LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Partial Differential Equations II
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Exercise Sheet 10

For the exercises below, recall the notation $e^{it\Delta}u_0 := \frac{e^{-|x|^2/(4it)}}{(4\pi it)^{d/2}} * u_0 = \left(e^{-4\pi^2 it|\xi|^2}\widehat{u_0}\right)^{\vee}$, see equation (4.2) in [LP].

Exercise 1. The goal of this exercise is to prove that there cannot be p, q, t with $1 \leq q < p < \infty$ and $t \in \mathbb{R} \setminus \{0\}$ such that the map $e^{it\Delta} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is continuous.

(i) Prove that $e^{it\Delta}$ commutes with translations. That is, if $\tau_h f$ is defined by $(\tau_h f)(x) := f(x-h)$ for some $h \in \mathbb{R}^d$, then $\tau_h e^{it\Delta} f = e^{it\Delta}\tau_h f$.

(ii) Prove that if $f \in L^r(\mathbb{R}^d)$ with $1 \leq r < \infty$, then

$$\lim_{|h|\to\infty} \|f + \tau_h f\|_r = 2^{1/r}\|f\|_r.$$

(iii) More generally, let $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ be any bounded linear operator that commutes with translations. Deduce from (ii) that

$$\|Tf\|_q \leq \|T\|_{p\to q} 2^{\frac{1}{p}-\frac{1}{q}}\|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^d),$$

where $\|T\|_{p\to q}$ denotes the operator norm of T. Conclude that $p \leq q$, unless T is the zero operator.

(Compare Exercise 4.8 on p. 90 in [LP].)

Exercise 2. The goal of this exercise is to prove that if $f \in L^2(\mathbb{R}^d)$, then

$$\lim_{|t|\to\infty} \|e^{it\Delta} f - \frac{e^{i|\cdot|^2/(4t)}}{(4\pi it)^{d/2}} \widehat{f}(\frac{\cdot}{4\pi t})\|_2 = 0. \tag{1}$$

(i) Prove that for all $t \neq 0$, the operator $U(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by

$$U(t)f(x) := (4\pi it)^{-d/2} e^{i|x|^2/(4t)} \widehat{f}(\frac{x}{4\pi t}),$$

is an isometry, i.e. $\|U(t)f\|_2 = \|f\|_2$ for all $f \in L^2(\mathbb{R}^d)$. Conclude that it suffices to prove (1) for $f \in \mathcal{S}(\mathbb{R}^d)$.

(ii) Prove that

$$e^{it\Delta} f(x) - U(t)f(x) = \frac{e^{i|x|^2/(4t)}}{(4\pi it)^{d/2}} \widehat{F_t}(\frac{x}{4\pi t}),$$

where $F_t(y) = (e^{i|y|^2/(4t)} - 1)f(y)$.

(iii) Justify the estimate $|e^{i|x|^2/(4t)} - 1| \leq c\frac{|x|^2}{4t}$ and use it to complete the proof of (1).

(Compare Exercise 4.12 on p. 91 in [LP].)

For the following Exercises (3) and (4) you may use without proof the following fact, known as *Osgood's lemma*: Let $U \subset \mathbb{C}^d$ open. A continuous function $f : U \to \mathbb{C}$ is holomorphic if and only if for every $a \in U$ and $k = 1, ..., d$, the map $z \mapsto f(a_1, ..., a_{k-1}, z, a_{k+1}, ..., a_d)$ is holomorphic in $z = a_k$ as a function of one complex variable.

Exercise 3. (i) Prove: If $u_0 \in C_c(\mathbb{R}^d)$ (continuous functions with compact support), then $e^{it\Delta}u_0$ has an analytic extension to $\mathbb{C}^d$ for all $t \neq 0$.

Hint: Use formula (4.7) in [LP], which states that

$$(4\pi it)^{d/2} e^{-i|x|^2/(4t)} (e^{it\Delta}u_0)(x) = \left(\widehat{e^{i|\cdot|^2/(4t)}u_0}\right)\left(\frac{x}{4\pi t}\right), \tag{2}$$

and prove more generally that if $\varphi \in C_c^\infty(\mathbb{R}^d)$, then $\widehat{\varphi}$ has a holomorphic extension to $\mathbb{C}^d$.

(ii) Prove: If $u_0 \in C_c(\mathbb{R}^d)$, then $e^{it\Delta}u_0 \notin L^1(e^{\varepsilon|x|}\,\mathrm{d}x)$ for all $t \neq 0$, $\varepsilon > 0$, unless $u_0 \equiv 0$.

Hint: Use (2) *again and prove more generally that if $\varphi \in C_c(\mathbb{R}^d)$, then $\widehat{\varphi} \notin L^1(e^{\varepsilon|x|}\,\mathrm{d}x)$, unless $\varphi \equiv 0$. To achieve this, assume by contradiction that $\widehat{\varphi} \in L^1(e^{\varepsilon|x|}\,\mathrm{d}x)$. Under this assumption, prove that φ has a holomorphic extension to a neighborhood of $\mathbb{R}^d$ by expanding the Fourier inversion formula $\varphi(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\widehat{\varphi}(\xi)\,\mathrm{d}\xi$ into a power series.*

(Compare Exercise 4.5 on p. 89 in [LP].)

Exercise 4. Prove the *Paley-Wiener theorem*, which states the following:

(i) If $f \in C_c^\infty(\mathbb{R}^d)$ has support in $\{|x| \leq M\}$, then $\widehat{f}$ has a holomorphic extension to all of $\mathbb{C}^d$. Moreover,

$$\forall k \in \mathbb{N}_0 \ \exists c_k > 0 \ \text{such that} \ |\widehat{f}(\xi + i\eta)| \leq c_k \frac{e^{2\pi M|\eta|}}{(1 + |(\xi + i\eta)|)^k} \ \text{ for any } \xi, \eta \in \mathbb{R}^d, \tag{3}$$

where c_k only depends on k and f.

(ii) Conversely, if $F = F(\xi + i\eta)$ is an analytic function in $\mathbb{C}^d$ satisfying (3) for some $M > 0$, then F is (the holomorphic extension of) the Fourier transform of some function $f \in C_c^\infty(\mathbb{R}^d)$ with support in $\{|x| \leq M\}$.

Hint: Justify that $f(x) := \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\xi+i\eta)} F(\xi+i\eta)\,\mathrm{d}\xi$ is in fact independent of $\eta \in \mathbb{R}^d$, by using (3) *and the residue theorem.*

(Compare Exercise 1.10 on p. 21 in [LP].)

Solutions to this exercise sheet can be handed in for correction on UniWorX until 02.07.2019. Please only upload .pdf files (and not .jpg or other formats).
The sheet will be discussed in the Exercise Class 02.07.2019. No solutions will be provided online.