LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Prof. Dr. Thomas Ø. Sørensen
Lukas Emmert, Tobias König

Partial Differential Equations II
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Exercise Sheet 7

Exercise 1. For $f \in \mathcal{S}(\mathbb{R}^d)$, consider the initial value problem (IVP) associated to the *heat equation*,

$$\begin{cases} \partial_t u = \Delta u & \text{on } \mathbb{R}^d \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^d. \end{cases}$$

Its solution is given by $u(x,t) := (e^{t\Delta} f)(x) := (K_t * f)(x)$, $x \in \mathbb{R}^d$, $t > 0$, where $K_t(x) = (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}$. Moreover, recall the definition $(D^\rho f)(x) := [(2\pi|\xi|)^\rho \widehat{f}]^\vee(x)$ for every $\rho \geq 0$. Throughout the following, we consider a pair of exponents $1 \leq q < p < \infty$.

(i) Prove that for every $s \geq 0$ and $\rho \in [1,\infty)$, there is $c_{s,\rho} > 0$ such that

$$\|D^s K_t\|_\rho = c_{s,\rho} t^{-\frac{d}{2}(1-\frac{1}{\rho}) - \frac{s}{2}} \qquad \text{for all } t > 0.$$

Let $r \in (1,\infty)$ be given by $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Deduce that there is $C > 0$ such that

$$\|D_x^s u(\cdot,t)\|_p \leq C t^{-\frac{d}{2r} - \frac{s}{2}} \|f\|_q \qquad \text{for all } t > 0.$$

(ii) Consider $p \in (1,\infty)$ and $\rho \in [0,2)$ fixed. Let the number $\sigma = \sigma(q)$ be defined by

$$\frac{1}{\sigma} = \frac{d}{2}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\rho}{2}.$$

Deduce from (i) that the operator $\Omega : L^q(\mathbb{R}^d) \to L^{\sigma *}((0,\infty))$ given by $(\Omega f)(t) := \|D_x^\rho e^{t\Delta} f\|_p$ is well-defined and of weak type $(q, \sigma(q))$ for every $q \in (1,p)$ such that $\sigma(q) \geq 1$.

(iii) An off-diagonal generalization of the Marcinkiewicz interpolation theorem states that if a sublinear operator T is of weak type (q_0, σ_0) and of weak type (q_1, σ_1), then T is of weak type $(q_\theta, \sigma_\theta)$ for every $\theta \in (0,1)$. Moreover, T is of strong type $(q_\theta, \sigma_\theta)$ *provided that* $q_\theta \leq \sigma_\theta$ (where $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and likewise for σ_θ).
Use this theorem to prove that there is $C > 0$ such that

$$\left(\int_0^\infty \|D_x^\rho u(\cdot,t)\|_p^{\sigma(q)} \,\mathrm{d}t \right)^{\frac{1}{\sigma(\rho)}} \leq C \|f\|_q$$

whenever $q < p$ and $\sigma(q) \geq q$.

(Compare Exercise 2.13 on p. 41 in [LP].)

Exercise 2. For $f,g \in C_c^\infty(\mathbb{R}^d)$, let w be a solution to the IVP of the *wave equation*

$$\begin{cases} \partial_t^2 w = \Delta w & \text{on } \mathbb{R}^d \times (0,\infty), \\ w(x,0) = f(x) & \text{on } \mathbb{R}^d, \\ \partial_t w(x,0) = g(x) & \text{on } \mathbb{R}^d. \end{cases}$$

(i) Prove: If $d=1$, then *D'Alembert's formula* holds:

$$w(x,t) = \frac{f(x+t) - f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s.$$

Hint: Use the formula from Exercise 1.18 (i) in [LP] or consider the function v defined by $w(x,t) = v(x+t, x-t)$.

In the following, we assume that $d=3$, $f=0$ and g is radial, i.e. there is $G:(0,\infty)\to\mathbb{R}$ such that $g(x) = G(|x|)$ for all $x \in \mathbb{R}^3$.

(ii) Prove that w is radial and that $w(x,t) = \frac{1}{2|x|}\int_{||x|-t|}^{|x|+t} \rho G(\rho)\,\mathrm{d}\rho$.

Hint: Consider the function v defined by $w(x,t) = |x|^{-1} v(|x|,t)$.

(iii) Deduce from (ii) and the Hardy-Littlewood Theorem 2.5 that there is $C>0$ such that

$$\left(\int_{\mathbb{R}} \|w(\cdot,t)\|_\infty^2\,\mathrm{d}t\right)^{1/2} \le C\|g\|_2 \quad \text{for every } g \in C_c^\infty(\mathbb{R}^3).$$

(Compare Exercise 2.14 on p. 42 in [LP].)

Exercise 3. Prove: If $p \in (1,2]$ and $s > \frac{d}{p}$, then there is $C>0$ such that

$$\|f\|_\infty \le C\|f\|_p^{1-\frac{d}{ps}} \|D^s f\|_p^{\frac{d}{ps}} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d).$$

Hint: Argue similarly to the proof of Theorem 3.2 to obtain the additive inequality $\|f\|_\infty \le C(\|f\|_p + \|D^s f\|_p)$. Now plug in the family of functions $(f_\lambda)_{\lambda>0}$ given by $f_\lambda(x) = f(\lambda x)$ and optimize in λ.

(Compare Exercise 3.9 (i),(ii) on p. 58 in [LP].)

Exercise 4. Let $j,m \in \mathbb{N}_0$ with $j \le m$. Assume that $m = 2k$ for some $k \in \mathbb{N}_0$. Let $\theta \in [j/m, 1]$ and $p,q,r \in [1,\infty)$ such that the relation

$$\frac{1}{p} - \frac{j}{d} = \theta\left(\frac{1}{q} - \frac{m}{d}\right) + (1-\theta)\frac{1}{r}$$

is satisfied. Prove the *Gagliardo-Nirenberg inequality*: Then there is $C>0$ such that for all multiindices α with $|\alpha| = j$

$$\|\partial_x^\alpha f\|_p \le C \sum_{|\beta|=m} \|\partial_x^\beta f\|_q^\theta \|f\|_r^{1-\theta} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d).$$

Hint: Combine Exercises 2.10 and 2.11, as well as Theorem 2.6 from [LP].

(Compare Exercise 3.9 (iv) on p. 59 in [LP].)

Solutions to this exercise sheet can be handed in for correction on UniWorX until 11.06.2019. Please only upload .pdf files (and not .jpg or other formats).

Because Tuesday, June 11th, is lecture-free, **an Exercise Class discussing this sheet will be held on Wednesday, June 12th,** in place of the tutorial class. No solutions will be provided online.