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Partial Differential Equations II
SoSe 2019

Exercise Sheet 12

Exercise 1. Let $d > 2$, $\alpha > \frac{1}{2}$ and $f \in L^2(\mathbb{R}^d)$. Using the estimate (4.24) from [LP], prove that $(1+|x|)^{-\alpha} D_x^{1/2} e^{it\Delta} f \in L^2(\mathbb{R}^d)$ for a.e. $t \in \mathbb{R}$.

Hint: Argue similarly to the proof of Corollary 4.2 in [LP].

(Compare Exercise 4.20 on p. 92 in [LP].)

Exercise 2. Let $\alpha \in (1, 1+\frac{4}{d})$ and consider the solution $u \in C((-T,T) : L^2(\mathbb{R}^d))$, given by Theorem 5.2, to the IVP

$$u(t) = e^{it\Delta} u_0 + i\lambda \int_0^t e^{i(t-s)\Delta} (|u|^{\alpha-1} u)(s) \, \mathrm{d}s, \tag{1}$$

with $\lambda \in \mathbb{R}$ and initial datum $u_0 \in L^2(\mathbb{R})$. In this exercise, we shall prove that there is $c > 0$ only depending on α, λ and d such that the lifespan $T = T(u_0)$ of u can be taken to satisfy

$$T \geq c \|u_0\|_2^{-\beta}, \qquad \text{with } \beta = \frac{4(\alpha-1)}{4 - n(\alpha-1)}. \tag{2}$$

(i) Prove that if u is a solution to (1) with initial datum u_0, then for any $\mu > 0$, $u_\mu(x,t) = \mu^{\frac{2}{\alpha-1}} u(\mu x, \mu^2 t)$ is a solution to (1) with initial datum $u_{\mu,0}(x) = \mu^{\frac{2}{\alpha-1}} u(\mu x, 0)$. Use this to heuristically deduce the correct value of the power β appearing in (2).

(ii) Rigorously prove (2) by reviewing the proof of Theorem 5.2.

(Compare Exercise 5.4 on p. 122 in [LP].)

Exercise 3. For $f, g \in C_c^\infty(\mathbb{R}^d)$, let w be a solution to the IVP of the wave equation

$$\begin{cases} \partial_t^2 w = \Delta w & \text{on } \mathbb{R}^d \times (0,\infty), \\ w(x,0) = f(x) & \text{on } \mathbb{R}^d, \\ \partial_t w(x,0) = g(x) & \text{on } \mathbb{R}^d. \end{cases} \tag{3}$$

Recall from Exercise Sheet 6, Exercise 3, that the solution to the IVP (3) is given by $w(\cdot,t) = \cos(Dt)f + \frac{\sin(Dt)}{D}g$, where D is the Fourier multiplier given by $\widehat{Dh}(\xi) = 2\pi|\xi|\widehat{h}(\xi)$.

(i) Let $d = 3$ and $f = 0$. Prove that

$$w(x,t) = \frac{1}{4\pi t}\int_{\{|y|=t\}} g(x+y)\,\mathrm{d}S(y), \tag{4}$$

where $\mathrm{d}S(y)$ denote the surface measure on the sphere $\{|y| = t\} \subset \mathbb{R}^3$ induced by the standard scalar product on $\mathbb{R}^3$.

Hint: Derive and apply the identity

$$\int_{\{|y|=t\}} e^{2\pi i\xi\cdot y}\,\mathrm{d}S(y) = 4\pi t\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}, \qquad \textit{for all } \xi \in \mathbb{R}^3, \quad t > 0. \tag{5}$$

(ii) Let $d = 3$ and $g = 0$. Prove that

$$w(x,t) = \frac{1}{4\pi t^2}\int_{\{|y|=t\}} (f(x+y) + \nabla f(x+y)\cdot y)\,\mathrm{d}S(y). \tag{6}$$

Hint: Try to express w in terms of the solution from (i).

(Compare Exercise 1.18 (iii), (iv) on p. 23 in [LP].)

Exercise 4. (i) Let $d = 3$. Prove that for every $t \in \mathbb{R}\setminus\{0\}$ and every $p \in [1,\infty]\setminus\{2\}$, the function $\cos(2\pi|\xi|t)$ is not an $L^p(\mathbb{R}^3)$-multiplier, i.e. there is no $C > 0$ such that

$$\|(\cos(2\pi|\cdot|t)\widehat{f})^\vee\|_p \le C\|f\|_p \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^3).$$

Hint: Let $f = f(|x|) = h(|x|)/|x|$ be radial and supported in a suitable annulus $\{0 < a < |x| < b\}$. Using Sheet 7, Exercise 2, prove the identity $(\cos(2\pi|\cdot|t)\widehat{f})^\vee = \frac{h(r+t)-h(r-t)}{2r}$.

(ii) Let $d = 3$. Prove that there is $C > 0$ such that for every $t \neq 0$,

$$\|(\cos(2\pi|\cdot|t)\widehat{f})^\vee\|_\infty \le Ct^{-1}\left(\sum_{i,j}\|\partial_{x_i}\partial_{x_j}f\|_1\right) \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^3).$$

Hint: Use formula (6).

(iii) Let $d = 1$. Prove that for every $t \in \mathbb{R}$ and every $p \in [1,\infty]$, the function $\cos(2\pi|\xi|t)$ is an $L^p(\mathbb{R})$-multiplier.

(Compare Exercise 2.16 on p. 43 in [LP].)

Solutions to this exercise sheet can be handed in for correction on UniWorX until 16.07.2019. Please only upload .pdf files (and not .jpg or other formats).
The sheet will be discussed in the Exercise Class 16.07.2019. No solutions will be provided online.