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# Exercise Sheet 9

**Exercise 1.** Let $d \geq 3$ and let $f \in C_0^\infty(\mathbb{R}^d)$ be a radial function, i.e. $f(x) = f(y)$ whenever $|x| = |y|$. Prove that $f$ satisfies

$$|f(x)| \leq C|x|^{(2-d)/2}\|\nabla f\|_2 \qquad \text{for all } x \in \mathbb{R}^d \setminus \{0\}, \tag{1}$$

for some constant $C > 0$, independent of $f$.

(Compare Exercise 3.12 on p. 59 in [LP].)

**Exercise 2.** In this exercise we study compact and non-compact embeddings of Sobolev spaces into $L^p$-spaces. Recall that a linear operator $T : X \to Y$ between Banach spaces $X$ and $Y$ is *compact* if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $X$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a subsequence which converges in $Y$.

(i) Prove that, for every $p \in [2, \frac{2d}{d-2}]$, the embedding $H^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ is well-defined and continuous.

(ii) Prove that the embedding $H^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ is not compact.

We denote by

$$H^1_{\text{rad}}(\mathbb{R}^d) := \{f \in H^1(\mathbb{R}^d) \ : \ \text{For every rotation } R \in SO(d),\ f(y) = f(Ry) \text{ for a.e. } y \in \mathbb{R}^d\}$$

the subspace of $H^1(\mathbb{R}^d)$ consisting of radial functions.

(iii) Use Exercise 1 to prove that, for $p \in (2, \frac{2d}{d-2})$, the embedding $H^1_{\text{rad}}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ is compact.

*Hint: You may use without proof that the space of radial $C_0^\infty(\mathbb{R}^d)$-functions is dense in $H^1_{rad}(\mathbb{R}^d)$. Moreover, you may use without proof the Rellich-Kondrachov theorem which states the following: Let $p \in [1, \frac{2d}{d-2})$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. If for a sequence of functions $(u_k : \Omega \to \mathbb{R})_{k\in\mathbb{N}}$, the sequence $\left(\int_\Omega (u_k^2 + |\nabla u_k|^2)\right)_{k\in\mathbb{N}}$ is bounded, then there is a function $u \in L^p(\Omega)$ and a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ such that $u_{k_l} \to u$ in $L^p(\Omega)$. (In other words, the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact.)*

(iv) Prove that the embedding $H^1_{\text{rad}}(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ is not compact.

**Exercise 3.** Prove that for $s \in \mathbb{N}$, there exists $C > 0$ such that

$$\|fg\|_{s,2} \leq C\left(\|f\|_{s,2}\|g\|_\infty + \|f\|_\infty\|g\|_{s,2}\right)$$

for all $f, g \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

*Hint: Combine the Leibniz rule with the Gagliardo-Nirenberg inequality from the previous exercise sheets:* $\|\partial_x^\alpha f\|_p \leq C \sum_{|\beta|=m} \|\partial_x^\beta f\|_q^\theta \|f\|_r^{1-\theta}$ *for parameters* $p, q, r \in [1, \infty]$ *and* $\theta \in [j/m, 1]$ *related by* $\frac{1}{p} - \frac{|\alpha|}{d} = \theta(\frac{1}{q} - \frac{m}{d}) + (1-\theta)\frac{1}{r}$.

(Compare equation (3.13) on p. 52 in [LP].)

**Exercise 4.** The *Poisson kernel* is defined by $P(x) := c_d(1 + |x|^2)^{-\frac{d+1}{2}}$, where $c_d > 0$ is such that $\|P\|_{L^1(\mathbb{R}^d)} = 1$. Setting $P_t(x) = t^{-d}P(x/t)$, we define, for any function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the *Poisson maximal function* $\mathcal{M}_P f(x) := \sup_{t>0} |(P_t * f)(x)|$. Finally, for $1 \leq p < \infty$, we define the *Hardy space* $\mathcal{H}^p(\mathbb{R}^d) := \{f \in L^1_{\text{loc}}(\mathbb{R}^d) : \mathcal{M}_P f \in L^p(\mathbb{R}^d)\}$, equipped with the norm $\|f\|_{\mathcal{H}^p} := \|\mathcal{M}_P f\|_p$.

(i) For $1 < p < \infty$, prove that $\mathcal{H}^p(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ and that there exists $C > 0$ such that $\|f\|_p \leq \|f\|_{\mathcal{H}^p} \leq C\|f\|_p$ for all $f \in L^p(\mathbb{R}^d)$.

*Hint: By Proposition 2.4 in [LP], one has* $\mathcal{M}_P f(x) \leq \mathcal{M}f(x)$ *for every* $x \in \mathbb{R}^d$ *and every* $f \in L^1_{loc}(\mathbb{R}^d)$*, where* $\mathcal{M}f$ *is the Hardy-Littlewood maximal function.*

(ii) For $p = 1$, prove that $\mathcal{H}^1(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$. Moreover, prove that if $f \in \mathcal{H}^1(\mathbb{R}^d)$ with $f \geq 0$, then $f \equiv 0$. In particular, $\mathcal{H}^1(\mathbb{R}^d) \neq L^1(\mathbb{R}^d)$.

Solutions to this exercise sheet can be handed in for correction on UniWorX until 25.06.2019. Please only upload .pdf files (and not .jpg or other formats).
The sheet will be discussed in the Exercise Class 25.06.2019. No solutions will be provided online.