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Partial Differential Equations II
SoSe 2019

Exercise Sheet 11

Exercise 1. In this exercise we expand on the 'regularizing effects' of the group $(e^{it\Delta})_{t\in\mathbb{R}}$. To this end, for any $t \in \mathbb{R}$, we define the auxiliary operators

$$\Gamma_j := x_j + 2it\partial_{x_j}, \qquad j = 1, ..., d,$$

acting on functions $f = f(x,t)$ with $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. Moreover, for any multiindex $\alpha \in \mathbb{N}_0^d$ we define (notice that $\Gamma_j\Gamma_k = \Gamma_k\Gamma_j$ for all j,k)

$$\Gamma^\alpha := \prod_{j=1}^d \Gamma_j^{\alpha_j}.$$

(i) Prove: For all $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R})$, all $\alpha \in \mathbb{N}_0^d$ and all $t \neq 0$,

$$\Gamma^\alpha f = e^{i\frac{|x|^2}{4t}}(2it)^{|\alpha|}\partial_x^\alpha e^{-i\frac{|x|^2}{4t}} f = e^{it\Delta} x^\alpha e^{-it\Delta} f.$$

(ii) Prove that Γ_j commutes with $\partial_t - i\Delta$.

(iii) Assume $u_0 \in L^2(\mathbb{R}^d)$ and $x^\alpha u_0 \in L^2(\mathbb{R}^d)$. Prove that $t \mapsto (\Gamma^\alpha(e^{it\Delta}u_0))(t,\cdot)$ is in $C(\mathbb{R} : L^2(\mathbb{R}^d)$ and therefore

$$\left(t \mapsto \partial_x^\alpha(e^{-i\frac{|x|^2}{4t}} e^{it\Delta}u_0)\right) \in C(\mathbb{R}\setminus\{0\} \,;\, L^2(\mathbb{R}^d)).$$

Deduce that, in particular, $\partial_x^\alpha e^{it\Delta}u_0 \in L^2_{\text{loc}}(\mathbb{R}^d)$ for all $t \neq 0$.

(iv) Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Prove that $e^{it\Delta}u_0 \in \mathcal{S}(\mathbb{R}^d)$ for all $t \in \mathbb{R}$.

(Compare Exercise 4.4 on p. 89 in [LP].)

Exercise 2. Let $r \in [2,\infty)$ and let r' be such that $\frac{1}{r} + \frac{1}{r'} = 1$. Prove that there is $C > 0$ such that

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |e^{it\Delta}u_0(x)|^{3r} \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{3r}} \leq C\|\widehat{u_0}\|_{r'} \qquad \text{for all } u_0 \in \mathcal{S}(\mathbb{R}).$$

(Compare Exercise 4.11 on p. 91 in [LP].)

Exercise 3. (i) Prove that if $f \in L^2(\mathbb{R})$, then $e^{it\Delta} \in C(\mathbb{R})$ for almost every $t \in \mathbb{R}$.

Hint: Use the Strichartz estimate (4.14) in [LP] with $(p,q) = (\infty,\ 4)$ *together with a density argument.*

(ii) Prove that inequality (4.14) in [LP] cannot hold with a constant $c > 0$ independent of f if the exponents p, q do not satisfy the condition $\frac{2}{q} = \frac{d}{2} - \frac{d}{p}$.

Hint: For $u(x,t) = (e^{it\Delta} f)(x)$ *and any* $\lambda > 0$*, consider* $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$.

(Compare Exercise 4.9 on p. 90 in [LP].)

Exercise 4. Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and let $t \mapsto u(\cdot,t) \in C(\mathbb{R}\ ;\ \mathcal{S}(\mathbb{R}^d))$ be a solution to the *inhomogeneous* IVP

$$\begin{cases} \partial_t u(x,t) = i\Delta u(x,t) + F(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

wth the inhomogeneity $(t \mapsto F(\cdot,t)) \in C(\mathbb{R}\ ;\ \mathcal{S}(\mathbb{R}^d))$. Prove that u is given by *Duhamel's formula*:

$$u(x,t) = e^{it\Delta} u_0(x) + \int_0^t e^{i(t-s)\Delta} F(x,s) \mathrm{d}s\,, \qquad \text{for all } (x,t) \in \mathbb{R}^d \times \mathbb{R}.$$

Hint: Consider $v = v(x,t)$ *defined by* $u(x,t) = (e^{it\Delta} v)(x,t)$.

(Compare Exercise 4.15 on p. 91 in [LP].)

Solutions to this exercise sheet can be handed in for correction on UniWorX until 09.07.2019. Please only upload .pdf files (and not .jpg or other formats).
The sheet will be discussed in the Exercise Class 09.07.2019. No solutions will be provided online.