Prof. Dr. Thomas Ø. Sørensen
Lukas Emmert, Tobias König

Partial Differential Equations II
SoSe 2019

Exercise Sheet 8

Exercise 1. (i) Let $1 < p < \infty$. Prove *Hardy's inequality* in dimension $d = 1$: For all $f \in C_c^\infty((0,\infty))$ with $f \geq 0$,

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)\,\mathrm{d}s\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,\mathrm{d}x\,. \tag{1}$$

Hint: Define $F(x) = \int_0^x f(s)\,\mathrm{d}s$, integrate by parts and use Hölder's inequality.

(ii) Prove that equality in (1) holds only if $f = 0$ a.e..

Hint: Recall that equality in Hölder's inequality $\int_{\mathbb{R}} gh \leq \|g\|_p \|h\|_q$ holds if and only if there exists $\lambda \in \mathbb{R}$ such that $g(x)^p = \lambda h(x)^q$ for a.e. x. Use this to derive that any optimizer f must satisfy $f(x) = cx^\lambda$.

(iii) Prove that the constant $c_p = \left(\frac{p}{p-1}\right)^p$ is sharp, i.e. (1) is wrong if c_p is replaced by any $c < c_p$.

Hint: Use test functions of the form $cx^{-\lambda}$ found in (ii), with $0 < \lambda < 1/p$, truncated to an interval (a, b).

(Compare Exercise 1.5 on p. 19 in [LP].)

Exercise 2. (i) Let $d \in \mathbb{N}$ and $1 \leq p < \infty$ with $p \neq d$. Prove *Hardy's inequality* in dimension $d \geq 1$. For all $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$,

$$\int_{\mathbb{R}^d} \frac{|f(x)|^p}{|x|^p}\,\mathrm{d}x \leq \left|\frac{p}{p-d}\right|^p \int_{\mathbb{R}^d} |\nabla f(x)|^p\,\mathrm{d}x. \tag{2}$$

Prove that if $p \in [1, d)$, (2) holds even for all $f \in C_c^\infty(\mathbb{R}^d)$.

Hint: Pass to radial coordinates and argue as in Exercise 1.

(ii) Let $1 \leq p < \infty$, and $q \in [1, p]$ with $q \neq d$. Prove that for all $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$,

$$\int_{\mathbb{R}^d} \frac{|f(x)|^p}{|x|^q}\,\mathrm{d}x \leq \left|\frac{p}{d-q}\right|^q \|f\|_p^{p-q} \|\nabla f\|_p^q.$$

Hint: Apply (2) to $f^{p/q}$.

(Compare Exercise 3.13 on p. 59 in [LP].)

Exercise 3. Let $d \geq 3$ and $p \in [1, d)$.

(i) Using Hardy's inequality (2), give an alternative proof of the *Gagliardo-Nirenberg-Sobolev inequality* proved in the lecture: There is $C > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\left(\int_{\mathbb{R}^d} |f(x)|^{\frac{dp}{d-p}} \, \mathrm{d}x \right)^{\frac{d-p}{d}} \leq C \int_{\mathbb{R}^d} |\nabla f(x)|^p \, \mathrm{d}x \,. \tag{3}$$

Hint: You may assume the fact that due to rearrangement inequalities, it suffices to prove (3) *for f which are non-negative and radial-decreasing. For such f, justify the inequality $\int_{\mathbb{R}^d} f(x)^q \, \mathrm{d}x \geq C|y|^d f(y)^q$ for any $y \in \mathbb{R}^d$ and $q \geq 1$.*

(ii) Let $j, m \in \mathbb{N}_0$ with $j \leq m$. Let $\theta \in [j/m, 1]$ and $p, q, r \in [1, \infty)$ such that the relation

$$\frac{1}{p} - \frac{j}{d} = \theta\left(\frac{1}{q} - \frac{m}{d}\right) + (1 - \theta)\frac{1}{r} \tag{4}$$

is satisfied. On Exercise Sheet 7, Exercise 4, it was proved that

$$\|\partial_x^\alpha f\|_p \leq C \sum_{|\beta|=m} \|\partial_x^\beta f\|_q^\theta \|f\|_r^{1-\theta} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d), \tag{5}$$

provided that $m = 2k$ for some $k \in \mathbb{N}_0$. Use this result and the inequality (3) to prove (5) in case that $m = 2k + 1$ for some $k \in \mathbb{N}_0$, under the additional assumption that $q < d$.

(Compare Exercise 3.9 (iv) on p. 59 in [LP].)

Exercise 4. Let $0 < b < 1$. Recall the definition, for $f \in \mathcal{S}(\mathbb{R}^d)$, say, of

$$(D^b f)(x) := [(2\pi|\xi|)^b \widehat{f}]^\vee(x) \quad \text{and} \quad (\mathcal{D}_b f)(x) := \left(\int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2b}} \, \mathrm{d}y \right)^{1/2} .$$

(i) Prove that there exists $c_{d,b} > 0$ such that $\|D^b f\|_2 = c_{d,b} \|\mathcal{D}^b f\|_2$ for all $f \in \mathcal{S}(\mathbb{R}^d)$.

(ii) Prove that for all $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{D}^b(fg)(x) \leq \|f\|_\infty \mathcal{D}^b g(x) + |g(x)| \mathcal{D}^b f(x)$$

and

$$\|\mathcal{D}^b(fg)\|_2 \leq \|f \mathcal{D}^b g\|_2 + \|g \mathcal{D}^b f\|_2 \,.$$

(Compare Exercise 3.10 (i), (ii) on p. 59 in [LP].)

Solutions to this exercise sheet can be handed in for correction on UniWorX until 18.06.2019. Please only upload .pdf files (and not .jpg or other formats).
The sheet will be discussed in the Exercise Class 18.06.2019. No solutions will be provided online.