THE HITCHIN FIBRATION UNDER DEGENERATIONS TO NODED RIEMANN SURFACES

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Abstract. In this note we study some analytic properties of the linearized self-duality equations on a family of smooth Riemann surfaces Σ_R converging for $R \searrow 0$ to a surface Σ_0 with a finite number of nodes. It is shown that the linearization along the fibres of the Hitchin fibration $\mathcal{M}_d \to \Sigma_R$ gives rise to a graph-continuous Fredholm family, the index of it being stable when passing to the limit. We also report on similarities and differences between properties of the Hitchin fibration in this degeneration and in the limit of large Higgs fields as studied in [12]. Hitchin fibration and self-duality equations and noded Riemann surface

1. Introduction

Let Σ be a closed Riemann surface with complex structure J. We fix a hermitian vector bundle $(E, h) \to \Sigma$ of rank 2 and degree $d(E) \in \mathbb{Z}$. We furthermore fix a Kähler metric on Σ such that the associated Kähler form ω satisfies $\int_\Sigma \omega = 2\pi$. The main object of this article is the moduli space of solutions (A, Φ) of Hitchin's self-duality equations

$$\begin{aligned} F_A + [\Phi \wedge \Phi^*] &= -i\mu(E)\,\mathrm{id}_E\,\omega, \\ \bar{\partial}_A \Phi &= 0 \end{aligned} \tag{1}$$

for a unitary connection $A \in \mathcal{A}(E, h)$ and a **Higgs field** $\Phi \in \Omega^{1,0}(\Sigma, \mathrm{End}(E))$. We here denote by $\mu(E) = d(E)/2$ the slope of the complex rank-2 vector bundle E.

The set of smooth solutions to Eq. (1) is invariant under the action $g^*(A, \Phi) = (g^*A, g^{-1}\Phi g)$, where $g \in \mathcal{G}(E, h)$ is a unitary gauge transformation. The corresponding moduli space of solutions

$$\mathcal{M} := \frac{\{(A, \Phi) \in \mathcal{A}(E, h) \times \Omega^{1,0}(\Sigma, \mathrm{End}(E)) \mid (1)\}}{\mathcal{G}(E, h)}$$

has been widely studied in the literature, beginning with Hitchin's seminal work [8]. We note that for the definition of $\mathcal{M}$ the precise choice of the background hermitian metric h is irrelevant, since any two hermitian metrics are complex gauge equivalent. The dependence of (1) and hence $\mathcal{M}$ on the degree d of the complex vector bundle E is supressed from our notation.

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One interesting feature of $\mathcal{M}$ is the existence of a singular torus fibration over the complex vector space $\mathrm{QD}(\Sigma, J)$ of holomorphic quadratic differentials, called **Hitchin fibration**,

$$\det\colon \mathcal{M} \to \mathrm{QD}(\Sigma, J), \quad [(A, \Phi)] \mapsto \det \Phi.$$

The map det is well-defined since (A, Φ) is supposed to satisfy $\bar{\partial}_A \Phi = 0$. It is holomorphic with respect to the natural complex structure on $\mathrm{QD}(\Sigma, J)$ and the one on $\mathcal{M}$ given by

$$I(\alpha, \varphi) = (i\alpha, i\varphi)$$

for a pair $(\alpha, \varphi) \in \Omega^{0,1}(\Sigma, \mathfrak{sl}(E)) \oplus \Omega^{1,0}(\Sigma, \mathfrak{sl}(E))$ representing a tangent vector at some $[(A, \Phi)] \in \mathcal{M}$. It is furthermore shown in [8, §8] that the map det is surjective and proper with preimage $\det^{-1}(q)$ (for q with simple zeroes) being biholomorphically equivalent to the Prym variety of the double covering of Σ determined by $\sqrt{q}$.

The recent works [12, 13] and [18] have been concerned with two types of degenerations of $\mathcal{M}$ which are of quite different nature. In the first two mentioned articles the degeneration profile of solutions (A, Φ) of Eq. (1) has been studied in the limit of large Higgs fields, i.e. in the case where the L^2-norm of Φ tends to infinity. Since the map $[(A, \Phi)] \to \|\Phi\|^2_{L^2(\Sigma)}$ on $\mathcal{M}$ is well-known to be proper, this analysis was used in [12] to obtain a geometric compactification of $\mathcal{M}$ together with a parametrization of charts covering a large portion of the boundary of the compactified moduli space (corresponding to Higgs fields Φ satisfying the generic property that $\det \Phi$ has only simple zeroes). While so-far the Riemann surface (Σ, J) has been fixed, we consider in [18] $\mathcal{M}$ as being parametrized by the complex structure J and in particular focus on families of Riemann surfaces converging to a noded limit (this limit representing a boundary point of the Deligne-Mumford compactification of Teichmüller moduli space). Our main result, a gluing theorem, is described in further detail below.

The purpose of this note is to describe some geometric aspects of the Hitchin fibration under each of the above two degenerations. Concerning the first one, the results discussed here have been obtained in collaboration with Mazzeo, Weiß and Witt and will in full detail be presented in the forthcoming article [14], which is concerned with the large scale geometry of the natural complete L^2 hyperkähler metric on $\mathcal{M}$. Concerning the second degeneration, we here study the linearization of the self-duality equations (1), which gives rise to a family of (nonuniformly) elliptic operators degenerating to a so-called b-operator in the limit of a noded surface. This situation is not unlike to the one studied by Mazzeo and the author in [11], where the existence of a complete polyhomogeneous expansion of the Weil-Petersson metric on the Riemann moduli space of conformal structures was shown.

Further results concerning similar geometric aspects of this latter degeneration will be presented elsewhere.

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2. Preliminaries on the moduli space of solutions to Hitchin's self-duality equations

2.1. **Hitchin's self-duality equations.** We recall some basic facts concerning Higgs bundles and the self-duality equations (1), referring to [12] for further details. First we note that the second equation in (1) implies that any solution (A, Φ) determines a **Higgs bundle** $(\bar\partial, \Phi)$, i.e. a holomorphic structure $\bar\partial = \bar\partial_A$ on E for which Φ is holomorphic: $\Phi \in H^0(\Sigma, \mathrm{End}(E) \otimes K_\Sigma)$, $K_\Sigma \cong T^*_{\mathbb{C}}\Sigma$ denoting the canonical line bundle of Σ. Conversely, given a Higgs bundle $(\bar\partial, \Phi)$, the operator $\bar\partial$ can be augmented to a unitary connection A such that the first Hitchin equation holds provided $(\bar\partial, \Phi)$ is **stable**. Stability here means that $\mu(F) < \mu(E)$ for any Φ-invariant holomorphic line-bundle F, that is, $\Phi(F) \subset F \otimes K_\Sigma$.

In the following notation such as $\mathfrak{u}(E)$ refers to the bundle of endomorphisms of E which are hermitian with respect to the fixed hermitian metric h on E. This bundle splits as $\mathfrak{su}(E) \oplus i\underline{\mathbb{R}}$, the splitting being induced by the Lie algebra splitting $\mathfrak{u}(2) \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ into trace-free and pure trace summands. Consequently, the curvature F_A of a unitary connection A decomposes as

$$F_A = F_A^\perp + \frac{1}{2}\,\mathrm{Tr}(F_A) \otimes \mathrm{id}_E,$$

where $F_A^\perp \in \Omega^2(\Sigma, \mathfrak{su}(E))$ is its trace-free part and $\frac{1}{2}\,\mathrm{Tr}(F_A) \otimes \mathrm{id}_E$ is the pure trace or central part, see e.g. [9]. Note that $\mathrm{Tr}(F_A) \in \Omega^2(i\underline{\mathbb{R}})$ equals the curvature of the induced connection on the determinant line bundle $\det E$. From now on, we fix a background connection $A_0 \in \mathcal{A}(E, h)$ and consider only those connections A which induce the same connection on $\det E$ as A_0 does. Equivalently, such a connection A is of the form $A = A_0 + \alpha$ where $\alpha \in \Omega^1(\Omega, \mathfrak{su}(E))$, i.e. A is trace-free "relative" to A_0. We choose the unitary background connection A_0 on E such that $\mathrm{Tr}\, F_{A_0} = -i\deg(E)\omega$. This decomposition allows us to replace Eq. (1) with the slightly easier system of equations

$$\begin{aligned} F_A^\perp + [\Phi \wedge \Phi^*] &= 0, \\ \bar\partial_A \Phi &= 0 \end{aligned} \tag{2}$$

for A trace-free relative to A_0 and a trace-free Higgs field $\Phi \in \Omega^{1,0}(\Sigma, \mathrm{End}_0(E))$. The relevant groups of gauge transformations in this fixed determinant case are $\mathcal{G}^c = \Gamma(\Sigma, \mathrm{SL}(E))$ and $\mathcal{G} = \mathcal{G}(E, h) = \Gamma(\Sigma, \mathrm{SU}(E, h))$, the former being the complexification of the latter.

2.2. **The limit of large Higgs fields.** We discuss the results obtained in [12] concerning the degeneration of solutions (A,Φ) of Eq. (2) as $\|\Phi\|_{L^2(\Sigma)}\to\infty$. For this purpose we introduce the **rescaled self-duality equations**

$$\begin{aligned} F_A^{\perp}+t^2[\Phi\wedge\Phi^*] &= 0,\\ \bar\partial_A\Phi &= 0 \end{aligned} \tag{3}$$

for some parameter $t>0$.

Existence of limiting configurations. We describe the following local model for degenerations of solutions to Eq. (3) as $t\to\infty$. On $\mathbb{C}$ we consider the pair $(A_t^{\mathrm{fid}},\Phi_t^{\mathrm{fid}})$ given in coordinates $z=re^{i\theta}$ by

$$A_t^{\mathrm{fid}}=\left(\frac{1}{8}+\frac{1}{2}\frac{\partial h_t}{\partial r}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\left(\frac{dz}{z}-\frac{d\bar z}{\bar z}\right),\quad \Phi_t^{\mathrm{fid}}=\begin{pmatrix}0&r^{\frac12}e^{h_t}\\ r^{\frac12}e^{-i\theta}e^{-h_t}&0\end{pmatrix}dz.$$

It automatically satisfies the second equation in (3). The function $h_t\colon(0,\infty)\to\mathbb{R}$ is determined in such a way that also the first equation holds. The existence and precise properties of the 1-parameter family of functions h_t have been discussed in [12, 13]. We only mention here that it is smooth on $(0,\infty)$, has a logarithmic pole at $r=0$ and satisfies $\lim_{t\to\infty}h_t=0$ uniformly on compact subsets of $(0,\infty)$. Note that in this ansatz the determinant $\det\Phi_t^{\mathrm{fid}}=-z\,dz^2$ is independent of t and has a simple zero at $z=0$. We also point out that for $t\to\infty$ the family of smooth solutions $(A_t^{\mathrm{fid}},\Phi_t^{\mathrm{fid}})$ has a limit which is singular in $z=0$ and which satisfies the decoupled self-duality equations (4) below. Here and in the following we denote

$$\Sigma^\times:=\Sigma\setminus q^{-1}(0)$$

for some given Higgs field Φ where $q=\det\Phi$. We call the Higgs field **simple** if q has only simple zeroes, the number of which then equals $4(\gamma-1)$.

Definition 2.1. A **limiting configuration** is a pair (A_∞,Φ_∞) such that Φ_∞ is simple and which satisfies the decoupled self-duality equations

$$F_{A_\infty}^{\perp}=0,\quad [\Phi_\infty\wedge\Phi_\infty^*]=0,\quad \bar\partial_{A_\infty}\Phi_\infty=0 \tag{4}$$

on $\Sigma^\times$, and which furthermore agrees with $(A_\infty^{\mathrm{fid}},\Phi_\infty^{\mathrm{fid}})$ near each point of $(\det\Phi)^{-1}(0)$ with respect to some unitary frame for E and local holomorphic coordinate system such that $\det\Phi=-z\,dz^2$.

The main result for limiting configurations is the following.

Theorem 2.1. *Let (E,h,Φ) be a hermitian Higgs bundle with simple Higgs field. Let A_h denote the Chern connection associated with (E,h). Then in the complex gauge orbit of (A_h,Φ) over $\Sigma\setminus(\det\Phi)^{-1}(0)$ there exists a limiting configuration $(A_\infty,\Phi_\infty)=g_\infty^*(A_h,\Phi)$. It is unique up to a unitary gauge transformation. The limiting complex gauge transformation g_∞ is* *singular in the points of* $(\det\Phi)^{-1}(0)$*, near which it takes the form*

$$g_\infty = \begin{pmatrix} |z|^{-\frac{1}{4}} & 0 \\ 0 & |z|^{\frac{1}{4}} \end{pmatrix}$$

up to multiplication with a smooth unitary gauge transformation on Σ*.*

This result was proved in [12]. It relies on the Fredholm theory of conic elliptic operators, in this case for the twisted Laplacian Δ_{A_∞}.

Desingularization by gluing. We finally describe a partial converse, also shown in [12], to Theorem 2.1. It entails a global version of the observation that the family of smooth fiducial solutions $(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}})$ desingularizes the limiting fiducial solution $(A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})$. These solutions are obtained by gluing (A_∞, Φ_∞), which on the complement of some neighbourhood of $(\det\Phi)^{-1}(0)$ is a bounded solution to Eq. (3) for any t, to the model solution $(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}})$ for some large but finite t.

Theorem 2.2. *Suppose that* (A_∞, Φ_∞) *is a limiting configuration where* Φ_∞ *is a simple Higgs field. Then there exists a family of smooth solutions* (A_t, Φ_t) *to the rescaled self-duality equations* (3) *with* $A_t \to A_\infty$ *and* $\Phi_t \to \Phi_\infty$ *in* C^∞_{loc} *at exponential rate in* t *on the complement of* $(\det\Phi_\infty)^{-1}(0)$*.*

2.3. **The limit under degeneration to a noded surface.** Following [18] we introduce the setup for our study of the Hitchin fibration on a family of Riemann surfaces degenerating to a surface with one or more nodes.

Plumbing construction. We briefly recall the conformal plumbing construction for Riemann surfaces. A **Riemann surface with nodes** is a closed one-dimensional complex manifold with singularities Σ_0 where each point has a neighbourhood complex isomorphic to either a disk $\{|z| < \epsilon\}$ or to $U = \{zw = 0 \mid |z|, |w| < \epsilon\}$, in which case the point is called a node. A Riemann surface with nodes arises from an unnoded surface by pinching of one or more simply closed curves. Conversely, the effect of the so-called conformal plumbing construction is that it opens up a node by replacing the neighbourhood U by $\{zw = t \mid t \in \mathbb{C}, |z|, |w| < \epsilon\}$. To describe this construction in more detail, let (Σ_0, z, p) be a Riemann surface of genus $\gamma \geq 2$ with conformal coordinate z and a single node at p. Let $t \in \mathbb{C}\setminus\{0\}$ be fixed with $|t|$ sufficiently small. We then define a smooth Riemann surface Σ_t by removing the disjoint disks $D_t = \{|z| < |t|, |w| < |t|\} \subseteq U$ from Σ_0 and passing to the quotient space $\Sigma_t = (\Sigma_0 \setminus D_t)/_{zw=t}$, which is a Riemann surface of the same genus as Σ_0. In the following we allow for Riemann surfaces with a finite number of nodes, the set of which we denote by $\mathfrak{p} = \{p_1, \ldots, p_k\} \subset \Sigma$. The value of t may be different at different nodes. We let $R := \max_{p\in\mathfrak{p}} |t(p)|^2$ be the maximum of the squares of these absolute values. To deal with the case of multiple nodes in an efficient way we make the **convention** that in the notation Σ_R the dependence of the parameter $t \in \mathbb{C}$ on the point $p \in \mathfrak{p}$ is suppressed.

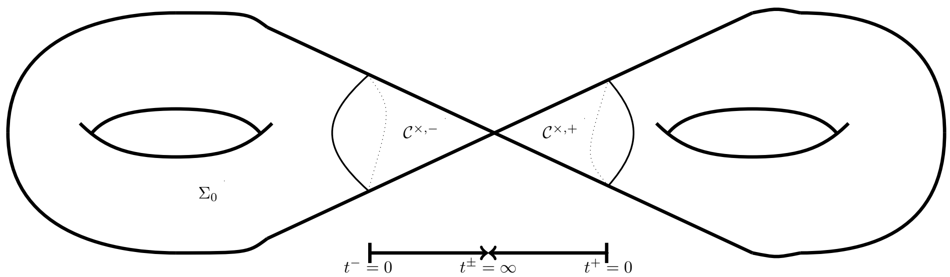

FIGURE 1. Degenerate Riemann surface Σ_0 with one node $p \in \mathfrak{p}$, which separates the two half-infinite cylinders $\mathcal{C}_0^\pm$.

Let $\rho = |t| < 1$ and consider the annuli

$$R_\rho^+ = \{z \in \mathbb{C} \mid \rho \le |z| \le 1\} \qquad \text{and} \qquad R_\rho^- = \{w \in \mathbb{C} \mid \rho \le |w| \le 1\}. \tag{5}$$

The above identification of R_ρ^+ and R_ρ^- along their inner boundary circles $\{|z| = \rho\}$ and $\{|w| = \rho\}$ yields a smooth cylinder C_t.

As before, we let $\mathrm{QD}(\Sigma) = H^0(\Sigma, K_\Sigma^2)$ denote the $\mathbb{C}$-vector space of holomorphic quadratic differentials on Σ. On a noded Riemann surface we will allow for quadratic differentials meromorphic with poles of order at most 2 at points in the subset $\mathfrak{p} \subset \Sigma$ of nodes. In this case, the corresponding $\mathbb{C}$-vector space of meromorphic quadratic differentials is denoted by $\mathrm{QD}_{-2}(\Sigma)$.

The local model. We briefly describe how to extend the setup of the self-duality equations to the case where the underlying manifold is a noded Riemann surface Σ_0. The complex rank-2 vector bundle in this situation is supposed to be a **cylindrical vector bundle** as discussed e.g. in [3, 16]. By this we mean that a pair of cylindrical coordinates $(\tau^\pm, \theta^\pm)$ with

$$\tau^\pm = \left|\log r^\pm\right|$$

is chosen, one for each of the two connected components $\mathcal{C}_0^\pm$ of the punctured neighbourhood $\mathcal{C}_0$ of $p \in \mathfrak{p}$. We then fix a smooth hermitian metric h on E in such a way that its restriction to $E|_{\mathcal{C}_0^\pm}$ is invariant under pullback by translations in the $\tau^\pm$-direction. We furthermore require that h is invariant under pullback via the isometric involution $(\tau^\pm, \theta^\pm) \mapsto (\tau^\mp, \arg t - \theta^\mp)$ interchanging the two half-infinite cylinders $\mathcal{C}^+$ and $\mathcal{C}^-$. The pair (E, h) induces a hermitian vector bundle on each surface Σ_R by restriction, which by the assumptions on h extends smoothly over the cut-locus $|z| = |w| = \rho$, cf. Figure 2.

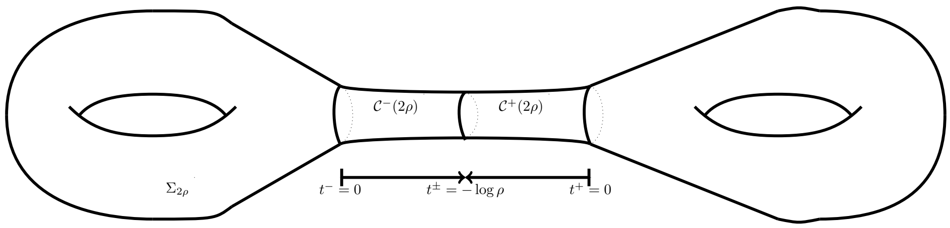

FIGURE 2. Setup for the gluing theorem (Thm. 2.3).

We next fix constants $\alpha \in \mathbb{R}$ and $C \in \mathbb{C}$. Then the pair

$$A^{\mathrm{mod}} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right), \quad \Phi^{\mathrm{mod}} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \frac{dz}{z} \tag{6}$$

provides a solution on $\mathbb{C}$ to the self-duality equations (1), which we call **model solution to parameters** (α, C). It is smooth outside the origin and has a logarithmic (first-order) singularity in $z = 0$, provided that α and C do not both vanish. It furthermore restricts to a smooth solution on each of the annuli $R_\rho^\pm$ defined in (5). For constants $t \in \mathbb{C}$ and $\rho = |t|$ such that $0 < \rho < 1$ let $\mathcal{C}_t$ denote the complex cylinder obtained from gluing the two annuli R_ρ^- and R_ρ^+. Since

$$\frac{dz}{z} = -\frac{dw}{w}$$

the two model solutions $(A_+^{\mathrm{mod}}, \Phi_+^{\mathrm{mod}})$ to parameters (α, C) over R_ρ^+ and $(A_-^{\mathrm{mod}}, \Phi_-^{\mathrm{mod}})$ to parameters $(-\alpha, -C)$ over R_ρ^- glue to a smooth solution $(A^{\mathrm{mod}}, \Phi^{\mathrm{mod}})$ on $\mathcal{C}_t$, again called model solution to parameters (α, C). In the following it is always assumed that $\alpha > 0$.

We now impose the following **assumptions**.

(A1) For each $p \in \mathfrak{p}$, the constant $C_p = C_{p,+}$ is nonzero.
(A2) For each $p \in \mathfrak{p}$, the constants $C_{p,+}$ and $C_{p,-}$ satisfy $C_{p,+} = -C_{p,-} = C_p$.
(A3) The meromorphic quadratic differential $q = \det \Phi$ has only simple zeroes.

The relevance of the model solutions is that any solution of the self-duality equations with logarithmic singularities in $\mathfrak{p}$ is exponentially close (with respect to the above cylindrical coordinates) to some model solution, a fact which is due to Biquard-Boalch [2, Lemma 5.3].

The main result shown in [18] is the following gluing theorem.

Theorem 2.3. *Let (Σ, J_0) be a Riemann surface with nodes in a finite set of points $\mathfrak{p} \subset \Sigma$. Let (A_0, Φ_0) be a solution of the self-duality equations with logarithmic singularities in $\mathfrak{p}$, thus representing a point in $\mathcal{M}(\Sigma, J_0)$. Suppose that (A_0, Φ_0) satisfies the assumptions (A1–A3) stated above. Let*

(Σ, J_i) be a sequence of smooth Riemann surfaces converging uniformly to (Σ, J_0). Then, for every sufficiently large $i \in \mathbb{N}$, there exists a smooth solution (A_i, Φ_i) of Eq. (2) on (Σ, J_i) such that $(A_i, \Phi_i) \to (A_0, \Phi_0)$ as $i \to \infty$ uniformly on compact subsets of $\Sigma \setminus \mathfrak{p}$.

3. The Hitchin fibration in the limit of large Higgs fields

Throughout this section it is supposed that the above assumption (A3) holds. Recall the Definition 2.1 of limiting configurations. Our first observation is that the second component Φ_∞ of a limiting configuration is completely determined up to a unitary gauge transformation by the holomorphic quadratic differential q. This is a consequence of the standard fact that any normal endomorphism is diagonalizable by some $g \in \mathrm{SU}(n)$. Now consider the space of unitary connections solving

$$\bar{\partial}_A \Phi_\infty = 0, \qquad F_A^\perp = 0; \tag{7}$$

the gauge freedom is the stabilizer of Φ_∞ in $\Gamma(\Sigma^\times, \mathrm{SU}(E))$, i.e. the group of unitary gauge transformations of the complex line bundle

$$L_{\Phi_\infty} := \{\gamma \in \mathrm{End}(E) \mid [\Phi_\infty \wedge \gamma] = 0\} \tag{8}$$

over $\Sigma^\times$. Fix a base solution A_∞ of Eq. (7) and write $A = A_\infty + \alpha$, where $\alpha \in \Omega^1(\Sigma^\times, \mathfrak{su}(E))$. The first equation gives that

$$[\alpha^{0,1} \wedge \Phi_\infty] = [\alpha \wedge \Phi_\infty] = 0,$$

so α takes values in the real line bundle $L_{\Phi_\infty}^{\mathbb{R}} := L_{\Phi_\infty} \cap \mathfrak{su}(E)$. This implies in particular that $[\alpha \wedge \alpha] = 0$. From the second equation of (7) we obtain $d_{A_\infty}\alpha = 0$, hence the ungauged deformation space at (A_∞, Φ_∞) can be identified with

$$Z^1(\Sigma^\times; L_{\Phi_\infty}^{\mathbb{R}}) := \{\alpha \in \Omega^1(\Sigma^\times, L_{\Phi_\infty}^{\mathbb{R}}) \mid d_{A_\infty}\alpha = 0\}.$$

Next consider the subgroup $\mathrm{Stab}_{\Phi_\infty}$ of unitary gauge transformations which fix Φ_∞. If $g \in \mathrm{Stab}_{\Phi_\infty}$ is of the form $g = \exp(\gamma)$ for some $\gamma \in \Omega^0(\Sigma^\times, L_{\Phi_\infty}^{\mathbb{R}})$, then g acts on $\alpha \in \Omega^1(\Sigma^\times, L_{\Phi_\infty}^{\mathbb{R}})$ by

$$\alpha^g = g^{-1}\alpha g + g^{-1}(d_{A_\infty} g) = \alpha + d_{A_\infty}\gamma.$$

We here use that $L_{\Phi_\infty}^{\mathbb{R}}$ is a parallel line subbundle of $\mathfrak{su}(E)$ with respect to A_∞, so $g^{-1}\alpha g = \alpha$ and $d_{A_\infty}\exp(\gamma) = \exp(\gamma) d_{A_\infty}\gamma$. Hence the gauged deformation space is

$$H^1(\Sigma^\times; L_{\Phi_\infty}^{\mathbb{R}}) = \frac{Z^1(\Sigma^\times; L_{\Phi_\infty}^{\mathbb{R}})}{B^1(\Sigma^\times; L_{\Phi_\infty}^{\mathbb{R}})},$$

where

$$B^1(\Sigma^\times; L_{\Phi_\infty}^{\mathbb{R}}) := \{d_{A_\infty}\gamma \mid \gamma \in \Omega^0(\Sigma^\times, L_{\Phi_\infty}^{\mathbb{R}})\}.$$

Lemma 3.1 (cf. [12])**.** *Suppose the assumption (A3) holds. Then*

$$\dim_{\mathbb{R}} H^1(\Sigma^\times; L^{\mathbb{R}}_{\Phi_\infty}) = 6(\gamma - 1),$$

where γ is the genus of Σ.

Proof. Working either with $\Sigma^\times$ or the homotopy equivalent space $M = \Sigma \setminus B_\varepsilon(\mathfrak{p})$ for some sufficiently small $\varepsilon > 0$ (so ∂M is a union of k circles, $k = |\mathfrak{p}|$), we note the following. First, there are no nontrivial parallel sections since $L^{\mathbb{R}}_{\Phi_\infty}$ is twisted near each p_i, so $H^0(\Sigma^\times; L^{\mathbb{R}}_{\Phi_\infty}) = 0$; by Poincaré duality, $H^2(\Sigma^\times; L^{\mathbb{R}}_{\Phi_\infty}) = H^0(M, \partial M; L^{\mathbb{R}}_{\Phi_\infty}) = 0$ as well. Recall also that since $L^{\mathbb{R}}_{\Phi_\infty}$ is a flat real line bundle, i.e., a local system of rank 1, it has Euler-Poincaré characteristic

$$\chi(\Sigma^\times; L^{\mathbb{R}}_{\Phi_\infty}) = \chi(\Sigma^\times) = 2 - 2\gamma - k.$$

These facts together give that

$$\dim_{\mathbb{R}} H^1(\Sigma^\times; L^{\mathbb{R}}_{\Phi_\infty}) = k + 2\gamma - 2 = 4\gamma - 4 + 2\gamma - 2 = 6\gamma - 6,$$

as claimed. ☐

We now turn to a discussion of the asymptotic behaviour of the natural L^2 Riemannian metric G on $\mathcal{M}$ in the limit of large Higgs fields. By definition

$$G_x((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = 2\,\mathrm{Re} \int_\Sigma \mathrm{Tr}(\alpha_1^* \wedge \alpha_2 + \varphi_1 \wedge \varphi_2^*) \tag{9}$$

for $x = [(A, \Phi)] \in \mathcal{M}$ and a pair $(\alpha_j, \varphi_j) \in \Omega^{0,1}(\Sigma, \mathfrak{sl}(E)) \oplus \Omega^{1,0}(\Sigma, \mathfrak{sl}(E))$ in unitary gauge, representing a pair of tangent vectors of $\mathcal{M}$ in x. The metric G has been introduced by Hitchin in [8]; remarkably, it is a complete (in the case where $d = \deg E$ is odd) hyperkähler metric. The definition (9) of G cannot be extended to points $x = (A_\infty, \Phi_\infty)$, the limiting configurations of §2.2. Namely, it turns out that infinitesimal variations of limiting configurations in directions transversally to the Hitchin fibration fail to have finite L^2 norm. In contrast, the restriction of the metric G to its fibres extends to the limit, in a way which we describe next.

For a simple holomorphic quadratic differential $q \in \mathrm{QD}(\Sigma)$ we let

$$\mathbb{T}_q := \{[(A_\infty, \Phi_\infty)] \mid (4) \text{ and } \det \Phi_\infty = q\}$$

denote the associated torus of limiting configurations, which by Lemma 3.1 has real dimension $6(\gamma - 1)$. We then set (supressing the dependence from the base point $x = [(A_\infty, \Phi_\infty)]$ from the notation)

$$G_q(\alpha_1, \alpha_2) = 2\,\mathrm{Re} \int_\Sigma \mathrm{Tr}(\alpha_1^* \wedge \alpha_2) \tag{10}$$

for

$$\alpha_1, \alpha_2 \in T_x\mathbb{T}_q = \{\alpha \in \Omega^1(\Sigma^\times, L^{\mathbb{R}}_{\Phi_\infty}) \mid \alpha \in \ker(d_{A_\infty} \oplus d^*_{A_\infty})\}.$$

Assuming finiteness of the integral on the right-hand side of (10) for a moment, Lemma 3.1 can now easily be used to deduce flatness of the Riemannian metric G_q. Namely, we observe that for any $x \in \mathbb{T}_q$ a tangent frame of

$T_x\mathbb{T}_q$ induces a flat local coordinate system on some neighbourhood of x. Indeed, let $(\beta, 0) \in T_x\mathbb{T}_q$ and put $B_\infty := A_\infty + \beta$. Then it follows that the unitary connection B_∞ satisfies

$$F_{B_\infty} = F_{A_\infty} + d_{A_\infty}\beta + \frac{1}{2}[\beta \wedge \beta] = F_{A_\infty} = 0,$$

since $[\beta \wedge \beta] = 0$ for any 1-form with values in L_{Φ_∞} and $d_{A_\infty}\beta = 0$ by assumption. Therefore (B_∞, Φ_∞) represents again a point in $\mathbb{T}_q$. It follows that the Riemannian metric G_q has constant coefficients with respect to the above local coordinate system. Therefore its curvature vanishes.

The outlined result has been obtained in collaboration with R. Mazzeo, H. Weiß and F. Witt as part of a much broader study of the asymptotic geometry of the moduli space $\mathcal{M}_d$, cf. the forthcoming article [14]. There it is shown that the restriction of the metric G to the fibre over tq of the Hitchin fibration converges to G_q as $t \to \infty$. This also explains finiteness of the integral in (10). Concerning the structure of the L^2 metric on $\mathcal{M}_d$ itself it is proven that it is asymptotically close to the well-studied semiflat hyperkähler metric G_{sf}, an incomplete Riemannian metric which is defined only on the region $\mathcal{M}^*$ comprised by the simpe Higgs fields. This metric stems from the data of an algebraic completely integrable system determined by restricting the Hitchin fibration to $\mathcal{M}^*$. The term 'semiflat' here refers to the fact that the fibres $\det^{-1}(q)$ are exactly flat with respect to G_{sf}. Moreover, this metric induces on the base $\mathrm{QD}(\Sigma)$ a Kähler metric with further interesting properties, a so-called special Kähler metric as studied e.g. by Freed [5]. Our investigation here is guided by the conjectural picture due to Gaiotto, Moore and Neitzke [6, 7] which describes the L^2 metric G as a perturbation series off the semi-flat metric G_{sf}.

4. The Hitchin fibration under degenerations of the Riemann surface

The aim of this section is to carry part of the preceding discussion over to the case of a family of Riemann surfaces degenerating to a noded limit. To start with, we recall that the linearization of the self-duality equations at a solution (A, Φ) gives rise to the operator

$$D_{(A,\Phi)} \colon (\alpha, \varphi) \mapsto \begin{pmatrix} d_A\alpha + [\Phi \wedge \varphi^*] + [\Phi^* \wedge \varphi] \\ \bar\partial_A\varphi + [\alpha^{0,1} \wedge \Phi] \end{pmatrix}. \tag{11}$$

Elements in the nullspace of $D_{(A,\Phi)}$ represent tangent vectors of $\mathcal{M}$ at $x = [(A, \Phi)]$ up to unitary gauge. We are here only interested in the directions tangential at x to the fibres of the Hitchin fibration, the space of which we denote by $\mathcal{V}_x$. It follows from (11) that the space

$$\mathcal{W}_x := \{\alpha \in \Omega^1(\Sigma, \mathfrak{su}(E)) \mid d_A\alpha = 0, d_A^*\alpha = 0, [\alpha^{0,1} \wedge \Phi] = 0\},$$

the equation $d_A^*\alpha = 0$ constituting a gauge-fixing condition, represents a subspace of $\mathcal{V}_x$. By comparing dimensions, we show below that over a smooth surface Σ both spaces do in fact coincide. Because of the decompositions

$$d_A = \partial_A + \bar\partial_A \qquad \text{and} \qquad d_A^* = -*\partial_A * - *\bar\partial_A *$$

we see that the nullspaces of the operators $d_A + d_A^*$ and $\bar\partial_A \colon \Omega^{1,0}(\Sigma, \mathfrak{sl}(E)) \to \Omega^{1,1}(\Sigma, \mathfrak{sl}(E))$ are in bijection to each other. It therefore suffices to consider instead of $d_A + d_A^*$ the simpler operator $\bar\partial_A$. Indeed, writing the connection A with respect to a local unitary frame over $U \subset \Sigma$ as $A = \beta\, d\bar z - \beta^*\, dz$ for some $\beta \in C^\infty(U, \mathfrak{sl}(2,\mathbb{C}))$ and using $(\alpha^{1,0})^* = -\alpha^{0,1}$ it follows that

$$d_A\alpha = \bar\partial\alpha^{1,0} + [\beta\, d\bar z \wedge \alpha^{1,0}] + \partial\alpha^{0,1} - [\beta^*\, dz \wedge \alpha^{0,1}] = 2\,\mathrm{Re}\,\bar\partial_A\alpha^{1,0},$$

and similarly

$$\begin{aligned} -*d_A^*\alpha &= \bar\partial * \alpha^{1,0} + [\beta\, d\bar z \wedge *\alpha^{1,0}] + \partial * \alpha^{0,1} - [\beta^*\, dz \wedge *\alpha^{0,1}] \\ &= \bar\partial(-i\alpha^{1,0}) + [\beta\, d\bar z \wedge (-i\alpha^{1,0})] + \partial(-i\alpha^{0,1}) - [\beta^*\, dz \wedge (-i\alpha^{0,1})] \\ &= 2\,\mathrm{Im}\,\bar\partial_A\alpha^{1,0}, \end{aligned}$$

and therefore $\bar\partial_A\alpha^{1,0}$ determines $(d_A + d_A^*)\alpha$ and vice versa.

Although it is at present not clear whether the equality $\mathcal{W}_x = \mathcal{V}_x$ continuous to hold for the noded limit Σ_0 of the smooth family Σ_R of Riemann surfaces (we here use the notation introduced in §2.3), it motivates our study of the behaviour of the family of operators $\bar\partial_A$ in the limit $R \searrow 0$. For a Higgs field Φ we set as before $\Sigma_R^\times = \Sigma_R \setminus (\det\Phi)^{-1}(0)$. We also recall Definition 2.1 of a limiting Higgs field Φ_∞ on $\Sigma_R^\times$, i.e. a Higgs field which satisfies $[\Phi_\infty \wedge \Phi_\infty^*] = 0$. We keep the above assumption (A3) that all the Higgs fields we consider have simple determinants.

Proposition 4.1. *Let Φ be a smooth Higgs field on Σ_R such that* $\det\Phi = q$. *Then there exists a limiting Higgs field Φ_∞ of the same determinant q, and moreover, for any such Φ_∞ the line bundles L_Φ and L_{Φ_∞} (cf.* (8)*) are isomorphic as complex vector bundles over $\Sigma_R^\times$.*

Proof. Since by assumption $q = \det\Phi$ has only simple zeroes, there exists a limiting Higgs field Φ_∞ as introduced in §3 of the same determinant q. Furthermore, as shown in [12, Lemma 4.2] one can choose a smooth section $g \in \Gamma(\Sigma_R^\times, \mathrm{SL}(E))$ such that $\Phi = g^{-1}\Phi_\infty g$. Since the Lie group $\mathrm{SL}(2,\mathbb{C})$ is homotopy-equivalent to the simply-connected manifold S^3 and $\Sigma_R^\times$ retracts onto a bouquet of circles there are no obstructions to the existence of a smooth path $g_t \colon t \mapsto g_t \in \Gamma(\Sigma_R^\times, \mathrm{SL}(E))$ satisfying $g_0 = \mathrm{Id}$ and $g_1 = g$. Therefore the complex line bundles L_Φ and L_{Φ_∞} are isomorphic. □

Proposition 4.2. *Let (A,Φ) be a smooth solution of the self-duality equations* (2) *on Σ_R, $R > 0$. Then the line bundle L_Φ over $\Sigma_R^\times$ is parallel with respect to the connection A.*

Proof. Let $\gamma \in \Omega^0(\Sigma_R^\times, L_\Phi)$ be a smooth section of L_Φ, i.e. $[\Phi \wedge \gamma] = 0$. Then $d_A\gamma$ is again a section of L_Φ. Indeed, since $\bar\partial_A \Phi = 0$ and $\partial_A \Phi = 0$ by degree reasons, hence $d_A\Phi = 0$, it follows that

$$[\Phi \wedge d_A\gamma] = -d_A[\Phi \wedge \gamma] + [d_A\Phi \wedge \gamma] = 0,$$

as asserted. □

In view of Proposition 4.2 it follows that the unitary connection A induces a connection (also denoted by A) on the line bundle $L_\Phi \to \Sigma_R^\times$ with associated operator $d_A = \partial_A + \bar\partial_A$ acting on L_Φ-valued differential forms. Furthermore, from Lemma 3.1 and Proposition 4.1 we deduce that for $R > 0$ the Fredholm index of the elliptic operator

$$\bar\partial_A \colon \Omega^{1,0}(\Sigma_R^\times, L_\Phi) \to \Omega^{1,1}(\Sigma_R^\times, L_\Phi) \tag{12}$$

equals $6(\gamma - 1)$.

We next study the behaviour of the operator family $\bar\partial_A$ in the limit $R \searrow 0$. This corresponds to the passage from a family of elliptic operators with smooth coefficients for $R > 0$ to the singular limiting operator $\bar\partial_A$ on $\Sigma_0^\times$, a so-called b-operator (cf. [15]). Following [17], a natural domain for the limiting-operator $\bar\partial_A$ is the space of sections $\gamma \in \Omega^{1,0}(\Sigma_0^\times, L_\Phi)$ which take the following form with respect to the local holomorphic coordinates $z = re^{i\theta}$ and $w = se^{i\psi}$ near $p \in \mathfrak{p}$ as introduced in §2.3. Namely,

$$\gamma(z) = u(z)\frac{dz}{z}, \qquad \text{and} \qquad \gamma(w) = v(w)\frac{dw}{w}$$

for some matrix-valued functions $u \in L^2(dr \wedge d\theta)$ and $v \in L^2(ds \wedge d\psi)$, respectively, which satisfy the matching condition $u(0) = -v(0)$. This choice of domain reflects the symmetry assumption made in (A2).

We now suppose that $R \geq 0$ is sufficiently small. Recalling that by a result due to Biquard and Boalch (cf. [2] and also [18, Lemma 3.1] for a description in terms of the present setup) every solution (A, Φ) is asymptotically close to some model solution $(A_R^{\mathrm{mod}}, \Phi_R^{\mathrm{mod}})$, where

$$\Phi_R^{\mathrm{mod}} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \frac{dz}{z} \qquad (C \neq 0),$$

we can pass to a local holomorphic frame in which the Higgs field Φ is diagonal in some neighbourhood of $\mathfrak{p}$. Since by Proposition 4.2 L_Φ is A-parallel, the $(0,1)$-part of the connection A must also be diagonal with respect to this frame, and therefore $\bar\partial_A$ acts as the standard (untwisted) Dolbeault operator $\bar\partial$ near each node.

From here on we may appeal to the analysis of the Dolbeault operator on degenerating Riemannian surfaces which has been carried out by Seeley and Singer [17]. It relies on the following functional analytic result due to Cordes and Labrousse [4]. Let H_1 and H_2 be Hilbert spaces, and D_t, $t \in \mathbb{R}$, be a family of closed operators $H_1 \supset \operatorname{dom} D_t \to H_2$ of which we assume the following. Let $G_t \subset \operatorname{dom}(D_t) \oplus H_2$ denote the graph of D_t and let $P_t \colon H_1 \oplus H_2 \to G_t$ be the orthogonal projection. The family D_t is called **graph-continuous** at $t_0 \in \mathbb{R}$ if P_t is norm-continuous at t_0.

Lemma 4.3. *Suppose D_t is a family of operators, graph-continuous at $t = 0$, such that D_0 is Fredholm. Then for all sufficiently small t, the operator D_t is Fredholm as well and* $\operatorname{ind} D_t = \operatorname{ind} D_0$.

Proof. For a proof we refer to [17, §2]. □

The main step now is to show graph-continuity of the R-dependent family of operators $\bar{\partial}_A$ on Σ_R at $R = 0$. This analysis has been carried out in [17] for the Dolbeault operator $\bar{\partial}$ acting on the canonical line bundle $K_\Sigma \cong T^*_{\mathbb{C}}\Sigma$. It consists of several steps, the first of which being the construction of a local parametrix of $\bar{\partial}$ on each 'neck' $\mathcal{C}^- \cup \mathcal{C}^+$ of Σ_R (cf. Figure 2). This family of local parametrices is then shown to converge in a suitable sense as $R \searrow 0$ to the local parametrix on Σ_0. These local parametrices are then glued to a R-independent interior parametrix to obtain graph-continuity of the family of operators $\bar{\partial}$ and the Fredholm property in the limit. This scheme of proof carries over without any serious changes to the family of operators $\bar{\partial}_A$ considered here. We therefore conclude that the latter family is graph-continuous and the limiting operator $\bar{\partial}_A$ on Σ_0 is Fredholm. Thus Lemma 4.3 implies the following result.

Theorem 4.4. *Let (A, Φ) be a solution of the self-duality equations* (2) *on the Riemann surface Σ_R, where $R \geq 0$. Then the operator $\bar{\partial}_A$ in* (12) *is a Fredholm operator of index* $6(\gamma - 1)$.

5. Concluding remarks

In this note we did not discuss the behaviour of the full linearized operator (11) in the limit $R \searrow 0$. This analysis can be carried out along similar lines, showing the stability of the Fredholm index of $D_{(A,\Phi)}$ in this limit. Such a result can then be used to show bijectivity of the gluing map which assigns to a singular solution (A_0, Φ_0) of the self-duality equations on Σ_0 a smooth solution on each nearby surface Σ_R, cf. Theorem 2.3.

Concerning properties of the L^2 metric on the family of moduli spaces, it is worthwhile to point out the difference to the situation considered in §3. There it turned out that restriction of the L^2 metric to the fibres of the Hitchin fibration persists in the limit $t \to \infty$ and induces a flat metric on each limiting torus $\mathbb{T}_q$. This is in contrast to what we encounter in the case of degenerating Riemann surfaces, where the L^2 metric is not defined on the limiting tori. Indeed, any α in the domain of $\bar{\partial}_A$ satisfies the decay condition

$$\lim_{z \to p} \alpha(z) = u_* \frac{dz}{z} \qquad (p \in \mathfrak{p})$$

for some u_*, the decay being at a polynomial rate in $r = |z|$. If $u_* \neq 0$ then α does not have finite L^2 norm since $|u_*/z|^2$ is not integrable with respect to the measure $r\,dr \wedge d\theta$. Conversely, if $u_* = 0$ then α has finite L^2 norm. The subspace of such 1-forms α equals the kernel of the operator $\bar{\partial}_A$ under the so-called Atiyah-Patodi-Singer (APS) boundary conditions. Therefore the APS index theorem permits us to determine the Fredholm index in this case. Using the well-known gluing properties of the APS index, it can explicitly be computed from the index of $\bar{\partial}_A$ on a smooth surface (where it equals $6(\gamma - 1)$) and the kernel of $\bar{\partial}_A$ acting on cross-sections of the cylindrical ends of $\Sigma_0 \setminus \mathfrak{p}$, cf. [10].

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