere. Thus, f is a diffeomorphism.
- **Lemma:** Given ψ as above, there exists a smooth function V on $\mathbb{R}^{2n}$ so that $(x_1, y_1) = \psi(x_0, y_0) = (f(x_0, y_0), g(x_0, y_0))$ if and only if

$$\begin{aligned} x_1 - x_0 &= \frac{\partial V}{\partial y}(x_1, y_0) \\ y_1 - y_0 &= -\frac{\partial V}{\partial x}(x_1, y_0). \end{aligned} \tag{18}$$

- **Proof:** There is a smooth inverse $U : (x_1, y_0) \longmapsto (u(x_1, y_0), y_0)$ of $F : (x_0, y_0) \longmapsto (f(x_0, y_0), y_0)$. Set

$$y_1 = g(x_1, y_0) = g(u(x_1, y_0), y_0).$$

Then $U^*(\psi^*(y_1 dx_1) + x_0 dy_0) = v dx_1 + u dy_0$ is closed since ψ is a symplectomorphism and both xdy and $-ydx$ are primitives of the same symplectic form. Since $\mathbb{R}^{2n}$ is simply connected there is a function $W : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ so that $dW = vdx_1 + udy_0$. The function $V(x_1, y_0) = \langle x_1, y_0 \rangle - W(x_1, y_0)$ is the desired function.
- There is a structural similarity between (18) and the Hamiltonian equation. This suggest viewing W as a discrete time analogue for Hamiltonian functions.

14. Lecture on December, 3 – Fixed points of Hamiltonians and critical points of a discrete action functional

- Let $H(t, x, y)$ be a time dependent family of C^2-functions on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ so that I is a compact interval and

$$\sup_{(t,z) \in \mathbb{R} \times \mathbb{R}^{2n}} \left(\left\| \mathrm{Hess}^{\mathbb{R}^{2n+1}} H_t(z) \right\| + \left\| d^{\mathbb{R}^{2n+1}} H(t, z) \right\| \right) < \infty$$

The boundedness of the first summand ensures that X_{H_t} is Lipschitz, the second part ensures that the length of X_{H_t} is bounded and varies in a Lipschitz fashion with t. This implies that solutions of initial value problems exist for all times, initial conditions and that these solutions are unique.

For $t_0 < t_1$ let $\phi_H^{t_0, t_1}$ be the Hamiltonian diffeomorphism defined by H_t (initial value problem with initial condition at $t = t_0$). For big N the diffeomorphisms

$$\phi_H^{\tau_j, \tau_{j+1}} \text{ with } \tau_j = t_0 + \frac{j}{N}(t_1 - t_0)$$

satisfy the condition (17) and

$$\underbrace{\phi_H^{t_0,t_1}}_{=\psi} = \underbrace{\phi_H^{\tau_{N-1},\tau_N}}_{\psi_{N-1}} \circ \underbrace{\phi_H^{\tau_{N-1},\tau_{N-2}}}_{\psi_{N-2}} \circ \ldots \circ \underbrace{\phi_H^{\tau_1,\tau_0}}_{\psi_0}.$$

For each $j = 0, \ldots, N-1$, there is a function V_j as in (18) which determines ψ_j. This can be used to describe solutions of $(x_N, y_N) = \psi(x_0, y_0)$ in variational terms.

- Let $\mathcal{P} = \mathbb{R}^{2N+1}n$. Points in $\mathcal{P}$ are discrete paths $(x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1}, x_N)$. Let

$$\Phi : \mathcal{P} \longrightarrow \mathbb{R}$$

(19)
$$(x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1}, x_N) \longmapsto \sum_{j=0}^{N-1} \left(\langle y_j, x_{j+1} - x_j\rangle - V_j(x_{j+1}, y_j)\right).$$

This functional is of course suggested by (14). y_N can be read of from (x_{N-1}, y_{N-1}) and V_{N-1} via (18).

- **Lemma:** $z \in \mathcal{P}$ is critical for Φ with respect to variations $\zeta = (\xi_0, \xi_1, \ldots, \xi_N, \eta_0, \ldots, \eta_{N-1})$ with $\xi_0 = \xi_N$ if and only if z satisfies the difference equations (18).
- **Proof:** This is a computation for $j = 1, \ldots, N-1$:

(20)
$$\begin{aligned}
\frac{\partial \Phi}{\partial x_j} &= -y_j + y_{j-1} - \frac{\partial V_{j-1}}{\partial x}(x_j, y_{j-1}) \\
\frac{\partial \Phi}{\partial y_j} &= x_{j+1} - x_j - \frac{\partial V_j}{\partial y}(x_{j+1}, y_j).
\end{aligned}$$

The second computation works also for $j = 0$.

- The analogy between V_j and Hamiltonians H_t goes further: One can show that if ψ admits a generating function S as on p. 19, then (x_0, x_N) determine the critical point $z = ((x_0, y_0), \ldots (x_N, y_N))$ uniquely. This defines a map $\mathbb{R}^{2n} \longrightarrow \mathcal{P}$ taking (x_0, x_N) to z and

$$S((x_0, x_N)) = \Phi(z).$$

- Let

$$\mathcal{P}_{per} = \{z \in \mathcal{P} \,|\, x_0 = x_N\} \simeq \{N- \text{ periodic sequences}\}$$

We extend ψ_j, V_j in a N-periodic fashion, $\psi_{j+N} = \psi_j$ and $V_{j+N} = V_j$.

- Recall that a fixed point x of a diffeomorphism ψ of M is non-degenerate if the graph of $d\psi$ in $T_xM \times T_xM$ is transverse to the diagonal. This is equivalent to the condition that 1 is not an eigenvalue of $d\psi$. Otherwise a fixed point is degenerate.
- **Lemma (ingredient # 1 for the Arnol'd conjecture for T^{2n}):** $z \in \mathcal{P}_{per}$ is a critical point of $\Phi : \mathbb{P}_{per} \longrightarrow \mathbb{R}$ if and only if (x_0, y_0) is a fixed point of ψ and $z_{j+1} = \psi_j(z_j)$. It is a non-degenerate critical point if and only of (x_0, y_0) is a non-degenerate fixed point.
- **Proof:** The identities (20) now hold for all j. If (x_0, y_0) is a fixed point of ψ and $(x_{j+1}, y_{j+1}) = \psi_j(x_j, y_j)$, then z is a critical point of Φ by (20) together with the defining property (18) of V_j. The converse is just as obvious.

For the second part we compute the Hessian of Φ at a critical point. This gives rise to a symmetric linear operator

$$
\begin{aligned}
\xi'_{j+1} &= \eta_j - \eta_{j+1} - \frac{\partial^2 V_j}{\partial x^2}(x_{j+1}, y_j)\xi_{j+1} - \frac{\partial^2 V_j}{\partial x \partial y}(x_{j+1}, y_j)\eta_j \\
\eta'_j &= \xi_{j+1} - \xi_j - \frac{\partial^2 V_j}{\partial x \partial y}(x_{j+1}, y_j)\xi_{j+1} - \frac{\partial^2 V_j}{\partial^2 y}(x_{j+1}, y_j)\eta_j
\end{aligned} \tag{21}
$$

so that

$$
(\ \widehat{\xi}^T \quad \widehat{\eta}^T)\ \mathrm{Hess}(\Phi) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = (\ \widehat{\xi}^T \quad \widehat{\eta}^T)\ \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}
$$

We want to determine the kernel of this operator in terms of ψ. Recall that from (18)

$$
\begin{aligned}
x_{j+1} - x_j &= \frac{\partial V_j}{\partial y}(x_{j+1}, y_j) \\
y_{j+1} - y_j &= -\frac{\partial V_j}{\partial x}(x_{j+1}, y_j).
\end{aligned} \tag{22}
$$

if and only if $(x_{j+1}, y_{j+1}) = \psi_j(x_j, y_j)$. Differentiating this we get $D\psi_j(\zeta_j) = \widehat{\zeta}_{j+1}$, or in other terms

$$
\begin{aligned}
\widehat{\xi}_{j+1} - \xi_j &= \frac{\partial^2 V_j}{\partial x \partial y}\widehat{\xi}_{j+1} + \frac{\partial^2 V_j}{\partial^2 y}\eta_j \\
\widehat{\eta}_{j+1} - \eta_j &= -\frac{\partial^2 V_j}{\partial x^2}\widehat{\xi}_{j+1} - \frac{\partial^2 V_j}{\partial x \partial y}\eta_j
\end{aligned} \tag{23}
$$

Because ψ_j is a diffeomorphism, the solutions $\widehat{\xi}_j, \widehat{\eta}_j$ of this linear system of equations (with given ξ_j, η_j) are unique and they exist. In particular, if $(\widehat{\xi}_N, \widehat{\eta}_N) = D\psi(\xi_0, \eta_0) = (\xi_0, \eta_0) = (\xi_N, \eta_N)$, then $(\xi_j, \eta_j) = (\widehat{\xi}_j, \widehat{\eta}_j)$ for all j.

By (23) the kernel of the operator at a critical point (implying that (22) holds) defined in (21) corresponds to of variations ζ so that $\zeta_{j+1} = d\psi_j(z_j)\zeta_j$. Applying the chain rule to $\psi = \psi_{N-1} \circ \ldots \circ \psi_0$ we get

$$
\zeta_0 = \zeta_{N+0} = d\psi(z_0)\zeta_0.
$$

Since z and ζ are periodic, this implies that the kernel of the Hessian is empty if and only if 1 is not an eigenvalue of $d\psi(z_0)$.

Conversely, assume that (ξ_0, η) is an eigenvector of $d(\psi)$ with eigenvalue 1. This corresponds to a solution of (23) with $\widehat{\xi}_j = \xi_j$ and $\widehat{\eta}_j = \eta_j$. This is equivalent to a non-trivial solution $(\xi', \eta') = 0$,i.e. the degeneracy of the Hessian of Φ.

- The previous Lemma relates properties of a quadratic form with properties of a linear map. These things transfrom differently. Since quadradic forms and linear maps transform differently one cannot expect very many such relationships.

15. Lecture on December, 6 – Flux homomorphism, Ham versus Symp

- The second step in the proof of the Arnol'd conjecture for T^{2n} is showing that if $\psi = \psi_{N-1} \circ \ldots \circ \psi_0 : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$, is a lift of a Hamiltonian diffeomorphism of T^{2n},then V_j are lifts of functions on T^{2n}. Note that do far, we only used that ψ is symplectic.)

This will be a consequence of a broader discussion of the difference between $\mathrm{Symp}_0(M,\omega)$ and $\mathrm{Ham}(M,\omega)$ and some properties of these groups.

- **Theorem (Weinstein):** If (M,ω) is closed, then $\mathrm{Symp}_0(M,\omega)$ is locally path connected in the C^1-topology.
- **Proof:** Let $\psi \in \mathrm{Symp}(M,\omega)$. Then the graph of ψ is a Lagrangian submanifold of $(M\times M, \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega)$.

  If ψ is C^0 close to the identity, then the graph is contained in a Weinstein neighborhood $N(\Delta) \longrightarrow \Delta$ with $N(\Delta)$ being symplectomorphic to a neighborhood $N(M \subset T^*M)$ of $M \subset (T^*M, d\lambda)$ of the diagonal Δ (=graph of the identity, and a Lagrangian). If the C^1-distance between ψ and id is small enough, then graph(Γ) is transverse to the fibers of the projection $N(\Delta) \longrightarrow \Delta$ so that the inclusion $M \longrightarrow N(\Delta) \simeq N(M \subset T^*M)$ is induced by a 1-form σ (this is true for $\psi = \mathrm{id}$ and we are dealing with an open condition). For example id corresponds to $\sigma \equiv 0$.

  Again restricting $d_{C^1}(\psi, \mathrm{id})$ we may assume that all 1-forms $s\sigma, s \in [0,1]$ intersect the fibers of the projection $\mathrm{pr}_2 : M \times M \longrightarrow M$ transversely, so that they define graphs of diffeomorphisms ψ_s of M. Since the graphs of all these 1-forms correspond to closed forms, so ψ_s is a symplectomorphism.
- **Consequence:** The connected component $\mathrm{Symp}_0(M,\omega)$ of id in $\mathrm{Symp}(M,\omega)$ is the path-connected component of the identity.
- **Consequence of the proof:** The proof shows more than stated, namely that $\mathrm{Symp}(M,\omega)$ is locally contractible. By standard theory of coverings this implies that $\mathrm{Symp}_0(M,\omega)$ has a universal covering

$$\widetilde{\mathrm{Symp}}_0(M,\omega) \longrightarrow \mathrm{Symp}_0(M,\omega)$$

  so that the covering projection is a homomorphism of groups, a local homeomorphism and $\widetilde{\mathrm{Symp}}_0(M,\omega)$ is simply connected. The group structure can be obtained in two equivalent ways:
  1. $[\phi_t]\cdot[\psi_t] = [\phi_t \circ \psi_t]$ using the product structure on G.
  2. $[\phi_t]\cdot[\psi_t]$ is represented by the concatination of the two paths $[\phi_t]$ and $[\phi_1 \circ \psi_t]$.

  These two ways yield equivalent (homotopic) results (consider $\phi_t \circ \psi_s$).
- **Proposition:** $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega)$ is a normal subgroup.
- **Proof:** Let $H : M \longrightarrow \mathbb{R}$ be smooth and $\phi \in \mathrm{Symp}(M,\omega)$. Then

$$\begin{aligned}
-d(H\circ\phi)(Y_x) &= (\phi^*(-dH))(Y_x) = (\phi^*(\omega(X_H,\cdot)))\,(Y_x)\\
&= \omega(X_H, \phi_*(Y_X)) = (\phi^{-1*}\omega)(X_H, \phi_*(Y_x))\\
&= \omega(\phi_*^{-1}X_H, Y_x) = \omega(\phi_*^{-1}(X_H(\phi(x)))\big(Y_x\big).
\end{aligned}$$

  Now assume that ψ_t is the Hamiltonian flow of H_t. Then $d/dt\psi_t = X_{H_t}\circ\psi_t$ and

$$\begin{aligned}
\frac{d}{dt}\left(\phi^{-1}\circ\psi_t\circ\phi\right)(x) &= \phi_*^{-1}\left(X_{H_t}(\psi_t\circ\phi(x))\right)\\
&= \phi_*^{-1}\left(X_{H_t}(\phi\circ\phi^{-1}\circ\psi_t\circ\phi(x))\right)\\
&= X_{H_t\circ\phi}(\phi^{-1}\circ\psi_t\circ\phi)(x).
\end{aligned}$$

  Thus, $H_t\circ\phi$ generates $\phi^{-1}\circ\psi_t\circ\phi$.
- **Remark:** According to a theorem of Banyaga, $\mathrm{Ham}(M,\omega)$ is a simple group when M is closed.

- **Definition:** Let $(M, d\lambda)$ be an exact symplectic manifold. Then $\mathrm{Symp}^c(M,\omega)$ is the group of compactly supported symplectomorhpisms topologized as direct limit the group $\mathrm{Symp}^K(M,\omega)$ symplectomorphisms with support in compact sets K. $\mathrm{Ham}^K(M,\omega)$ is generated by functions with support in K and $\mathrm{Ham}^c(M,\omega)$ is topologized in the same way as $\mathrm{Symp}^c(M,\omega)$.
- **Proposition:** Let (M,ω) be connected exact symplectic and ϕ_t an isotopy connecting $\phi = \phi_1$ to $\mathrm{id} = \phi_0$. Then ϕ_t is a symplectic isotopy if and only if $\phi_t^*\lambda - \lambda$ is closed for all $t \in [0,1]$. It is a Hamiltonian isotopy generated by H_t if and only if $\phi_t^*\lambda - \lambda = dF_t$ for a smooth family of smooth functions $F_t : M \longrightarrow \mathbb{R}$ and

$$F_t = \int_0^t (i_{X_s}\lambda - H_s)\, ds \tag{24}$$

up to a function which depends only on t.
- **Proof:** The first part is obvious. Assume that ϕ_t is generated by the functions H_t with Hamiltonian vector fields X_t. Then by (1)

$$\begin{aligned}\frac{d}{dt}\phi_t^*\lambda &= \phi_t^*\left(L_{X_t}\lambda\right)\\ &= -d\phi_t^*(H_t - i_{X_t}\lambda)\\ \frac{d}{dt}\phi_t^*\lambda &= \frac{d}{dt}\left(\lambda + dF_t\right)\\ &= d\dot{F}_t\end{aligned}$$

This is implies (24).
- The previous proposition characterizes Hamiltonian isotopy among symplectic isotopies (and symplectic isotopies among smooth isotopies) when the symplectic form is exact.
- **Definition:** The *flux homomorphism* of a closed symplectic manifold is

$$\begin{aligned}\mathrm{Flux} : \widetilde{\mathrm{Symp}}_0(M,\omega) &\longrightarrow H^1_{dR}(M)\\ [\phi_t] &\longmapsto \left[\int_0^1 i_{X_t}\omega\, dt\right]\end{aligned} \tag{25}$$

where ϕ_t is a symplectic isotopy of (M,ω), $[\phi_t]$ its homotopy class relative end points and X_t satisfies

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

- **Lemma:** This is well defined.
- **Proof:** $i_{X_t}\omega$ is closed because ϕ_t is a symplectic isotopy. For $\gamma : S^1 \longrightarrow M$

$$\int_{S^1}\int_0^1 (\gamma^*(i_{X_t}\omega)\, dt) = \int_0^1\int_0^1 \omega(X_t(\gamma(s)), \dot\gamma(s))\, dt\, ds$$

depends only on the free homotopy class of γ. Define $\beta : S^1 \times [0,1] \longrightarrow M$ as $\beta(s,t) = \phi_t^{-1}(\gamma(s)) \Leftrightarrow \phi_t(\beta(s,t)) = \gamma(s)$. Then

$$\begin{aligned}\dot\gamma(s) &= d\phi_t(\beta(s,t))\frac{\partial\beta}{\partial s}(s,t)\\ \frac{d}{dt}\phi_t(\beta(s,t)) &= X_t(\beta(s,t)) + d\phi_t(\beta(s,t))\frac{\partial\beta}{\partial t}(s,t)\\ &= 0.\end{aligned}$$

We equip $S^1 \times [0,1]$ with the product orientation. Then

$$\begin{aligned}
\mathrm{Flux}(\phi_t) &= \int_0^1 \int_0^1 \omega\left(X_t(\gamma(s)), \dot{\gamma}(s)\right) dt\, ds \\
&= \int_0^1 \int_0^1 \omega\left(-d\phi_t(\beta(s,t))\frac{\partial \beta}{\partial t}, d\phi_t(\beta(s,t))\frac{\partial \beta}{\partial s}\right) dt\, ds \\
&= -\int_0^1 \int_0^1 (\phi_t^*\omega)\left(\frac{\partial \beta}{\partial t}, \frac{\partial \beta}{\partial s}\right) dt\, ds \\
&= \int_0^1 \int_0^1 \omega\left(\frac{\partial \beta}{\partial s}, \frac{\partial \beta}{\partial t}\right) dt\, ds.
\end{aligned}$$

Since ω is closed this depends only on the homotopy class of β so that the boundary of the image consists of the fixed closed curves $\gamma(s), s \in [0,1]$, $\phi_1(\gamma(s)), s \in [0,1,]$ and $\beta(1,t) = \beta(0,t)$ for all t. This is a standard application of Stokes theorem (and the fact that integrals of 2-forms over closed curves vanish).

- The last expression allows a geometric interpretation of the Flux: $\langle \mathrm{Flux}([\phi_t]), \gamma\rangle$ is the symplectic area swept out by the cylinder $\phi_t(\gamma_s))$.
- **Lemma:** Flux is a homomorphism of groups.
- **Proof:** Let $[\phi_t], [\psi_t] \in \widetilde{\mathrm{Symp}}_0(M,\omega)$ and view $[\phi_t]\cdot[\psi_t]$ as concatination of paths $[\phi_t]$ and $[\phi_1 \circ \psi_t]$. Then the interpretation of the Flux and the fact that ϕ_1 is a symplectomorphism implies that $\langle \mathrm{Flux}(\cdot), \gamma\rangle$ is a homomorphism. This implies Flux is a homomorphism.
- **Example:** Let α be a closed form and X so that $\omega(X_\alpha, \cdot) = -\alpha$ and ϕ_t the flow of X_α. Then $\mathrm{Flux}([\phi_t]) = -\alpha$. In particular, Flux is surjective.

16. Lecture on December, 10 – More Flux homomorphism, Ham versus Symp

- The Flux-homomorphism can be used to characterize Hamiltonian diffeomorphisms.
- **Theorem:** $\phi \in \mathrm{Symp}_0(M,\omega)$ is Hamiltonian if and only if there is a symplectic isotopy ϕ_t so that $\phi_0 = \mathrm{id}$, $\phi_1 = \phi$ and $\mathrm{Flux}([\phi_t]) = 0$.
- **Proof:** Assume ϕ is Hamiltonian, i.e. there is a family of functions H_t so that X_{H_t} generates ϕ_t with $\phi = \phi_1$. Then

$$\mathrm{Flux}([\phi_t]) = \left[\int_0^1 i_{X_t}\omega dt\right] = \left[\int_0^1 (-dH_t)dt\right] = 0 \in H^1_{dR}(M).$$

Conversly: Assume that ϕ_t is a symplectic isotopy with $\phi_1 = \phi$ and $\mathrm{Flux}([\phi_t]) = 0$. Using this last fact we want to modify the symplecic isotopy $[\phi_t]$ so that the result ϕ'_t is Hamiltonian for all t and $\phi'_t = \phi_t$ for $t = 0, 1$.

1. Step: Reduction from $\int_0^1 i_{X_t}\omega dt = dF$ to $\int_0^1 i_{X_t}\omega dt = 0$. Let X be the Hamiltonian vector field of F and ψ_t its Hamiltonian flow. We view $[\phi_t]\cdot[\psi_t]$ as a concatination of paths. Then the flux of this vanishes not only in H^1 but already in Ω^1. If we find ϕ'_t for the new isotopy $[\phi_t \cdot \psi_t]$ with the desired properties, then we achieved our goal for $\phi_1 = \phi$ since the difference between $\phi_1 \circ \psi_1$ and ϕ_1 is Hamiltonian.

From now one we may assume that

$$\int_0^1 \omega(X_t,\cdot)dt \equiv 0 \Leftrightarrow \int_0^1 X_t\,dt \equiv 0.$$

2. Step: Let $Y_t = -\int_0^t X_\tau d\tau$. This is a symplectic vector field as an integral of symplectic vector fields. For t fixed let ϑ_t^s be the flow of Y_t (with time parameter s). Then $\phi_t' := \vartheta_t^1 \circ \phi_t$ has constant vanishing flux since Flux is a homomorphism: For $0 \le T \le 1$ and by the homotopy invariance of the flux in the first/second line

$$\begin{aligned}\mathrm{Flux}([\phi_t'], 0 \le t \le T) &= \mathrm{Flux}([\vartheta_t^1, 0 \le t \le T]) + \mathrm{Flux}([\phi_t, 0 \le t \le T]) \\ &= \mathrm{Flux}([\vartheta_T^s, 0 \le s \le 1]) + \left[\int_0^T i_{X_t}\omega\,dt\right] \\ &= \left[i_{Y_T}\omega + \int_0^T i_{X_t}\omega\,dt\right] \\ &= 0.\end{aligned}$$

ϕ_t' is homotopic to ϕ_t through symplectomorphisms relative to the endpoints (for this note that $Y_1 \equiv 0 \equiv Y_0$ so that the corresponding flows are constant).

3. Step: If $[\phi_t, 0 \le t \le T]$ has constant flux 0, then

$$0 = \left[\frac{d}{dT}\mathrm{Flux}([\phi_t, 0 \le t \le T])\right] = [i_{X_T}\omega]\,.$$

Therefore, X_T is a Hamiltonian vector field for all T, the normalized primitive depends smoothly on T.

- **Fact:** Assume that M is connected. There is an obvious exact sequence of Lie-algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow (C^\infty(M), \{\cdot,\cdot\}) \longrightarrow (\chi(M,\omega), [\cdot,\cdot]) \longrightarrow H^1_{dR}(M) \longrightarrow 0$$

Here $\chi(M,\omega) = \{X \in \Gamma TX \mid L_X\omega = 0\}$ is the space of symplectic vector fields with the standard Lie bracket. The map to $H^1_{dR}(M)$ is $X \longmapsto i_X\omega$ while the map from $C^\infty(M)$ assigns Hamiltonian vector fields to smooth functions.

The previous proposition allows to promote this sequence to groups.

- **Corollary:** Let M be closed. There is an exact sequence of simply connected topological groups

$$1 \longrightarrow \widetilde{\mathrm{Ham}}(M,\omega) \longrightarrow \widetilde{\mathrm{Symp}}_0(M,\omega) \longrightarrow H^1_{dR}(M) \longrightarrow 0$$

The first map is the inclusion, the second map is the Flux-homomorphism.

- Originally, we were interested in $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega)$ and not in universal covers. For this we will use yet another exact sequence:
- **Lemma:** Let (M,ω) be closed and symplectic. Then there is an exact sequence

$$0 \longrightarrow \pi_1(\mathrm{Ham}(M,\omega) \longrightarrow \pi_1(\mathrm{Symp}_0(M,\omega)) \longrightarrow \Gamma_\omega \longrightarrow 0.$$

Here loops are viewed as paths in the universal covering, the map to from $\pi_1(\mathrm{Symp}_0(M,\omega))$ is induced by the flux homomorphism and the image of the restriction of Flux to $\pi_1(\mathrm{Symp}_0) \subset \widetilde{\mathrm{Symp}}_0$ has image Γ_ω. The map from $\pi_1(\mathrm{Ham}(M,\omega))$ is induced by inclusion. We use a topology on $\mathrm{Ham}(M,\omega)$ which could be finer a priori than the C^1-topology/the subspace topology of

$\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega)$: Namely, ϕ is ϵ-close to id if there is a time dependent Hamiltonian flow ϕ_t so that $\phi_1 = \phi, \phi_0$id and ϕ_t is ε-close (in the C^1-norm) to the identity. It turns out that this topology coincides (this is related to the flux-conjecture which is no longer a conjecture) with the subspace topology. What matters here is that the inclusion map $\mathrm{Ham}(M,\omega)$ is continuous.

Using the topology we described, defining the universal cover of $\mathrm{Ham}(M,\omega)$ is standard since every point has a contractible neighborhood.

- **Proof:** The only non-trivial statement is the injectivity of the inclusion induced map. Assume that $[\gamma] \in \pi_1(\mathrm{Ham}(M,\omega))$ is a loop which is null homotopic in $\mathrm{Symp}(M,\omega)$. Then there is a continuous family of paths $\gamma_s^{(t)}, s,t \in [0,1]$ so that $\gamma_0^{(t)} = \mathrm{id}$ and $\gamma_1(t) = \gamma(t)$. Each of the curves γ_s ends at a symplectomorphism with vanishing flux. The proof of the theorem allows to replace the path of symplectomorphisms $\gamma_s^{(t)}, s \in [0,1]$, with a path $\hat{\gamma}_s^{(t)}$ in $\mathrm{Ham}(M,\omega)$ with the same endpoints.

  The proof of the theorem implies that one can choose $\hat{\gamma}_s^{(t)}$ continuously in t, so γ is null homotopic in $\mathrm{Ham}(M,\omega)$ if it is null homotopic in $\mathrm{Symp}_0(M,\omega)$.
- Finally, we can establish a precise statement about the relationship between $\mathrm{Ham}(M,\omega)$ and $\mathrm{Symp}_0(M,\omega)$. It is an immediate consequence of what we know.
- **Proposition:** There is an exact sequence

$$0 \longrightarrow \mathrm{Ham}(M,\omega) \longrightarrow \mathrm{Symp}_0(M,\omega) \longrightarrow \frac{H^1_{dR}(M)}{\Gamma_\omega} \longrightarrow 0. \tag{26}$$

  where the first map is inclusion and the second map is induced by the Flux-homomorphism.
- **Example:** We will study $\mathrm{Symp}(T^{2n},\omega_{st})$ (not only the connected component of id). We fix the universal cover $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}/\mathbb{Z}^{2n} = T^{2n}$.
  1. **Observation:** Let $\phi \in \mathrm{Symp}(T^{2n},\omega_{st})$ and $\widetilde{\phi} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ a lift. For $l \in \mathbb{Z}^{2n}$,
  $$\widetilde{\phi}(w+l) - \widetilde{\phi}(w) = A(l) \in \mathbb{Z}^{2n}$$
  is constant and A is $\mathbb{Z}$-linear (look at $\widetilde{\phi}(w+(l+m)) - \widetilde{\phi}(w) = \widetilde{\phi}(w+(l+m)) - \widetilde{\phi}(w+l) - (\widetilde{\phi}(w+l) - \widetilde{\phi}(w))$. Since $\widetilde{\phi}$ is invertible, the same is true for A, i.e. $A \in \mathrm{Gl}(2n,\mathbb{Z})$.
  A is the identity iff ϕ acts trivially on $H_1(T^{2n},\mathbb{Z})$.
  2. **Observation:** A is a symplectic: If A represents the homomorphism $\phi_* : \pi_1(T^{2n}) \longrightarrow \pi_1(T^{2n}) \simeq H_1(T^{2n},\mathbb{Z})$ with respect to the standard basis of $\mathbb{Z}^{2n}$, so A^T represents the dual morphism on $H^1(T^{2n},\mathbb{Z})$ with respect to the dual basis $dx_1, dy_1, \ldots, dx_n, dy_n$ (via integration of forms). $H^1(T^{2n},\mathbb{R})$ has a symplectic structure
  $$\Omega([\alpha],[\beta]) = \int_{T^{2n}} \alpha \wedge \beta \wedge \omega_{st}^{n-1}.$$
  Since $\phi^*\omega_{st} = \omega_{st}$
  $$\int_{\phi(T^{2n})} \alpha \wedge \beta \wedge \omega_{st}^{n-1} = \int_{T^{2n}} \Phi^*(\alpha \wedge \beta \wedge \omega_{st}^{n-1})$$
  implies $\Omega([\alpha],[\beta]) = \Omega(\phi^*[\alpha],\phi^*[\beta])$.

3. **Proposition:** Assume ϕ_t is a symplectic isotopy and $\widetilde{\phi}_t$ is the unique lift with $\widetilde{\phi}_0 = \mathrm{id}_{\mathbb{R}^{2n}}$. Then $\widetilde{\phi}_t(w+l) = \widetilde{\phi}_t(w) + l$ with $l \in \mathbb{Z}^{2n}$ and

(27)
$$\mathrm{Flux}([\phi_t]) = \left[\sum_{j=1}^{2n} a_j dw_j\right]$$

with

$$a = (a_1, \ldots, a_{2n}) = J_0 \int_{T^{2n}} \underbrace{\left(\widetilde{\phi}_1(w) - \widetilde{\phi}_0(w)\right)}_{=\text{lifts of functions on } T^{2n}} \omega^n.$$

Proof: Fix a family of Hamiltonian functions H_t for $\widetilde{\phi}_t$ ($\frac{d}{dt}\phi_t$ is not a Hamiltonian vector field on T^{2n}, but $\frac{d}{dt}\widetilde{\phi}_t$ is Hamiltonian on $\mathbb{R}^{2n}$). These functions do not descend to T^{2n}, in general. However, $H_t(w+l) - H_t(w)$ is a constant function on $\mathbb{R}^{2n}$ for all $l \in \mathbb{Z}^{2n}$. So, there is a time dependent vector $h(t) = (h_1(t), \ldots, h_{2n}(t))$ such that

(28)
$$H_t(w+l) - H_t(w) = \langle h, l \rangle \text{ for all } l \in \mathbb{Z}^{2n}$$

which measures the obstruction for H_t being the lift of a Hamiltonian function on T^{2n}. For the Flux of $[\phi_t]$ we get

$$\begin{aligned}
\mathrm{Flux}([\phi_t]) &= \int_0^1 [i_{X_t}\omega_{st}] \\
&= \int_0^1 \underbrace{[dH_t]}_{\text{exact on } \mathbb{R}^{2n}, \text{ not } T^{2n}} dt \\
&= \left[\sum_{j=1}^{2n} \left(\int_0^1 h_j(t)dt\right) dw_j\right].
\end{aligned}$$

Now compute (sometimes T^{2n} denotes a fundamental domain of the $\mathbb{Z}^{2n}$ action on $\mathbb{R}^{2n}$ or the $2n$-torus).

$$\begin{aligned}
a &= J_0 \int_{T^{2n}} (\widetilde{\phi}_1(w) - \widetilde{\phi}_0(w))\omega_{st}^n \\
&= J_0 \int_{T^{2n}} \int_0^1 \left(\frac{d}{dt}\widetilde{\phi}_t\right)(\widetilde{\phi}_t(w))\, dt\, \omega_{st}^n \\
&= J_0 \int_{T^{2n}} \int_0^1 \left(-J_0 \nabla H_t(\widetilde{\phi}_t(w))\right) dt\, \omega_{st}^n \\
&= \int_{T^{2n}} \int_0^1 \nabla H_t(w)\, dt\, \omega_{st}^n \\
&= \int_0^1 h(t) dt
\end{aligned}$$

For the step from the third to the fourth line one uses that ϕ_t is a symplectomorphism, for the next step one uses (28). This confirms (27).

4. **Observation:** The last statement implies $H^1_{dR}(M, \mathbb{Z}) \subset \Gamma_{\omega_{st}}$. The vector a can be interpreted as average displacement of points in T^{2n} by the symplectomorphism ϕ_1.

5. **Consequence:** Assume that ϕ_t is a loop in $\mathrm{Ham}(T^{2n},\omega)$. Then the loop $t \longmapsto \phi_t(q)$ in T^{2n} is contractible for all $q \in T^{2n}$.
Proof: Choose the lift $\widetilde{\phi}_t$ of ϕ_t to $\mathbb{R}^{2n}$ so that $\widetilde{\phi}_0 = \mathrm{id}$. Since ϕ_t is Hamiltonian, $\mathrm{Flux}([\phi_t]) = 0$. Because ϕ_t is a loop, there is $l \in \mathbb{Z}^{2n}$ so that $\widetilde{\phi}_1(w) = w + l$. Consider the linear curve γ which represents $J_0 \cdot l$. Then using notation from above ($\beta(s,t) = \phi_t^{-1}(\gamma_s)$)
$$\begin{aligned}\langle \mathrm{Flux}([\phi_t]), \gamma\rangle &= \int_\beta \omega \\ &\neq 0\end{aligned}$$
because the cylinder β is homotopic relative to the boundary to a cylinder in T^{2n} whose lift to $\mathbb{R}^{2n}$ is contained in a plane parallel to the plane spanned be $l, J_0 l$. The symplectic form is an area form on such planes and the cylinder has two different boundary components. Thus, $l = 0$. This implies the claim.

17. Lecture on December, 13 – Generating functionis for Hamiltonian diffeo's of T^{2n}

- **Definition:** A symplectomorphism ϕ of (T^{2n}, ω_{st}) is called *exact* if it acts trivial on homology ($A = \mathrm{id}$ in the notation from above) and admits a lift $\widetilde{\phi}$ so that
$$\int_{T^{2n}} (\widetilde{\phi}(w) - w)\omega_{st}^n = 0. \tag{29}$$
The last integral is over a fundamental domain. ϕ acts trivially on homology if and only if it is homotopic to the identity.
- **Fact:** We have shown above that Hamiltonian diffeomorphisms of T^{2n} are exact. (We did not determine Γ_ω completely.) This will be used in the following Lemma.
- **Lemma:** Let $\widetilde{\phi} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be the lift of an exact symplectomorphism of T^{2n}. If $\widetilde{\phi}$ is sufficiently close to id in the C^1-topology, then there is a smooth function $V : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ which generates ψ and satisfies
$$V(x+k, y+l) = V(x,y)$$
for all $k, l \in \mathbb{Z}^n$.
Recall that *generates* means that $\widetilde{\phi}(x_0, y_0) = (x_1, y_1)$ is equivalent to
$$\begin{aligned} x_1 - x_0 &= \frac{\partial V}{\partial y}(x_1, y_0) \\ y_1 - y_0 &= -\frac{\partial V}{\partial x}(x_1, y_0). \end{aligned}$$
- **Proof:** Since $\widetilde{\phi}$ is homotopic to the identity, there are functions $p, q : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$ which are periodic ($p(x+k, y+l) = p(x,y)$ for $k, l \in \mathbb{Z}^n$) and
$$\widetilde{\phi}(x,y) = (x + p(x,y), y + q(x,y)).$$
(29) implies
$$\int_{T^{2n}} p\,\omega_{st}^n = 0 = \int_{T^{2n}} q\,\omega_{st}^n.$$

Recall that $(x,y) \longmapsto (x+p(x,y),y)$ is a diffeomorphism of $\mathbb{R}^{2n}$. Hence, it induces a diffeomorphism of T^{2n}. Because

$$p(x,y) = \frac{\partial V}{\partial y}(x+p(x,y),y)$$

and $p(x,y)$ descends to an $\mathbb{R}^n$-valued function on T^{2n}, the same is true for $\partial V/\partial y$. The same argument shows that $\partial V/\partial y$ is the lift of a function on T^{2n} with values in $\mathbb{R}^n$. Therefore, there are vectors $a,b\in\mathbb{R}^n$ so that

(30) $$V(x,y) = \langle (a,b),(x,y)\rangle + W(x,y)$$

for a function W which is 1-periodic in all variables. The fact that the average displacement of $\widetilde{\phi}$ vanishes allows to determine (a,b) (this vector has to vanish). Here are the details: Differentiating (30) with respect to x and y we get

$$\begin{aligned} a + \frac{\partial W}{\partial x}(x+p(x,y),y) &= -q(x,y)\\ b + \frac{\partial W}{\partial y}(x+p(x,y),y) &= p(x,y). \end{aligned}$$

The second identity implies (differentiate by x):

$$\frac{\partial^2 W}{\partial y \partial x}(x+p(x,y),y)\left(\mathbb{E} + \frac{\partial p}{\partial x}\right) = -\frac{\partial p}{\partial x}$$

Multiplying with $(\mathbb{E}+\partial p/\partial x)^{-1}$ (the matrix is invertible by the C^1-smallness assumption (17)) we get

$$\mathbb{E} - \frac{\partial^2 W}{\partial x \partial y}(x+p(x,y),y) = \left(\mathbb{E} + \frac{\partial p}{\partial x}(x,y)\right)^{-1}$$

This allows to perform a change of variables $(x',y') = (x+p(x,y),y)$ after integrating the identity involving a over a fundamental domain of the Z^{2n} action on $\mathbb{R}^{2n}$ we get

$$\begin{aligned} a &= -\int_{T^{2n}} \frac{\partial W}{\partial x}(x+p(x,y),y)\,dx\,dy\\ &= -\int_{T^{2n}} \frac{\partial W}{\partial x'}(x',y')\frac{dx'\,dy'}{\det\left(\mathbb{E}+\frac{\partial p}{\partial x}(x,y)\right)}\\ &= -\int_{T^{2n}} \frac{\partial W}{\partial x'}(x',y')\det\left(\mathbb{E} - \frac{\partial^2 W}{\partial x'\partial y'}(x',y')\right)dx'\,dy'\\ &= -\int_{T^{2n}} \det\left(\mathbb{E} - \frac{\partial^2 W}{\partial x'\partial y'}(x',y')\right) dW\wedge dx'_2\wedge\ldots dx'_n\wedge dy'_1\wedge\ldots\wedge dy_n \end{aligned}$$

Again all integrals are over fundamental domains of the $\mathbb{Z}^{2n}$-action on $\mathbb{R}^{2n}$ (it does not matter which fundamental domain is used). The expression of the last line is the pull-back of the $2n$-form

(31) $$(dW)(x''_1,\ldots,y''_n)\wedge dx''_2\wedge\ldots dx''_n\wedge dy''_1\wedge\ldots\wedge dy''_n$$

under the map

$$\begin{aligned} \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{2n}\\ (x',y') &\longmapsto \left(x', y' - \frac{\partial W}{\partial x'}\right) = (x'',y''). \end{aligned}$$

This is a diffeomorphism of $\mathbb{R}^{2n}$ because its first derivatives (i.e. the second derivatives of W) are as close to id as the first derivatives of $\partial V \partial y$. Moreover, it is $\mathbb{Z}^{2n}$ periodic.

Since the $2n$-form given in (31) descends to an exact $2n$-form manifold $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. This implies $a = 0$. $b = 0$ follows from a similar computation.

18. Lecture on December 17 ,– Conley index, non-degenerate case of the Arnol'd conjecture

- **Corollary:** Let ϕ be a Hamiltonian diffeomorphism of T^{2n}. If ϕ is sufficiently C^1-close to id, then the Arnol'd conjecture holds for ϕ. Moreover, the discretized action (19) functional is defined on $\mathcal{P}_{per}/\mathbb{Z}^{2n}$.
- **Proof:** The first part follows from the second if $N = 1$, i.e. is ϕ is so small that it does admit a generating function V. For the second part recall that the action is a linear combination of function $V_j(x_{j+1}, y_j)$ and $\sum_{j=0}^{N-1} \langle y_j, x_{j+1} - x_j \rangle$. $\mathbb{Z}^{2n}$ acts by translation on P_{per}, i.e. for $w \in \mathbb{Z}^{2n}$ and $((x_0, y_0), \dots, (x_{N-1}, y_{N-1}))$ (with the convention $x_{j+N} = x_j, y_{j+N} = y_j$)

$$w \cdot ((x_0, y_0), \dots, (x_{N-1}, y_{N-1})) = ((x_0, y_0) + w, \dots, (x_{N-1}, y_{N-1}) + w).$$

  So both the V_j-summands and the part $\sum_{j=0}^{N-1} \langle y_j, x_{j+1} - x_j \rangle$ are invariant under this action. Thus, the discretized action functional descends to

$$\Phi : \mathcal{P}_{per}/\mathbb{Z}^{2n} \longrightarrow \mathbb{R}. \tag{32}$$

- **Remark:** The source is diffeomorphic to $T^{2n} \times \mathbb{R}^{2n(N-1)}$. To finish the proof of the Arnold conjecture for T^{2n} we have to show that if Φ has only non-degenerate critical points, then there are at least 2^{2n} of them and at least $2n + 1$ critical points in general and these critical points correspond to geometrically distinct fixed points of Φ on T^{2n}.

  Morse-theory is the standard tool to relate critical points of functions with non-degenerate critical points (Morse functions) with the topology of the underlying space. To estimate the number of critical points of a general function from below one used the Lusternik-Schnirelman category.

  Both these tools need to be adapted from their standard setting (discussed for example in the first chapters of [Mi-M]) to the present one because Φ is not bounded from below.

  Let V be $-\nabla\Phi$. This vector field is complete so that we consider the flow. We will relate dynamical properties of V to the topology of $X = \mathbb{R}^{2nN}/\mathbb{Z}^{2n} = M$.
- **Morse-theory and Conley index:** Let V be a complete vector field on the manifold M and f_t its flow. It is not required that V be a gradient vector field until later. However, it should be noted that gradient flows are simpler than general flows because all recurrent trajectories (i.e. flowlines that accumulate on themselves, for example closed leaves) are singular points.
- **Definition:** $\Lambda \subset M$ is *invariant* if $f_t(\Lambda) = \Lambda$. An invariant set is *isolated* if there is a neighborhood N so that $\Lambda = \cap_t f_t(N)$.
- **Definition:** Let Λ be an isolated invariant set. A pair (N, L) (with $L \subset N \subset M$) of compact sets is an *index pair* for Λ if
  1. $N \setminus L$ isolates Λ, i.e. $\Lambda = \cap_t f_t(N \setminus L)$.
  2. L is positively invariant, i.e. for all $x \in L$ and $t > 0$ so that $f_{[0,t]}(x) \in N$ implies $f_{[0,t]} \subset L$.

3. a point in N passes trough L before leaving N, i.e. for all $x \in N \setminus L$ there is $\varepsilon > 0$ so that $f_{[0,\varepsilon]}(x) \subset N$.

- **Theorem:** For every isolated invariant set Λ, there is an index pair.
- **Lemma:** Given two index pairs $(N, L), (N', L')$ for Λ, then the spaces N/L and N'/L' are homotopy equivalent.
- **Definition:** An index pair (N, L) is *regular* if L is a deformation retract of neighborhood of L in N.

  If (N, L) is regular, then $H_*(N, L) = \widetilde{H}_*(N/L)$. This is a measure for the topological complexity of Λ which is stable under deformations of V as long as (N, L) is still a (regular) index pair. It is convenient to organize information about $\widetilde{H}_*(N/L)$ in the *index polynomial*

$$p_\lambda(s) = \sum_k \underbrace{\dim(H_k(N, L; \mathbb{R}))}_{b_k(\Lambda)} s^k.$$

  In all cases we are concerned with, (N, L) will be a compact CW-pair. This implies that (N, L) is regular and that $H_*(N, L; \mathbb{R})$ has finite dimension. A standard properties if $H_*(\cdot, \cdot)$

$$p_{\Lambda \cup \Lambda'} s = p_\Lambda(s) + p_{\Lambda'}(s)$$

  when the invariant sets Λ and Λ' have disjoint isolating neighborhoods.
- **Remark:** The case when $V = -\nabla f$ is the gradient flow of a smooth function (assuming that the flow of V exists forever, i.e. that V is complete). Then singular points are invariant sets and all other invariant sets are unions of singular points and flow lines "connecting" them.
- **Example:** Let $f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2$ be the standard Morse singularity of index k on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Write $I = [-1, 1]$ and $V = -\nabla f$. 0 is a singular point of V and $N = I^n$ isolates 0. Tacking $L = (\partial I^k) \times I^{n-k}$ we get an index pair (N, L) and $N/L \sim I^k/(\partial I^k) \simeq S^k$ where $\sim$ denotes homotopy equivalence. Recall

$$\widetilde{H_l}(S^k; \mathbb{R}) \simeq \begin{cases} \mathbb{R} & l = k \\ 0 & l \neq k. \end{cases}$$

  k is the number of negative eigenvalues of the Hessian of f at a critical point.
- The following theorem is a standard result from Morse-theory which allows to estimate the number of singular points of an invariant set of a gradient flow of a Morse-function Φ on a manifold of dimension n in terms of the topology of an isolating set of Λ. Let

$$c_k(\Lambda) = |\{x \in \Lambda \,|\, x \text{ is a critical point of index } k\}|.$$

- **Theorem (Morse inequalities):** For $0 \leq k \leq n$

$$c_k(\Lambda) - c_{k-1}(\Lambda) + \ldots (-1)^{k+1} c_0(\Lambda) \geq b_k(\Lambda) - b_{k-1}(\Lambda) + \ldots + (-1)^{k+1} b_0(\Lambda). \tag{33}$$

  Equality holds for $k = n$. By induction, this implies that

$$\sum_k c_k \geq \sum_k b_k$$

- The proof of this relies on rebuilding (the homotopy type of) N from L using the flow and information about the singular points.
- The right hand side of (33) does not really depend on Λ but on the index pair (N, L) for Λ.

19. Lecture on December, 20 – Arnol'd conjecture, finally

- **Theorm (Arnol'd conjecture), non-degenerate case:** Let ϕ be a Hamiltonian diffeomorphism of T^{2n} and assume that all fixed points are non-degenerate (i.e. 1 is not an eigenvalue of $d\phi : T_xT^{2n} \longrightarrow T_xT^{2n}$ for a fixed point x of ϕ). Then there are at least 2^{2n} fixed points.
- **Proof:** Let $\widetilde{\phi}$ be the endpoint of the lift of the the Hamiltonian flow of the Hamiltonian functions defining ϕ to $\mathbb{R}^{2n} \longrightarrow \mathbb{T}^{2n}$. For big enough N, there are functions V_j generating symplectomorphisms ϕ_j of $\mathbb{R}^{2n}$ so that

$$\widetilde{\phi} = \phi_{N-1} \circ \phi_{N-2} \circ \ldots \circ \phi_0$$

and the fixed points of ϕ correspond to (equivalence classes of) critical points of the action functional (under the $\mathbb{Z}^{2n}$-action)

$$\begin{aligned} \Phi : \mathcal{P}_{per}/\mathbb{Z}^{2n} &\longrightarrow \mathbb{R} \\ [x_0, y_0, \ldots, x_{N-1}, y_{N-1}] &\longmapsto \sum_{j=0}^{N-1} \left(\langle y_j, x_{j+1} - x_j \rangle - V_j(x_{j+1}, y_j) \right) \end{aligned}$$

The domain of Φ is diffeomorphic to $T^{2n} \times \mathbb{R}^{2n(N-1)}$ via

$$\begin{aligned} z_0 &= z_0 \in T^{2n} \\ \xi_j &= x_j - x_{j-1} \in \mathbb{R}^n \\ \eta_j &= y_j - y_{j-1} \in \mathbb{R}^n \end{aligned}$$

for $j = 1, \ldots, N-1$. We denote $\zeta = (\xi, \eta)$. In these coordinates, Φ becomes

$$\begin{aligned} \Phi : T^{2n} \times \mathbb{R}^{2n(N-1)} &\longrightarrow \mathbb{R} \\ (z_0, \zeta) &\longmapsto \langle \zeta, P\zeta \rangle + W(z_0, \zeta_1, \ldots, \zeta_{N-1}). \end{aligned}$$

The matrix P is of the form $P = \begin{pmatrix} 0 & -B \\ -B^T & 0 \end{pmatrix}$ with

$$B = \underbrace{\begin{pmatrix} \mathbb{E}_n & \mathbb{E}_n & \ldots & \mathbb{E}_n \\ 0 & \mathbb{E}_n & \ldots & \mathbb{E}_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbb{E}_n \end{pmatrix}}_{(N-1) \text{ columns}}$$

The matrix P is symmetric and represents a non-degenerate quadratic form with $\det(P) = \pm 1$. We want to determine the index of P (the index is the difference between the number of positive eigenvalues and the number of negative eigenvalues).

P is homotopic through non-degenerate symmetric matrices to

$$\begin{pmatrix} 0 & -\mathbb{E}_{2n(N-1)} \\ -\mathbb{E}_{2n(N-1)} & 0 \end{pmatrix}.$$

This matrix has index 0, so the same is true for P. Hence, there is a decomposition $E_+ \oplus E_-$ of $\mathbb{R}^{2n(N-1)}$ so that the quadratic form P is positive definite on E_+ and negative definite on E_-. From this we get an index pair for the unique critical point 0 of the quadratic form: For all $K > 0$

$$N_K = \{(e_+, e_-) \in E_+ \oplus E_- \mid \|e_+\| \leq K, \|e_-\| \leq K\}$$
$$L_K = \{(e_+, e_-) \in E_+ \oplus E_- \mid \|e_+\| \leq K, \|e_-\|{=}K\}$$

is an index pair for $\Lambda = \{0\} \in \mathbb{R}^{2n(N-1)}$ of the negative gradient flow of the quadratic form. N_K is a product of two $n(N-1)$-dimensional balls, L_K is the product of such a ball with a sphere of one dimension less. If $V_j = 0$ for all j, then $T^{2n} \times (N, L)$ is an index pair for the negative gradient flow of Φ. If one chooses K so that the negative gradient flow points out of N_K precisely along L_K, then $T^{2n} \times (N, L)$ is an index pair for the negative gradient flow of Φ with no-trivial given V_j. Such a K exists because V_j and its derivatives are bounded functions on T^{2n} for all j. Hence, the Conley index of the union of all bounded flow lines is

$$T^{2n} \times N_K / T^{2n} \times L_K$$

One can compute the homology of this index pair:

$$b_{k+n(N-1)}(T^{2n} \times N_K, T^{2n} \times L_K) = \binom{2n}{k}$$

This implies the claim.

- We still have to discuss the algebraic topology for the degenerate/topological case. This is more delicate because we can only assume that the number of critical points of Φ is finite, so that the critical points are isolated. The following definitions provide tools to estimate the number of critical points from below.
- **Definition:** An open subset U of a manifold M is *cohomologically trivial* if the inclusion induced map

$$i^k : H^k_{dR}(M) \longrightarrow H^k_{dR}(U)$$

is zero for $k \geq 1$. The *Ljusternik-Schnirelmann category* of a subset $A \subset M$ is

$$\nu_{LS} = \text{minimal number } N \text{ so that there are cohomologically trivial}$$
$$U_1, \ldots, U_N \subset M \text{ so that } A \subset \cup_j U_j.$$

This defines a map $\nu_{LS} : \text{PowerSet}(M) \longrightarrow \mathbb{N}_0$ satisfying a the following rules:

1. (continuity): For all $A \subset N$ there is an open set U so that $\nu_{LS}(U) = \nu_{LS}(A)$.
2. (monotonicity): $A \subset B$ implies $\nu_{LS}(A) \leq \nu_{LS}(B)$.
3. (subadditive): $\nu_{LS}(A \cup B) \leq \nu_{LS}(A) + \nu_{LS}(B)$.
4. (normalization): $\nu_{LS}(\emptyset) = 0$ and $\nu_{LS}(\{x_1, \ldots, x_N\}) = 1$ for all finite subsets of M.
5. (naturaltiy): If $\phi : M \longrightarrow M$ is a homeomorphism, then $\nu_{LS}(\phi(A)) = \nu_{LS}(A)$.

- **Lemma:** If M is of dimension n, then $\nu_{LS}(A) \leq n + 1$ for all $A \subset M$.
- **Sketch of Proof:** Fix a triangulation $\mathcal{T}$ of M and consider its first barycentric subdivision $\mathcal{T}^1$. The vertices of $\mathcal{T}^1$ can be grouped into $n + 1$ classes $V_j, j = 0, \ldots n$ according to the dimension of the face they are a barycenter of. For example, V_0 are the vertices of $\mathcal{T}$, V_1 are mid points of edges of $\mathcal{T}$ etc.

  The open star of a vertex of a triangulation is the union of all (open) simplices whose closure contains the vertex. All open stars of a vertex in $\mathcal{T}^1$ are contractible and the open stars of vertices in V_j (for fixed j) are pairwise disjoint.

- **Definition:** The cup-length $cl(M)$ of the manifold M is the smallest number N so that for all $\alpha_i \in H^{k_i}_{dR}(M)$ with $k_i \geq 1$
$$\alpha_1 \wedge \ldots \wedge \alpha_N = 0.$$
- **Remark:** If M is n-dimensional, then $cl(M) \leq n+1$.
- **Example:** The cup-length of T^{2n} is $2n+1$ since $dx_1 \wedge \ldots \wedge dy_n$ is a non-trivial class which is the product of $2n$ forms of degree 1.
- **Lemma:** Let M be compact. Then $\nu_{LS}(M) \geq cl(M)$.
- **Proof:** This is an application of the exactness of the Mayer-Vietoris sequence: Let $U, V \subset M$ be open. Then there is an exact sequence

$$\ldots H^{k-1}_{dR}(U \cap V) \longrightarrow H^k_{dR}(U \cup V) \longrightarrow H^k_{dR}(U) \oplus H^k_{dR}(V) \longrightarrow H^k_{dR}(U \cap V) \longrightarrow H^{k+1}_{dR}(U \cup V) \ldots$$

The second map is induced by inclusion, the third map is the difference of the maps induced two restriction maps. The connecting homomorphisms are constructed using a partition of unity subordinate to the open cover U, V of $U \cup V$.

Claim: Let $\alpha \in \Omega^k(U \cup V)$ be exact on U and on V. Let $\beta \in \Omega^l(U \cup V)$ be closed on U and exact on V. Then $\alpha \wedge \beta$ is exact on $U \cup V$.

Proof of claim: $\alpha|_U = d\sigma_U, \alpha_V = d\sigma_V, \beta_V = d\tau_V$ by assumption. Choose an extension (denoted by σ_U) of σ_U to $U \cup V$. Now the $k+l-aq$-form ρ is defined by

$$\rho|_U = \sigma_U \wedge \beta \text{ on } U$$
$$\rho|_V = \sigma_V \wedge \beta + (-1)^{k-1} d((\sigma_U - \sigma_V) \wedge \tau_V) \text{ on } V.$$

The forms ρ_U, ρ_V coincide on $U \cup V$, so they define a form on $U \cup V$. One can check that
$$d\rho = \alpha \wedge \beta.$$

Let $U_1, \ldots, U_N$ be an open covering of M by cohomologically trivial sets and $\alpha_1, \ldots, \alpha_N$ forms of positive degree. Using induction we will show that $\alpha_1 \wedge \ldots \alpha_k$ is exact on $U_1 \cup \ldots U_k$ for $1 \leq k \leq N$. This is obvious for $k=1$. For the inductive step apply the claim to

$$\alpha = \alpha_1 \wedge \alpha_k \text{ on } U = U_1 \cup \ldots U_k$$
$$\beta = \alpha_{k+1} \text{ on } V = U_{k+1}.$$

- **Theorem (Lusternik-Schnirelmann):** Let M be a compact manifold. Then a (negative) gradient flow has at least $\nu_{LS}(M)$ critical points.
- **Proof:** Let $f : M \longrightarrow \mathbb{R}$ be smooth, and $V = -\nabla f$ (using some Riemannian metric). Let ϕ_t be the flow of V. We assume that f has finitely many critical points so that they are isolated.

For $c \in \mathbb{R}$ let $M^c := f^{-1}((-\infty, c])$. If c is not critical, then for sufficiently small $\varepsilon > 0$ the flow ϕ_t allows to construct a homeomorphisms of M mapping $M^{c+\varepsilon}$ heomeomorphically onto $M^{c+\varepsilon}$. Then
$$\nu_{LS}(M^{c+\varepsilon}) = \nu_{LS}(M^{c-\varepsilon})$$
by naturality. Let $c_j = \sup\{c \in \mathbb{R} \,|\, \nu_{LS}(M^c) < j\}$ for $j = 1, \ldots, \nu_{LS}(M)$. Then $c_1 = \min(f(M))$, and c_j is a critical value of f for all j.

We are done once we show that the critical levels c_j are pairwise distinct, i.e. $c_{j+1} > c_j$. Because there are only finitely many critical points, the critical level $f^{-1}(c_j)$ contains only finitely many of them: $x_1, \ldots, x_m$. There are contractible neighborhoods V_i of x_i and $\varepsilon > 0$ so that $M^{c_j+\varepsilon} \backslash \cup_i V_i$ can be pushed into $M^{c_j-\varepsilon}$ by the flow. Hence,

$$\begin{aligned}\nu_{LS}(M^{c_j+\varepsilon}) &\leq \nu_{LS}(M^{c_j-\varepsilon}) + \nu(\cup_i V_i)\\ &= \nu_{LS}(M^{c_j+\varepsilon}) + 1\\ &< j+1.\end{aligned}$$

This implies $c_{j+1} > c_j$.

- **Remark:** We have used deRham cohomology to explain the Lusternik Schnirelmann category because it is assumed that we all know it. It would work in the same way with most other cohomology theories (for example singular). It turns out that Alexander-Spanier homology is better suited than singular or deRham cohomology.

  The reason for this is that in general Λ is a complicated space (not a manifold, not a CW-complex). In the non-degenerate case this is not as big of a problem because the singularities and the flow lines connecting them are very simple (for a generic choice of Riemannian metric defining the gradient flow, we are interested in the number of critical points which are not affected be the choice of a metric.)

  In the Alexander theory (like in singular cohomology) it is possible to define a map

  $$i^* : \check{H}^*(N) \longrightarrow \check{H}^*(\Lambda)$$

  which is induced by the inclusion $\Lambda \longrightarrow N$ where (N, L) is an index pair. In the Alexander theory, and not int the singular theory, one can define several long exact sequences allowing to prove properties of i^*. For details see [Co], p. 74ff.
- **Idea of proof of the Arnol'd conjecture in the degenerate case:** Let $\pi : T^{2n} \times \mathbb{R}^{2n(N-1)} \longrightarrow T^{2n}$ be the projection. One has to show that

  $$i^* \circ \pi^* : \check{H}^*(T^{2n}) \longrightarrow \check{H}^*(\Lambda)$$

  is injective. This is clear when $\Phi = \mathrm{id}$, i.e. when $V_j \equiv 0$ for all j and properties of Alexander cohomology allow to prove the general case. The cup length can be defined for all cohomology theories (and not only on manifolds but also on compact metric spaces). The same holds for the notion of category. Also, the proof of the Lusternik Schnirelmann theorem can be adapted to the setting of flows on metric spaces like Λ (gradients are not really defined there, but one can still talk about Lyapunov functions) .

  Because the cup length of $\check{H}^*(T^{2n})$ is $2n+1$ this implies that Λ contains at least $2n+1$ critical points.
- Remark: This is only the beginning of a long story, see [McDS], Section 11.

20. Lecture on January, 7 – Symplectic Capacities

- In the following symplectic manifolds are allowed to have boundary, they are of dimension $2n$. We denote

$$B(r) = \{(x,y) \in \mathbb{R}^{2n} \mid |x|^2 + |y|^2 < 1\} \subset \left(\mathbb{R}^{2n}, \omega_0 = \sum_i dy_i \wedge dx_i\right)$$

$$Z(r) = \{(x,y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < 1\} \subset \left(\mathbb{R}^{2n}, \omega_0\right).$$

The symplectic form ω_0 looks slightly non-standard, but it the standard symplectic form on $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$. The almost complex structure $J = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$ is adapted to ω_0 and $\omega(-JX, Y) = \omega(X, JY)$ is the standard Euclidean metric on $\mathbb{R}^{2n}$. The Hamiltonian vector field is then $X_H = J\nabla H$.

- **Definition:** A *symplectic capacity* c is a map $(M,\omega) \longmapsto c(M,\omega) \in \mathbb{R}_0^+ \cup \{\infty\}$ so that
  1. (Monotonicity): If $\psi : (N, \omega_N) \longrightarrow (M, \omega_M)$ is a symplectic embedding, then
  $$c(N, \omega_N) \leq c(M, \omega_M).$$
  2. (Conformal): $c(M, \alpha\omega) = |\alpha| c(M, \omega)$ for all $\alpha \neq 0$.
  3. (Nontrivial): $c(B(1), \omega_0) = \pi = c(Z(1), \omega_0)$.

  Sometimes one replaces the last property by the weaker requirement
  $$0 < c(B(1), \omega_0) \text{ and } c(Z(1), \omega_0) < \infty.$$
  Other restrictions, e.g. considering only open subsets in $\mathbb{R}^{2n}$ are common.
- **Remark:** As it turns out there are many different capacities, in particular when $n > 1$. The existence of a capacity is a non-trivial fact that we assume for now.
- **Remark:** If $n = 1$, then $c(M, \omega) = \int_M \omega$ is a symplectic capacity. If $n > 1$, then
  $$(M, \omega) \longmapsto \left(\int_M \omega^n \right)^{1/n}$$
  is monotone and conformal, but $\int_{Z(1)} \omega^n = \infty$.
- **Remark:** If $f : (M, \omega) \longrightarrow (N, \mu)$ is a symplectomorphism, then $c(M, \omega) = c(N, \mu)$. This is immediate from the monotonicity.
- **Lemma:** For $U \subset (\mathbb{R}^{2n}, \omega_0)$ and $\alpha \neq 0$
  $$c(\alpha U, \omega_0) = \alpha^2 c(U, \omega_0)$$
  where $\alpha U = \{\alpha u \,|\, u \in U\} \subset \mathbb{R}^{2n}$.
- **Proof:** The map
  $$\psi : (\alpha U, \omega_0) \longrightarrow (U, \alpha^2 \omega_0)$$
  $$(x, y) \longmapsto \frac{(x, y)}{\alpha}$$
  is a symplectomorphism. Hence, $c(\alpha U, \omega_0) = c(U, \alpha^2 \omega_0) = \alpha^2 c(U, \omega_0)$ by conformality.
- In particular, $c(B(r)) = r^2 c(B(1)) = r^2 \pi$ and $c(Z(r)) = r^2 \pi$. If $n = 1$, $c(B(1), \omega_0) = \mathrm{area}(B(1))$. Since $B(r) \subset \overline{B(r)} \leq B(r + \varepsilon)$ for $\varepsilon > 0$, one has $c(\overline{B(r)}, \omega_0) = c(B(r), \omega_0)$.
- **Proposition:** If $D \subset \mathbb{R}^2$ is compact, connected domain with smooth boundary, then $c(D, \omega_0) = \mathrm{area}(D)$.
- **Proof Sketch:** After removing a finite collection of curves we obtain D_0 so that $\mathring{D}_0$ is diffeomorphic to a disc, i.e. there is a diffeomorphism $\phi : B(r) \longrightarrow \mathring{D}_0$ and $\mathrm{area}(D) = \mathrm{area}(B(r))$. For $\varepsilon > 0$ one finds $r_1 < r$ so that $\mathrm{area}(\phi(\overline{B(r_1)})) \geq \mathrm{area}(D) - \varepsilon$.

By the proof of the Moser theorem, there is an ambient isotopy ψ_t of $\mathbb{R}^2$, supported in D so that $\psi_1 \circ \phi$ is symplectic on $\overline{B(r_1)}$. Then

$$\text{area}(D)-\varepsilon \leq \text{area}(\psi_1\circ\phi(\overline{B(r_1)})) = \text{area}(\overline{B(r_1)}) = c(\overline{B(r_1)}) \leq c(\psi_1\circ\phi(\overline{B(r_1)})) \leq c(D).$$

Conversely, for all ε there is a diffeomorphism

$$\phi : D \longrightarrow \overline{B(r)} \setminus \{\text{finite set of balls with total area} \leq \varepsilon\}$$

such that $\text{area}(B(r)) - \varepsilon = \text{area}(D)$ By Moser's theorem one can again isotope ϕ so that it is symplectic and has image contained in a disc $B(R)$ so that $\text{area}(B(R)) \leq \text{area}(D) + \varepsilon$. Then

$$c(D) \leq c(B(R)) \leq \text{area}(D) + \varepsilon.$$

Thus, $\text{area}(D) - \varepsilon \leq c(D) \leq \text{area}(D) + \varepsilon$.

- **Example:** Let $n > 1$ and $\hat{Z}(1) = \{(x, y) \in \mathbb{R}^{2n} \mid x_1^2 + x_2^2 \leq 1\} \subset \mathbb{R}^{2n}$. Then $c(\hat{Z}, \omega_0) = \infty$. This is true since for all N, there is a symplectic embedding of $B(N)$ into $\hat{Z}$, take $(x, y) \longmapsto (x/N, Ny)$, for example. This is generalized in
- **Proposition:** Let U be bounded, open, nonempty and $W \subset \mathbb{R}^{2n}$ a subspace of codimension 2. Then

$$c(U + W) = \infty \text{ if } W^{\perp_\omega} \text{ is isotropic}$$
$$0 < c(U + W) < \infty \text{ if } W^{\perp_\omega} \text{ is not isotropic}$$

- Of course, the concept of symplectic capacity grew out of an important theorem:
- **Gromov's non-squeezing theorem:** There is a symplectic embedding

$$\psi : B(r) \longrightarrow Z(R)$$

if and only if $r \leq R$.

- **Proof assuming the existence of a symplectic capacity:** If $r \leq R$, then $B(r) \subset Z(R)$. Conversely, if there is a symplectic embedding $\psi : B(r) \longrightarrow Z(R)$, then

$$r^2\pi = c(B(r)) = c(\psi(B(r)) \leq c(Z(R)) = R^2\pi.$$

- **Proposition:** Fix $n = 2$. Let $0 < r_1 \leq r_2$ and $0 < s_1 \leq s_2$. There is a symplectic diffeomorphism $\psi : B^2(r_1) \times B^2(r_2) \longrightarrow B^2(s_1) \times B^2(s_2) \subset \mathbb{R}^4$ if and only if $r_1 = s_1$ and $r_2 = s_2$. (B^2 denotes a disc in the x_1, y_1 or the x_2, y_2 plane.)
- **Proof assuming the existence of a symplectic capacity:** If ψ is such a diffeomorphism, then since $B^4(r_1) \subset B^2(r_1) \times B^2(r_2)$. Moreover, $B^2(s_1) \times B^2(s_2) \subset Z(s_1)$. Hence,

$$r_1^2\pi = c(B^4(r_1)) \leq c(B^2(r_1) \times B^2(r_2)) \leq c(B^2(s_1) \times B^2(s_2)) \leq c(Z(s_1)) = s_1^2\pi.$$

This implies $r_1 \leq s_1$. Applying the same argument to ψ^{-1} we get $s_1 \leq r_1$,i.e. $r_1 = s_1$. Since ψ is also volume preserving, we get $r_1r_2 = s_1s_2$, so $r_2 = s_2$.

- So far, we have no idea what a symplectic capacity might measure. Still assuming the existence a capacity, the following theorem gives another, explicit capacity called the *Gromov width.* It is still unclear whether there is a single capacity for $n \geq 2$.
- **Theorem:**

$$(M, \omega) \longmapsto D(M, \omega) = \sup\left\{\pi r^2 \mid \text{there is a symp. embedding } (B(r), \omega_0) \longrightarrow (M, \omega)\right\}$$

is a symplectic capacity. It is minimal in the sense that $D(M,\omega) \leq c(M,\omega)$ for all symplectic capacities c. For all compact symplectic manifolds (M,ω) one has $D(M,\omega) < \infty$.

- **Proof:** Monotonicity is clear. Let $\psi : (B(r),\omega_0) \longrightarrow (M,\alpha\omega)$ be a symplectic embedding. Then

$$\begin{aligned} \hat{\psi} : (B(r/\sqrt{|\alpha|}),\omega_0) &\longrightarrow (M,\omega) \\ (x,y) &\longmapsto \psi(\sqrt{|\alpha|}(x,y)) \end{aligned}$$

is a symplectic embedding for $\alpha > 0$ since $\hat{\psi}^*\omega = \sqrt{|\alpha|}^2\psi^*\omega = \psi^*\left(\sqrt{|\alpha|}^2\omega\right) = \frac{|\alpha|}{\alpha}\psi^*(\alpha\omega) = \frac{|\alpha|}{\alpha}\omega_0$. If $\alpha < 0$, one precomposes $\hat{\psi}$ with the anti-symplectic involution $(x,y) \longmapsto (-x,y)$. This implies that to each embedding $(B(r),\omega_0) \longrightarrow (M,\alpha\omega)$ there is a symplectic embedding $(B(r)/\sqrt{|\alpha|},\omega_0) \longrightarrow (M,\omega)$ and vice versa. This implies conformality.

  $D(B(1),\omega_0) = \pi$: This is easy since the identity is a symplectic embedding $B(1) \longrightarrow B(1)$, hence $D(B(1),\omega_0) \geq \pi$. The opposite inequality follows from the fact that symplectic embeddings are volume preserving, i.e. if there is a symplectic embedding $\psi : B(R) \longrightarrow B(1)$, then $R \leq 1$.

  $D(Z(1),\omega_0) = \pi$: This is where the existence of a capacity is used via the Gromov's non-squeezing theorem. Assume that $B(r) \longrightarrow Z(1)$ is a symplectic embedding. Then $r \leq 1$ by non-squeezing. The inequality $D(Z(1),\omega_0) \geq \pi$ is obvious.

  Let c be any capacity. If $\psi : (B(r),\omega_0) \longrightarrow (M,\omega)$ is a symplectic embedding, then $c(B(r),\omega_0) \leq c(M,\omega)$. The second to last claim follows.

  The last claim follows from the fact that compact symplectic manifolds have finite volume.
- **Remark:** Let $\psi_j : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be a family of symplectic diffeomorphisms which C^1-converges (locally) to a map $\psi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$. Then since

$$d\psi_j(x)^T J_0 d\psi(x) = J_0$$

for all x, the limit ψ is also symplectic. Here $J_0 = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$ is an almost complex structure compatible with ω_0 yielding the standard Euclidean structure. The following theorem is a vast generalization of this statement:
- **Theorem (Gromov, Eliashberg):** Let $\psi_j : (M,\omega) \longrightarrow (M,\omega)$ be a sequence of symplectic diffeomorphisms such that ψ_j converges locally uniformly to the *diffeomorphism* ψ, then ψ is symplectic.
- By the Darboux theorem, this is a Corollary of the following
- **Theorem:** Let $\psi_j : (B(1),\omega_0) \longrightarrow (\mathbb{R}^{2n},\omega_0)$ be a sequence of symplectic embeddings converging to $\psi : B(1) \longrightarrow M$ which is differentiable at the origin. Then $d\phi(0)$ is symplectic.
- **Convention:** We extend a symplectic capacity to all subsets of $\mathbb{R}^{2n}$ via

$$c(A) = \inf\{c(U) \mid A \subset U \subset \mathbb{R}^{2n} \text{ open }\}.$$

The monotonicity of c implies $A \subset B \Rightarrow c(A) \leq c(B)$. By invariance $c(A) = c(\psi(A))$ if ψ is a symplectic diffeomorphism defined on an open set containing A.
- **Proof of the last Theorem:**

  Let $H = d\psi(0)$ and assume $\psi(0) = 0$.

1. **ψ is measure preserving with respect to the standard symplectic volume $\mu = \omega_0^n/n!$:** This follows from uniform convergence and
$$\int_U \psi^*\mu := \int_{\psi(U)} \mu = \lim_{j\to\infty} \int_{\psi_j(U)} \mu = \lim_{j\to\infty} \int_U \psi_j^*\mu = \int_U \mu.$$
for all open sets in $B(1)$.
2. **H is an isomorphism:** Since ψ is differentiable at 0, there is a map h from a neighborhood of the origin to $\mathbb{R}^{2n}$ with $h(0) = 0$ and $|h(x)|/|x| \to 0$ as $x \to 0$ in the complement of the origin so that
$$\psi(x) = Hx + h(x).$$
Then
$$\lim_{\varepsilon\to 0} \frac{\mu(\psi(B(\varepsilon))}{\mu(B(\varepsilon))} = |\det(H)|.$$
The previous step implies $\det(H) = \pm 1$.
3. **Lemma (Eliashberg):** Assume H is a linear isomorphism which is *not* conformally symplectic, i.e. $H^*\omega_0 \neq \lambda\omega_0$ for all $\lambda \in \mathbb{R}$. Then for all $a > 0$ there are symplectic matrices U, V so that
$$U^{-1}HV = \left(\begin{array}{cc|c} a & 0 & \\ 0 & a & 0 \\ \hline \multicolumn{2}{c|}{*} & * \end{array}\right)$$
This will be postponed.
4. **H is conformally symplectic:** Assume not. Then by the Lemma we can find $a > 0$ and symplectic matrices U, V so that $U^{-1}HV$ has the form given above and $U^{-1}HV(B(1)) \subset Z(1/8)$ when $a > 0$ is sufficiently small. Then $U^{-1} \circ \psi \circ V$ maps $B(\varepsilon)$ into $Z(\varepsilon/4)$ for $\varepsilon > 0$ sufficiently small. Because $U^{-1} \circ \psi_j \circ V$ converges uniformly to $U^{-1} \circ \psi \circ V$ on a compact neighborhood of 0, we get
$$U^{-1} \circ \psi_j \circ V(B(\varepsilon)) \subset Z(\varepsilon/2)$$
for $B(\varepsilon)$ contained in that neighborhood for sufficiently big j (such that $U^{-1} \circ \psi_j \circ V$ is $\varepsilon/2$ close to $U^{-1} \circ \psi \circ V$ on the neighborhood). This is a contradiction to Gromov's non-squeezing theorem. Thus, $H^*\omega_0 = \lambda\omega_0$ for some $\lambda \in \mathbb{R}$.
5. **$\lambda = 1$:** Consider the symplectic embeddings $\psi_j \circ \mathrm{id} : B^{2n}(1) \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with the product symplectic structure. This converges locally uniformly to $\hat\psi$ satisfying the assumptions of the theorem and
$$D\hat\psi(0) = \begin{pmatrix} H & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$
This has to be conformally symplectic. Hence, $\lambda = 1$.
6. **Proof of the Lemma:** This requires some clever linear algebra. Let B be the ω_0-symplectic adjoint of H uniquely defined by
$$\omega_0(Hx, y) = \omega_0(x, By)$$
If B were conformally symplectic for some λ, then $B \circ H = \lambda \cdot \mathrm{id}$, so H would be conformally symplectic since B is an isomorphism (like H). Let $\omega = B^*\omega_0$.
Claim: There is $x \in \mathbb{R}^{2n}$ so that $\omega(x, \cdot)$ is not a multiple of $\omega_0(x, \cdot)$.

Proof of Claim: Assume not. Then for all $x \neq 0$ there is $\lambda(x)$ so that $\omega(x, \cdot) = \lambda(x)\omega_0(x, \cdot)$. Let $\xi \in \mathbb{R}^{2n}$ so that $\omega_0(x, \xi) \neq 0$. Then

$$\begin{aligned} \lambda(\xi)\omega_0(\xi, y) &= -\omega(y, \xi) \\ &= -\lambda(y)\omega(x, \xi) = \lambda(y)\omega_0(\xi, x). \end{aligned}$$

for y in a neighborhood of x where $\omega(y, \xi) \neq 0$. Then $\lambda(y)$ is constant on that neighborhood. Since $\mathbb{R}^{2n} \setminus \{0\}$ is connected, this implies that λ is constant on $\mathbb{R}^{2n} \setminus \{0\}$. This concludes the proof of the claim.
For x from the claim the map

$$\begin{aligned} \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^2 \\ u &\longmapsto (\omega_0(x, u), \omega(x, u)) \end{aligned}$$

is surjective (since $\omega_0(x, \cdot)$ and $\omega(x, \cdot)$ are linearly independent). For all $a > 0$ there is y so that

$$\omega_0(x, y) = 1 \text{ and } \omega(x, y) = a^2.$$

Since $\omega_0(Bx, By) = \omega(x, y)$ there are ω_0 symplectic bases (e_i, f_i) and (e_i', f_i') with

$$e_1 = x \qquad f_1 = y$$

$$e_1' = \frac{Bx}{a} \qquad f_1' = \frac{By}{a}$$

i.e. $Be_1 = ae_1'$ and $Bf_1 = af_1'$. Since $A = -J_0 B^T J_0$ the map A in terms of the basis (e_i, f_i) and (e_i', f_i') has the desired form.

21. Lecture on January, 10 – Existence of symplectic capacities

- For a smooth function H on a symplectic manifold the Hamiltonian vector field X_H is defined as the unique vector field such that $\omega(X_H, \cdot) = -dH(\cdot)$. What follows is the definition of the *Hofer-Zehnder capacity.*
- **Definition:** Let $\mathcal{H}(M, \omega)$ be the set of smooth functions H on M so that
  1. H is constant (this constant is called oscillation and denoted by $m(H)$) outside of a compact set contained in the interior of M,
  2. There is a nonempty open set where H vanishes.
  3. $0 \leq H(x) \leq m(H)$ for all $x \in M$.

  A function $H \in \mathcal{H}(M, \omega)$ is *admissible* if all periodic solutions of X_H are either constant or have period $T > 1$.
- **Definition:** Let (M, ω) be symplectic. The Hofer-Zehnder capacity of (M, ω) is

  $$c_{HZ}(M, \omega) = \sup\{m(H) \mid H \in \mathcal{H}(M, \omega) \text{ is admissible}\}.$$
- **Theorem (Hofer-Zehnder):** This is a symplectic capacity.
- The proof of this will take some time. The most difficult step is to show $c(Z(1), \omega_0) \leq \pi$ which requires to establish the existence of non-trivial periodic orbits with period ≤ 1 for all Hamiltonian vector fields for functions $H \in \mathcal{H}(Z(1), \omega_0)$ with $m(H) > \pi$.
- **Lemma:** c_{HZ} is monotone.

- **Proof:** For a symplectic embedding $\psi : (M,\omega) \longrightarrow (N,\sigma)$ define $\psi_* : \mathcal{H}(M,\omega) \longrightarrow \mathcal{H}(N,\sigma)$ via

$$(\psi_*(H))(x) = \begin{cases} m(H) & \text{if } x \notin \psi(M) \\ H(\psi^{-1}(x)) & \text{if } x \in \psi(M) \end{cases}$$

Because ψ is symplectic, one has $X_{\psi_* H}(x) = 0$ on the complement of $\psi(M)$ and $X_{\psi_* H}(x) = \psi_*(X_H(\psi^{-1}(x)))$ for $x \in \psi(M)$. Therefore, non-trivial periodic orbits of $X_{\psi_* H}$ are in one-to-one correspondence with non-trivial periodic orbits of X_H and the correspondence preserves periods. Therefore, $\psi_* H$ is admissible when H is admissible, so

$$c_{HZ}(M,\omega) \leq c_{HZ}(N,\sigma).$$

- **Lemma:** c_{HZ} is conformal.
- **Proof:** Let $\alpha \neq 0$ and consider

$$\psi_* : \mathcal{H}(M,\omega) \longrightarrow \mathcal{H}(M,\alpha\omega)$$
$$H \longmapsto |\alpha| H.$$

Clearly, $X^{\alpha\omega}_{|\alpha|H} = \frac{|\alpha|}{\alpha} X^{\omega}_H$. So admissible functions in $\mathcal{H}(M,\omega)$ map to admissible functions in $\mathcal{H}(M,\alpha\omega)$.
- **Lemma:** $c_{HZ}(B(1),\omega) \geq \pi$.
- **Proof:** For $\varepsilon > 0$ we have to construct functions H without fast periodic orbits so that $m(H) \geq \pi - \varepsilon$. We pick a smooth function

$$f : [0,1] \longrightarrow [0,\infty)$$
$$t \longmapsto \begin{cases} 0 & t \text{ near } 0 \\ \pi - \varepsilon & t \text{ near } 1 \end{cases}$$

so that $0 \leq f'(t) < \pi$ for all t. Set $H(x) = f(|x|^2)$ for $x \in B(1)$. Then $m(H) = \pi - \varepsilon$, H vanishes on a neighborhood of the origin and $H \leq \pi - \varepsilon$. Then

$$X_H = J\nabla H = Jf'(|x|^2)2x$$

so that Hamiltonian flow of H preserves the levels of the function $x \longmapsto |x|^2$. Therefore, $X_H(x) = a(J_0 x)$ with $0 \leq a = 2f'(|x|^2) < 2\pi$ constant along orbits. Solutions of this ODE are either constant (when $a = 0$) or periodic with period $T = 2\pi/a > 1$ when $a \neq 0$. Therefore, H is admissible with $m(H) = \pi - \varepsilon$.
- **Remark:** By monotonicity, $c_{HZ}(Z(1),\omega_0) \geq c_{HZ}(B(1),\omega_0) \geq \pi$. We are done once we show $\pi \geq c_{HZ}(Z(1),\omega_0)$.
- We will show that if $m(H) > \pi$ for $H \in \mathcal{H}(Z(1),\omega_0)$, then H is not admissible, i.e. there is a non-trivial periodic orbit with period $T \leq 1$. Fix such an H. First, we do some simple reductions and extend H.
- Without loss of generality, we may assume that H vanishes on a neighborhood of the origin. This is because symplectomorphisms act transitively on connected manifolds. Since H is constant near $\partial Z(1)$ it would be easy to extend H to a function on $\mathbb{R}^{2n}$ constant outside of $Z(1)$ but we want a different, modification.
- **Choice of extension of H to $\mathbb{R}^{2n}$:** For $N \in \mathbb{N}$ sufficiently large $H \in \mathcal{H}(Z(1),\omega_0)$ is constant outside of the ellipsoid

$$E_N = \left\{ z = (x_1,\ldots,y_n) \in \mathbb{R}^{2n} \mid q_N(z) = x_1^2 + y_1^2 + \frac{1}{N^2}\sum_i (x_i^2 + y_i^2) < 1 \right\}.$$

We assumed that $m(H) > \pi$, i.e. there is $\varepsilon > 0$ so that $\pi + \varepsilon < m(H)$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be smooth so that

$$\begin{aligned} f(s) &= m(H) \text{ for } s \leq 1 \\ f(s) &\geq (\pi+\varepsilon)s \text{ for all } s \in \mathbb{R} \\ f(s) &= (\pi+\varepsilon)s \text{ for } s \text{ sufficiently large} \end{aligned} \tag{34}$$

and $0 < f'(s) \leq \pi + \varepsilon$ when $s > 1$. The modification $\overline{H}$ of H is

$$\overline{H}(z) = \begin{cases} H(z) & \text{for } z \in E \\ f(q_N(z)) & \text{for } z \notin E. \end{cases}$$

Clearly, $\overline{H}$ is quadratic outside of a compact set, therefore the solutions of the Hamiltonian equations of $\overline{H}$ are defined on $\mathbb{R}$ (i.e. the Hamiltonian vector field is complete). From now on we denote $\overline{H}$ be H. Note that did not really extend H, but we will refer to the new function as extension nevertheless.

- **1-Periodic orbits of the Hamiltonian system as critical points of an action:** Informal discussion: We consider a smooth Hamiltonian function, smooth symplectic form etc. Thus, solutions of the Hamiltonian system are smooth. A 1-periodic solution can be viewed as element of the loop space $\Omega = C^\infty(S^1, \mathbb{R}^{2n})$ where $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. With the smooth topology this a Frechet space, we would like to deal with a Hilbert space later, with the C^1-topology it is incomplete. The tangent space of this vector space (point wise addition etc.) will be identified with the space itself. We define

$$\begin{aligned} \Phi : \Omega &\longrightarrow \mathbb{R} \\ x(t) &\longmapsto \int_0^1 \left(\langle -J\dot{x}(t), x(t) \rangle - H(x(t)) \right) dt. \end{aligned} \tag{35}$$

Computing the derivative of Φ with respect to the variation $x(t) + \varepsilon y(t)$ of $x(t)$ we get

$$\begin{aligned} \Phi'(x)(y) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi(x(t) + \varepsilon y(t)) \\ &= \int_0^1 \left(\frac{1}{2}(\langle -J\dot{y}(t), x(t) \rangle + \langle -J\dot{x}(t), y(t) \rangle) - \langle \nabla H(x(t)), y(t) \rangle \right) dt \\ &= \int_0^1 \left(\frac{1}{2}(-\langle -Jy(t), \dot{x}(t) \rangle + \langle -J\dot{x}(t), y(t) \rangle) - \langle \nabla H(x(t)), y(t) \rangle \right) dt \\ &= \int_0^1 \langle -J\dot{x}(t) - \nabla H(x(t)), y(t) \rangle \, dt. \end{aligned}$$

For the third equality one uses partial integration and the fact that $x(t), y(t)$ are loops. Thus, critical points of $\Phi : \Omega \longrightarrow \mathbb{R}$ are precisely those loops which satisfy the Hamiltonian equation $-J\dot{x}(t) = \nabla H(x(t))$.

- **Example:** Consider the loops

$$x_k(t) = \cos(2\pi kt)x_0 + \sin(2\pi kt)Jx_0$$

with $\|x\|_0 = 1$. Then

$$\int_0^1 \frac{1}{2}\langle -J\dot{x}_k(t), x_k(t) \rangle dt = \pi k$$

and the L^2-norm of $x_m(t)$ is one. Because the second summand of $\Phi(x(t))$ is bounded by $m(H)$, the functional is not bounded from above or below. Finding minimizers to establish the existence of critical points will not work.

- **Characterization of periodic orbits with period $T = 1$ in E_N:** Let $x(t)$ be a 1-periodic solution of $\dot{x} = X_H(x)$. If

$$\Phi(x) = \int_0^1 \left(\frac{1}{2} \langle -J\dot{x}, x \rangle - H(x(t)) \right) dt > 0$$

then x is not constant and $x(t) \subset E_N$ for all t. Thus, $x(t)$ is a non-constant, 1-periodic solution of the original Hamiltonian system (for non-extended H) in $Z(1)$.

- **Proof:** If $x(t)$ is constant then the action is non-positive since $H \geq 0$. Since $dH = 0$ near ∂E_N, the Hamiltonian vector field vanishes there. This implies that a solution of the Hamiltonian system which starts in E_N stays inside E_N for ever.

  Solutions $x(t)$ of the Hamiltonian system which live outside of E_N satisfy

$$-J\dot{x}(t) = (\nabla H)(x(t)) = f'(q_N(x(t))(\nabla q_N)(x(t))$$

where ∇ is the ordinary gradient with respect to the standard Euclidean structure on $\mathbb{R}^{2n}$. Outside of E_N, the quadratic form q_N is constant along solutions of the Hamiltonian system, i.e. $q_N(x(t)) = \tau$ for a constant τ depending on the solution $x(t)$. Because of

$$\langle \nabla q_N(z), z \rangle = 2q_N(z)$$

and of the definition of H outside of E_N this implies for a solution $x(t)$ outside of E_N that

$$\begin{aligned} \Phi(x) &= \int_0^1 \left(\frac{1}{2} \langle -J\dot{x}, x \rangle - H(x(t)) \right) dt \\ &= \int_0^1 (\tau f'(\tau) - f(\tau))\, dt \\ &= \tau f'(\tau) - f(\tau) \leq 0 \end{aligned}$$

by the second condition in (34).

22. Lecture on January, 14 - Minimax, Analytic setting

- **Reminder:** Let E be a Hilbert space. A function $f : E \longrightarrow \mathbb{R}$ is differentiable in $e \in E$ if for all $e \in E$ there is a linear form $df \in E^*$ (continuous) such that $f(e') = f(e) + df(e' - e) + h(e' - e)$ for a function h defined on a neighborhood of e such that $\lim_{e'-e \to 0} \frac{h(e'-e)}{\|e'-e\|} = 0$. Moreover, f is C^1 if it is differentiable everywhere and the derivative depends continuously on e (in the operator norm topology on E^*).

  Since E is a Hilbert space, there is a unique $\nabla f \in E$ so that $df(\cdot) = \langle \nabla f, \cdot \rangle$. Critical points of f are defined to be zeroes of ∇f. If f is C^1, then the negative gradient flow $\varphi_t : E \longrightarrow E$ is the solution of

$$\begin{aligned} \varphi_0(x) &= x \\ \frac{d}{dt}(\varphi_t(x)) &= -\nabla f(x(t)) \end{aligned}$$

provided that this solution exists and is unique. A sufficient condition for this is that ∇f is Lipschitz continuous. As usual, $f(x(t))$ is decreasing along the negative gradient flow of f. More precisely, for positive t,

$$\begin{aligned} f(\varphi_t(x)) - f(x) &= \int_0^t \frac{d}{ds} f(\varphi_s(x)) ds = \int_0^t \left\langle \nabla f(\varphi_s(x)), \frac{d}{ds}\varphi_s(x) \right\rangle ds \\ &= -\int_0^t \|\nabla f(\varphi_s(x))\|^2 ds. \end{aligned} \tag{36}$$

A condition which is useful for finding critical points is the Palais-Smale condition:

(PS) f satisfies PS is every sequence $x_j \in E$ with

$$\begin{aligned} &\nabla f(x_j) \to 0 \in E \\ &|f(x_j)| < c \text{ for some constant } c \end{aligned}$$

has a convergent subsequence.

The limit of this subsequence is a critical point since f is C^1.

- **Minimax Lemma:** Let $\mathcal{F}$ be a family of subsets of E and $f \in C^1(E, \mathbb{R})$ such that
  - f satisfies (PS),
  - the negative gradient flow of f is well-defined,
  - $\mathcal{F}$ is positively invariant, i.e. if $A \in \mathcal{F}$, then $\varphi_t(A) \in \mathcal{F}$ for all $t > 0$,
  - the minimax $c(f, \mathcal{F})$ of f with respect to $\mathcal{F}$ is a real number, i.e.

  $$-\infty < c(f, \mathcal{F}) := \inf_{A \in \mathcal{F}} \left(\sup_{x \in A} f(x) \right) < \infty.$$

  Then $c(f, \mathcal{F})$ is a critical value.
- **Example (Minimizers):** Assume that f is bounded from below, satisfies (PS) and has complete negative gradient flow. Then $c(f, \mathcal{F}) = \inf f$ and choosing $\mathcal{F}$ to consist of sets in E where f is bounded the Minimax Lemma ensures the existence of a critical point x_0 with $f(x_0) = c(f, \mathcal{F})$.
- **Example (Mountain Pass Lemma):** $R \subset E$ is a *mountain range relative to* $f \in C^1(E)$ if $E \setminus R$ is not connected, $f|_R$ is bounded from below by some α so that each connected component of $E \setminus R$ contains a point e with $f(e) < \alpha$. We assume that f satisfies (PS) and that the negative gradient flow is well-defined. Then there is a critical point x_R with $f(x_R) \geq \inf_R f$.

  For the proof we need to find $\mathcal{F}$ suitable for the application of the Minimax Lemma. This consists of images of paths in E passing from one side of the mountain range to another. More precisely, let E_0, E_1 be two connected components of $E \setminus R$ and α a lower bound for $f|_R$. Let Γ of continuous paths in E which start in $E_0 \cap f^{-1}((-\infty, \alpha))$ and end in $E_1 \cap f^{-1}((-\infty, \alpha))$ (both sets are non-empty). Then set $\mathcal{F} = \{\text{image}(\gamma) \mid \gamma \in \Gamma\}$.
- **Non-example:** Consider the function $f(x, y) = e^{-x} - y^2$ with the mountain range $R = f^{-1}([0, \infty))$. This does not admit a mountain pass for lack of (PS).
- **Proof of Minimax Lemma:** Let $c = c(f, \mathcal{F})$. We first observe that for all $\varepsilon > 0$ the set $f^{-1}([c - \varepsilon, c + \varepsilon])$ contains a point x_ε so that $\|\nabla f(x_\varepsilon)\| < \varepsilon$. Assume this is not true. Then for some $\varepsilon_0 > 0$ the gradient of f is bounded away from 0 on $f^{-1}([c - \varepsilon_0, c + \varepsilon_0])$. This implies that this set will be moved away from itself by the negative gradient flow in finite positive time (see (36)).

Thus, every set $A \in \mathcal{F}$ will be transported into another set $A' \in \mathcal{F}$ such that $f < c - \varepsilon$ on A'. This contradicts the definition of the minimax $c = c(f, \mathcal{F})$.

Applying this observation of $\varepsilon_j = 1/j$ we obtain a sequence x_j such that $\|\nabla f(x_j)\| < 1/j$ with $|f(x) - x| \le 1/j$. By (PS) this sequence contains a subsequence converging to a critical point of f.

- **Analytical setting:** We want to find a Hilbert space E
  1. containing the set of closed loops $C^\infty(S^1, \mathbb{R}^{2n})$,
  2. for which there is a convenient extension of Φ (defined in (35)) from $C^\infty(S^1, \mathbb{R}^{2n})$ to E.

  Recall that *smooth loops* $x \in C^\infty(S^1, \mathbb{R}^{2n})$ can be represented as Fourier series
  $$x(t) = \sum_{k \in \mathbb{Z}} e^{2\pi k J t} x_k$$
  with Fourier coefficients $x_k \in \mathbb{R}^{2n}$ such that the series converges uniformly to $x(t)$ and the Fourier series of all derivatives obtained by differentiating the Fourier series. For $x, y \in C^\infty(S^1, \mathbb{R}^{2n})$ let
  $$a(x, y) = \frac{1}{2} \int_0^1 \langle -J\dot{x}(t), y(t) \rangle dt$$
  be the dominant (symplectic) part of Φ viewed as bilinear form (in Φ, one has $x(t)$ in the place of $y(t)$). This can be computed in terms of Fourier coefficients of x, y and using the orthonormality relation
  $$\int_0^1 \left\langle e^{2\pi k J t} x_k, e^{2\pi l J t} y_l \right\rangle dt = \delta_{kl} \langle x_k, y_l \rangle$$
  one obtains
  $$a(x, y) = \pi \left(\sum_{k > 0} |k| \langle x_k, y_k \rangle - \sum_{k < 0} |k| \langle x_k, y_k \rangle \right).$$
  This will motivate the choice of Hilbert space to which we will extend a and Φ.
- **Definition:** The Sobolev space $H^s(S^1, \mathbb{R}^{2n})$ with $s \ge 0$ is

$$H^s(S^1, \mathbb{R}^{2n}) = \left\{ x = \sum_{k=-\infty}^{\infty} e^{2\pi k J t} x_k \;\middle|\; \sum_{k=-\infty}^{\infty} |k^{2s}| |x_k|^2 < \infty \text{ for } x_k \in \mathbb{R}^{2n} \right\} \subset L^2(S^1, \mathbb{R}^{2n}).$$

The inner product is
$$\langle x, y \rangle_s = \langle x_0, y_0 \rangle + 2\pi \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{2s} \langle x_k, y_k \rangle.$$

The form of a in terms of Fourier coefficients suggests using $s = 1/2$ and $E = H^{1/2}(S^1, \mathbb{R}^{2n})$. E has an orthogonal decomposition $E = E^0 \oplus E^+ \oplus E^-$. Let $P^\pm, P^0$ be the corresponding projections. We decompose $x = x^0 + x^+ + x^-$ in these terms and get
$$a(x, y) = \frac{1}{2}(\langle x^+, y^+ \rangle_{H^{1/2}} - \langle x^-, y^- \rangle_{H^{1/2}}).$$
This is continuous and bilinear. The associated quadratic form $a(x) := a(x, x) = \frac{\|x^+\|^2 - \|x^-\|^2}{2}$ is a C^1-function on E with derivative
$$da(x)(y) = 2a(x, y) = \langle x^+ - x^-, y \rangle_{H^{1/2}}.$$

Thus, $\nabla a(x) = x^+ - x^- \in E$. This is clearly Lipschitz and the negative gradient flow of a is complete. The critical points of a are the constant functions.

We will need several standard facts from the theory of Sobolev spaces. There are stronger/more general versions of the following statements.

- **Rellich Lemma:** The embeddings $H^t \longrightarrow H^s$ with $t > s \geq 0$ map bounded sets to precompact sets (i.e. sets with compact closure).

  For example, a bounded sequence in $H^{1/2}$ contains an L^2-convergent subsequence.
- **Sobolev embedding:** if $s > 1/2$ and $x \in H^s$, then x is continuous and there is a constant c_s depending only on s so that
$$\sup_{t\in S^1} |x(t)| \leq c_s \|x\|_s \text{ for all } x \in H^s.$$
  Moreover, if $s > 1/2 + r$, then $x \in H^s$ is a C^r-function and there is a constant depending only on s so that
$$\sup_{t\in S^1, 0\leq k\leq r} |D^k x(t)| \leq c_s \|x\|_s \text{ for all } x \in H^s.$$
- Let $j : H^{1/2} \longrightarrow L^2$ be the inclusion. The adjoint j^* of j is defined via
$$(j(x), y)_{L^2} = \langle x, j^*(y)\rangle_{1/2}$$
  for all $x \in H^{1/2}$ and $y \in L^2$. We can express $x \in H^{1/2}(S^1, \mathbb{R}^{2n})$ and $y \in L^2(S^1, \mathbb{R}^{2n})$ in terms of Fourier coefficients
$$\sum_{k\in\mathbb{Z}} \langle x_k, y_k\rangle_{\mathbb{R}^{2n}} = \langle x_0, (j(y))_0\rangle_{\mathbb{R}^{2n}} + 2\pi \sum_{k\neq 0} |k| \langle x_k, (j^*(y))_k\rangle_{\mathbb{R}^{2n}}$$
  This holds for all $x \in H^{1/2}$. Therefore,
$$j^*(y) = y_0 + \sum_{k\neq 0} \frac{1}{2\pi|k|} e^{2\pi k J t} y_k$$
  This implies $\|j^*(y)\|_{H^1} \leq \|y\|_{L^2}$ and the continuity of j^*
$$j^*(L^2) \subset H^1 \text{ and } \|j^*(y)\|_1 \leq \|y\|_0 = \|y\|_{L^2}. \tag{37}$$
  In particular, $j^* : L^2 \longrightarrow H^1 \longrightarrow H^{1/2}$ is a compact operator (Rellich Lemma). This will be used for the study of the second summand of Φ.
- **Not so elementary estimate:** If $u \in H^{1/2}(S^1, \mathbb{R})$, then $u \in L^p(S^1, \mathbb{R})$ for all $1 \leq p < \infty$ and the embeddings $H^{1/2}(S^1, \mathbb{R}) \hookrightarrow L^p(S^1, \mathbb{R})$ are continuous and even compact.
- The second part of $\Phi(x)$ is (up to sign)
$$b(x) = \int_0^1 H(x(t))dt. \tag{38}$$
  We want to determine the $H^{1/2}$-gradient of b.
- b **as function on** L^2**:** Before doing this, note that $|H(z)| \leq M|z|^2$, so if $x \in L^2(S^1, \mathbb{R}^{2n})$. Then, the integral (38) is well-defined. In particular, it defined for $x \in H^{1/2}$ but it is natural study b on L^2.

  Because H is differentiable
$$H(z + \zeta) = H(t) + \langle \nabla H(z), \zeta\rangle + \underbrace{\int_0^1 \langle \nabla H(z + t\zeta) - \nabla H(z), \zeta\rangle dt}_{=:R(x,\zeta)}$$

Since $\nabla H(z) \leq M|z|$ (recall that H vanishes near the origin and is quadratic far out),

$$\begin{aligned} L^2(S^1, \mathbb{R}^{2n}) &\longrightarrow \mathbb{R} \\ \zeta &\longmapsto \int_0^1 \langle \nabla H(x(t)), \zeta(t) \rangle dt \end{aligned}$$

is well-defined. We hope that this is the L^2-derivative of b. This is the case since the last term $R(x, \zeta)$ of $H(x + \zeta)$ satisfies

$$|R(x, \zeta)| \leq M \|\zeta\|_{L^2}^2$$

For this recall that H is smooth and the coefficients of the Hessian of H are uniformly bounded because H coincides with a fixed quadratic form outside of E_N. This yields the above estimate by the mean value theorem. In order to show that b is a C^1-function on L^2, we have to verify that ∇b depends continuously on x. Moreover, we would like to show that the gradient flow of b is well-defined.

- **Lemma:** ∇b is Lipschitz on $L^2(S^1, \mathbb{R}^{2n})$.
- **Proof:**

$$\begin{aligned} \left| \underbrace{\int_0^1 \langle \nabla H(x(t)), \zeta(t) \rangle dt}_{=d_{L^2} b(x)(\zeta)} - \int_0^1 \langle \nabla H(y(t)), \zeta(t) \rangle dt \right| &= \left| \int_0^1 \langle \nabla H(x(t)) - \nabla H(y(t)), \zeta(t) \rangle dt \right| \\ &\leq \int_0^1 M |x(t) - y(t)| \cdot |\zeta(t)| dt \\ &\leq M \|x - y\|_{L^2} \cdot \|\zeta\|_{L^2}. \end{aligned}$$

This shows that ∇b is Lipschitz, so b is a C^1 function on L^2 and the gradient flow exists forever.

- **b as function on $H^{1/2}$:** We now denote by b the function on L^2. The function we are really interested in is the composition $b \circ j$ with $j : H^{1/2} \longrightarrow L^2$ the inclusion. Since j is linear, this map is smooth and coincides with its own derivative. By the chain rule

$$\begin{aligned} \langle \nabla_{H^{1/2}} (b \circ j)(x), y \rangle_{H^{1/2}} &= d(b \circ j)(x)(y) \\ &= db(j(x))(j(y)) \\ &= \langle (\nabla_{L^2} b)(j(x)), j(y) \rangle_{L^2} = \langle j^* (\nabla_{L^2} b)(j(x)), y \rangle_{H^{1/2}}. \end{aligned}$$

Again, we want to ensure that $\nabla(b \circ j)$ is a C^1-function on $H^{1/2}$.

- **Lemma:** $\nabla(b \circ j)$ is Lipschitz and maps bounded sets to relatively compact sets.
- **Proof:** This follows from the last Lemma and $\nabla_{H^{1/2}}(b \circ j) = j^*(\nabla_{L^2} b)$ and the fact that j^* is compact.
- **Remark:** We have extended Φ from $C^\infty(S^1, \mathbb{R}^{2n})$ to a C^1-function on $H^{1/2}(S^1, \mathbb{R}^{2n})$ with Lipschitz continuous gradient

$$\nabla_{H^{1/2}} \Phi(x) = x^+ - x^- - j^*(\nabla_{L^2} b(j(x))). \tag{39}$$

Thus, the negative gradient flow of Φ is well-defined on $H^{1/2}(S^1, \mathbb{R}^{2n})$. We look for critical points of Φ, i.e. for zeroes of $\nabla_{H^{1/2}} \Phi$. The most important condition we need to establish for this is the Palais-Smale condition. Before we will do this, we show that the critical points of the extended functional are all smooth loops, i.e. they are critical points of the original functional on $C^\infty(S^1, \mathbb{R}^{2n})$.

23. Lecture on January, 14 - Smoothness of critical points, Palais Smale condition for Φ

- **Lemma:** A critical point $x \in H^{1/2}(S^1, \mathbb{R}^{2n})$ of Φ lies in $C^\infty(S^1, \mathbb{R}^{2n})$ and satisfies Hamilton's equation.
- **Proof:** By assumption x and $\nabla H(x)$ lie in L^2, so we represent these functions by Fourier series:

$$x = \sum_{k \in \mathbb{Z}} e^{2\pi k J t} x_k$$
$$\nabla H(x) = \sum_{k \in \mathbb{Z}} e^{2\pi k J t} a_k$$

with $a_k, x_k \in \mathbb{R}^{2n}$. Because x is a critical point of Φ

$$\begin{aligned} \langle x^+ - x^- - j^* b(j(x)), \zeta \rangle_{H^{1/2}} &= \langle x^+ - x^-, \zeta \rangle_{H^{1/2}} - \langle \nabla H(j(x)), j(\zeta) \rangle_{L^2} \\ &= 0 \end{aligned}$$

for all $\zeta \in H^{1/2}(S^1, \mathbb{R}^{2n}) \subset L^2(S^1, \mathbb{R}^{2n})$. We are free to pick ζ, and we choose $\zeta_j(t) = e^{2\pi i j J} \zeta_0$ for $\zeta_0 \in \mathbb{R}^{2n}$. Using the definition of the $H^{1/2}$-scalar product we get

$$\begin{aligned} 2\pi |j| x_j &= a_j \text{ for } j \neq 0 \\ 0 &= a_0 \end{aligned} \tag{40}$$

Since a_j is square summable that same is true for $j x_j$.

So far, we only knew $x \in H^{1/2}(S^1, \mathbb{R}^{2n})$. **Now, we know** $\mathbf{x} \in \mathbf{H^1}(\mathbf{S^1}, \mathbb{R}^{\mathbf{2n}})$. Moreover, x is in $C^0(S^1, \mathbb{R}^{2n})$ by the Sobolev embedding theorems. Then $\nabla H(x(t))$ is continuous, so

$$\xi^\pm(t) = \int_0^t J(\nabla H(x(t))^\pm) dt \tag{41}$$

is a C^1-function on $\mathbb{R}$. The Fourier coefficients of $\xi^\pm$ coincide with those of $x^\pm$ by (40) except for the zeroth coefficient, i.e. up to a constant ξ coincides with x. **This implies** $\mathbf{x(t)} \in \mathbf{C^1}(\mathbf{S^1}, \mathbb{R}^{\mathbf{2n}})$ and $x(t)$ solves $\dot{x}(t) = J \nabla H(x(t))$. **Therefore,** $\mathbf{x} \in \mathbf{C^2}(\mathbf{S^1}, \mathbb{R}^{\mathbf{2n}})$. Iterating the argument starting at (41) we obtain $x \in C^\infty(S^1, \mathbb{R}^{2n})$.
- **Remark:** The following Lemma is stronger than (PS).
- **Lemma:** Assume that $x_k \in H^{1/2}$ satisfies $\lim_k \nabla_{H^{1/2}} \Phi(x_k) = 0$. Then x_k contains a convergent subsequence.
- **Proof:** If x_k is bounded, then $j^*(\nabla b(j(x_k)))$ contains a convergent subsequence by the compactness of j^*. We assume that $j^*(\nabla b(x_k))$ converges. This implies that $x_k^+ - x_k^-$ converges to the same value, i.e. $x_k^\pm$ converges. x_k^0 is bounded, like x_k itself. So we have a convergent subsequence of x_k so that the limit is a critical point.

We want to show that x_k is bounded. This will be done by contradiction, so we assume $\|x_k\|_{H^{1/2}} \to \infty$ and denote $y_k = x_k / \|x_k\|$. Then since $\nabla_{L^2} b(j(\zeta)) =$

$(\nabla H(x),\zeta)_{L^2}$

(42) $$\lim_k\left(y_k^+-y_k^--j^*\left(\frac{1}{\|x_k\|}\nabla H(x_k)\right)\right)=0\text{ in }H^{1/2}$$

By the choice of H outside of E_N and near the origin there is M so that $|\nabla H(z)|\le M|z|$. Therefore, $\frac{1}{\|x_k\|}\nabla H(x_k)$ is bounded, j^* is compact. As above, this implies that y_k has a convergent subsequence. The convergence is in $H^{1/2}$ (and therefore also in L^2) and therefore $y^+=\lim_k y_k^+$. We assume that y_k itself converges and denote the limit by y. Recall that H is equal to

$$Q(x_1,\dots,y_n)=(\pi+\varepsilon)\underbrace{\left(x_1^2+y_1^2+\frac{1}{N^2}\sum_{m=2}^{n}(x_m^2+y_m^2)\right)}_{=q_N}.$$

outside of a compact set. The Hamiltonian system $\dot{x}=\nabla Q(x)$ has no periodic solutions with period 1 except for the constant solution at the origin. We will compare the flow of $\dot{x}=\nabla Q(x)$ with x_k which are an unbounded sequence of solutions of $\dot{x}=\nabla H(x)$: Recall that $y_k=x_k/\|x_k\|$

$$\left\|\frac{\nabla H(x_k)}{\|x_k\|}-\nabla Q(y)\right\|_{L^2}\le\frac{1}{\|x_k\|}\|\nabla H(x_k)-\nabla Q(x_k)\|_{L^2}+\|\nabla Q(y_k)-\nabla Q(y)\|_{L^2}$$

Note that ∇Q is linear, and since $y_k\to y$ in L^2 the last summand goes to zero. Moreover, $|\nabla H(x_j)-\nabla Q(x_j)|$ is uniformly bounded because $Q=H$ outside of a compact set. Since **we assume** $\|x_j\|\to\infty$ we get

$$\frac{\nabla H(x_j)}{\|x_j\|}\to\nabla Q(y)\text{ in }L^2(S^1,\mathbb{R}^{2n}).$$

Because j^* is continuous we obtain

$$j^*\left(\frac{\nabla(b\circ j)(x_k)}{\|x_j\|}\right)=j^*\left(\frac{\nabla H(x_j)}{\|x_j\|}\right)\to j^*\left(\nabla Q(y)\right)\text{ in }H^{1/2}(S^1,\mathbb{R}^{2n}).$$

By (42) this implies that y solves $y^+-y^--j^*(\nabla Q(y))=0$ and $\|y\|_{H^{1/2}}=1$. By the previous Lemma for Q instead of H implies that y is smooth and a solution of $\dot{y}=J\nabla Q(y)$ with period 1. Which is not the zero solution. But this equation has no non-constant 1-periodic solutions except the zero solution which does not satisfy $\|y\|=1$. This is the desired contradiction, so $\|x_j\|$ is bounded.

- **Remark:** This Lemma made use of the choice of H outside of a compact set but you could probably talk your way out of this if you wished to.
- **Remark:** If H is constant, we can solve the differential equation $\dot{x}=-(x^+-x^-)$ explicitly. According to the following lemma, the difference between this solution and the actual solution differ by a map with a compactness property.

24. Lecture on January, 21 - Set up for Minimax

- **Lemma:** The flow $\dot{x}=\nabla_{H^{1/2}}\Phi$ is

(43) $$x\cdot t=\underbrace{e^tx^-+x^0+e^{-t}x^++K(x,t)}_{=:y(t)\text{ for fixed x}}$$

where $K:\mathbb{R}\times H^{1/2}\longrightarrow H^{1/2}$ is continuous and maps bounded sets to precompact sets.

- **Proof:** One makes the following Ansatz for K

$$\begin{aligned} K(t,x) &= -\int_0^t \left(e^{t-s}P^- + P^0 + e^{-(t-s)}P^+\right)(\nabla(b\circ j))(x\cdot s)ds \\ &= -j^*\int_0^t \left(e^{t-s}P^- + P^0 + e^{-(t-s)}P^+\right)(\nabla H)(j(x\cdot s))ds \end{aligned}$$

One verifies that $\xi(t) = y(t) - x\cdot t$ solves the initial value problem $\dot{\xi}(t) = \xi^+(t) - \xi^-(t)$ with $\xi(0) = 0$. Therefore, $\xi(t) \equiv 0$.
- **Remark:** This was somewhat technical. The remaining part of the proof of the existence of periodic orbit consists in applying minimax method cleverly to establish the existence of a critical point x of Φ with $\Phi(x) > 0$. We will find two subsets $\partial\Sigma$ and Γ in the Hilbert space $H^{1/2}$ which are linked in way that allows applying minimax methods.
- **The set $\partial\Sigma$:** $\Sigma = \Sigma_\tau$ is defined as

$$\Sigma_\tau = \left\{x \;\middle|\; x = x^- + x^0 + se^+ \text{ with } 0 \le s \le \tau \text{ and } \|x^- + x^0\| \le \tau\right\}$$

where $e^+ = e^{2\pi Jt}e_1 \in E^+$. This vector lies in E^+ and $\|e^+\|_{H^{1/2}} = 2\pi$, $\|e^+\|_{L^2} = 1$. The set Σ is some kind of infinite dimensional square, and we take $\partial\Sigma$ to be the union of its faces, i.e. the set theoretic boundary of Σ in $E^- \oplus E^0 \oplus \mathbb{R}e_1$.
- **Lemma:** There is τ_0 so that Φ is non-positive on $\partial\Sigma$.
- **Proof:** Recall that $a(x) = \frac{1}{2}\left(\|x^+\|^2_{H^{1/2}} - \|x^-\|^2_{H^{1/2}}\right)$. **Since $H \ge 0$,** this implies the claim for $\Phi|_{E^-\oplus E^0}$ (for $s = 0$). **By the choice of H outside of E_N**

$$H(z) \ge (\pi+\varepsilon)q_N(z) - \gamma$$

for some constant $\gamma > 0$. Then

$$\begin{aligned} \Phi(x) &\le a(x) - (\pi+\varepsilon)\int_0^1 q_N(x(t))dt \\ &= \frac{1}{2}\left(\|se^+\|^2_{H^{1/2}} - \|x^-\|^2_{H^{1/2}}\right) - (\pi+\varepsilon)\int_0^1 q_N(x^-(t) + x^0(t) + se^+)dt + (\pi+\varepsilon)\gamma \\ &= \frac{1}{2}\left(2\pi s^2 - \|x^-\|^2_{H^{1/2}}\right) - (\pi+\varepsilon)\int_0^1 q_N(x^-(t) + x^0(t) + se^+)dt + (\pi+\varepsilon)\gamma \\ &= \frac{1}{2}\left(2\pi s^2 - \|x^-\|^2_{H^{1/2}}\right) - (\pi+\varepsilon)\int_0^1 q_N(x^-(t)) + q_N(x^0(t)) + s^2q_N(e^+(t)))dt + (\pi+\varepsilon)\gamma \end{aligned}$$

droping some negative terms

$$\begin{aligned} &\le \frac{1}{2}\left(2\pi s^2 - \|x^-\|^2_{H^{1/2}}\right) - (\pi+\varepsilon)s^2\|e^+\|_{L^2} - (\pi+\varepsilon)q_N(x^0) + (\pi+\varepsilon)\gamma \\ &= -\varepsilon s^2 - \|x^-\|_{H^{1/2}} - (\pi+\varepsilon)q_N(x^0) + (\pi+\varepsilon)\gamma. \end{aligned}$$

This is non-positive on $\{s = \tau\} \subset \Sigma$ when τ is sufficiently large and negative on $\{\|x^- + x^0\| = \tau\}$ again for τ sufficiently large.
- **The set Γ:** For $\alpha > 0$ let $\Gamma_\alpha = \{x \in E^+ \mid \|x\|_{H^{1/2}} = \alpha\}$.
- **Lemma:** For suitable α there is $\beta > 0$ so that $\Phi|_{\Gamma_\alpha} \ge \beta > 0$.
- **Proof:** By the not so elementary estimate, there are constants M_p so that

$$\|x\|_{L^p} \le M_p\|x\|_{H^{1/2}}$$

for $p \geq 1$ and suitable constant M_p. Since H **vanishes on a neighborhood of the origin**, there is a constant c so that $|H(z)| \leq c|z|^3$. This implies that

$$\int_0^1 H(x(t))dt \leq c\|x\|^3_{L^3} \leq cM_3\|x\|_{H^{1/2}}.$$

For $x \in E^+$ we get $\Phi(x) = \frac{1}{2}\|x\|^2_{H^{1/2}} - cM_3\|x\|^2_{H^{1/2}}$. This implies the claim.

- Let φ_t be the flow of $-\nabla_{H^{1/2}}\Phi$ on $E = H^{1/2}$. Clearly, $\Gamma \cap \Sigma \neq \emptyset$, and one is inclined to believe that $\Gamma \cap \varphi_t(\Sigma) \neq \emptyset$ for all t. We assume this for now and show that c_{HZ} is a symplectic capacity.
- **Assumption:** We choose $\alpha < \tau_0$. Then Γ and Σ intersect exactly once.

25. Lecture on January, 23 - End of Proof, Leray-Schauder degree

- **Proof of the existence of a critical point x of Φ with $\Phi(x) > 0$:** Apply the minimax method to the family $\mathcal{F} = \{\varphi_t(\Sigma) | t \geq 0\}$ of subsets of Φ. The minimax of this family is

$$c(\Phi, \mathcal{F}) = \inf_{t \geq 0} (\sup\{\Phi(x) \,|\, x \in \varphi_t(\Sigma)\})$$

The sup is $\geq \beta$ since $\varphi_t(\Sigma)$ contains a point of Γ. Moreover, because the flow maps bounded sets into bounded sets, the sup is also $< \infty$. Since φ_t is a complete flow and Φ satisfies the (PS) condition, there is a critical point x with $\Phi(x) \geq \beta$. This concludes the proof under the assumption $\Gamma \cap \varphi_t(\Sigma) \neq \emptyset$ for $t \geq 0$.
- **Remark:** The Leray-Schauder degree will appear in this last step.
- **Review of Leray-Schauder degree:** Let X be a Banach space and $F : X \longrightarrow X$ continuous so that F maps bounded sets to sets with compact closure. We go through the definition of the Leray-Schauder degree, but do not prove anything beyond the facts needed to define it (omitting a proof that the Leray-Schauder degree is well-defined). Following [De], we take the point of view an analyst might want to take.
  1. Let $V \subset E$ have compact closure. For all $\varepsilon > 0$, there is a finite dimensional subspace $X_\varepsilon \subset X$ and projection $P_\varepsilon : V \longrightarrow X_\varepsilon$ so that $|P_\varepsilon x - x| \leq \varepsilon$.
     Proof: There are finitely many points $x_1, \ldots, x_m \in V$ so that $V \subset \cup_i B(x_i, \varepsilon)$. Let $g_i(x) = \max\{0, \varepsilon - |x - x_i|\}$. These functionals are continuous and $\sum_i g_i > 0$ on V and we can define $\lambda_i(x) = g_i(x)/\left(\sum_j g_j(x)\right)$. Define X_ε to be the span of $x_1, \ldots, x_m$ and

$$\begin{aligned} P_\varepsilon : V &\longrightarrow X_\varepsilon \\ x &\longmapsto \sum_i \lambda_i(x) x_i. \end{aligned}$$

     Then $|P_\varepsilon(x) - x| < \varepsilon$ since $\lambda_i(x) = 0$ for $x \notin B(x_i, \varepsilon)$.
  2. This implies that on a bounded set Ω the map F can be approximated by a map with finite dimensional image: For all $\varepsilon > 0$ and the set $V = \overline{F(\Omega)}$ we pick X_ε and P_ε from above. Then $F_\varepsilon = P_\varepsilon \circ F$ satisfies

$$\sup_{x \in \Omega} |F(x) - F_\varepsilon(x)| < \varepsilon.$$

3. The Leray-Schauder degree $\deg(\Omega, G = \mathrm{Id} - F, y)$ is defined when $y \notin G(\partial\Omega)$ as follows. Since $G(\partial\Omega)$ is compact, there is $\alpha > 0$ so that $\operatorname{dist}(y, G(\partial\Omega)) = \alpha$. According to the above, there is $F_1 : \overline{\Omega} \longrightarrow X_1$ with X_1 of finite dimension (pick $\varepsilon = \alpha/2$) and we may assume that $y \in X_1$ and $\Omega_1 = \Omega \cap X_1$ is not empty and open with compact closure. The degree of
$$G_1 = \mathrm{id} - F_1 : \Omega_1 \longrightarrow X_1$$
with respect to y is defined via the signed count of solutions of $G_2(x) = y$ with $x \in \Omega_1$ provided that y is a regular value of a smooth approximation G_2 of G_1 so that $\sup_{\Omega_1} |G_2 - G_1| \leq \alpha/2$.
4. There are many things to be shown here: Most pressing are independence of choices, and homotopy invariance. The latter infers that the degree is constant when F varies through compact operators so that y never lies in the image of the boundary of Ω.

- **Theorem:** The Leray-Schauder degree has the following properties:
  1. $\deg(\Omega, \mathrm{id}, y) = 1$ if $y \in \Omega$ and $= 0$ if $y \notin \Omega$.
  2. If $\deg(\omega, \mathrm{id} + F, y) \neq 0$, then there is $x \in \Omega$ with $x + F(x) = y$.
  3. If $H : [0,1] \times \overline{\Omega} \longrightarrow X$ is a homotopy mapping bounded sets to compact ones so that $y \notin H([0,1] \times \partial\Omega)$, then
  $$\deg(\Omega, \mathrm{Id} + H(0,x), y) = \deg(\Omega, \mathrm{Id} + H(1,x), y)$$
  4. If $\overline{\Omega} = \cup_i \overline{\Omega}_i$ with $\Omega_i \subset \Omega, i = 1, \ldots, m$, open and pairwise disjoint so that $y \notin \partial\Omega_i$ for all i, then
  $$\deg(\Omega, \mathrm{Id} + F, y) = \sum_{i=1}^{m} \deg(\Omega_i, \mathrm{Id} + F, y).$$
- **Lemma:** $\Gamma \cap \varphi_t(\Sigma) \neq \emptyset$ for $t \geq 0$.
- **Proof:** Points $x \in \Gamma \cap \varphi_t(\Sigma)$ solve the following equation

$$\begin{aligned} x &\in \Sigma \\ P^- \varphi_t(x) = P^0 \varphi_t(x) &= 0 \\ \|\varphi_t(x)\|_{H^{1/2}} &= \alpha \end{aligned} \tag{44}$$

Since $\varphi_t(x) = e^t x^- + x^0 + e^{-t} x^+ + K(t,x)$, (44) is equivalent to (multiply the E^--coordinate by e^{-t}):

$$\begin{aligned} x^+ &= s e^+ \text{ with } 0 \leq s \leq \tau \\ \|x^- + x^0\| &\leq \tau \\ x^- + x^0 + (e^{-t} P^- + P^0) K(t,x) &= 0 \\ \|\varphi_t(x)\| &= \alpha \end{aligned} \tag{45}$$

We define

$$B(t,x) = (e^{-t} P^- + P^0) K(t,x) + P^+ \left((\|\varphi_t(x)\| - \alpha) e^+ - x \right). \tag{46}$$

for $x \in E^- \oplus E^0 \oplus \mathbb{R}e^+ = F$. Note that $B(t,x) \in F$ for $x \in F$.

Solutions of $x + B(t,x) = 0$ with $x \in E^+ \oplus E^0 \oplus \mathbb{R}e^+$ are solutions of the third and fourth equation of (45): The third equation follows from the $E^+ \oplus E^0$-component of $x + B(t,x) = 0$, the fourth equation is the E^+-component. Moreover, $x \in E^+ \oplus E^0 \oplus \mathbb{R}e^+$ reflects $x \in \Sigma$ partially.

We want to show that the equation $x + B(t,x) = 0$ has a solution in $\Sigma \subset F$. Note that $B(t,x)$ is the sum of two maps. The first maps bounded sets to precompact sets. The second maps bounded sets to bounded sets in $\mathbb{R}e^+$. Thus, $B(t,x)$ maps bounded sets to precompact sets.

For operators of the form $\mathrm{id} + B(t,\cdot)$ where B is compact one can use the Leray-Schauder degree to show the existence of a solution of $0 = x + B(t,x)$. First, not that this equation cannot have a solution $x \in \partial\Sigma$ because $\Phi(x) \geq \beta > 0$ for $x \in \Gamma_\alpha$ while $\Phi(\varphi_t(x)) \leq \Phi(x) \leq 0$ for all $t \geq 0$ and $x \in \partial\Sigma$. By homotopy invariance of the degree

$$\deg(\Sigma, \mathrm{id} + B(t,\cdot), 0) = \deg(\Sigma, \mathrm{id} + B(0,\cdot), 0)$$

for $t \geq 0$. This simplifies matters because $K(0,x) = 0$. It remains to count solutions of $x + P^+ ((\|\varphi_0(x)\| - \alpha)e^+ - x) = 0$. The term in brackets is still non-linear, but it is homotopic to a constant map (through compact maps):

$$\begin{aligned}
\deg(\Sigma, \mathrm{id} + B(0,\cdot), 0) &= \deg\left(\Sigma, \mathrm{id} + P^+\left((\|\varphi_0(x)\| - \alpha)e^+ - x\right), 0\right) \\
&= \deg\left(\Sigma, \mathrm{id} + P^+\left((\sigma\|x\| - \alpha)e^+ - \sigma x\right), 0\right) \text{ for } 0 \leq \sigma \leq 1 \\
&= \deg\left(\Sigma, \mathrm{id} - \alpha e^+, 0\right) \\
&= 1
\end{aligned}$$

when $\tau > \alpha$.

- **Remark:** The proof above can be modified to prove the next theorem.

  For this, we consider first an embedded surface S and a function H so that S is a regular level set if H. Then ∇H is orthogonal to S, so the line field $J\nabla H$ is independent of the choice if H (as long as S is a regular level set.) This line field generates a foliation of rank 1 (given by $TS^{\perp_\omega}$), called the characteristic foliation.

  In the next theorem, will consider an embedding $\psi : (-1,1) \times S^{2n-1} \longrightarrow \mathbb{R}^{2n}$ of a thickened closed hypersurface and a smooth function H so that $H(\psi(s,p)) = 1 + \sigma(s)$ for a strictly monotone function σ defined on a neighborhood of 0 with $\sigma(0) = 0$

  We seek closed leaves of this rank-1-foliation on S_s for varying s. The action of such a leaf x is now defined to be

$$A(x) = \frac{1}{2}\int_0^T \langle -J\dot{x}, x\rangle dt$$

  when x has period T.

- **Theorem (Hofer-Zehnder,** [HZ2]**):** Let $\psi : (-1,1) \times S \longrightarrow \mathbb{R}^{2n}$ be a smooth embedding of a closed, connected $2n-1$-manifold S. Then there is a constant $d(\psi)$ such that for all $0 < \delta < 1$ there is ε with $|\varepsilon| < \delta$ so that $S_\varepsilon = \psi(\{\varepsilon\} \times S)$ has a periodic orbit x with $0 < A(x) < d$.
- **Proof:** One has to modify the previous proof slightly. We fix a function H on $\psi((-1,1) \times S) = U$.
  1. **Modification of** H**:** The complement of U has two connected components, we denote the bounded component by B and the unbounded one by A. Let $\gamma = \mathrm{diam}(U)$ and choose r, b so that

$$\begin{aligned}
\gamma &< r < 2\gamma \\
\frac{3}{2}\pi r^2 &< b < 2\pi r^2.
\end{aligned}$$

Pick a smooth function $f : (-1,1) \longrightarrow \mathbb{R}$ vanishing on $(-1,-\delta]$ and $f \equiv b$ on $[\delta, 1)$ which is positive derivative on $(-\delta,\delta)$. In order to define $\overline{H}$ we choose $g : (0,\infty) \longrightarrow \mathbb{R}$

$$\begin{aligned} g(s) &= b \text{ for } s \le r \\ g(s) &= \frac{3}{2}\pi s^2 \text{ for } s \text{ large} \\ g(s) &\ge \frac{3}{2}\pi s^2 \text{ for } s > r \end{aligned}$$

so that $0 < g'(s) \le 3\pi s$ for $s > r$. We now choose $\overline{H} \in C^\infty(\mathbb{R}^{2n},\mathbb{R})$ so that

$$\overline{H}(x) = \begin{cases} 0 & \text{for } x \in B \\ f(\varepsilon) & \text{for } x = \psi(\varepsilon, p) \text{ with } p \in S, -\delta \le \varepsilon \le \delta \\ b & \text{for } x \in A \text{ with } |x| \le r \\ g(|x|) & \text{for } |x| > r. \end{cases}$$

Notice that

$$-b + \frac{3}{2}\pi|x|^2 \le H(x) \le b + \frac{3}{2}\pi|x|^2.$$

2. **Characterization of periodic orbits as critical points of a functional:** Critical points $x \in \mathbb{C}^\infty(S^1,\mathbb{R}^{2n})$ of the functional

$$\overline{\Phi}(x) = \int_0^1 \big(\langle -J\dot{x}, x\rangle/2 - \overline{H}(x(t))\big)\, dt$$

are solutions of Hamilton's equation $\dot{x}(t) = J\nabla\overline{H}(x(t))$.
3. **Characterization of periodic orbits of the original H we are interested in:** If x is 1-periodic solution of $\dot{x}(t) = J\nabla\overline{H}(x(t))$ with $\overline{\Phi}(x) > 0$, then $x(t) \in S_\varepsilon$ for some $|\varepsilon| < \delta$.
The proof of this is an exercise.
4. **Analytical setting:** Remains unchanged, critical points of the functional $\overline{\Phi}$ associated to $\overline{H}$ are smooth solutions of Hamiltons equation. The (PS) condition holds for $\overline{\Phi}$.
5. **Description of the negative gradient flow:** As before.
6. **Choice of Γ:** As before, $\Phi|_\Gamma| > \beta > 0$ for $\Gamma = \Gamma_\alpha$.
7. **Choice of Σ_τ:** As before, exercise. One obtains the estimate $\overline{\Phi} \le 0$ on $\partial\Sigma$ and

$$\overline{\Phi}(x) \le b \text{ on } \Sigma \tag{47}$$

8. **Existence of periodic orbit:** As before, we find a periodic orbit with $\Phi(x) \ge \beta > 0$.
9. **Estimate for $A(x)$:**By construction $\overline{\Phi}(x) \ge \beta$. Morover, by the choice of $\overline{H}$ we have $0 \le \overline{H}(x(t))$. Therefore,

$$\begin{aligned} A(x) &= \overline{\Phi}(x) + \int_0^1 \overline{H}(x(t))dt \\ &\ge \beta + 0 > 0. \end{aligned}$$

By (47) and the fact that $\overline{H} \leq b$ on U one obtains

$$\begin{aligned} A(x) &= \overline{\Phi}(x) + \int_0^1 \overline{H}(x(t))dt \\ &\leq b + b \leq 16\pi\gamma^2 = d(\psi). \end{aligned}$$

26. Lecture on January 28 – Closed characteristics, Hofer metric on $\widetilde{\text{Ham}}$ and Ham

- **Remark:** The previous theorem can be paraphrased as follows. Given a family of hypersurfaces foliating a tubular neighborhood of S. Then the hypersurfaces containing closed characteristics (i.e. closed leaves of $TS_\varepsilon^{\perp_\omega} \subset TS_\varepsilon$) is dense. Of course, we would like to know when a **given** hypersurface contains a closed characteristic.
- **Reminder:** Let X be a Liouville vector field on $(\mathbb{R}^{2n}, \omega = dy \wedge dx)$, i.e. $L_X\omega = \omega$. Then the flow ϕ_t of X expands ω, so $\phi_t^*\omega = e^t\omega$. If S is a fixed closed hypersurface and X a Liouville vector field which is transverse to the closed hypersurface S. Then $\psi(t,p) = \varphi_t(p)$ parametrizes a family of hypersurfaces as above.

  Because X is a Liouville vector field, the characteristic line field $TS^{\perp_\omega}$ is mapped to the characteristic line field on $\varphi_t(S)$ by φ. Thus, a closed characteristic on $\varphi_t(S)$ corresponds to a closed characteristic on the original surface S.

  An example of this situation is when S is convex or star shaped with respect to a point (wlog the origin). in $\mathbb{R}^{2n}$. For this note that

  $$X = \frac{1}{2}\sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right)$$

  is a Liouville vector field on $\mathbb{R}^{2n}$.
- **Corollary:** A starshaped closed hypersurface in $\mathbb{R}^{2n}$ contains a closed characteristic.
- **Remark:** The previous theorem admits a simpler proof, the one outlined was useful to recast the difficult part of the proof of the fact that c_{HZ} is a symplectic capacity.
- **Alternative, partial argument:** Assume that no hypersurface in the family $S_\varepsilon, -\delta < \varepsilon < \delta$ contains a closed characteristic and choose R so that $B_R(0)$ contains the image of ψ. Using a function $H \in \mathcal{H}(B_R(0), \omega)$ which is constant outside of $\cup_{-\delta<\varepsilon<\delta} S_\varepsilon$ and so that all surfaces S^ε are contained in level sets of H. Since there are no closed characteristics, there are no non-constant periodic orbits of $J\nabla H$ of period 1. This would imply $c_{HZ}(B_R(0), \omega) = \infty$ which is not true, hence the assumption was wrong.
- **Corollary:** Let $S \subset (M, \omega)$ be a closed hypersurface and $N(S)$ a tubular neighborhood of S which is parametrized by $\psi : I \times S \longrightarrow N(S)$. If $c_{HZ}(U, \omega) < \infty$, then there is a dense subset $J \subset I$ so that S_ε has a closed characteristic.
- **Example (Zehnder):** Let $M = T^3 \times [0,1]$ with $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. We will construct a symplectic structure on M from a modification of ω_0 on $\mathbb{R}^4$. Pick an antisymmetric matrix $A \in \mathrm{Gl}(4, \mathbb{R})$. Then

  $$\omega(X, Y) = \langle AX, Y\rangle$$

is a symplectic form on $\mathbb{R}^4$ (with constant coefficients when expressed in terms of the standard framing of $\Lambda^2 T^*\mathbb{R}^4$). The Hamiltonian vector field of a function H is

$$X_H = -A^{-1}\nabla H.$$

We pick

$$A = \begin{pmatrix} 0 & 1 & -\alpha_2 & 0 \\ -1 & 0 & \alpha_1 & 0 \\ \alpha_2 & -\alpha_1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 0 & -1 & 0 & -\alpha_1 \\ 1 & 0 & 0 & -\alpha_2 \\ 0 & 0 & 0 & -1 \\ \alpha_1 & \alpha_2 & 1 & 0 \end{pmatrix}$$

(with $\det(A) = 1$). The symplectic form is then

$$\omega^* = dx_2 \wedge dx_1 + dx_4 \wedge dx_3 + \alpha_1 dx_3 \wedge dx_2 + \alpha_2 dx_1 \wedge dx_3$$

The Hamiltonian vector field of the function $H(x_1, x_2, x_3, x_4) = x_4$ is

$$X = (\alpha_1, \alpha_2, 1, 0)^T.$$

If $\alpha_1, \alpha_2, 1$ are linearly independent over $\mathbb{Q}$, then the flow lines of X are dense in the level set containing them (Kronecker's theorem). In view of the above corollary, we conclude that

$$c_{HZ}(M, \omega^*) = \infty.$$

In particular, there is no symplectic embedding $(T^3 \times [0,1], \omega^*)$ into $(\mathbb{R}^4, \omega_{st})$. (One can find a smooth embedding $M \longrightarrow \mathbb{R}^4$.) To see this one could also observe, that the symplectic structure ω^* on M is not exact (since the closed 2-torus corresponding to the x_1, x_2-plane has non-vanishing symplectic area).

- The above observations justify why closed characteristics on hypersurfaces in symplectic manifolds are important. Unless a hypersurface has
  - a neighborhood with finite Hofer-Zehnder capacity, and
  - some type of stability property

  it is difficult to infer that $S \subset (M, \omega)$ contains a closed characteristic. Our methods so far allow only to establish existence of closed characteristics exist on S_ε for $\varepsilon \in J$ a dense subset of I and an embedding $\psi : I \times S \longrightarrow M$. With a little more work one can establish that $J \subset I$ has full measure.

- **Convention:** In the following, (M, ω) is either closed with $\omega \in H^2$ rational, or $(\mathbb{R}^2 n, \omega_0)$. More generally, a symplectic manifold is tame when there is an almost complex structure such that the injectivity radius is bounded from below, the resulting manifold is complete and the curvature is bounded.
- **References:** [Po, HZ] for the Hofer norm/metric
- **Reminder:** The group of Hamiltonian diffeomorpisms $\mathrm{Ham}(M, \omega)$ with compact support in the interior of M is indeed a group.
- **Proposition:** Let H, K be time dependent smooth functions with compact support in the interior and φ_t, ψ_t be the corresponding Hamiltonian diffeomorphisms and ϑ a symplectomorphism of M. Then several group theoretic constructions from φ_t, ψ_t are also Hamiltonian:

$$\begin{aligned} \varphi_t^{-1} &\text{ is Hamiltonian for } \overline{H}(t,x) := -H(t, \varphi_t(x)) \\ \varphi_t \circ \psi_t &\text{ is Hamiltonian for } (H\#K)(t,x) = H(t,x) + K(t, \varphi_t^{-1}(x)) \\ \vartheta \circ \varphi_t \circ \vartheta^{-1} &\text{ is Hamiltonian for } H(t, \vartheta^{-1}(x)) \\ \varphi_t^{-1} \circ \psi_t &\text{ is Hamiltonian for } \overline{H}\#K. \end{aligned}$$

- **Consequence:** Let φ_t, ψ_t be Hamiltonian flows generated by time independent normalized functions H, K. If $\varphi_t \circ \psi_t = \psi_t \circ \varphi_t$, then $\{H, K\} \equiv 0$.
- **Proof:** $\varphi_t \circ \psi_t$ resp. $\psi_t \circ \varphi_t$ is generated by

$$H(x) + K(\varphi_t^{-1}(x)) \text{ resp. } K(x) + H(\psi_t^{-1}(x))$$

Since the flows are the same, these functions (both are normalized) agree. Differentiating the resulting equality with respect to t, one gets

$$-\{K, H\} = dK(X_H) = dH(X_K) = -\{H, K\}.$$

Hence, $\{K, H\} \equiv 0$.
- **Proposition:** Let $U \subset (M, \omega)$ be open. Then there exist $\varphi, \psi \in \mathrm{Ham}_c(M, \omega)$ which do not commute.
- **Proof:** Pick tangent vectors $X, Y \in T_xM, x \in U$ so that $\omega(X, Y) \neq 0$ and choose germs of functions H, K whose symplectic gradients at x are X, Y. Then extend to normalized Hamiltonians. The resulting flows do not commute.
- **Remark:** The third operation shows that $\mathrm{Ham}(M, \omega)$ is normal in $\mathrm{Symp}_0(M, \omega)$. Moreover, we know already that both groups are sufficiently connected to have universal coverings. We will be focusing on the standard contact structure on $\mathbb{R}^{2n}$. In this case $\mathrm{Ham}(\mathbb{R}^{2n}, \omega)$ is simply connected.
- **Sketch of Proof:** Let $p \in M$. Then there is a map $\mathrm{Ham}(\mathbb{R}^{2n}, \omega) \longrightarrow \mathbb{R}^{2n}$ given by evaluation on p. This map is a fibration. We denote the subgroup of $\mathrm{Ham}(M, \omega)$ which fix p by $\mathrm{Ham}(M, \omega, p)$. For p, q we choose Hamiltonian diffeomorphisms $\psi(q)$ which move q to p along the straight line from q to p so that $\psi(p) = \mathrm{id}$ and so that $\psi(q)$ depends continuously on q. Then the map

$$\begin{aligned} \mathrm{Ham}(M, \omega) &\longrightarrow \mathrm{Ham}(M, \omega, p) \\ \varphi &\longrightarrow \psi(\varphi(p)) \circ \varphi \end{aligned}$$

is a retraction. It also shows that $\mathrm{Ham}(M, \omega, p)$ is homotopy equivalent to $\mathrm{Ham}(M, \omega)$. The long exact sequence for homotopy groups of fibrations implies that $\mathrm{Ham}(\mathbb{R}^{2n}, \omega)$ is simply connected. The same proof works for contractible (M, ω).
- **Remark:** In the following we will consider the Hofer metric on $\mathrm{Ham}(\mathbb{R}^{2n}, \omega)$. For general (M, ω) one considers $\widetilde{\mathrm{Ham}}(M, \omega)$ to obtain analogous statements.
- **Definition:** The *oscillation* is

$$\begin{aligned} \| \cdot \| : C_c^\infty(\mathbb{R}^{2n}) &\longrightarrow \mathbb{R}_0^+ \\ H &\longmapsto \sup_x\{H(x)\} - \inf_x\{H(x)\}. \end{aligned}$$

This defines a $\mathrm{Diff}(\mathbb{R}^{2n})$-invariant norm on compactly supported smooth functions. If $\varphi_t, t \in [0, 1]$ is a Hamiltonian isotopy generated by smooth family of functions $H(t, \cdot)$, then we define the length of φ_t as

$$L(\varphi_t) := \|H\| := \int_0^1 \|H(t, \cdot)\| dt.$$

Finally, we define the Hofer norm

$$\begin{aligned} E = \| \cdot \| : \widetilde{\mathrm{Ham}}(M, \omega) &\longrightarrow \mathbb{R}_0^+ \\ \psi &\longmapsto \inf \left\{ \|\psi_t\| \;\middle|\; \begin{array}{l} \psi_t \text{ is generated by a smooth family of Hamiltonian} \\ \text{functions and } \psi_1 = \psi \end{array} \right\}. \end{aligned}$$

The following series of facts is a consequence of the previous proposition.

- **Facts:** The energy function satisfies
  1. positivity: $\|\varphi\| \geq 0$ for all $\varphi \in \widetilde{Ham}(M,\omega)$ and $E(\mathrm{id}) = 0$.
  2. symmetry: $\|\varphi\| = \|\varphi^{-1}\|$ for all $\varphi \in \widetilde{Ham}(M,\omega)$ (with group theoretic inverse, not path reversal)
  3. invariance: $\|\vartheta \circ \varphi \circ \vartheta^{-1}\| = \|\varphi\|$.
  4. triangle inequality: $\|\varphi \circ \psi\| \leq \|\varphi\| + \|\psi\|$ (composition, not concatination).

  For the last one:
$$\begin{aligned} l(\varphi_{H\#K}) &= \int_0^1 \left(\sup_x (H\#K)_t - \inf_x (H\#K)_t \right) dt \\ &\leq \int_0^1 \left(\sup_x (H_t) - \inf_x (H_t) \right) dt + \int_0^1 \left(\sup_x (K_t) - \inf_x (K_t) \right) dt \end{aligned}$$
- **Theorem:** If $\|\varphi\| = 0$, then $\varphi = \mathrm{id}$.
- **Remark:** On $\mathrm{Ham}(M,\omega)$ one has the uniform topology (after the choice of a metric) It turns out that $\|\cdot\|$ is continuous with respect to this topology.

27. Lecture on January 31 – Displacement energy, Non-degeneracy of the Hofer norm for $(\mathbb{R}^{2n}, \omega_0)$

- **Lemma:** $\|[\varphi,\psi]\| \leq 2\min\{\|\varphi\|, \|\psi\|\}$.
- **Proof:** One uses the facts from above:
$$\begin{aligned} \|[\varphi,\psi]\| &= \|\varphi \circ \psi \circ \varphi^{-1} \circ \psi^{-1}\| \\ &\leq \|\varphi \circ \psi \circ \varphi^{-1}\| + \|\psi^{-1}\| \\ &\leq 2\|\psi\|. \end{aligned}$$
  A similar computation implies $\|[\varphi,\psi]\| \leq \|\varphi\|$.
- The following Lemma is useful when one tries to estimate the energy from above. For this it is useful to recall that diffeomorphism with disjoint support commute.
- **Proposition:** Let $U \subset M$ be open and bounded. If $\vartheta \in \mathrm{Ham}(M,\omega)$ satisfies $\vartheta(U) \cap U = \emptyset$, then
$$\|[\varphi,\psi]\| \leq 4\|\vartheta\| \tag{48}$$
  for all $\varphi, \psi \in \mathrm{Ham}_{comp}(U,\omega)$.
- **Proof:** Define $\gamma = [\varphi, \vartheta^{-1}]$. On U this map equals φ. Hence $[\varphi,\psi] = [\gamma,\psi]$ on U. Outside of U, the first commutator is obviously the identity. The second commutator is also the identity
$$[\gamma,\psi](x) = \varphi \circ \vartheta^{-1} \circ \varphi^{-1} \circ \vartheta \circ \psi \circ \underbrace{\vartheta^{-1} \circ \varphi \circ \vartheta}_{\mathrm{supp} \subset \vartheta^{-1}(U)} \circ \varphi^{-1} \circ \psi^{-1}(x)$$
  outside of U. Hence, $[\varphi,\psi] = [\gamma,\psi]$ everywhere. This implies
$$\begin{aligned} \|[\varphi,\vartheta]\| &= \|[\gamma,\psi]\| \\ &\leq 2\|\gamma\| \leq 4\|\vartheta\|. \end{aligned}$$
- **Remark:** This suggests that the following quantity is interesting.
- **Definition:** Let $A \subset (M,\omega)$ be a bounded set. The *displacement energy* of A is
$$e(A) = \inf\{\|\varphi\| \,|\, \varphi \in \mathrm{Ham}(M,\omega) \text{ with } \varphi(A) \cap A = \emptyset\}.$$

- **Examples:** If M is closed and $\mathrm{vol}(A) > \mathrm{vol}(M)$, then $e(A) = \infty$. If $A \subset B^{2n-1} \subset \mathbb{R}^{2n-1} \times \{0\} \subset \mathbb{R}^{2n}$, then $e(A) = 0$. Below we will show that closed Lagrangian submanifolds L of $\mathbb{R}^{2n}$ have $0 < e(L) < \infty$.
- **Remark:** The computations in the previous Lemmas do not rely on the specific nature of E. They depend only on properties like invariance etc. It is legitimate to ask for other norms. For example, one can consider

$$E_p(\varphi_t) = \int_0^1 (|H_t(x)|^p \omega^n)^{1/p}\, dt$$

for $\infty > p \geq 1$ where H_t is normalized. It turns out that E_p is degenerate in the sense that $E_p(\varphi) = 0$ does *not* imply $\varphi = \mathrm{id}$.

In order to see this, consider a Darboux chart containing a small ball B which can be displaced from itself by the partially defined Hamiltonian flow φ_t of the function x_1. Consider the spheres $S_t = \varphi_t(\partial B)$. Multiplying x_1 with a time dependent cut-off function one make the p-norm of the resulting function arbitrarily small so that the new function equals x_1 on a neighborhood of S_t (this is not true for the L^∞-norm). Thus, for all $\varepsilon > 0$ there is a Hamiltonian flow ϑ_p which when restricted to S produces the same family of spheres S_t as the original function and $E_p(\vartheta_p)$. In particular, B is displaced from itself and $E_p(\vartheta_p) \leq C \cdot \varepsilon$.

By (48), this implies that $E_p([\varphi, \psi]) = 0$ for some non-commuting pair φ, ψ with support in B.
- **Theorem:** The map

$$\begin{aligned} d : \mathrm{Ham}(M,\omega) \times \mathrm{Ham}(M,\omega) &\longrightarrow \mathbb{R}_0^+ \\ (\varphi, \psi) &\longmapsto \|\varphi^{-1} \circ \psi\| \end{aligned}$$

defines a bi-invariant norm (i.e. $d(\vartheta \circ \varphi, \vartheta \circ \psi) = d(\varphi, \psi) = d(\varphi \circ \vartheta, \psi \circ \vartheta)$).
- We will outline a proof of the fact that the Hofer norm is non-degenerate on $\mathrm{Ham}(\mathbb{R}^{2n}, \omega)$. It is based on a theorem by Sikorav which is itself based on pseudo-holomorphic curves introduced by Gromov.
- **Theorem (Sikorav):** Let $L \subset \mathbb{R}^{2n} \times B^2(r)$ be a closed Lagrangian. Then

$$\gamma(L) \leq \pi r^2. \tag{49}$$

- The following theorem estimates the displacement energy of a Lagrangian $L \subset R^{2n}$ from below in terms of $\gamma(L)$. Since every non-trivial Hamiltonian diffeomorphism displaces a ball from itself and since this ball contains a split Lagrangian torus with $\gamma(L) > 0$ this implies that the Hofer norm on $\mathrm{Ham}(M, \omega)$ is non-degenerate.
- **Theorem:** Let $L \subset \mathbb{R}^{2n}$ be a Lagrangian submanifold. Then

$$\gamma(L) \leq 2e(L). \tag{50}$$

- **Proof:** Let φ_t be a family of Hamiltonian isotopies generated by H_t so that $\varphi_1(L) \cap L = \emptyset$. From this data and every $\varepsilon > 0$ we will construct an embedding $L'' \longrightarrow \mathbb{R}^{2n} \times B^2(r)$ so that
  1. $\gamma(L) = \gamma(L'')$, and
  2. $\pi r^2 \leq 2l(\varphi_t) + 10\varepsilon$.

Then (49) implies (50). The construction of the embedding of L'' is done in several steps. We pick $\varepsilon > 0$.

- **Step 0:** Reparametrize φ_t to obtain a path φ'_t so that $\varphi_t = \mathrm{id}$ for $0 \leq t \leq \varepsilon$ and $\varphi_t = \varphi_1$ for $1-\varepsilon \leq t \leq 1$. For this note that if $b : [0,1] \longrightarrow [0,1]$ is smooth, then $\varphi_{b(t)}$ is generated by $\frac{db}{dt}(t)H_{b(t)}$. This does not change the energy. We denote the new parametrized path/Hamiltonian functions by φ_t, H_t.
- **Step 1:** Consider that following loop of Hamiltonian diffeomorphisms
$$g_t = \begin{cases} \varphi_t & \text{for } t \in [0,1] \\ \varphi_{2-t} & \text{for } t \in [1,2]. \end{cases}$$
This is generated by the loop of Hamiltonian functions
$$G_t = \begin{cases} H_t & \text{for } t \in [0,1] \\ H_{2-t} & \text{for } t \in [1,2]. \end{cases}$$
We identify $S^1 = \mathbb{R}/2\mathbb{Z}$ and apply the Lagrangian suspension construction we obtain
$$\begin{aligned} L \times S^1 &\longrightarrow \mathbb{R}^{2n} \times T^*S^1 \\ (x,t) &\longmapsto \left(g_t(x), -G_t(g_t(x))dt \in T^*_t S^1\right). \end{aligned}$$
This is a Lagrangian submanifold L' with $\gamma(L') = \gamma(L)$ (one uses product symplectic structure of the standard symplectic structures on $\mathbb{R}^{2n}$ and T^*S^1). To check this verify $\int_0^2 G_t(g_t(x))dt = 0$ for all x since $[S^1]$ and $H_1(L)$ generate $H_1(L' = L \times S^1)$. The Lagrangian submanifold $L' = L \times S^1$ is contained in $\mathbb{R}^{2n} \times C$ with $C = \{\alpha dt \in T^*_t S^1 \mid a_+(t) < \alpha < a_-(t)\}$ and
$$\begin{aligned} a_+(t) &= -\min_x G_t(x) + \varepsilon \\ a_-(t) &= -\max_x G_t(x) - \varepsilon. \end{aligned}$$
The area of $C \subset T^*S^1$ is
$$\mathrm{Area}(C) = \int_0^2 (a_+(t) - a_-(t))dt = 2l(\varphi_t) + 4\varepsilon.$$
- **Step 2:** To obtain L'' we compose the Lagrangian suspension with
$$\mathrm{id}_{\mathbb{R}^{2n}} \times \vartheta : \mathbb{R}^{2n} \times T^*S^1 \longrightarrow \mathbb{R}^{2n} \times \mathbb{R}^2$$
where $\vartheta : T^*S^1 \longrightarrow \mathbb{R}^2$ is a symplectic *immersion* which is exact (i.e. the $\vartheta^*\lambda_{\mathbb{R}^2}$ differs from $\lambda_{T^*S^1}$ by a closed form). For this it is sufficient to ensure that the restriction of ϑ to the zero section is an exact Lagrangian immersion and ϑ is area preserving. We choose ϑ so that the zero section maps to a figure eight where the self intersection point is the image of $0, 1 \in S^1 = [0,2]/0 \sim 2$. Furthermore, we may assume that ϑ is an embedding outside of $t \in [0,\varepsilon] \cup [2-\varepsilon, 2] \cup [1-\varepsilon, 1+\varepsilon]$ and that every point has at most two preimages.
We can choose the immersion so that the image of C is contained in a ball of area $\mathrm{Area}(C) + 2\varepsilon$.
While $\mathrm{id} \times \vartheta$ is an immersion, its restriction to $L' = L \times S^1$ is an embedding since $\varphi_1(L) \cap L = \emptyset$.
- **Step 3:** Apply Sikorav's theorem to conclude: $\gamma(L) \leq \mathrm{Area}(C) + 2\varepsilon \leq 2l(\varphi_t) + 6\varepsilon$. This implies (50).

- **Remark:** We still have to show (49). This result is an improvement of a remark in the seminal paper [Gr] and relies on solutions of the Cauchy-Riemann boundary value problem with $D^2 \longrightarrow \mathbb{C}^n \simeq \mathbb{R}^{2n}$. We will not go through the analytic setup nor prove one of the main "technical" results underlying this theory (called compactness for pseudoholomorphic curves). It is interesting enough to state these things.
- **Preliminaries for Sikorav:**
  1. **The boundary value problem :** This makes sense for smooth maps
  $$f : (\Sigma, j) \longrightarrow (M, J)$$
  where (Σ, j) is a Riemann surface (i.e. a real two-dimensional manifold with an atlas $(\psi_i : U_i \longrightarrow \psi_i(U_i) \subset \mathbb{C})$ so that all transition functions are holomorphic). Furthermore, (M, J) is a manifold with an almost complex structure J. One imposes boundary conditions when Σ has boundary, usually $f(\partial\Sigma) \subset L$ where $L \subset M$ is a Lagrangian submanifold. Also, a topological requirement is usually made, namely $f_*([L]) = \alpha \in H_2(M, L; \mathbb{Z})$ for a given class α.
  The symplectic form ω is not required for the definition of the Cauchy-Riemann operator, but ω is needed for compactness results. Of course one requires that J is a adapted to ω (in other words ω *tames* J). Then
  $$\overline{\partial} f = \frac{1}{2}\left(df + J \circ df \circ j\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + J\frac{\partial}{\partial y}\right)$$
  in terms of holomorphic coordinates x, y on Σ is the Cauchy-Riemann operator. If $\overline{\partial} f = 0$, then df is complex linear, i.e. $df \circ j = J \circ df$. For us, the following case is relevant:
     - $(\Sigma, j) = (D^2 \subset \mathbb{C}, j)$ (by the uniformization theorem, every holomorphic structure on D^2 is equivalent to this),
     - $(M, J) = (\mathbb{C}^n, i)$ which is tamed by the standard symplectic structure ω_0,
     - $\overline{\partial} f = g(z, f(z))$ with $g : D^2 \times \mathbb{C}^n \longrightarrow \mathbb{C}$ smooth,
     - L is the given Lagrangian in Sikorav's theorem, and
     - $\alpha = 0$.

  Generically, solutions of this boundary value problem appear in finite dimensional families and the dimension can be computed from the given data.
  2. **Wirtinger inequalities:** The following is the result of a clever application of the Cauchy-Schwarz inequality written for convenience using a global coordinate system x, y on $D^2 = \Sigma$
  $$\left|\int_{D^2} f^*\omega\right| \leq \mathrm{Area}(f) \leq 2\int_{D^2} |\overline{\partial} f|^2\, dx\, dy + \int_{D^2} f^*\omega. \tag{51}$$
  Note that $\int_{D^2} f^*\omega$ depends only on $\alpha \in H_2(M, L; \mathbb{Z})$ and that f has minimal area when $\overline{\partial} f = 0$. In particular, f is constant for $\overline{\partial} f = 0$ and $\alpha = 0$. The Wirtinger inequality implies that holomorphic curves have non-negative symplectic area. If the corresponding homology class is non-trivial, then the symplectic area is positive.
  3. **Cusp-solutions:** A cusp solution is given by following data:

- A decomposition $\alpha = \alpha' + \sum_i \beta_i$ with $\beta_i, \alpha' \in H_2(M, L; \mathbb{Z})$ and $\beta_i \neq 0$,
- a solution of the boundary value problem $f' : (D^2, \partial D^2) \longrightarrow (M, L)$ with $f_*([D^2]) = \alpha'$ and $\overline{\partial} f(z) = g(z, f(z))$, (note that in a more general setting solutions with $\Sigma = S^2$ may appear, they are irrelevant for us since $\pi_2(\mathbb{C}^n) = 0$ and non-trivial holomorphic spheres cannot be null-homologouos by the Wirtinger inequalities.)
- and solutions (holomorphic discs) $h_i : (D^2, \partial D^2) \longrightarrow (M, L)$ with $h_i([D^2]) = \beta_i$ and $\overline{\partial} h_j = 0$.

Gromov's compactness theorem states that for a generic set of functions g_s a sequence of solutions f_s of our boundary value problem with g_s either (sub-)converges to a solution f_{s_0} when $s \to s_0$ or to a cusp solution such that

$$\mathrm{Area}(f_s) \to \mathrm{Area}(f_{s_0}) + \sum_i \mathrm{Area}(h_i).$$

In particular, if our problem has a solution for $g_0 = 0$ but does not have a solution for g_1, then generically, non-trivial holomorphic discs must appear when one interpolates between 0 and g_1.

- **Lemma:** Let $g_s = (s, 0, \ldots, 0)$. For $L \subset \mathbb{C}^n \times B^2(r)$, our boundary value problem has no solution for $r < |s|$.
- **Proof:** Let f_s be a solution for a given s and ϕ its last component. Then

$$\begin{aligned}\pi s &= \int_{D^2} \overline{\partial} \phi \, dx \, dy \\ &= \frac{1}{2} \int_{D^2} d(\phi dx - i \phi dy) \\ &= \frac{1}{2} \int_{\partial D^2} \phi (dx - i dy)\end{aligned}$$

  From $|\phi| \leq r$ one obtains $|\sigma| < r$ after integration and taking absolute values.
- **Remark:** Solutions of $\overline{\partial} f_s = g_s$ with g_s constant are harmonic maps (i.e. have a special geometry).
- **Proof of Sikorav's theorem:** Our boundary value problem has a solution for $s = 0$, namely all constant solutions. If $|s| > r$, then there is no solution. We pretend that g_s is generic in the sense that sequences of solutions of our boundary value problem converge to a cusp solution $((\alpha', \beta_1, \ldots, \beta_k), (f'_{s_0}, h_1, \ldots, h_k)), k \geq 1$, before they cease to exist when $s \to s_0$ from below. By the first part of the Wirtinger inequality

$$\mathrm{Area}(f'_{s_0}) \geq |\omega(f'_{s_0})| = \left| \int_{\alpha'} \omega \right|$$

  and since $\alpha' + \sum_i \beta_i = 0$

$$\left| \int_{\alpha'} \omega \right| = \left| \sum_i \underbrace{\int_{\beta_i} \omega}_{>0} \right| \geq \gamma(L).$$

because there is at least one summand. Thus

$$\operatorname{Area}(f'_{s_0}) + \sum_i \operatorname{Area}(h_i) \geq 2\gamma(L)$$

Let f_n be the sequence of solutions of the g_{s_n} converging to the cusp solution for s_0. By the second part of the Wirtinger inequality

$$\operatorname{Area}(f_n) \leq 2\int_{D^2} s_n \, dx\, dy + \int_{\alpha=0} \omega \leq 2\pi s_0^2 \leq 2\pi r^2.$$

By the convergence of area, we get

$$2\gamma(L) \leq 2\pi r^2,$$

this is the desired inequality.

- We did not give a complete proof. The analytic set up , the compactness theorem as well as the fact that solutions of the boundary value problem come in families are missing. A reference which tries to give a clean proof of Gromov's non-squeezing theorem using holomorphic curves is [ABKLR]. A lot more detail can be found in the the following, much thicker references: [AD, AL, McDS2].

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