Morse Homology for the Yang-Mills Gradient Flow

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Abstract

We use the Yang-Mills gradient flow on the space of connections over a closed Riemann surface to construct a chain complex. The chain groups are generated by Yang-Mills connections. The boundary operator is defined by counting the elements of appropriately defined moduli spaces of Yang-Mills gradient flow lines that converge asymptotically to Yang-Mills connections.

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1. Introduction

Let $(\Sigma, g)$ be a closed oriented Riemann surface. Let $G$ be a compact Lie group, $\mathfrak{g}$ its Lie algebra, and $P$ a principal $G$-bundle over $\Sigma$. On $\mathfrak{g}$ we choose an ad-invariant inner product $\langle \cdot, \cdot \rangle$. The Riemannian metric $g$ induces for $k \in \{0, 1, 2\}$ the Hodge star operator $\ast : \Omega^k(\Sigma) \to \Omega^{2-k}(\Sigma)$ on differential $k$-forms. We denote by $A(P)$ the affine space of $\mathfrak{g}$-valued connection $1$-forms on $P$. The underlying vector space is the space $\Omega^1(\Sigma, \text{ad}(P))$ of sections of the adjoint bundle $\text{ad}(P) := P \times \text{Ad} \mathfrak{g}$. The curvature of a connection $A \in A(P)$ is the $\text{ad}(P)$-valued $2$-form $F_A = dA + \frac{1}{2}[A \wedge A]$. On $A(P)$ we consider the perturbed Yang-Mills functional defined by

$$\mathcal{YM}^V(A) = \frac{1}{2} \int_{\Sigma} \langle F_A \wedge \ast F_A \rangle + V(A)$$

(1)

for a gauge-invariant perturbation $V : A(P) \to \mathbb{R}$, the precise form of which will be defined later. The corresponding Euler-Lagrange equation is the second order partial differential equation $d^*_A F_A + \nabla V(A) = 0$, called perturbed Yang-Mills equation. The (negative) $L^2$ gradient flow equation formally associated with the functional $\mathcal{YM}^V$ is the PDE

$$\partial_s A + d^*_A F_A + \nabla V(A) = 0.$$ 

(2)

The group $G(P)$ of principal $G$-bundle automorphisms of $P$ acts on the space $A(P)$ by gauge transformations, i.e. as $g^*A := g^{-1} Ag + g^{-1}dg$. The functional $\mathcal{YM}^V$ is invariant under such gauge transformations, and hence are the solutions of the perturbed Yang-Mills (gradient flow) equations. The action is not free. The occurring stabilizer subgroups are Lie subgroups of $G$, hence finite-dimensional. Restricting the action to the group $G_0(P)$ of so-called based gauge transformations, i.e. those transformations which fix a prescribed fibre of $P$ pointwise, one indeed obtains a free group action. For this reason we will study solutions to the gradient flow equation (2) only up to based gauge transformations, cf. however the comment below regarding a $G$-equivariant extension of
the theory.

The study of the Yang-Mills functional over a Riemann surface from a Morse theoretical point of view has been initiated by Atiyah and Bott in [5]. One essential observation made there is that the based gauge-equivalence classes of (unperturbed) Yang-Mills connections come as a family of finite-dimensional closed submanifolds of $A(P)/G_0(P)$. As discussed in detail in [5], the unperturbed Yang-Mills functional satisfies the so-called Morse-Bott condition. In our context this condition asserts that, for any Yang-Mills connection $A \in A(P)$, the kernel of the Hessian $H_{YM}$ coincides with the subspace of $T_A A(P)$ comprising the tangent vectors at $A$ to the critical manifold containing $A$. Equivalently, the restriction of $H_{YM}$ to the normal space at $A$ of this critical manifold is non-degenerate. Furthermore, the spectrum of $H_{YM}$ consists solely of eigenvalues, with a finite number of negative ones. Hence the situation one encounters for the functional $YM$ over a Riemann surface parallels the one for Morse-Bott functions on finite-dimensional manifolds. The Morse theoretical approach taken by Atiyah and Bott indeed turned out to be very fruitful and had remarkable applications e.g. to the cohomology of moduli spaces of stable vector bundles over $\Sigma$ (cf. e.g. [13] for a review of these results). However, the literature so far still lacked a proper treatment of the analytical aspects of such a Morse-Bott theory and the underlying $L^2$ gradient flow (2). The present article aims to close this gap and to introduce and work out in full analytical detail a Yang-Mills Morse homology theory over $\Sigma$.

Let us now briefly describe our setup. We shall work with a Banach space $Y$ of so-called abstract perturbations $V : A(P) \to \mathbb{R}$. The space $Y$ is generated by a countable set of gauge-invariant model perturbations $V_\ell$ of the form

$$V_\ell(A) := \rho(\|\alpha(A)\|^2_{L^2(\Sigma)}) \langle \eta, \alpha(A) \rangle,$$

with $\rho = \rho(\ell) : \mathbb{R} \to \mathbb{R}$ a cut-off function, $\eta = \eta(\ell) \in \Omega^1(\Sigma, \text{ad}(P))$ a fixed $\text{ad}(P)$-valued 1-form, and $\alpha(A) = g^* A - A_0$. Here $g \in G(P)$ is chosen such that the local slice condition $d_A^* A_0 = 0$ is satisfied with respect to some reference connection $A_0 = A_0(\ell)$. Our construction of model perturbations relies crucially on the recent $L^2$ local slice theorem by Mrowka and Wehrheim [17]. The space $Y$ of perturbations is sufficiently flexible to achieve transversality of Fredholm sections as we shall describe below. This approach to transversality draws from ideas successfully used by Weber [33] in the related situation of the heat flow for loops on a compact manifold. Let

$$\mathcal{P}(a) := \{ A \in A(P) \mid d_A F_A = 0 \text{ and } \mathcal{V}_{YM}(A) \leq a \}$$

denote the set of Yang-Mills connections of energy at most $a$. On $\mathcal{P}(a)$ we fix a Morse function $h : \mathcal{P}(a) \to \mathbb{R}$, i.e. a smooth function $h$ with isolated non-degenerate critical points whose stable and unstable manifolds intersect transversally. To a critical point $x$ of $h$ (which in particular is a critical point
of $\mathcal{YM}^V$) we assign the non-negative number
\[
\text{Ind}(x) := \text{ind}_{\mathcal{YM}^V}(x) + \text{ind}_h(x),
\]
where $\text{ind}_h(x)$ is the usual Morse index of $x$ with respect to $h$ and $\text{ind}_{\mathcal{YM}^V}(x)$ denotes the number of negative eigenvalues (counted with multiplicities) of the Yang-Mills Hessian $H_x \mathcal{YM}^V$. For a regular value $a$ of $\mathcal{YM}^V$ we consider the $\mathbb{Z}_2$ module
\[
CM^a_a(\mathcal{YM}^V, h) := \bigoplus_{x \in \text{crit}(h) \cap P(a)} \langle x \rangle
\]
generated by the critical points of $h$ of Yang-Mills energy at most $a$. This module is graded by the index $\text{Ind}$. Under certain transversality assumptions (which resemble the usual Morse-Bott transversality required in finite-dimensional Morse theory) there is a well-defined boundary operator
\[
\partial_* : CM^a_a(\mathcal{YM}^V, h) \to CM^a_{a-1}(\mathcal{YM}^V, h)
\]
which arises from counting so-called cascade configurations of (negative) $L^2$ gradient flow lines, i.e. finite tuples of solutions of
\[
\partial_s A + d^*_xF_A + \nabla V(A) = 0,
\]
whose asymptotics as $s \to \pm \infty$ obey a certain compatibility condition.

1.1. Main results

The purpose of the present work is to prove the following result.

**Theorem 1.1 (Main result).** Let $a \geq 0$ be a regular value of $\mathcal{YM}^V$. For any Morse function $h : P(a) \to \mathbb{R}$ and generic perturbation $V \in Y$, the map $\partial_*$ satisfies $\partial_k \circ \partial_{k+1} = 0$ for all $k \in \mathbb{N}_0$ and thus there exist well-defined homology groups
\[
HM^a_* (\mathcal{A}(P)) = \ker \partial_k / \text{im} \partial_{k+1}.
\]
The homology $HM^a_* (\mathcal{A}(P))$ is called Yang-Mills Morse homology. It is independent of the choice of perturbation $V$ and Morse function $h$.

A proof of this result is given in Section 8. The preceding sections are of preparatory nature and comprise the more technical parts of the paper. In Section 2 we recall some known facts about the unperturbed Yang-Mills functional and introduce the Banach space $Y$ of perturbations needed later to make the transversality theory work. A collection of relevant properties and estimates involving the perturbations is contained in the appendix. In Section 3 the moduli space problem for Yang-Mills gradient flow lines with prescribed asymptotics as $s \to \pm \infty$ is put into an abstract Banach manifold setting. The moduli space
\[
\mathcal{M}(C^-, C^+) \text{ of gradient flow lines connecting a given pair } (C^-, C^+) \text{ of Yang-Mills critical manifolds is exhibited as the zero set of a section } F \text{ of a suitably defined Banach space bundle. The necessary Fredholm theory for the differential operator obtained by linearizing } F \text{ is developed in Section 5. In the preceding Section 4 we show exponential decay of any finite energy solution } A : \mathbb{R} \to \mathcal{A}(P) \text{ of (8) to a pair } A^\pm \text{ of Yang-Mills connections as } s \to \pm \infty. \text{ The issue of compactness of the moduli space is addressed in Section 6. The main result obtained there is the compactness Theorem 6.2 for sequences of solutions of (2) on finite time intervals. The proof of this result relies on certain a priori } L^p \text{ estimates for the curvature form } F_A, \text{ the weak Uhlenbeck compactness theorem (cf. [31, 32]), and a combination of elliptic and parabolic regularity estimates. Transversality of the section } F \text{ at a zero } x \in F^{-1}(0) \text{ is discussed in Section 7. We show, along the usual lines involving Sard’s lemma, that surjectivity of the linearized operator holds for generic perturbations } V \in Y.
\]

1.2. Further comments and related results

Equivariant theory

For the ease of presentation we develop here a non-equivariant Yang-Mills Morse theory on the space \( \mathcal{A}(P)/G_0(P) \) of based gauge equivalence classes of connections. Alternatively, it seems possible to take a \( G \)-equivariant approach by extending the setup to the space \( \mathcal{A}(P) \times E_nG \), with \( E_nG \) a suitable finite-dimensional approximation to the classifying space \( EG \). This space carries a free action by the full group \( G(P) \) of gauge transformations via

\[
g^*(A, \lambda) = (g^*A, \hat{g}\lambda),
\]

where the map \( g \mapsto \hat{g} \in G \) is given by evaluating \( g \) at some fixed \( p \in P \). By extending \( \mathcal{Y}\mathcal{M}^V \) in a suitable \( G(P) \)-invariant way, we expect our construction of Morse homology groups to carry over almost literally to the quotient manifold \( (\mathcal{A}(P) \times E_nG)/G(P) \).

Connection with Morse homology of loop groups

Yang-Mills Morse homology is strongly related to heat flow homology, at least in the case of the sphere \( \Sigma = S^2 \). This connection is due to the following result, cf. [28, 29]. For a compact Lie group \( G \) we let \( \Omega G \) denote the associated based loop group, i.e. the space \( \Omega G := \{ \gamma \in C^\infty(S^1, G) \mid \gamma(1) = 1 \} \) with group structure given by pointwise multiplication. Then as a special case of Weber’s heat flow homology [33] there is a chain complex \( CM_*\left( \Omega G \times E_nG \right) \) generated by the critical points of the classical action functional

\[
\mathcal{E} : \Omega G \to \mathbb{R}, \quad \gamma \mapsto \frac{1}{2} \int_0^1 \| \partial_t \gamma(t) \|^2 \, dt,
\]

suitably extended to the product \( \Omega G \times E_nG \).
**Theorem 1.2.** For any compact Lie group $G$ and any principal $G$-bundle $P$ over $\Sigma = S^2$, there exists a natural chain homomorphism

$$\Theta : CM_*(\frac{A(P) \times E_n G}{G(P)}) \to CM_*(\frac{\Omega G \times E_n G}{G}).$$

It induces an isomorphism

$$[\Theta] : HM_*(\frac{A(P) \times E_n G}{G(P)}) \to HM_*(\frac{\Omega G \times E_n G}{G}).$$

between Yang-Mills Morse homology and heat flow homology.

It would be interesting to work out a similar correspondence in the case where $\Sigma$ is a Riemann surface of arbitrary genus.

**Products**

In finite dimensional Morse homology it is well known how to implement a module structure, cf. the monograph [25]. In infinite dimensional situations one often encounters similar algebraic structures, like e.g. the quantum product in Floer homology or the Chas-Sullivan loop product in the Morse homology of certain loop spaces, cf. [3, 4, 6, 8]. Using finite-dimensional Morse homology as a guiding principle, one should be able to implement a natural product structure in the setup presented here. In a subsequent step one could ask how this relates to products in loop space homology of $\Omega G$.

**Related work**

For finite dimensional manifolds, the construction of a Morse homology theory from the set of critical points of a Morse function and the isolated flow lines connecting them goes back to Thom [30], Smale [26] and Milnor [16], and had later been rediscovered by Witten [34]. For a historical account we refer to the survey paper by Bott [7]. In infinite dimensions the same sort of ideas underlies the construction of Floer homology of compact symplectic manifolds (cf. the expository notes [22]), although the equations encountered there are of elliptic rather than parabolic type. More in the spirit of classical finite dimensional Morse homology is the aforementioned heat flow homology for the loop space of a compact manifold due to Weber [33], which is based on the $L^2$ gradient flow of the classical action functional. For another approach via the theory of ODEs on Hilbert manifolds and further references, see Abbondandolo and Majer [2]. The cascade construction of Morse homology in the presence of critical manifolds satisfying the Morse-Bott condition is due to Frauenfelder [12].

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2. Yang-Mills functional

2.1. Preliminaries

Let $(\Sigma, g)$ be a compact oriented Riemann surface. Let $G$ be a compact Lie

group with Lie algebra $\mathfrak{g}$. On $\mathfrak{g}$ we fix an ad-invariant inner product $(\cdot, \cdot)$,

which exists by compactness of $G$. Let $P$ be a principal $G$-bundle over $\Sigma$. A
gauge transformation is a section of the bundle $\text{Ad}(P) := P \times_G G$ associated
to $P$ via the action of $G$ on itself by conjugation $(g, h) \mapsto g^{-1}hg$. Let $\text{ad}(P)$
denote the Lie algebra bundle associated to $P$ via the adjoint action

$$(g, \xi) \mapsto \left. \frac{d}{dt} \right|_{t=0} g^{-1} \exp(t\xi) g \quad \text{for } g \in G, \xi \in \mathfrak{g}$$

of $G$ on $\mathfrak{g}$. The space of smooth $\text{ad}(P)$-valued differential $k$-forms is denoted
by $\Omega^k(\Sigma, \text{ad}(P))$, and by $\mathcal{A}(P)$ the space of smooth connections on $P$. The
latter is an affine space over $\Omega^1(\Sigma, \text{ad}(P))$. The group $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$ by
gauge transformations and on $\Omega^k(\Sigma, \text{ad}(P))$ by conjugation. We call a connection
$A \in \mathcal{A}(P)$ irreducible if the stabilizer subgroup $\text{Stab}_A \subseteq \mathcal{G}(P)$ is trivial.
Otherwise it is called reducible. It is easy to show that $\text{Stab} A$ is a compact Lie

group, isomorphic to a subgroup of $G$. Let $z \in \Sigma$ be arbitrary but fixed. We let
$\mathcal{G}_0(P) \subseteq \mathcal{G}(P)$ denote the group of based gauge transformation, i.e. those gauge
transformations which leave the fibre $P_z \subseteq P$ above $z$ pointwise fixed. It is a
well-known fact that $\mathcal{G}_0(P)$ acts freely on $\mathcal{A}(P)$.

The curvature of the connection $A$ is the $\text{ad}(P)$-valued 2-form $F_A = dA + \frac{1}{2}[A \wedge
A]$. It satisfies the Bianchi identity $d_F A = 0$. Covariant differentiation with
respect to the Levi-Civita connection associated with the metric $g$ and a connection
$A \in \mathcal{A}(P)$ defines an operator $\nabla_A : \Omega^k(\Sigma, \text{ad}(P)) \to \Omega^{k+1}(\Sigma) \otimes \Omega^1(\Sigma, \text{ad}(P))$.

Its antisymmetric part is the covariant exterior differential operator

$$d_A : \Omega^k(\Sigma, \text{ad}(P)) \to \Omega^{k+1}(\Sigma), \quad \alpha \mapsto d\alpha + [A \wedge \alpha].$$

The formal adjoints of these operators are denoted by $\nabla_A^*$ and $d_A^*$. The covariant
Hodge Laplacian on forms is the operator $\Delta_A := d_A^*d_A + d_A d_A^*$, the covariant
Bochner Laplacian on forms is $\nabla_A^*\nabla_A$. They are related through the Bochner-
Weitzenb"ock formula

$$\nabla_A = \nabla_A^* \nabla_A + \{F_A, \cdot \} + \{Rg, \cdot \}.$$
We also make use of the notation \( H := d_A^*d_A + \ast [\ast F_A \wedge \cdot] \). For a definition of Sobolev spaces of sections of vector bundles, of connections, and of gauge transformations we refer to the book [32, Appendix B]. We employ the notation \( W^{k,p}(\Sigma) \) and \( W^{k,p}(\Sigma, T^\ast \Sigma \otimes \text{ad}(P)) \) for the Sobolev spaces of \( \text{ad}(P) \)-valued sections, respectively \( \text{ad}(P) \)-valued 1-forms whose weak derivatives up to order \( k \) are in \( L^p \). Similarly, the notation \( A^{k,p}(P) \) indicates the Sobolev space of connections on \( P \) of class \( W^{k,p} \). We shall also use the parabolic Sobolev spaces

\[ W^{1,2;p}(I \times \Sigma, \text{ad}(P)) := L^p(I, W^{2,2;p}(\Sigma, \text{ad}(P))) \cap W^{1,p}(I, L^p(\Sigma, \text{ad}(P))) \]

of \( \text{ad}(P) \)-valued sections over \( I \times \Sigma \), with \( I \subseteq \mathbb{R} \) an interval (and similarly for parabolic Sobolev spaces of connections and for \( \text{ad}(P) \)-valued 1-forms). Further notation which is used frequently is \( \dot{A} := \partial_s A := dA/ds \), etc. for derivatives with respect to time.

2.2. Perturbations

Our construction of a Banach space of perturbations is based on the following \( L^2 \) local slice theorem due to T. Mrowka and K. Wehrheim [17]. We fix \( p > 2 \) and let

\[ S_{A_0}(\varepsilon) := \{ A = A_0 + \alpha \in A^{0,p}(\Sigma) \mid d_A^*\alpha = 0, ||\alpha||_{L^2(\Sigma)} < \varepsilon \} \]

denote the set of \( L^p \)-connections in the local slice of radius \( \varepsilon \) with respect to the reference connection \( A_0 \).

**Theorem 2.1 (\( L^2 \) local slice theorem).** Let \( p > 2 \). For every \( A_0 \in A^{0,p}(\Sigma) \) there are constants \( \varepsilon, \delta > 0 \) such that the map

\[ m : (S_{A_0}(\varepsilon) \times G^{1,p}(\Sigma)) / \text{Stab} A_0 \to A^{0,p}(\Sigma), \]

\[ [(A_0 + \alpha, g)] \mapsto (g^{-1})^* (A_0 + \alpha) \]

is a diffeomorphism onto its image, which contains an \( L^2 \) ball,

\[ B_\delta(A_0) := \{ A \in A^{0,p}(\Sigma) \mid ||A - A_0||_{L^2(\Sigma)} < \delta \} \subseteq \text{im} \ m. \]

**Proof:** For a proof we refer to [17, Theorem 1.7].

We fix the following data.

(i) A dense sequence \( (A_i)_{i \in \mathbb{N}} \) of irreducible smooth connections in \( A(P) \).

(ii) For every \( A_i \) a dense sequence \( (\eta_{ij})_{j \in \mathbb{N}} \) of smooth 1-forms in \( \Omega^1(\Sigma, \text{ad}(P)) \) satisfying \( d_A^*\eta_{ij} = 0 \) for all \( j \in \mathbb{N} \).

(iii) A smooth cutoff function \( \rho : \mathbb{R} \to [0, 1] \) such that \( \rho = 1 \) on \([-1, 1] \), \( \text{supp} \rho \subseteq [-4, 4] \), and \( ||\rho'||_{L^\infty(\mathbb{R})} < 1 \). Set \( \rho_k(r) := \rho(k^2r) \) for \( k \in \mathbb{N} \).
Let $\delta = \delta(A_i) > 0$ be as in Theorem 2.1 and assume that for $A \in \mathcal{A}^{0,p}(\Sigma)$ there exists $g \in G^{1,p}(P)$ with $\|g^* A - A_i\|_{L^2(\Sigma)} < \delta(A_i)$. It then follows from Theorem 2.1 that there exists a unique $\alpha = \alpha(A) \in L^p(\Sigma, \text{ad}(P))$ which satisfies for some $g \in G^{1,p}(P)$ the conditions

$$g^* A - A_i = \alpha \quad \text{and} \quad d_A^* \alpha = 0.$$  \hspace{1cm} (4)

Hence the map

$$V_\ell : \mathcal{A}(P) \to \mathbb{R}, \quad A \mapsto \rho_k(\|\alpha(A)\|_{L^2(\Sigma)}^2)\langle \alpha(A), \eta_j \rangle \hspace{1cm} (5)$$

is well-defined for every triple $\ell = (i,j,k) \in \mathbb{N}^3$ with $k > \frac{4}{\delta(A_i)}$. Note that $V_\ell$ is invariant under gauge transformations.

**Remark 2.2.** In the following only perturbations $V_\ell$, $\ell = (i,j,k) \in \mathbb{N}^3$, with $k > \frac{4}{\delta(A_i)}$ and hence

$$\text{supp} \rho_k \subseteq \left[0, \frac{\delta(A_i)^2}{4}\right]$$

will be relevant. We henceforth consider only indices $k$ which satisfy this condition, and renumber the corresponding triples $(i,j,k)$ by integers $\ell \in \mathbb{N}$.

Given $\ell \in \mathbb{N}$, we fix a constant $C_\ell > 0$ such that the following three conditions are satisfied.

(i) $\sup_{A \in \mathcal{A}(P)} |V_\ell(A)| \leq C_\ell$,

(ii) $\sup_{A \in \mathcal{A}(P)} \|\nabla V_\ell(A)\|_{L^2(\Sigma)} \leq C_\ell$,

(iii) $\|\nabla V_\ell(A)\|_{L^p(\Sigma)} \leq C_\ell(1 + \|F_A\|_{L^4(\Sigma)})$ for all $1 < p < \infty$ and $A \in \mathcal{A}(P)$.

The existence of the constant $C_\ell$ follows from Proposition Appendix A.4. The **universal space of perturbations** is the normed linear space

$$Y := \left\{ V := \sum_{\ell=1}^{\infty} \lambda_\ell V_\ell \Big| \lambda_\ell \in \mathbb{R} \text{ and } \|V\| := \sum_{\ell=1}^{\infty} C_\ell |\lambda_\ell| < \infty \right\}. \hspace{1cm} (6)$$

It is a separable Banach space isomorphic to the space $\ell^1$ of summable real sequences. Some relevant properties of the perturbations $V_\ell$ are discussed in Appendix A.

2.3. **Critical manifolds**

We introduce some further notation and recall several results from [5, 18, 19] concerning the set of critical points of the unperturbed Yang-Mills functional $\mathcal{Y}\mathcal{M}$. Let

$$\text{crit}(\mathcal{Y}\mathcal{M}) := \{ A \in \mathcal{A}^{1,p}(\Sigma) \mid d_A^* F_A = 0 \}$$

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denote the set of critical points of $YM$, the equation $d^* A F A = 0$ being understood in the weak sense. Similarly, the notation $\text{crit}(YM^\nu)$ refers to the set of critical points of the perturbed Yang-Mills functional. We let $CR$ denote the set of connected components of $\text{crit}(YM)$. The group $G_0^{2,p}(P)$ of based gauge transformations of class $W^{2,p}$ acts freely on each $C \in CR$. The quotient $C/G_0^{2,p}(P)$ is a compact finite-dimensional smooth manifold, cf. [18, Section 2]. The set of critical values of $YM$ is a discrete subset of $\mathbb{R}_{\geq 0}$. The following proposition shows that below a fixed level $a \geq 0$, there are at most finitely many critical manifolds.

**Proposition 2.3.** For $a \geq 0$ let $CR^a \subseteq CR$ denote the set of critical manifolds of Yang-Mills energy at most $a$. Then $CR^a$ is a finite set, for every real number $a \geq 0$.

**Proof:** Assume by contradiction that there exists an infinite sequence $(C_\nu) \subseteq CR^a$ of pairwise different critical manifolds $C_\nu$. For each $\nu$ we fix a Yang-Mills connection $A_\nu \in C_\nu$. Then by Uhlenbeck’s strong compactness theorem (cf. [32, Theorem E]), after modifying the sequence $A_\nu \in C_\nu$ by suitable gauge transformations and passing to a subsequence, the sequence $A_\nu$ converges uniformly with all derivatives to a smooth connection $A_\ast \in A(P)$. The limit connection $A_\ast$ itself is Yang-Mills and has energy at most $a$, hence $A \in C_\ast \in CR^a$ for some critical manifold $C_\ast$. But $C_\ast$ is by construction not isolated within the set $CR^a$ (with respect to every $C_k^\ast(\Sigma)$ topology), contradicting the fact that it satisfies the Morse-Bott condition. \hfill \Box

In the following we shall also make use of the fact that the Yang-Mills functional $YM$ satisfies the Palais-Smale condition in dimension 2 (which holds true also in dimension 3 but not in higher dimensions).

**Definition 2.4.** A sequence $(A_i) \subseteq A(P)$ is said to be a Palais-Smale sequence if there exists $M > 0$ such that $\|F_{A_i}\|_{L^2(\Sigma)} < M$ for all $i$, and

$$\|d^*_{A_i} F_{A_i}\|_{W^{-1,2}(\Sigma)} \to 0 \quad \text{as} \quad i \to \infty.$$ 

**Theorem 2.5 (Equivariant Palais-Smale condition).** For any Palais-Smale sequence $(A_i) \subseteq A(P)$ there exists a subsequence, which we also denote $(A_i)$, and a sequence $(g_i) \subseteq G(P)$ such that $g_i^* A_i$ converges in $W^{1,2}(\Sigma)$ to a Yang-Mills connection $A_\ast \in A(P)$ as $i \to \infty$.

**Proof:** For a proof we refer to [19, Theorem 1]. \hfill \Box

**Proposition 2.6.** Let $\varepsilon, M > 0$. There exists a constant $\delta > 0$ with the following significance. If $A \in A(P)$ satisfies $\|F_A\|_{L^2(\Sigma)} < M$ and

$$\|A - A_0\|_{L^2(\Sigma)} > \varepsilon$$

for every Yang-Mills connection $A_0 \in A(P)$ then it follows that

$$\|d^*_{A_0} F_A\|_{W^{-1,2}(\Sigma)} > \delta.$$
Proof: Assume by contradiction that there exists a sequence \((A_i) \subseteq A(P)\) satisfying \(\|F_{A_i}\|_{L^2(\Sigma)} < M\) and (7) with

\[
\lim_{i \to \infty} \|d_{A_i}^* F_{A_i}\|_{W^{-1,2}(\Sigma)} = 0.
\]

Then by Theorem 2.5 there exist a subsequence, still denoted \((A_i)\), a sequence of gauge transformations \((g_i)\), and a Yang-Mills connection \(A_\ast\) with

\[
\lim_{i \to \infty} g_i^* A_i = A_\ast
\]

in \(L^2(\Sigma)\) (even in \(W^{1,2}(\Sigma)\)). Therefore, for \(i\) sufficiently large, the Yang-Mills connection \(g_i^* A_\ast\) satisfies

\[
\| (g_i^{-1})^* A_\ast - A_i \|_{L^2(\Sigma)} < \varepsilon,
\]

contradicting (7).

Proposition 2.7. For every \(\varepsilon, M > 0\) there exists a constant \(\delta > 0\) with the following significance. Assume the perturbation \(V\) satisfies the conditions \(\|V\| < \delta\) and

\[
\text{supp} V \subseteq A(P) \setminus \bigcup_{A \in \text{crit}(YM)} B_\varepsilon(A),
\]

where \(B_\varepsilon(A) := \{ A_1 \in A(P) \mid \| A_1 - A \|_{L^2(\Sigma)} < \varepsilon \}\). Then the perturbed Yang-Mills functional \(YM^V\) has the same set of critical points as the functional \(YM\) below the level \(M\), i.e. it holds that

\[
\text{crit}(YM^V) \cap \{ A \in A(P) \mid YM(A) < M \} = \text{crit}(YM) \cap \{ A \in A(P) \mid YM(A) < M \}.
\]

Proof: The inclusion \(\text{crit}(YM) \subseteq \text{crit}(YM^V)\) is clear because \(V\) is supposed to be supported away from \(\text{crit}(YM)\). It remains to show that the set \(\text{crit}(YM^V) \cap \{ A \in A(P) \mid YM(A) < M \}\) only contains points that are also critical for \(YM\). Thus let \(A \in A(P) \setminus \bigcup_{A \in \text{crit}(YM)} B_\varepsilon(A)\). By Proposition 2.6 there exists a constant \(\delta_1(\varepsilon, M) > 0\) with \(\|\nabla YM(A)\|_{W^{-1,2}(\Sigma)} \geq \delta_1\). Choosing \(\delta < \delta_1(\varepsilon, M)\) it follows that

\[
\|\nabla YM^V(A)\|_{W^{-1,2}(\Sigma)} \geq \|\nabla YM(A)\|_{W^{-1,2}(\Sigma)} - \|\nabla V(A)\|_{W^{-1,2}(\Sigma)}
\]

\[
\geq \|\nabla YM(A)\|_{W^{-1,2}(\Sigma)} - \|\nabla V(A)\|_{L^2(\Sigma)} > \delta_1(\varepsilon, M) - \delta > 0.
\]

The second inequality is by Proposition Appendix C.3. The third one is a consequence of the definition of \(\|V\|\) and condition (ii) in Section 2.2 from which it follows that

\[
\|\nabla V(A)\|_{L^2(\Sigma)} \leq \sum_{i=1}^{\infty} |\lambda_{\ell}| : \|\nabla V_{\ell}(A)\|_{L^2(\Sigma)} \leq \sum_{i=1}^{\infty} |\lambda_{\ell}| C_{\ell} = \|V\| < \delta.
\]

Hence \(A \notin \text{crit}(YM^V)\), and this proves the remaining inclusion.
To prove Theorem 1.1 we need to consider the Yang-Mills gradient flow for connections of Yang-Mills energy below a fixed sublevel set \( a \geq 0 \). For a given regular value \( a \geq 0 \) of \( \mathcal{YM} \) and each critical manifold \( C \in \mathcal{CR}^a \) we fix a closed \( L^2 \) neighborhood \( U_C \) of \( C \) such that \( U_{C_1} \cap U_{C_2} = \emptyset \) whenever \( C_1 \neq C_2 \). From Proposition 2.3 it follows that such a choice is possible. We then restrict the universal Banach space \( Y \) of perturbations as follows.

**Definition 2.8.** We call a perturbation \( \mathcal{V} = \sum_{\ell=1}^{\infty} \lambda_{\ell} V_{\ell} \in Y \) admissible if it satisfies

\[
\text{supp} \ V_{\ell} \cap U_C \neq \emptyset \quad \text{for some} \quad C \in \mathcal{CR}^a \implies \lambda_{\ell} = 0.
\]

It is straightforward to show that the space of admissible perturbation is a closed subspace of the Banach space \( Y \). Finally we remark that by Proposition 2.7 the perturbed functional \( \mathcal{YM}_{\mathcal{V}} \) has the same critical points as \( \mathcal{YM} \), for any admissible perturbation \( \mathcal{V} \in Y \) with \( \|\mathcal{V}\| \) small enough.

3. Yang-Mills gradient flow

In the following it will turn out to be convenient to allow for time-dependent gauge transformations to act on paths \( s \mapsto A(s) \in \mathcal{A}(P) \) of connections. We therefore denote by \( G(\hat{P}) \) the group of smooth gauge transformations of the principle \( G \)-bundle \( \hat{P} := \mathbb{R} \times P \) given by trivial extension of \( P \) to the base manifold \( \mathbb{R} \times \Sigma \).

**Definition 3.1.** Let \( \mathcal{V} \in Y \) be a regular perturbation. The perturbed Yang-Mills gradient flow is the nonlinear PDE

\[
0 = \partial_s A + d_A^* F_A - d_A \Psi + \nabla \mathcal{V}(A)
\]  

(8)

for paths \( A : s \mapsto A(s) \in \mathcal{A}(P) \) of connections and \( \Psi : s \mapsto \Psi(s) \in \Omega^0(\Sigma, \text{ad}(P)) \) of 0-forms.

The term \(-d_A \Psi\) plays the role of a gauge fixing term needed to make equation (8) invariant under time-dependent gauge transformations.

3.1. Moduli spaces

Let \( a \geq 0 \) denote the regular value of \( \mathcal{YM} \) fixed at the end of Section 2.3. We fix a pair \( (\hat{C}^-, \hat{C}^+) \in \mathcal{CR}^a \times \mathcal{CR}^a \) of critical manifolds of Yang-Mills energy at most \( a \), and denote \( C^\pm := \frac{\hat{C}^\pm}{w_0(\hat{P})} \). We also fix numbers \( p > 3 \) and \( \delta > 0 \).

Central to the construction of Morse homology groups (which will be carried out in Section 8) will be the moduli space of gradient flow lines between \( C^- \) and \( C^+ \). Let us define

\[
\mathcal{M}(\hat{C}^-, \hat{C}^+) := \left\{ (A, \Psi) \in \mathcal{A}_\delta^{1,2;\mu}(P) \times W^{1,p}_\delta(\mathbb{R} \times \Sigma) \mid (A, \Psi) \text{satisfies (8),} \right\}
\]

\[
\lim_{s \to \pm \infty} A(s) = A^\pm \text{ for some } A^\pm \in \hat{C}^\pm \right\}
\]
The group $G^2_{\delta}(\hat{P})$ acts freely on $\hat{\mathcal{M}}(\hat{C}^-,\hat{C}^+)$. For a precise definition of the involved Sobolev spaces of connections and groups of gauge transformations we refer to the next section. The moduli space of gradient flow lines between $\mathcal{C}^-$ and $\mathcal{C}^+$ is the quotient

$$\mathcal{M}(\mathcal{C}^-,\mathcal{C}^+) := \hat{\mathcal{M}}(\hat{C}^-,\hat{C}^+)/G^2_{\delta}(\hat{P}). \quad (9)$$

The aim of the next section is to reveal $\hat{\mathcal{M}}(\hat{C}^-,\hat{C}^+)$ as the zero set $F^{-1}(0)$ of an equivariant section $F$ of a suitably defined Banach space bundle $\mathcal{E}$ over a Banach manifold $\mathcal{B}$. After showing that the vertical differential $d_xF$ at any such zero $x \in F^{-1}(0)$ is a surjective Fredholm operator, the implicit function theorem applies and allows us to conclude that the moduli space $\mathcal{M}(\mathcal{C}^-,\mathcal{C}^+)$ is a finite-dimensional smooth manifold.

### 3.2. Banach manifolds

In this section we introduce the setup which will allow us to view the moduli space defined in (9) as the zero set of a Fredholm section of a certain Banach space bundle. These Banach manifolds are modeled on weighted Sobolev spaces in order to make the Fredholm theory work. We therefore choose a number $\delta > 0$ and a smooth cut-off function $\beta$ such that $\beta(s) = -1$ if $s < 0$ and $\beta(s) = 1$ if $s > 1$. We define the $\delta$-weighted $W^{k,p}$ Sobolev norm (for $k \in \mathbb{N}_0$ and $1 < p < \infty$) of a measurable function $u$ over $\mathbb{R} \times \Sigma$ to be the usual $W^{k,p}$ Sobolev norm of the function $e^{\delta\beta(s)}u$. In particular, we shall work with the space $\mathcal{A}^{1,2;p}_{\delta}(P)$ of time-dependent connections on $P$ which are locally of class $W^{1,2;p}$ and for which there exist limiting connections $A^\pm \in \hat{\mathcal{C}}^\pm$ and times $T^\pm \in \mathbb{R}$ such that the 1-forms $\alpha^\pm := A - A^\pm$ satisfy

$$\alpha^- \in W^{1,p}_{\delta}((\mathbb{R},\Sigma),L^p(T^-) \cap L^p_\delta((-\infty, T^-], W^{2,p}(\Sigma))) = W^{1,p}((-\infty, T^-], W^{2,p}(\Sigma)) \cap L^p_\delta((-\infty, T^-] \times \Sigma)$$

$$\alpha^+ \in W^{1,p}_{\delta}([T^+, \infty),L^p(\Sigma)) \cap L^p_\delta([T^+, \infty) \times \Sigma).$$

Similarly, let $\mathcal{G}^{2;p}_{\delta}(\hat{P})$ denote the group of based gauge transformations which are locally of class $W^{2;p}$ and in addition satisfy the following two conditions.\footnote{For a definition of Sobolev spaces of gauge transformations we refer to [32, Appendix B].}

The $\text{ad}(P)$-valued 1-form $g^{-1}dg$ satisfies

$$g^{-1}dg \in L^p_\delta(\mathbb{R}, W^{2;p}(\Sigma, T^*\Sigma \otimes \text{ad}(P))),$$

and there exist limiting based gauge transformations $g^\pm \in \mathcal{G}^{2;p}_{0}(P)$, numbers $T^\pm \in \mathbb{R}$, and $\text{ad}(P)$-valued 1-forms

$$\gamma^- \in W^{2;p}_{\delta}((-\infty, T^-] \times \Sigma)$$

and

$$\gamma^+ \in W^{2;p}_{\delta}([T^+, \infty) \times \Sigma).$$
with
\[ g(s) = \exp(\gamma^{-}(s))g^{-} \quad (s \leq T^{-}), \quad g(s) = \exp(\gamma^{+}(s))g^{+} \quad (s \geq T^{+}). \]

For a pair \((C^{-}, C^{+})\) of critical manifolds and numbers \(p > 3\) and \(\delta > 0\) denote by \(B := B(C^{-}, C^{+}, \delta, p)\) the Banach manifold of pairs
\[
(A, \Psi) \in A_{\delta}^{1,2,p}(P) \times W_{\delta}^{1,p}(\mathbb{R} \times \Sigma).
\]

The group \(G_{\delta}^{2,p}(\hat{P})\) acts smoothly and freely on \(\hat{B}\) via \(g \cdot (A, \Psi) = (g^{\ast}A, g^{-1}\Psi g + g^{-1}g\)). The resulting quotient space
\[
\hat{B} := B(C^{-}, C^{+}, \delta, p) := \hat{B}(C^{-}, C^{+}, \delta, p) / G_{\delta}^{2,p}(\hat{P})
\]
is again a smooth Banach manifold. The tangent space at the point \([(A, \Psi)] \in \hat{B}\) splits naturally as a direct sum
\[
T_{[(A, \Psi)]}\hat{B} = T_{[(A, \Psi)]}^{0}\hat{B} \oplus \mathbb{R}^{\dim C^{-}} \oplus \mathbb{R}^{\dim C^{+}},
\]
where \(T_{[(A, \Psi)]}^{0}\hat{B}\) can be identified with pairs
\[
(\alpha, \psi) \in W_{\delta}^{1,2,p}(\mathbb{R} \times \Sigma) \oplus W_{\delta}^{1,p}(\mathbb{R} \times \Sigma)
\]
which satisfy the gauge fixing condition
\[
L_{(A, \Psi)}^{\ast}(\alpha, \psi) := \partial_{s}\psi + [\Psi, \psi] - d_{A}^{\ast}\alpha = 0.
\]

Thus a tangent vector of the quotient space \(\hat{B}\) is identified with its unique lift to \(T\hat{B}\) which is perpendicular to the gauge orbit. We furthermore define the Banach space bundle \(\hat{E} = \hat{E}(C^{-}, C^{+}, \delta, p)\) over \(\hat{B}\) as follows. Let \(\hat{E}\) be the Banach space bundle over \(\hat{B}\) with fibres
\[
\hat{E}_{(A, \Psi)} := L^{p}_{\delta}(\mathbb{R}, L^{p}(\Sigma, T^{\ast}\Sigma \otimes \text{ad}(P)))/G_{\delta}^{2,p}(\hat{P})).
\]
The action of \(G_{\delta}^{2,p}(\hat{P})\) on \(\hat{B}\) lifts to a free action on \(\hat{E}\). We denote the respective quotient space by \(\mathcal{E}\). We finally define the smooth section \(\mathcal{F} : \hat{B} \to \mathcal{E}\) by
\[
\mathcal{F} : [(A, \Psi)] \mapsto [\partial_{s}A + d_{A}^{\ast}F_{A} - d_{A}\Psi + \nabla V(A)].
\]

Note that the moduli space defined in (9) is precisely the zero set \(\mathcal{F}^{-1}(0)\).

4. Exponential decay

The aim of this section is to establish exponential decay towards Yang-Mills connections for finite energy solutions of the Yang-Mills gradient flow equation (8). We shall prove the following result.
Theorem 4.1 (Exponential decay). For a solution $A : \mathbb{R} \to \mathcal{A}(P)$ of the Yang-Mills gradient flow equation (8) the following statements are equivalent.

(i) The solution $A$ has finite energy

$$E(A) = \int_{-\infty}^{\infty} \|\partial_s A\|_{L^2(\Sigma)}^2 \, ds.$$ 

(ii) There are positive constants $\lambda$ and $c_\ell$, $\ell \in \mathbb{N}_0$, such that the inequality

$$\|\partial_s A\|_{C^\ell([T, \infty) \times \Sigma)} + \|\partial_s A\|_{C^\ell((-\infty, -T] \times \Sigma)} \leq c_\ell e^{-\lambda T}$$ 

(11)

is satisfied for every $T \geq 1$.

If (i) or (ii) is satisfied, then there exist Yang-Mills connections $A^\pm \in \mathcal{A}(P)$ such that it holds exponential convergence

$$\lim_{s \to \pm \infty} A(s) = A^\pm$$ 

in the $C^\infty(\Sigma)$-topology.

For the proof of Theorem 4.1 we need a number of auxiliary results.

Proposition 4.2. For every $M > 0$, $\rho > 0$, $\kappa > 0$, and $p > 1$ there exists $\varepsilon > 0$ such that the following holds. If $A : [-\rho, \rho] \to \mathcal{A}(P)$ is a solution of (8) with $\|F_{A(0)}\|_{L^2(\Sigma)} \leq M$ and

$$\int_{-\rho}^{\rho} \|\partial_s A\|_{L^2(\Sigma)}^2 \, ds < \varepsilon$$

then there is a Yang-Mills connection $A^\infty \in \mathcal{A}(P)$ such that

$$\|A(0) - A^\infty\|_{W^{1,p}(\Sigma)} + \|\partial_s A(0)\|_{L^\infty(\Sigma)} < \kappa.$$ 

(12)

Proof: Assume by contradiction that this is wrong for some $\rho, M > 0$ and $p > 1$. Then there exists a constant $\kappa > 0$ and a sequence $A^\nu : [-\rho, \rho] \to \mathcal{A}(P)$ of solutions of (8) with $\|F_{A^\nu(0)}\|_{L^2(\Sigma)} \leq M$ for all $\nu$ and

$$\lim_{\nu \to \infty} \int_{-\rho}^{\rho} \|\partial_s A^\nu\|_{L^2(\Sigma)}^2 \, ds = 0,$$ 

(13)

but (12) fails. Hence by Theorem 6.2 there exists a sequence of gauge transformations $g^\nu \in \mathcal{G}(\hat{P}_I)$ such that (after passing to a subsequence) $(g^\nu)^*(A^\nu, 0) = (g^\nu)^*(A^\nu, (g^\nu)^{-1}\partial_s g^\nu)$ converges in $W^{2,p}(I \times \Sigma)$ to a solution $(A^\infty, \Psi^\infty)$ of (8). After modifying the sequence $g^\nu$ we may assume that $\Psi^\infty = 0$ and thus

$$\lim_{\nu \to \infty}(g^\nu)^{-1}\partial_s g^\nu = 0.$$ 

Hence to every sufficiently large $\nu \geq \nu_0$ we can apply a further gauge transformation to put the connection $(g^\nu)^*(A^\nu, 0)$ in temporal gauge (i.e. to achieve that $(g^\nu)^{-1}\partial_s g^\nu$ vanishes), and $(g^\nu)^*A^\nu$ still converges to
A^\infty$. By (8) and (13) it follows that $A^\infty$ is a Yang-Mills connection. It holds that
\[
\lim_{\nu \to \infty} \|A''(0) - (g^\nu)^{-1} A^\infty\|_{W^{1,\infty}(\Sigma)} = \lim_{\nu \to \infty} \|((g^\nu)^* A' - A^\infty)(0)\|_{W^{1,\infty}(\Sigma)} = 0,
\]
\[
\lim_{\nu \to \infty} \|\partial_\nu A''(0)\|_{L^\infty(\Sigma)} = \lim_{\nu \to \infty} \|\partial_\nu ((g^\nu)^* A') (0)\|_{L^\infty(\Sigma)} = \|\partial_\nu A^\infty\|_{L^\infty(\Sigma)} = 0.
\]
Hence the assumption that (12) fails was wrong. This proves the proposition. □

Let $C \in CR^\alpha$ be a critical manifold. For any connection $A \in A(P)$ sufficiently close to $C$ with respect to the $W^{1,\infty}$-topology (this assumption is needed in Proposition 4.4 below) there exists a Yang-Mills connection $A_0 \in C$ such that $\alpha := A - A_0 \in (T_0 C)$. This follows from the local slice Theorem 2.1. We remark that $\alpha \in (\ker H_{A_0})^\perp$ holds by the Morse-Bott condition. Note also that $A$ lies not in the support of any admissible perturbation $V \in Y$, provided it is $L^2$ close enough to $C$, which we assume. Hence for such $A$, $\nabla_\alpha MA^\alpha(A) = \nabla_\alpha MA^\alpha(A) = d_A^* F_A$. We decompose the Yang-Mills gradient at $A$ orthogonally as $d_A^* F_A = \beta_0 + \beta_1$ with $\beta_0 \in \im H_{A_0}$ and $\beta_1 \in \ker H_{A_0}$.

**Proposition 4.3.** For every $\varepsilon > 0$ there exists a constant $\delta > 0$ such that the term $\beta_1$ in the above decomposition satisfies
\[
\|\beta_1\|_{L^2(\Sigma)} \leq \varepsilon \|d_A^* F_A\|_{L^2(\Sigma)},
\]
whenever $\|\alpha\|_{W^{1,\infty}(\Sigma)} < \delta$.

**Proof:** We expand $\beta_0 + \beta_1 = d_A^* F_A = d_{A_0}^* F_{A_0} + \alpha$ as
\[
\beta_0 + \beta_1 = H_{A_0} \alpha + \frac{1}{2} d_{A_0}^* [\alpha \wedge \alpha] - [\ast \alpha \wedge \ast (d_{A_0} \alpha + \frac{1}{2} [\alpha \wedge \alpha])] =: H_{A_0} \alpha + R(\alpha). \quad (14)
\]
Note that there exists a constant $c > 0$ such that
\[
\|R(\alpha)\|_{L^2(\Sigma)} \leq c \|\alpha\|_{W^{1,\infty}(\Sigma)} \|\alpha\|_{L^2(\Sigma)}.
\]
From (14) it follows that $H_{A_0} \alpha \in (\ker H_{A_0})^\perp$ and hence
\[
\|\beta_1\|_{L^2(\Sigma)} \leq \|R(\alpha)\|_{L^2(\Sigma)} \leq c\delta \|\alpha\|_{L^2(\Sigma)}.
\]
Denoting by $\lambda > 0$ the smallest (in absolute value) non-zero eigenvalue of $H_{A_0}$ it furthermore follows that $\|H_{A_0} \alpha\|_{L^2(\Sigma)} \geq \lambda \|\alpha\|_{L^2(\Sigma)}$. It now follows that (we drop subscripts after $\|\cdot\|$)
\[
\|\beta_1\|_{d_A^* F_A} \leq c\delta \|\alpha\|_{H_{A_0} \alpha} + \|R(\alpha)\|_{H_{A_0} \alpha} \leq \frac{c\delta \|\alpha\|}{\lambda} + \frac{c\delta \|R(\alpha)\|}{\lambda} \leq \frac{c\delta}{\lambda} + \frac{c\delta}{\lambda^2} \|R(\alpha)\| \leq \frac{c\delta}{\lambda} + \frac{c\delta}{\lambda^2} e^{-\delta - 1 - \lambda}.
\]
Choose $\delta > 0$ small enough such that $\frac{c\delta}{\lambda} + \frac{c\delta}{\lambda^2} e^{-\delta - 1 - \lambda} < \varepsilon$ is satisfied. The claim then follows. □
Proposition 4.4. Let $\mathcal{C} \subseteq \mathcal{A}(P)$ be a Yang-Mills critical manifold and let $A$ and $A_0$ be connections as described before Proposition 4.3. Then there exists a constant $c(A_0) > 0$ such that the estimate
\[ \|\beta\|_{L^2(\Sigma)} + \|\nabla A\beta\|_{L^2(\Sigma)} \leq c(A_0)\|H_A\beta\|_{L^2(\Sigma)} \tag{15} \]
is satisfied for every $\beta \in (\text{ker } H_{A_0})^\perp$.

Proof: Because $\mathcal{C}$ satisfies is a non-degenerate critical manifold it follows that the restriction of the Yang-Mills Hessian $H_{A_0}$ to $(\text{ker } H_{A_0})^\perp$ is a bijective operator

\[ H_{A_0}|(\text{ker } H_{A_0})^\perp : W^{2,2}(\Sigma, T^*\Sigma \otimes \text{ad}(P)) \cap (\text{ker } H_{A_0})^\perp \rightarrow L^2(\Sigma, T^*\Sigma \otimes \text{ad}(P)) \cap (\text{ker } H_{A_0})^\perp. \]

Thus there holds the estimate
\[ \|\beta\|_{L^2(\Sigma)} + \|\nabla A\beta\|_{L^2(\Sigma)} \leq c(A_0)\|H_{A_0}|(\text{ker } H_{A_0})^\perp\|H_A\beta\|_{L^2(\Sigma)} \]
for a constant $c(A_0) > 0$ and every $\beta \in (\text{ker } H_{A_0})^\perp$. Now the difference $H_A - H_{A_0}$ is the operator
\[ H_{A_0} + \alpha - H_{A_0} = -d_{A_0} * [\alpha \wedge *] + [\alpha \wedge d_{A_0}^* \cdot] - *[\alpha \wedge d_{A_0} \cdot] + d_{A_0}^* [\alpha \wedge \cdot] \]
\[ - [\alpha \wedge *[\alpha \wedge * \cdot]] - *[\alpha \wedge * [\alpha \wedge \cdot]] + *([d_{A_0} \alpha + \frac{1}{2}[\alpha \wedge \alpha]) \wedge \cdot], \]
which converges to $0$ in $\mathcal{L}(W^{1,2}(\Sigma), L^2(\Sigma))$ as $\alpha \rightarrow 0$ in $W^{1,\infty}(\Sigma)$. Thus for $\|\alpha\|_{W^{1,\infty}(\Sigma)}$ sufficiently small the term $\|(H_A - H_{A_0})\beta\|_{L^2(\Sigma)}$ can be absorbed in the left-hand side of (16). This proves the proposition. \hfill $\Box$

Lemma 4.5 ($L^2$ exponential decay of the gradient). Let $s \mapsto A(s)$ with $s \in \mathbb{R}$ be a solution of the Yang-Mills gradient flow equation (8) such that $\lim_{s \rightarrow \pm \infty} \|\partial_s A(s)\|_{L^\infty(\Sigma)} = 0$ and such that for a constant $T > 0$ the following condition is satisfied. There exist Yang-Mills critical manifolds $\mathcal{C}^\pm$ such that the conclusion of Proposition 4.4 applies to all $A(s)$ with $|s| > T$. Then exponential decay $\|\partial_s A\|_{L^2(\Sigma)} \rightarrow 0$ for $s \rightarrow \pm \infty$ holds, i.e. there exists a constant $\lambda > 0$ such that
\[ \|\partial_s A(s)\|_{L^2(\Sigma)} \leq e^{\lambda(s+T)} \|\partial_s A(-T)\|_{L^2(\Sigma)}^2 \tag{17} \]
is satisfied for all $s \leq -T$. An analogue decay estimate holds for $s \geq T$.

Proof: We use Lemma Appendix C.2 to show exponential decay. By the Yang-Mills gradient flow equation (8) it follows the identity
\[ \dot{A} = -\partial_s d_A^* F_A = -d_A^* d_A \dot{A} + *[\dot{A} \wedge *F_A] = -H_A \dot{A}. \]
at every large enough $|s| > T$ such that $\nabla V(A(s)) = 0$. We furthermore calculate

$$\partial_s (H_A \dot{A}) = H_A \dot{A} + d_A^* [\dot{A} \wedge \dot{A}] - \ast \{ \dot{A} \wedge \ast d_A \dot{A} \} + \ast \{ \ast d_A \dot{A} \wedge \dot{A} \}.$$ 

It then follows that

\begin{align*}
\frac{d^2}{ds^2} \frac{1}{2} \| \dot{A} \|^2_{L^2(\Sigma)} &= \frac{d}{ds} (\dot{A}, \dot{A}) \\
&= \| \dot{A} \|^2_{L^2(\Sigma)} - \langle \partial_s (H_A \dot{A}), \dot{A} \rangle \\
&= 2 \| H_A \dot{A} \|^2_{L^2(\Sigma)} - \langle \dot{A}, d_A^* [\dot{A} \wedge \dot{A}] \rangle + \langle \dot{A}, \ast \{ \dot{A} \wedge \ast d_A \dot{A} \} \rangle \\
&\quad + \langle \dot{A}, \ast \{ \dot{A} \wedge \ast d_A \dot{A} \} \rangle \\
&= 2 \| H_A \dot{A} \|^2_{L^2(\Sigma)} - 3 \langle d_A \dot{A}, [\dot{A} \wedge \dot{A}] \rangle. \tag{18}
\end{align*}

We use the orthogonal decomposition of $\dot{A}$ as described before Proposition 4.3 into $\dot{A} = \beta_0 + \beta_1$ where $\beta_0 \in \text{im} H_{A_0}$ and $\beta_1 \in \ker H_{A_0}$. The last term in (18) can then be estimated as

$$\| \langle d_A \dot{A}, [\dot{A} \wedge \dot{A}] \rangle \| \leq \| \dot{A} \|_{L^\infty(\Sigma)} \left( \| d_A \beta_0 \|^2_{L^2(\Sigma)} + \| d_A \beta_1 \|^2_{L^2(\Sigma)} + \| \beta_0 \|^2_{L^2(\Sigma)} + \| \beta_1 \|^2_{L^2(\Sigma)} \right).$$

With $\beta_1$ satisfying $H_{A_0} \beta_1 = 0$, hence $d_{A_0}^* d_{A_0} \beta_1 = - * [ * F_{A_0} \wedge \beta_1 ]$ we find that

$$\| d_A \beta_1 \|^2_{L^2(\Sigma)} \leq 2 \| d_{A_0} \beta_1 \|^2_{L^2(\Sigma)} + 2 \| [ \alpha \wedge \beta_1 ] \|^2_{L^2(\Sigma)} \leq 2 \| \beta_1 \|_{L^2(\Sigma)} \| d_{A_0} \beta_1 \|_{L^2(\Sigma)} + 2 \| [ \alpha \wedge \beta_1 ] \|^2_{L^2(\Sigma)} = \| \beta_1 \|_{L^2(\Sigma)} \| [ * F_{A_0} \wedge \beta_1 ] \|_{L^2(\Sigma)} + 2 \| [ \alpha \wedge \beta_1 ] \|^2_{L^2(\Sigma)} \leq c \| \beta_1 \|^2_{L^2(\Sigma)} \left( \| F_{A_0} \|_{L^\infty(\Sigma)} + \| \alpha \|^2_{L^\infty(\Sigma)} \right). \tag{19}$$

Using Proposition 4.3 and the estimate (15) with constant $c(A_0)$, the right-hand side of (18) can now further be estimated as follows. We put $\delta := 2 - 3c(A_0)^2 \| \beta \|_{L^\infty(\Sigma)} > 0$. Then,

\begin{align*}
\frac{d^2}{ds^2} \frac{1}{2} \| \dot{A} \|^2_{L^2(\Sigma)} &= 2 \| H_A \dot{A} \|^2_{L^2(\Sigma)} - 3 \langle d_A \dot{A}, [\dot{A} \wedge \dot{A}] \rangle \\
&\geq 2 \| H_A \beta_0 \|^2_{L^2(\Sigma)} - 2 \| H_A \beta_1 \|^2_{L^2(\Sigma)} - 3 \| \dot{A} \|_{L^\infty(\Sigma)} \left( \| d_A \beta_0 \|^2_{L^2(\Sigma)} + \| \beta_0 \|^2_{L^2(\Sigma)} \right) \\
&\quad - c \| \dot{A} \|_{L^\infty(\Sigma)} \left( 1 + \| F_{A_0} \|_{L^\infty(\Sigma)} + \| \alpha \|^2_{L^\infty(\Sigma)} \right) \| \beta_1 \|^2_{L^2(\Sigma)} \\
&\geq \delta \| H_A \beta_0 \|^2_{L^2(\Sigma)} - 2 \| H_A \beta_1 \|^2_{L^2(\Sigma)} - c \| \dot{A} \|_{L^\infty(\Sigma)} \left( 1 + \| F_{A_0} \|_{L^\infty(\Sigma)} + \| \alpha \|^2_{L^\infty(\Sigma)} \right) \| \beta_1 \|^2_{L^2(\Sigma)} \\
&\geq \delta \| H_A \beta_0 \|^2_{L^2(\Sigma)} - 2 \| H_A \beta_1 \|^2_{L^2(\Sigma)} - c_1(A_0) \varepsilon^2 \| \dot{A} \|^2_{L^2(\Sigma)} \\
&\geq \delta \| H_A \beta_0 \|^2_{L^2(\Sigma)} - 2 \| H_A - H_{A_0} \|^2_{L^1(\Sigma), L^2(\Sigma)} \| \beta_1 \|^2_{L^1(\Sigma)} \| \beta_1 \|^2_{W^{1,2}(\Sigma)} \\
&\quad - c_1(A_0) \varepsilon^2 \| \dot{A} \|^2_{L^2(\Sigma)}. 
\end{align*}
Thanks to Proposition Appendix B.5 we can bound the term \( \| \beta_1 \|^2_{W^{1, 2}(\Sigma)} \) as
\[
\| \beta_1 \|^2_{W^{1, 2}(\Sigma)} \leq c (\| d A \beta_1 \|^2_{L^2(\Sigma)} + \| d' A \beta_1 \|^2_{L^2(\Sigma)}) + c \| F_A \|_{L^\infty(\Sigma)} \| \beta_1 \|^2_{L^2(\Sigma)}
\leq c \| \beta_1 \|^2_{L^2(\Sigma)} (1 + \| \alpha \|^2_{L^\infty(\Sigma)} + \| F_{A_0} \|^2_{L^\infty(\Sigma)} + \| F_A \|^2_{L^\infty(\Sigma)})
\leq c_2 (A_0) \varepsilon^2 \| \hat{A} \|^2_{L^2(\Sigma)}.
\]
In the second line we used (19) and the assumption \( d' A_0 \beta_1 = 0 \), hence \( d' \beta_1 = -\ast (\alpha \wedge \beta_1) \). The last line is by Proposition 4.3. Let us denote \( K := K(\alpha) := H_A - H_{A_0} \) and \( \| K \| := \| K \|_{C(W^{1, 2}, L^2)} \). It then follows for \( \varepsilon > 0 \) sufficiently small, for the constant \( \delta_1 := \delta c(A_0)^2 > 0 \), and with \( \| \hat{A} \|^2_{L^2(\Sigma)} = \| \beta_0 \|^2_{L^2(\Sigma)} + \| \beta_1 \|^2_{L^2(\Sigma)} \)
that
\[
\frac{d^2}{ds^2} \frac{1}{2} \| \hat{A} \|^2_{L^2(\Sigma)} \geq \delta \| H_A \beta_0 \|^2_{L^2(\Sigma)} - c \varepsilon^2 \| K \|^2 \| \hat{A} \|^2_{L^2(\Sigma)} - c_1 (A_0) \varepsilon^2 \| \hat{A} \|^2_{L^2(\Sigma)}
\geq \delta_1 \| \beta_0 \|^2_{L^2(\Sigma)} - c \varepsilon^2 \| K \|^2 \| \hat{A} \|^2_{L^2(\Sigma)} - c_1 (A_0) \varepsilon^2 \| \hat{A} \|^2_{L^2(\Sigma)}
\geq \delta_1 \| \hat{A} \|^2_{L^2(\Sigma)} - \delta_1 \varepsilon^2 \| \hat{A} \|^2_{L^2(\Sigma)} - c \varepsilon^2 \| K \|^2 \| \hat{A} \|^2_{L^2(\Sigma)} - c_1 (A_0) \varepsilon^2 \| \hat{A} \|^2_{L^2(\Sigma)}
\geq \delta_1 \| \hat{A} \|^2_{L^2(\Sigma)} - \delta_1 c \varepsilon^2 \| \hat{A} \|^2_{L^2(\Sigma)} - c \varepsilon^2 \| K \|^2 \| \hat{A} \|^2_{L^2(\Sigma)} - c_1 (A_0) \varepsilon^2 \| \hat{A} \|^2_{L^2(\Sigma)}
\geq \frac{\delta_1}{2} \| \hat{A} \|^2_{L^2(\Sigma)}.
\]
The exponential decay estimate (17) now follows from Lemma Appendix C.2.

Under the assumptions of Lemma 4.5 we immediately infer \( L^2 \) convergence of solutions \( A : \mathbb{R} \to A(P) \) of (8) to limit connections \( A^\pm \) as \( s \to \pm \infty \). Namely, the limits are given by
\[
A^- := A(-T) - \int_{-T}^\infty \partial_s A(s) \, ds, \quad A^+ := A(T) + \int_T^\infty \partial_s A(s) \, ds, \quad (20)
\]
for fixed \( T > 0 \) large enough. These integrals converge due to the exponential decay of \( \| \partial_s A(s) \|^2_{L^2(\Sigma)} \) as \( s \to \pm \infty \). Let us now turn to the proof of Theorem 4.1.

\textbf{Proof: (Theorem 4.1) (i) \Rightarrow (ii).} With \( A \) being a solution of (8) it follows that
\[
\mathcal{Y}\mathcal{M}^V(A(s_0)) - \mathcal{Y}\mathcal{M}^V(A(s_1)) = \int_{s_0}^{s_1} \| \partial_s A \|^2_{L^2(\Sigma)} \, ds \leq E(A)
\]
for all \( s_0 \leq s_1 \). From the assumption \( E(A) < \infty \) we thus infer the bound \( \mathcal{Y}\mathcal{M}^V(A(s)) \leq M(A) \) for a constant \( M(A) \) independent of \( s \). Again the assumption \( E(A) < \infty \) yields for any \( \rho > 0 \), \( \varepsilon > 0 \) the existence of a number \( T > 0 \) such that
\[
\int_{-\varepsilon - \rho}^{-s_1 - \rho} \| \partial_s A \|^2_{L^2(\Sigma)} \, ds + \int_{s_1 - \rho}^{s_1 + \rho} \| \partial_s A \|^2_{L^2(\Sigma)} \, ds < \varepsilon
\]
holds for every $s \geq T$. Hence $A$ satisfies the assumptions of Proposition 4.2 and thus estimate (12) holds for some constant $\kappa > 0$ which can be chosen arbitrarily small. Now Lemma 4.5 applies and shows that $A(s)$ converges, as $s \to \pm \infty$, to limit connections $A^\pm$ given by (20). It moreover follows $L^2$ exponential decay of $\partial_s A$ with constant $\lambda > 0$, i.e.

$$\|\partial_s A(s)\|_{L^2(\Sigma)}^2 \leq e^{k_s + T}\|\partial_s A(-T)\|_{L^2(\Sigma)}^2$$  \hspace{1cm} (21)$$
holds for all $s \leq -T$, and similarly for all $s \geq T$. Estimate (21) now allows us to prove the asserted forward and backward exponential decay (11). Consider first backward exponential decay. Set $\alpha := \partial_s A$. Then with $d^2_\alpha \alpha = 0$ we find that $\alpha$ satisfies the linearized Yang-Mills gradient flow equation

$$(\partial_s + \Delta_A)\alpha = -* [*F_A \wedge \alpha].$$  \hspace{1cm} (22)$$
As follows from (21), the term on the right-hand side of (22) is contained in $L^2((-\infty, -T] \times \Sigma)$. Standard parabolic estimates (cf. e.g. [15, Theorem 7.13]) apply and yield via induction on $\ell$ the estimate

$$\|\partial_s A\|_{W^{\ell,2}((-\infty, s] \times \Sigma)} \leq c\ell \|\partial_s A\|_{L^2((-\infty, s+\ell] \times \Sigma)}$$  \hspace{1cm} (23)$$
for a constant $c\ell = c\ell(A)$. Now fix $m \geq 0$. We apply to each interval $(s-j, s-j+1)$, $j \in \mathbb{N}$, and for sufficiently large $\ell = \ell(m)$ the Sobolev embedding

$$W^{\ell, 2}((s-j, s-j+1) \times \Sigma) \hookrightarrow C^m((s-j, s-j+1) \times \Sigma)$$
to obtain from (23) and a constant $c_{\ell, m}$ independent of $j$ the bound

$$\|\partial_s A\|_{C^m((-\infty, s] \times \Sigma)} \leq c_{\ell, m} \|\partial_s A\|_{L^2((-\infty, s+\ell] \times \Sigma)}.$$  \hspace{1cm} (24)$$
Finally, integrate estimate (21) over $(-\infty, s+\ell]$ and combine it with (24) to complete the proof of backward exponential decay. Forward exponential decay is obtained in a completely analogous manner. This proves the first implication of Theorem 4.1.

(ii) ⇒ (i). The assumption (11) for $\ell = 0$ and $T = 1$ implies

$$E(A) = \int_{-\infty}^{1} \|\partial_s A\|_{L^2(\Sigma)}^2 ds$$

$$= \int_{-1}^{1} \|\partial_s A\|_{L^2(\Sigma)}^2 ds + \int_{-\infty}^{-1} \|\partial_s A\|_{L^2(\Sigma)}^2 ds + \int_{1}^{\infty} \|\partial_s A\|_{L^2(\Sigma)}^2 ds$$

$$\leq C + |\Sigma| \int_{-\infty}^{-1} \frac{c_0^2 e^{2\lambda s}}{\lambda} ds + |\Sigma| \int_{1}^{\infty} c_0^2 e^{-2\lambda s} ds$$

$$= C + \frac{|\Sigma| c_0^2 e^{-2\lambda}}{\lambda}$$

and shows (i). It remains to prove exponential convergence $A(s) \to A^\pm$ in $C^\ell(\Sigma)$ as $s \to \pm \infty$. By (ii) we have that $\|\partial_s A(s)\|_{C^\ell(\Sigma)}$ converges to zero exponentially for every $\ell \geq 0$ as $s \to \pm \infty$. Hence

$$\|A(s_0) - A(s_1)\|_{C^\ell(\Sigma)} \leq \int_{s_0}^{s_1} \|\partial_s A\|_{C^\ell(\Sigma)} ds \leq \int_{s_0}^{s_1} c_\ell e^{\lambda s} ds \leq \frac{c_\ell e^{\lambda s_1}}{\lambda}$$
holds for constants \( c \), \( \lambda > 0 \) and all \( s_0 \leq s_1 \leq -1 \). Letting \( s_0 \to -\infty \) we obtain backward exponential decay in \( C^c(\Sigma) \). Forward exponential decay follows similarly. This completes the proof of Theorem 4.1. \( \square \)

5. Fredholm theory

5.1. Yang-Mills Hessian

We let \( \mathcal{H}_A \) denote the augmented Yang-Mills Hessian defined by

\[
\mathcal{H}_A := \left( \begin{array}{cc} d_A^* d_A + \ast [d_A \wedge \cdot] & -d_A \\ -d_A^* & 0 \end{array} \right) : \Omega^1(\Sigma, \text{ad}(P)) \oplus \Omega^0(\Sigma, \text{ad}(P)) \to \Omega^1(\Sigma, \text{ad}(P)) \oplus \Omega^0(\Sigma, \text{ad}(P)).
\]

In order to find a domain which makes the subsequent Fredholm theory work, we fix a smooth connection \( A \in \mathcal{A}(P) \) and decompose the space \( \Omega^1(\Sigma, \text{ad}(P)) \) of smooth \( \text{ad}(P) \)-valued 1-forms as the \( L^2(\Sigma) \) orthogonal sum

\[
\Omega^1(\Sigma, \text{ad}(P)) = \ker (d_A^* : \Omega^1(\Sigma, \text{ad}(P)) \to \Omega^0(\Sigma, \text{ad}(P))) \\
\oplus \text{im} (d_A : \Omega^0(\Sigma, \text{ad}(P)) \to \Omega^1(\Sigma, \text{ad}(P))).
\]

Let \( W^{2,p}_0 \) and \( W^{1,p}_1 \) denote the completions of the first component, respectively of the second component with respect to the Sobolev \( (k,p) \) norm \((k = 1,2)\). We set \( W^{2,p}_A(\Sigma) := W^{2,p}_0 \oplus W^{1,p}_1 \) and endow this space with the sum norm. Note that the this norm depends on the connection \( A \). For \( p > 1 \) we consider the operator

\[
\mathcal{H}_A : W^{2,p}_A(\Sigma) \oplus W^{1,p}(\Sigma, \text{ad}(P)) \to L^p(\Sigma, T^* \Sigma \otimes \text{ad}(P)) \oplus L^p(\Sigma, \text{ad}(P)).
\]

In the case \( p = 2 \) this is a densely defined symmetric operator on the Hilbert space \( L^2(\Sigma, T^* \Sigma \otimes \text{ad}(P)) \oplus L^2(\Sigma, T^* \Sigma) \) with domain

\[
\text{dom} \mathcal{H}_A := W^{2,2}_A(\Sigma) \oplus W^{1,2}(\Sigma, \text{ad}(P)). \tag{25}
\]

We show in Proposition 5.1 below that it is self-adjoint. For the further discussion of the operator \( \mathcal{D}_A \) it will be convenient to also decompose each \( \beta \in \text{im} \mathcal{H}_A \) as \( \beta = \beta_0 + \beta_1 \), where \( d^*_A \beta_0 = 0 \) and \( \beta_1 = d_A \omega \) holds for some 0-form \( \omega \). A short calculation shows that for \( \alpha = \alpha_0 + \alpha_1 = \alpha_0 + d_A \phi \) (with \( d^*_A \alpha_0 = 0 \)) this decomposition is given by

\[
\mathcal{H}_A \alpha = \beta_0 + d_A \omega,
\]

where \( \omega \) is a solution of

\[
\Delta_A \omega = \ast [d_A \ast F_A \wedge \alpha]. \tag{26}
\]
As $\Delta_A$ might not be injective due to the presence of $\Delta_A$-harmonic 0-forms, the solution $\omega$ of (26) need not be unique. This ambiguity however is not relevant, as only $d_A\omega$ enters the definition of $\beta_0$ and $\beta_1$. With respect to the above decomposition of the space of 1-forms the augmented Hessian $\mathcal{H}_A$ takes the form

$$
\mathcal{H}_A \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta_A \alpha_0 + \ast [\ast F_A \wedge \alpha_0] + [d_A^* F_A \wedge \varphi] - d_A \omega \\ -d_A \psi + d_A \omega \\ -d_A^* \alpha_1 \end{pmatrix}, \quad (27)
$$

with $\alpha_1 = d_A \varphi$ and $\omega$ a solution of (26). Note that the first line of (27) does not depend on the choice of $\varphi$ used to satisfy the condition $\alpha_1 = d_A \varphi$ because

$$
[d_A^* F_A \wedge \varphi] = - \ast d_A \ast [F_A \wedge \varphi] + \ast [F_A \wedge d_A \varphi] = 0
$$

holds for all $\varphi \in \ker d_A$.

**Proposition 5.1.** For every $A \in \mathcal{A}(P)$, the operator $\mathcal{H}_A$ with domain $\text{dom} \, \mathcal{H}_A$ as defined in (25) is self-adjoint. It satisfies for all $(\alpha, \psi) \in \text{dom} \, \mathcal{H}_A$ and $p > 1$ the elliptic estimate

$$
\|\alpha\|_{W^1,p(\Sigma)} + \|\psi\|_{W^{1,p}(\Sigma)} \leq c(\|\mathcal{H}_A(\alpha, \psi)\|_{L^p(\Sigma)} + \|\alpha, \psi\|_{L^p(\Sigma)}) \quad (28)
$$

with constant $c = c(A, p)$. If $A$ is a Yang-Mills connection, then the number of negative eigenvalues (counted with multiplicities) of $\mathcal{H}_A$ equals the Morse index of the Yang-Mills Hessian $\mathcal{H}_A \mathcal{YM}$ as given by (3).

**Proof:** We show estimate (28). All norms are with respect to the domain $\Sigma$, so we drop this in our notation. As remarked before, we may assume that $\varphi \in (\ker d_A)^\perp$. Hence the elliptic equation $\Delta_A \varphi = d_A^* \alpha_1$ holds and implies the estimate

$$
\|\varphi\|_{W^{1,p}} \leq c(A, p)\|d_A^* \alpha_1\|_{W^{-1,p}} \leq c(A, p)\|\alpha_1\|_{L^p}. \quad (29)
$$

Let $(\beta_0, \beta_1, \gamma) = \mathcal{H}_A(\alpha_0, \alpha_1, \psi)$ and assume that $(\beta_0 + \beta_1, \gamma) \in L^p(\Sigma, T^* \Sigma \otimes \text{ad}(P)) \oplus L^p(\Sigma, \text{ad}(P))$. Then by ellipticity of the operator $\Delta_A$ on $\Omega^*(\Sigma, \text{ad}(P))$ we obtain from (26), (29) and the first line of (27) the estimate

$$
\|\alpha_0\|_{W^{2,p}} \leq c(A, p)(\|\alpha_0\|_{L^p} + \|\varphi\|_{L^p} + \|\beta_0\|_{L^p} + \|\omega\|_{W^{1,p}}) \leq c(A, p)(\|\alpha\|_{L^p} + \|\beta_0\|_{L^p}).
$$

Applying $d_A^*$ to the second equation in (27) a similar elliptic estimate shows that

$$
\|\psi\|_{W^{1,p}} \leq c(A, p)(\|\psi\|_{W^{-1,p}} + \|d_A^* \beta_1\|_{W^{-1,p}} + \|\Delta_A \omega\|_{W^{-1,p}}) \leq c(A, p)(\|\psi\|_{L^p} + \|\beta_1\|_{L^p} + \|\alpha\|_{L^p})
$$

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We rewrite the third equation in (27) as $-\Delta A \varphi = \gamma$ and obtain again by elliptic regularity

$$\|\alpha_1\|_{W^{1,p}} \leq c(A,p)\|\varphi\|_{W^{2,p}} \leq c(A,p)\|\gamma\|_{L^p}.$$ 

Putting the previous three estimates together, (28) follows and implies self-adjointness in the case $p = 2$. To prove the assertion on the index, let $A$ be a Yang-Mills connection and $(\alpha, \psi)^T$ be an eigenvector of $H_A$ with corresponding eigenvalue $\lambda < 0$. Let $\alpha = \alpha_0 + \alpha_1$ be as above the Hodge decomposition of $\alpha$ with $d_A^* \alpha_0 = 0$ and $\alpha_1 = d_A \varphi$. Then the eigenvalue equation for $H_A$ together with the first line of (27) yields $H_A \alpha_0 = \lambda \alpha_0$. This uses that $d_A \omega = 0$ and $d_A^* F_A = 0$ as $A$ is assumed to be Yang-Mills. Hence $\lambda$ is a negative eigenvalue of $H_A$. Conversely, if the eigenvalue equation $H_A \alpha = \lambda \alpha$ is satisfied for some $\lambda < 0$ and $\alpha = \alpha_0 + \alpha_1$, then necessarily $\alpha_1 = 0$ because $\alpha_1 \in \ker H_A$. Thus $H_A \alpha_0 = \lambda \alpha_0$, and this equation implies that $\lambda$ is also an eigenvalue of $H_A$ with eigenvector $(\alpha_0, 0)^T$. 

5.2. Linearized operator

We next discuss the linearization of the perturbed Yang-Mills gradient flow equation (8). Since any solution $(A, \Psi)$ of the Yang-Mills gradient flow is gauge equivalent under $G_{2,p} \delta (\hat{P})$ to a solution satisfying $\Psi \equiv 0$, it suffices to consider the linearization along such trajectories only. We thus fix a solution $(A, \Psi) = (A, 0)$ of (8) and define for $p > 1$ the Banach spaces

$$Z_A^{k,p} := (W_\delta^{1,p}(\mathbb{R}, L^p(\Sigma, T^* \Sigma \otimes \text{ad}(P))) \cap L_\delta^p(\mathbb{R}, W_A^{2,p}(\Sigma))) \oplus W_\delta^{1,p}(\mathbb{R} \times \Sigma, \text{ad}(P))$$

and

$$L_A^{k,p} := L_\delta^p(\mathbb{R} \times \Sigma, T^* \Sigma \otimes \text{ad}(P)) \oplus L_\delta^p(\mathbb{R} \times \Sigma, \text{ad}(P)).$$

**Definition 5.2.** The horizontal differential at $(A, 0)$ of the section $F$ as in (10) (assuming $V = 0$) is the linear operator

$$D_A = \frac{d}{ds} + H_A : Z_A^{k,p} \rightarrow L_A^{k,p},$$

and the equation

$$D_A \left( \begin{array}{c} \alpha \\ \psi \end{array} \right) = 0$$

is called **linearized Yang-Mills gradient flow equation**.

We show in Section 5.3 that $D_A$ is a Fredholm operator and determine its index.
Remark 5.3. (i) From the definition of $\mathcal{F}$ as a section $\mathcal{F} : \mathcal{B} \to \mathcal{E}$ (cf. (10)) it follows that its linearization $d\mathcal{F}(A)$ acts on the space of pairs $(\alpha, \psi)$ where $\alpha(s)$ converges exponentially to some $\alpha^{\pm} \in T_{A^{\pm}} \mathcal{C}^{\pm}$ as $s \to \pm \infty$, i.e. on the space $\mathcal{Z}^{\delta, p}_{A} \oplus \mathbb{R}^{\dim \mathcal{C}^{-}} \oplus \mathbb{R}^{\dim \mathcal{C}^{+}}$. This is in contrast to the definition of $\mathcal{F}$ given in 5.2. However, it is easy to see that $d\mathcal{F}(A)$ is Fredholm if and only if this property holds for $\mathcal{D}_{A}$, and that the Fredholm indices are related via the formula

$$\text{ind } d\mathcal{F}(A) = \text{ind } \mathcal{D}_{A} + \dim \mathcal{C}^{-} + \dim \mathcal{C}^{+}.$$  

To see this, we view $d\mathcal{F}(A)$ as a compact perturbation of the operator $\mathcal{D}_{A}$, extended trivially to $\mathcal{Z}^{\delta, p}_{A} \oplus \mathbb{R}^{\dim \mathcal{C}^{-}} \oplus \mathbb{R}^{\dim \mathcal{C}^{+}}$.

(ii) The operator $\mathcal{D}_{A}$ arises as the linearization of the unperturbed Yang-Mills gradient flow equation (8). The Fredholm theory for general perturbations $V \in Y$ can be reduced to the unperturbed case because the terms involving $V$ contribute only compact perturbations to the operator $\mathcal{D}_{A}$.

5.3. Fredholm theorem

We fix a pair $\hat{\mathcal{C}}^{\pm} \in \mathcal{C} \mathcal{R}^{a}$ of critical manifolds and denote $\mathcal{C}^{\pm} = \hat{\mathcal{C}}^{\pm} / \mathcal{G}_{0}(P)$. We define the constant $\delta_{0}(\mathcal{C}^{-}, \mathcal{C}^{+})$ to be the infimum of the set

$$\{ |\lambda| \in \mathbb{R} \mid \lambda \neq 0 \text{ and } \lambda \text{ is eigenvalue of } \mathcal{H}_{A} \text{ for some } A \in \hat{\mathcal{C}}^{-} \cup \hat{\mathcal{C}}^{+} \}.$$  

We remark that $\delta_{0}(\mathcal{C}^{-}, \mathcal{C}^{+})$ is positive as follows from compactness of the manifolds $\mathcal{C}^{\pm}$. In the following we fix $p > 1$ and $0 < \delta < \delta_{0}(\mathcal{C}^{-}, \mathcal{C}^{+})$.

Theorem 5.4 (Fredholm theorem). Let $[(A, 0)] \in \mathcal{F}^{-1}(0)$ be a zero of the section $\mathcal{F}$ as given by (10) such that the asymptotic conditions

$$\lim_{s \to \pm \infty} A(s) = A^{\pm}$$

are satisfied in the $C^{\infty}(\Sigma)$ topology for Yang-Mills connections $A^{\pm} \in \hat{\mathcal{C}}^{\pm}$. Then the operator $\mathcal{D}_{A} = \frac{d}{ds} + \mathcal{H}_{A} : Z^{\delta, p}_{A} \to L^{\delta, p}$ associated with $A$ is a Fredholm operator of index

$$\text{ind } \mathcal{D}_{A} = \text{ind } H_{A-} \mathcal{Y} \mathcal{M} - \text{ind } H_{A+} \mathcal{Y} \mathcal{M} + \dim \mathcal{C}^{-}. \quad (30)$$

A proof of this theorem will be given below.

Weighted theory

Because the Hessians $\mathcal{H}_{A^{\pm}}$ will in general (i.e. if $\dim \mathcal{C}^{\pm} \neq 0$) have non-trivial zero eigenspaces, we cannot apply directly standard theorems on the spectral flow to prove Theorem 5.4. As an intermediate step we therefore use the Banach space isomorphisms

$$\nu_{1} : Z^{\delta, p}_{A} \to Z^{0, p}_{A} =: Z^{p}_{A} \quad \text{and} \quad \nu_{2} : L^{\delta, p} \to L^{0, p} =: L^{p}$$

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given by multiplication with the weight function $e^{\delta \beta(s)}$, where $\beta$ denotes the cut-off function introduced at the beginning of Section 3.2. Then the assertion of Theorem 5.4 is equivalent to the analogous one for the operator

$$D^\delta_A := \nu_2 \circ D_A \circ \nu_1^{-1} : Z^p_A \to L^p,$$

which we shall prove instead. Note that the operator $D^\delta_A$ takes the form

$$D^\delta_A = \frac{d}{ds} + \mathcal{H}_A + (\beta + \beta's)\delta,$$

and hence, by our choice of $\delta$, the operator family $s \mapsto \mathcal{H}_A + (\beta + \beta's)\delta$ converges to the invertible operators $\mathcal{H}_{A(s)} \pm \delta$ as $s \to \pm \infty$. Thus the upward spectral flow of this operator family is given by the right-hand side of (30).

**Case $p = 2$**

In this subsection we show Theorem 5.4 in the case $p = 2$, where it follows from well-known results on the spectral flow for families $B(s) : \text{dom } B(s) \to H$, $s \in \mathbb{R}$, of self-adjoint operators in Hilbert space $H$, cf. [21]. Since the operators we are concerned with have time-varying domains, we need an extension of this theory as outlined in [24, Appendix A]. The case of general Sobolev exponents $p > 1$ will afterwards be reduced to the Hilbert space case.

Set $H := L^2(\Sigma, T^*\Sigma \otimes \text{ad}(P))$. In the following it will be convenient to use the time-varying Hodge decomposition

$$H = X_0(s) \oplus X_1(s)$$

$$:= \{ \alpha \mid d^{A(s)}_\alpha \alpha = 0 \} \oplus \{ \alpha = d_{A(s)} \varphi \text{ for some } \varphi \in \Omega^0(\Sigma, \text{ad}(P)) \}$$

with respect to the connection $A(s)$. Recall from (25) that the domain of the operator $\mathcal{H}_{A(s)}$ is given by

$$\text{dom } \mathcal{H}_{A(s)} = W^{2,2}_A(\Sigma) \oplus W^{1,2}(\Sigma, \text{ad}(P)) =: W(s) \oplus W^{1,2}(\Sigma, \text{ad}(P)).$$

We fix $s_0 \in \mathbb{R}$ and set $A_0 := A(s_0)$. In the following we let $\beta(s) := A(s) - A_0$. Let $H = X_0 \oplus X_1$ be the Hodge decomposition corresponding to $A_0$ and denote $W_0 := W(s_0)$. For $s \in \mathbb{R}$ sufficiently close to $s_0$ we define the map $Q(s) : H \to H$ as follows. Let $\alpha \in H$ be decomposed as $\alpha = \alpha_0 + \alpha_1 \in X_0 \oplus X_1$. Then set

$$Q(s)\alpha := \text{pr}_{X_0(s)} \alpha_0 + \text{pr}_{X_1(s)} \alpha_1,$$

where $\text{pr}_{X_i(s)}$ denotes orthogonal projection onto the subspace $X_i(s)$, $i = 1, 2$. A short calculation shows that

$$Q(s)\alpha = \alpha_0 + d_{A(s)} \tau,$$

where $\tau$ solves the elliptic equation

$$\Delta_{A(s)} \tau = \Delta_{A_0} \varphi_0 + \ast [\beta(s) \wedge \ast (\alpha_0 - \alpha_1)],$$

with $\varphi_0$ being any solution of $d_{A_0} \varphi_0 = \alpha_1$. A solution $\tau$ of (33) exists and is unique up to adding an element of $\ker d_{A(s)}$. 

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Lemma 5.5. There exists \( \varepsilon = \varepsilon(A_0) > 0 \) such that the map \( Q(s) : H \to H \) has the following properties for every \( s \in (s_0 - \varepsilon, s_0 + \varepsilon) \).

(i) \( Q(s) \) is a Hilbert space isomorphism.

(ii) \( Q(s) \) preserves the Hodge decomposition of \( H \), i.e. it holds that \( Q(s)X_0 = X_0(s) \) and \( Q(s)X_1 = X_1(s) \).

(iii) The restriction of \( Q(s) \) to \( W_0 \) yields an isomorphism \( Q(s) : W_0 \to W(s) \).

Proof: Note that for \( \beta = 0 \) the map \( Q \) defined in (32) is the identity map on \( H \). Thus \( Q(s) \) is bijective for all \( s \) with \( \beta(s) \) sufficiently small.\(^1\) To show (ii) we first observe that \( Q(s)X_\iota \subseteq X_\iota(s) \), \( i = 1, 2 \), holds by definition of \( Q(s) \) in (31). Now let \( \alpha \in H \) be given and assume that \( \hat{\alpha} := Q(s)\alpha \) satisfies \( d_{A(s)}^\alpha \hat{\alpha} = 0 \). It then follows that

\[
0 = d_{A(s)}^\alpha \alpha_0 + \Delta_{A(s)}^\tau
= -\ast [\beta(s) \wedge \ast \alpha_0 + \Delta_{A_\iota} \varphi_0] + \ast [\beta(s) \wedge (\ast \alpha_0 - \ast d_{A_\iota} \varphi_0)]
= \Delta_{A_\iota} \psi_0 - \ast [\beta(s) \wedge \ast d_{A_\iota} \psi_0].
\]

Now the kernel of the operator \( \varphi \mapsto \Delta_{A_\iota} \varphi - \ast [\beta(s) \wedge \ast d_{A_\iota} \varphi] \) contains \( \ker d_{A(s)} \) and for \( \|\beta(s)\|_{C^0(\Sigma)} \) sufficiently small it is not larger. It thus follows that \( \alpha_1 = d_{A_\iota} \varphi_0 = 0 \) and hence \( \alpha = \alpha_0 \). This shows surjectivity of the map \( Q(s) : X_0 \to X_0(s) \). It follows similarly that also \( Q(s) : X_1 \to X_1(s) \) is surjective and completes the proof of (ii). To prove (iii) we introduce the notation \( X_{0,2}^\alpha(s) := \text{pr}_0(W(s)) \) (with \( \text{pr}_0 \) denoting projection onto the first summand of \( X_0(s) \oplus X_1(s) \)) and \( X_{0,2}^\alpha := X_{0,2}^\alpha(s_0) \). We have to verify that \( Q(s) \) maps the space \( X_{0,2}^\alpha \) bijectively to \( X_{0,2}^\alpha(s) \). Thus let \( \alpha \in X_{0,2}^\alpha \). Then \( \alpha_1 = d_{A_\iota} \varphi_0 = 0 \) and \( \tau \in W^{4,2}(\Sigma, \text{ad}(P)) \) as follows from (33) by elliptic regularity. Hence \( Q(s)\alpha = \alpha + d_{A(s)} \tau \in X_{0,2}^\alpha(s) \), as claimed. The opposite inclusion \( Q^{-1}(s)X_{0,2}^\alpha(s) \subseteq X_{0,2}^\alpha \) follows similarly. \( \square \)

Proof: (Theorem 5.4 in the case \( p = 2 \)) Lemma 5.5 shows that the disjoint union \( \bigsqcup_{s \in \mathbb{R}} W(s) \) is a locally trivial Hilbert space subbundle of \( \mathbb{R} \times H \) in the sense of [24, Appendix A]. Moreover, for \( s \to \pm \infty \) there holds convergence \( A(s) \to A^{\pm} \) in \( C^\infty(\Sigma) \). Hence the operators \( Q(s) \) can be chosen near the ends in such a way that \( Q(s) \to Q^\pm \) in \( \mathcal{L}(H) \) as \( s \to \pm \infty \) for appropriate Hilbert space isomorphisms \( Q^\pm : H \to H \). From this, one easily checks that the operators

\[
(Q(s) \oplus \text{id})^{-1} \circ \mathcal{H}_{A(s)} \circ (Q(s) \oplus \text{id}) : W_0 \oplus W^{1,2}(\Sigma, \text{ad}(P)) \to H \oplus L^2(\Sigma, \text{ad}(P))
\]

converge in \( \mathcal{L}(W_0, H) \), as \( s \to \pm \infty \), to the invertible operators \( (Q^\pm \oplus \text{id})^{-1} \circ \mathcal{H}_{A^{\pm}} \circ (Q^\pm \oplus \text{id}) \). Now Theorem A.4 of [24] applies and shows that the operator

\[\text{One easily checks that } \|Q(s) - Q(s_0)\|_{\mathcal{L}(H)} \leq c\|\beta(s)\|_{C^1(\Sigma)} \text{ which converges to 0 as } s \to s_0.\]
$D_A^\delta$ is Fredholm with index equal to the upward spectral flow of the operator family $s \mapsto H_A + (\beta + \beta')\delta$. As argued in Subsection 5.3, this spectral flow equals the right-hand side of (30). This proves the theorem in the case $p = 2$. □

Case $1 < p < \infty$

The proof of Theorem 5.4 in this case reduces by standard arguments to the case $p = 2$.

**Proof:** (Theorem 5.4 in the case $1 < p < \infty$) We outline the proof. Full details can be found in [29]. Combining via a standard cut-off function argument the estimate (C.2) with bijectivity of the operator $D_A$ for stationary paths $A(s) \equiv A^+ = A^-$ we obtain the estimate

$$\norm{\xi}_{L^p} \leq c(A) \left( \norm{D_A \xi}_{L^p} + \norm{\xi}_{L^p(I \times \Sigma)} \right)$$

for a constant $c(A)$ and some compact interval $I \subseteq \mathbb{R}$. Hence it follows from the abstract closed range lemma (cf. e.g. [29, 33]) that the operator $D_A$ has finite-dimensional kernel and closed range. Similarly, one can show that $\text{coker} \, D_A$ is also finite-dimensional, and that the dimensions of the kernel and cokernel do not depend on $p$. This proves Theorem 5.4 in the general case. □

6. Compactness

Throughout this section we identify the pair

$$(A, \Psi) \in C^\infty(\mathbb{R}, \mathcal{A}(P)) \times C^\infty(\mathbb{R}, \Omega^0(\Sigma, \text{ad}(P)))$$

with the connection $A = A + \Psi \, ds$ over the 3-dimensional manifold $\mathbb{R} \times \Sigma$. As such, its curvature is given by

$$F_A = F_A + (d_A \Psi - \partial_s A) \wedge ds.$$  (34)

For an interval $I \subseteq \mathbb{R}$ let $\hat{P}_I := I \times \mathbb{R}$ denote the trivial extension of the principle $G$-bundle $P$ to the base manifold $I \times \Sigma$, and denote as before $\hat{P} := \hat{P}_{\mathbb{R}}$. We use the symbols $\hat{\ast}, \hat{d}_A, \text{etc.}$ for the Hodge and differential operators acting on $\Omega^*(\mathbb{R} \times \Sigma, \text{ad}(\hat{P}_I))$. In particular, $\hat{d}_A$ and $\hat{d}_A^\ast$ act on 1-forms $\alpha + \psi \, ds$ as

$$\hat{d}_A(\alpha + \psi \, ds) = d_A \alpha + (d_A \psi + \partial_s \alpha - [\Psi, \alpha]) \wedge ds, \quad (35)$$

$$\hat{d}_A^\ast(\alpha + \psi \, ds) = d_A^\ast \alpha - \partial_s \psi - [\Psi, \psi]. \quad (36)$$

The Laplace operator $\hat{\Delta}_A$ on 0-forms $\psi \in \Omega^0(\mathbb{R} \times \Sigma, \text{ad}(\hat{P}_I))$ is given by

$$\hat{\Delta}_A \psi = \hat{d}_A \hat{d}_A^\ast \psi = (\Delta_A - \partial_s^2) \psi - \partial_s(\Psi, \psi) - [\Psi, \partial_s \psi + [\Psi, \psi]]. \quad (37)$$

From (36) it follows that the connection $A = A + \Psi \, ds$ is in local slice with respect to the reference connection $A_0 = A_0 + \Psi_0 \, ds$ if it satisfies

$$\hat{d}_{A_0}(A - A_0) = d_A^\ast(A - A_0) - \partial_s(\Psi - \Psi_0) - [\Psi_0, \Psi] = 0. \quad (38)$$
Remark 6.1. In addition to the space $W^{1,2;p}(I \times \Sigma)$ defined in Section 2.1, we shall in the following also use the analogously defined parabolic Sobolev spaces

$$W^{0,1;p}(I \times \Sigma) := L^p(I, W^{1,p}(\Sigma)),$$

$$W^{1,3;p}(I \times \Sigma) := L^p(I, W^{3,p}(\Sigma)) \cap W^{1,p}(I, L^p(\Sigma)),$$

cf. also [14, Chap. 1.3, 2.2].

The aim of this section is to prove the following compactness theorem. In the following, $\mathcal{V} \in \mathcal{Y}$ denotes an arbitrary but fixed regular perturbation.

**Theorem 6.2 (Compactness).** Let $A^\nu = A^\nu + \Psi^\nu ds$, $\nu \in \mathbb{N}$, be a sequence of solutions to the perturbed Yang-Mills gradient flow equation

$$\partial_s A + d_A^* F_A - d_A \Psi + \nabla \mathcal{V}(A) = 0. \quad (39)$$

Assume there exist critical manifolds $C^\pm \in \mathcal{C}R$ such that every $A^\nu$ is a connecting trajectory between $C^-$ and $C^+$. Then for every $1 < p < \infty$ and every compact interval $I \subseteq \mathbb{R}$ there exists a sequence $g^\nu \in \mathcal{G}(P_I)$ of gauge transformations such that a subsequence of the gauge transformed sequence $(g^\nu)^* A^\nu$ converges weakly in $W^{2,p}(I \times \Sigma)$ to a solution $A^\ast$ of (39).

We start with the following weaker statement.

**Theorem 6.3.** Let $A^\nu = A^\nu + \Psi^\nu ds$, $\nu \in \mathbb{N}$, be a sequence of solutions of (39). Assume there exist critical manifolds $C^\pm \in \mathcal{C}R$ such that every $A^\nu$ is a connecting trajectory between $C^-$ and $C^+$. Then for every $1 < p < \infty$ and every compact interval $I \subseteq \mathbb{R}$ there exists a constant $C(I, p)$, a smooth connection $A^\infty = A^\infty + \Psi^\infty \wedge ds \in \mathcal{A}(P_I)$, and a sequence $(g^\nu) \subseteq \mathcal{G}(P_I)$ of gauge transformations such that (after passing to a subsequence) the difference

$$\beta^\nu := (g^\nu)^* A^\nu - A^\infty, \quad \psi^\nu := (g^\nu)^* \Psi^\nu - \Psi^\infty$$

satisfies the uniform bound

$$\|\beta^\nu\|_{W^{1,2;p}(I \times \Sigma)} + \|\psi^\nu\|_{W^{2,p}(I \times \Sigma)} \leq C(I, p)$$

for all $\nu \in \mathbb{N}$.

**Proof:** The proof, which we divide into three steps, is based on Uhlenbeck’s weak compactness theorem, cf. the exposition [32] for details.

**Step 1.** Let $1 < p < 4$. There exists a constant $C(p)$ such that the curvature bound $\|F_{A^\nu}\|_{L^p(I \times \Sigma)} \leq C(p)$ is satisfied for all $\nu \in \mathbb{N}$.

Since the estimate is invariant under gauge transformations it suffices to prove it for $\Psi^\nu = 0$. Then, using equations (34) and (39), the curvature is given by

$$F_{A^\nu} = F_A + (d_A^* F_A + \nabla \mathcal{V}(A^\nu)) ds.$$
Uniform $L^p$ bounds for the terms $F_{A^\nu}$ and $d'_{A^\nu}F_{A^\nu}$ hold by Lemmata Appendix B.7 and Appendix B.11. With $V = \sum_{\ell=1}^{\infty} \lambda_t \nu_\ell \in Y$, a uniform estimate for $\nabla V(A^\nu)$ is provided by condition (iii) in Section 2.2 from which it follows that

$$\| \sum_{\ell=1}^{\infty} \lambda_\ell \nabla V_\ell (A^\nu) \|_{L^p(I \times \Sigma)} \leq \sum_{\ell=1}^{\infty} |\lambda_\ell| \cdot \| \nabla V_\ell (A^\nu) \|_{L^p(I \times \Sigma)}$$

$$\leq \sum_{\ell=1}^{\infty} |\lambda_\ell| \cdot \left( \int_I C_{2p}^p \left( 1 + \| F_{A^\nu(s)} \|_{L^4(\Sigma)} \right)^p ds \right)^{\frac{1}{p}}$$

$$= \sum_{\ell=1}^{\infty} C_\ell |\lambda_\ell| \cdot |I| + \sum_{\ell=1}^{\infty} C_\ell |\lambda_\ell| \cdot \| F_{A^\nu} \|_{L^p(I,L^4(\Sigma))}$$

$$= \left( |I| + \| F_{A^\nu} \|_{L^p(I,L^4(\Sigma))} \right) \| V \|.$$

Again by Lemma Appendix B.7, the term $\| F_{A^\nu} \|_{L^p(I,L^4(\Sigma))}$ is uniformly bounded.

**Step 2.** Let $3 < p < 4$ and choose $\varepsilon > 0$. There exists a sequence $g^\nu \in G_{2p}(\hat{P}_1)$ of gauge transformations and a smooth reference connection $A^\infty = A^\infty + \Psi^\infty ds$ such that (up to extraction of a subsequence) the sequence $(g^\nu)^* A^\nu$ satisfies the following three conditions.

(i) Each connection $(g^\nu)^* A^\nu$ is in local slice with respect to $A^\infty$.

(ii) The difference $\beta^\nu + \psi^\nu ds := (g^\nu)^* A^\nu - A^\infty$ is uniformly bounded in $W^{1,p}(I \times \Sigma)$.

(iii) The sequence $\beta^\nu + \psi^\nu ds$ satisfies the uniform bound

$$\| \beta^\nu \|_{C^0(I \times \Sigma)} + \| \psi^\nu \|_{C^0(I \times \Sigma)} < \varepsilon.$$

The sequence $A^\nu$ satisfies a uniform $L^p$ curvature bound by Step 1. Hence Uhlenbeck’s weak compactness theorem (cf. [32, Theorem 7.1]) yields a sequence $g^\nu \in G_{2p}(\hat{P}_1)$ of gauge transformations such that a subsequence of $(g^\nu)^* A^\nu$ converges weakly in $W^{1,p}(I \times \Sigma)$ to a limit connection $A' = A' + \Psi' ds$. This sequence is in particular bounded in $W^{1,p}(I \times \Sigma)$ and contains (by compactness of the embedding $W^{1,p}(I \times \Sigma) \hookrightarrow C^0(I \times \Sigma)$ for $p > 3$) a subsequence which converges in $C^0(I \times \Sigma)$ to $A'$. We label this subsequence again by $\nu$. It hence follows from the local slice theorem (cf. [32, Theorem 8.1]) that for every large enough $\nu$, the connection $(g^\nu)^* A^\nu$ can be put in local slice with respect to any smooth reference connection $A^\infty = A^\infty + \Psi^\infty ds$ sufficiently close in $W^{1,p}(I \times \Sigma)$ to $A'$. Therefore condition (i) is satisfied. Moreover, the local slice theorem asserts that this can be done preserving the uniform bound in $W^{1,p}(I \times \Sigma)$ and the uniform bound (with constant $\varepsilon$) in $C^0(I \times \Sigma)$. Thus also conditions (ii) and (iii) are satisfied.

**Step 3.** Proof of the theorem.
After applying a smooth gauge transformation to the sequence \((A^\nu)\), we may assume that the assertions of Step 2 continue to hold with \(\Psi^\infty = 0\). For convenience we drop the index \(\nu\) in the subsequent calculations. Expanding \(d_A, d_A^*\) and \(F_A\) as

\[
d_A = d_A^\infty + [\beta \wedge \cdot], \quad d_A^* = d_A^\infty - *[\beta \wedge * \cdot],
\]

\[
F_A = F_A^\infty + d_A^\infty \beta + \frac{1}{2} [\beta \wedge \beta],
\]

equation \((39)\) reads

\[
0 = \partial_s A^\infty + \partial_s \beta + d_A^\infty F_A^\infty - *[\beta \wedge *(F_A^\infty + d_A^\infty \beta + \frac{1}{2} [\beta \wedge \beta])]
\]

\[
+ d_A^\infty d_A^\infty \beta + \frac{1}{2} d_A^\infty [\beta \wedge \beta] - d_A^\infty \partial_s \psi - [\beta \wedge \psi] + \nabla V(A). \tag{40}
\]

We combine this equation with the local slice condition \((38)\) to obtain for \(\beta\) the parabolic PDE

\[
\partial_s \beta + \Delta_A^\infty \beta = -\partial_s A^\infty - d_A^\infty F_A^\infty - \frac{1}{2} d_A^\infty [\beta \wedge \beta] + d_A^\infty \partial_s \psi
\]

\[
+ *[\beta \wedge *(F_A^\infty + d_A^\infty \beta + \frac{1}{2} [\beta \wedge \beta])] + d_A^\infty \psi + [\beta \wedge \psi] - \nabla V(A). \tag{41}
\]

Applying \(d_A^\infty\) to both sides of equation \((40)\) and substituting

\[
d_A^\infty \partial_s \beta = \partial_s^2 \psi + *[\partial_s A^\infty \wedge * \beta]
\]

according to \((38)\), and using that

\[
d_A^\infty \nabla V(A) = d_A^\infty \nabla V(A) + *[\beta \wedge \nabla V(A)] = *[\beta \wedge \nabla V(A)]
\]

yields for \(\psi\) the elliptic PDE

\[
\hat{\Delta}_A^\infty \psi = d_A^\infty \partial_s A^\infty + *[\partial_s A^\infty \wedge * \beta] + *[\beta \wedge d_A^\infty *(F_A^\infty + \frac{1}{2} [\beta \wedge \beta])]
\]

\[
+ *[\beta \wedge d_A^\infty d_A^\infty \beta] - \frac{1}{2} [\beta \wedge d_A^\infty \partial_s \psi] - d_A^\infty d_A^\infty [\beta \wedge \psi] + *[\beta \wedge \nabla V(A)]. \tag{42}
\]

Let \(p > 1\) arbitrary. In the following, all norms are to be understood with respect to the domain \(I \times \Sigma\). From equation \((41)\) it follows by standard parabolic regularity theory that, for a constant \(c = c(I, p)\),

\[
c^{-1} \|\beta\|_{W^{1,2,p}} \leq 1 + \|\beta\|_{L^p} + \|\{\beta, [\beta \wedge \beta]\}\|_{L^p} + \|\{\nabla A^\infty = \beta, \beta\}\|_{L^p}
\]

\[
+ \|\{\beta, \nabla V(A)\}\|_{L^p} + \|d_A^\infty \partial_s \psi\|_{L^p} + \|\nabla V(A)\|_{L^p}. \tag{43}
\]

From equation \((42)\) and elliptic regularity we obtain for a constant \(c = c(I, p)\) the estimate

\[
c^{-1} \|\psi\|_{W^{2,p}} \leq 1 + \|\beta\|_{L^p} + \|\beta\|_{L^p} + \|\{\beta, \beta\}\|_{L^p} + \|\nabla A^\infty = \beta\|_{L^p} + \|\nabla A^\infty [\beta \wedge \psi]\|_{L^p}
\]

\[
+ \|d_A^\infty \{\beta, [\beta, \beta]\}\|_{L^p} + \|\{\beta, [d_A^\infty d_A^\infty \beta]\}\|_{L^p} + \|\beta \wedge \nabla V(A)\|_{L^p}. \tag{44}
\]
Now let $3 < p < 4$. By Step 2 there holds a uniform bound for $\|\beta\|_{C^0}$ and $||\beta||_{W^{1,p}}$. The term $\|\nabla V(A)\|_{L^p}$ is uniformly bounded as shown in Step 1. It thus follows that each term on the right-hand side of (43), except the term $||d_A \partial_x \psi||_{L^p}$, is uniformly bounded. It is estimated using (44). Note that the expression
\[
\|\{\beta, d_A^{\alpha}d_A^{\beta}\}\|_{L^p} \leq c\|\beta\|_{C^0}\|d_A^{\alpha}d_A^{\beta}\|_{L^p}
\]
appearing in (44) becomes absorbed by the left-hand side of (43) after fixing $\varepsilon$. Thus indeed the second line is by the Sobolev embedding $W^{2,p}$ in condition (iii) of Step 2 sufficiently small. Hence it follows that $\psi$ is uniformly bounded in $W^{2,p}$ and $\beta$ is uniformly bounded in $W^{1,2,p}$. This yields for $\nabla A^{\alpha}\beta$ and exponents $p_1$ and $r$ with $p_1 = \frac{5}{2} + r > 4$ the uniform bound
\[
||\nabla A^{\alpha}\beta||_{L^{p_1}(I \times \Sigma)} \leq ||\nabla A^{\alpha}\beta||_{L^{5/2}(I \times \Sigma)} + ||\nabla A^{\alpha}\beta||_{L^{5/2}(I \times \Sigma)}^{2p_1}
\]
\[
\leq ||\nabla A^{\alpha}\beta||_{L^{5/2}(I \times \Sigma)} + ||\nabla A^{\alpha}\beta||_{L^{5/2}(I \times \Sigma)}^{2p_1}.
\]
In the first line we used Hölder’s inequality with exponents $r = \frac{6}{5}$ and $s = 6$, and the second line is by the Sobolev embedding $W^{1,2}(\Sigma) \hookrightarrow L^{6r}(\Sigma)$. Both terms in the last line are uniformly bounded for $r < 2$. For the last one this follows by what we have already shown, while for the first one we use interpolation (with parameter $\theta = \frac{1}{3}$, cf. Lemma Appendix C.6) and Sobolev embedding to obtain
\[
||\nabla A^{\alpha}\beta||_{L^{5/2}(I \times \Sigma)} \leq ||\nabla A^{\alpha}\beta||_{W^{1/2}(I \times \Sigma)} \leq ||\beta||_{W^{1/2}(I \times \Sigma)}.
\]
Thus indeed $\nabla A^{\alpha}\beta$ is uniformly bounded in $L^{p_1}(I \times \Sigma)$ for all $p_1 < \frac{9}{2}$ and we can repeat the previous line of arguments with $p$ replaced by $p_1$ to get uniform bound for $\psi$ in $W^{2,p_1}$ and for $\beta$ in $W^{1,2,p_1}$. Repeating this argument a finite number of times, we inductively obtain uniform such bounds for every $p < \infty$. This completes the proof. \hfill \square

**Proposition 6.4.** Let $1 < r < \infty$. Assume $A = A + \Psi ds$ satisfies (39) on $I \times \Sigma$. Fix a smooth reference connection $A^\infty + \Psi^\infty ds \in \mathcal{A}(P_1)$ and denote $\beta := A - A^\infty$ and $\psi := \Psi - \Psi^\infty$. Then there exist constants $c(I, r)$ and $p \leq 4r$ such that
\[
\|F_A\|_{W^{1,2r}(I \times \Sigma)} \leq c(I, r)\left(1 + \|\beta\|_{W^{1,2r}(I \times \Sigma)} + \|\psi\|_{W^{1,2r}(I \times \Sigma)}^2\right)
\]
and
\[
\|F_A\|_{W^{1,2r}(I \times \Sigma)}^2 \leq c(I, r)\left(1 + \|\beta\|_{W^{1,2r}(I \times \Sigma)}^4 + \|\psi\|_{W^{1,2r}(I \times \Sigma)}^3 + \|F_A\|_{W^{1,2r}(I \times \Sigma)}^2\right).
\]
Proof: Let \( \mathcal{L}_A := \frac{d}{dt} + \Delta_A \) denote the heat operator induced by \( A^\infty \). From (39) it follows that \( F_A \) satisfies the evolution equation

\[
\mathcal{L}_A F_A = d_A \dot{A} + \Delta_A F_A = -d_A d_A^* F_A + d_A d_A \Psi - d_A \nabla \langle A \rangle + \Delta_A F_A
\]

\[
= -[d_A^\infty \beta \wedge F_A] - 2[\beta \wedge d_A^\infty F_A] + [\beta \wedge [\ast \beta \wedge F_A]] + [F_A \wedge \Psi] - d_A \nabla \langle A \rangle.
\]

(47)

Standard parabolic regularity results imply that \( \|F_A\|_{W^{1,2,r}} \) is controlled by the \( L^r \) norm of the right-hand side of (47). Using the expansion \( F_A = F_{A^\infty} + d_A^\infty \beta + \frac{1}{2}[\beta \wedge \beta] \) and Lemma Appendix A.4 (to bound the term \( d_A \nabla \langle A \rangle \)), it is easily checked that for \( p = 4r \) the \( L^r \) norm in turn is controlled by \( 1 + \|\beta\|_{W^{1,2,r}(I \times \Sigma)}^4 + \|\Psi\|_{W^{1,2,r}(I \times \Sigma)}^2 \). This shows (45). Estimate (46) follows analogously. Parabolic regularity (cf. [14, Chap. 5.1]) shows that \( \|F_A\|_{W^{1,3,r}(I \times \Sigma)} \) is bounded in terms of the \( W^{0,1;r} \) norm of the right-hand side of (47). It can be estimated in the same way as before with the only difference that now terms involving two space derivatives of \( F_A \) have to be estimated, hence the additional summand \( \|F_A\|_{W^{1,2,p}} \). The \( W^{0,1;r} \) norm of \( d_A \nabla \langle A \rangle \) is taken care of by Proposition Appendix A.5. This shows (46) and completes the proof. \( \square \)

Proposition 6.5. Let the sequence \( A^\nu = A^\nu + \Psi^\nu \, ds, \, \nu \in \mathbb{N} \), satisfy the assumptions of Theorem 6.3. Then for every \( 1 < r < \infty \) there exists a constant \( C(I, r) \) such that

\[
\|F_{A^\nu}\|_{W^{1,r}(I \times \Sigma)} \leq C(I, r)
\]

for all \( \nu \in \mathbb{N} \).

Proof: From (34) and (39) it follows that the curvature of \( A^\nu \) is

\[
F_{A^\nu} = F_{A^\nu} + (d_A^* F_{A^\nu} + \nabla \langle A^\nu \rangle) \wedge ds.
\]

The required uniform \( L^r \) bound for one space derivative of \( F_{A^\nu} \) follows from estimate (45), the right-hand side of which is uniformly bounded by Theorem 6.3. The same argument applies to \( \frac{d}{ds} F_{A^\nu} \). It remains to give an \( L^r \) bound for

\[
\frac{d}{ds} d_A^* F_{A^\nu} = d_A^* d_A \dot{A}^\nu - [\dot{A}^\nu \wedge F_A^\nu]
\]

\[
= -d_A^* d_A (d_A^* F_{A^\nu} - d_A \Psi^\nu + \nabla \langle A^\nu \rangle) - [\dot{A}^\nu \wedge F_A^\nu].
\]

This bound follows by combining estimate (46) with (45), and then using Theorem 6.3 as before. The relevant \( L^r \) estimate for \( d_A^* d_A \nabla \langle A^\nu \rangle \) is provided by Proposition Appendix A.5. \( \square \)

Proof: (Theorem 6.2) By Proposition 6.5 the sequence \( A^\nu \) has curvature uniformly bounded in \( W^{1,p}(I \times \Sigma) \) for every \( p < \infty \). Hence Uhlenbeck’s weak
compactness theorem (with one derivative more, cf. [18]) applies and shows that
after modifying the sequence by suitable gauge transformations and passing to
a subsequence, there holds weak convergence \( \lim_{\nu \to \infty} A^\nu = A^* \) in \( W^{2,p}(I \times \Sigma) \)
and strong convergence in \( W^{k,p}(I \times \Sigma) \) (for any \( k < 2 \), after passing to a further
subsequence) to some limiting connection \( A^* \), as claimed.

7. Transversality

7.1. Universal moduli space

For the definition of the Banach space \( Y \) and the notion of regular perturba-
tion we refer to Section 2.2 and Definition 2.8. Further properties of the space
\( Y \) are discussed in Appendix 2.2. Throughout we fix a pair of critical manifolds
\( \check{C}^\pm \in CR^a \). We let \( C^\pm := \check{C}^\pm G_0(P) \) and consider the smooth Banach space bundle
\( E(\delta, C^-, C^+); \) \( \times Y \), cf. Section 3.2 for definitions. We define the smooth section
\( F \) of \( E \) by

\[
F : ([A, \Psi, V]) \mapsto [\partial_s A + \partial_A F_A - dA \Psi + \nabla V(A)],
\]

and call its zero set \( M_{\text{univ}}(C^-, C^+) := F^{-1}(0) \) the universal moduli space. Thus
the perturbation \( V \) which had been kept fixed so far is now allowed to vary over
the Banach space \( Y \).

**Theorem 7.1 (Transversality).** The horizontal differential \( \partial_u F \) of the map
\( F \) is surjective for every \( u \in F^{-1}(0) \).

We give a proof in section 7.2. Assuming Theorem 7.1 for now, it follows from
the implicit function theorem that the universal moduli space \( M_{\text{univ}}(C^-, C^+) \) is
a smooth Banach manifold. Let \( \pi : M_{\text{univ}}(C^-, C^+) \to Y \) denote the projection
to the second factor. It is a smooth Fredholm map whose index is given by the
Fredholm index of \( D_A \). Hence we may apply to \( \pi \) the Sard-Smale theorem for
Fredholm maps between Banach manifolds, cf. the book [1, Theorem 3.6.15],
from which it follows that the set of regular values

\[
\mathcal{R} := \{ V \in Y \mid \text{\( d_u \pi \) is surjective for all } u \in M(C^-, C^+; V) \} \subseteq Y
\]
is residual in \( Y \). Hence in particular, there exists a regular value \( V_{\text{reg}} \in \mathcal{R} \)
in every arbitrarily small ball \( B_\varepsilon(0) \) (with respect to the norm on \( Y \)) around
zero. For every such \( V_{\text{reg}} \), the moduli space \( M(C^-, C^+; V_{\text{reg}}) \) is a submanifold
of \( M_{\text{univ}}(C^-, C^+) \) of dimension equal to \( \text{ind} D_A \).
7.2. Surjectivity of linearized operators

Let \((A, \Psi, V) \in \mathcal{F}^{-1}(0)\), where the section \(\mathcal{F}\) has been defined in (48). After applying a suitable gauge transformation we may assume \(\Psi = 0\). We let \(\hat{\mathcal{D}}_{(A, V)} := d_{(A, 0, V)}\mathcal{F}\) denote its horizontal differential at the point \((A, 0, V)\). The setup for the discussion of the operator \(\hat{\mathcal{D}}_{(A, V)}\) parallels the one introduced in Section 5.2. We put

\[
\hat{\mathcal{D}}_{(A, V)} : \mathbb{Z}^{\delta,p}_A \times Y \rightarrow \mathcal{L}^{\delta,p}, \quad (\alpha, \psi, v) \mapsto \mathcal{D}_A(\alpha, \psi) + \nabla v(A).
\]

Note that \(\hat{\mathcal{D}}_{(A, V)}\) is the sum of the Fredholm operator \(\mathcal{D}_A\) and the bounded operator \(v \mapsto \nabla v(A)\), and therefore has closed range. The Fredholm property of \(\mathcal{D}_A\) has been shown in Theorem 5.4. With \(v = \sum_{t=1}^{\infty} \lambda_t \mathcal{V}_t \in Y\), the assertion on boundedness follows from the estimate

\[
\|\nabla v(A)\|_{L^p(R \times \Sigma)} = \|\nabla v(A)\|_{L^p([-T,T] \times \Sigma)} \leq 2T + \|F_{A^*}\|_{L^p([-T,T],L^4(\Sigma))}\|v\|.
\]

The first identity holds for some constant \(T = T(A) < \infty\) because \(A(s)\) is contained in the support of \(v\) only for some finite time interval (by construction, supp \(v\) is contained in the complement of some \(L^2\) neighbourhood of \(\tilde{C}^- \cap \tilde{C}^+\)). The last inequality was shown in Step 1 of the proof of Theorem 6.3.

**Proposition 7.2.** The image of the operator \(\hat{\mathcal{D}}_{(A, V)} : \mathbb{Z}^{\delta,p}_A \times Y \rightarrow \mathcal{L}^{\delta,p}\) is dense in \(\mathcal{L}^{\delta,p}\), for every zero \((A, 0, V) \in \mathcal{F}^{-1}(0)\).

**Proof:** Density of the range is equivalent to triviality of its annihilator. This means that, given \(\eta \in (\mathcal{L}^{\delta,p})^* = \mathcal{L}^{-\delta,q}\) (where \(p^{-1} + q^{-1} = 1\)) with

\[
\langle \mathcal{D}_A(\alpha, \psi, v), \eta \rangle_{R \times \Sigma} = 0 \quad \text{for all} \quad (\alpha, \psi, v) \in \mathbb{Z}^{\delta,p}_A \times Y, \quad (49)
\]

then \(\eta = 0\). Condition (49) is equivalent to

\[
\langle \mathcal{D}_A(\alpha, \psi), \eta \rangle_{R \times \Sigma} = 0 \quad \text{and} \quad \langle \nabla v(A), \eta \rangle_{R \times \Sigma} = 0 \quad (50)
\]

for all \((\alpha, \psi, v) \in \mathbb{Z}^{\delta,p}_A \times Y\). Assume by contradiction that there exists \(0 \neq \eta \in \mathcal{L}^{-\delta,q}\) which satisfies both conditions in (50). Then it follows from the identity

\[
0 = \langle \eta, \mathcal{D}_A \xi \rangle_{R \times \Sigma} = \int_{-\infty}^{\infty} \langle D_{A(s)}^* \eta(s), \xi(s) \rangle \, ds
\]

that \(D_A^* \eta = 0\), where \(D_A^* := -\frac{d}{ds} + \mathcal{H}_A\). Hence Proposition 7.5 below applies and yields the existence of a model perturbation \(\mathcal{V}_0\) with the property that \(\langle \nabla \mathcal{V}_0(A), \eta \rangle_{R \times \Sigma} > 0\). By construction of \(\mathcal{V}_0\) from the data \(A_0 \in \mathcal{A}(P)\), \(\eta_0 \in \Omega^1(\Sigma, T^* \Sigma \otimes \text{ad}(P))\), and \(\varepsilon > 0\) there exists a close enough perturbation \(v \in Y\) such that \(\langle \nabla v(A), \eta \rangle_{R \times \Sigma} > 0\). This contradicts the second equation in (50). Hence \(\eta = 0\), completing the proof of the proposition. \(\Box\)

**Proof:** (Theorem 7.1) As seen above, the operator \(\hat{\mathcal{D}}_{(A, V)}\) has closed range. By Proposition 7.2 the range is dense in \(\mathcal{L}^{\delta,p}\). These two properties together
Proposition 7.3 (Slicewise orthogonality). Let $\eta \in \mathcal{L}^{-\delta,\eta}$ satisfy $D^*_A\eta = 0$ on $\mathbb{R} \times \Sigma$. Then for all $s \in \mathbb{R}$ there holds the relation $\langle \dot{A}(s), \eta(s) \rangle = 0$.

Proof: Set $\beta(s) := \langle \dot{A}(s), \eta(s) \rangle$. One easily checks that $A$ satisfies $D_A \dot{A} = 0$. Hence it follows that

$$\dot{\beta} = \langle \dot{A}, \eta \rangle + \langle \dot{A}, A\eta \rangle = \langle \dot{A}, \mathcal{H}_A\eta \rangle + \langle -\mathcal{H}_A \dot{A}, \eta \rangle = 0.$$

Thus $\beta$ is constant. Since $\lim_{s \to -\infty} \dot{A} = 0$ it follows that $\beta$ vanishes identically. \qed

For the proof of Proposition 7.5 (which we state below) we need the following auxiliary results.

Proposition 7.4 (No return). Let $A$ be a solution of (8) satisfying $A^\pm := \lim_{s \to \pm \infty} A(s) \in \mathring{C}^\pm$. Then there exists $s_0 \in \mathbb{R}$ such that for the reference connection $A_0 := A(s_0)$ and every $\delta > 0$ there is a constant $\varepsilon > 0$ with

$$\|\alpha(A(s))\|_{L^2(\Sigma)} < 3\varepsilon \quad \Rightarrow \quad s \in (s_0 - \delta, s_0 + \delta).$$

Proof: Let $\kappa > 0$ be such that $\text{dist}_{L^2(\Sigma)}(\mathring{C}^-, \mathring{C}^+) > 3\kappa$. From $\lim_{s \to \pm \infty} A(s) = A^\pm \in \mathring{C}^\pm$ it follows the existence of $s_0 \in \mathbb{R}$ such that $A_0 := A(s_0)$ satisfies

$$\text{dist}_{L^2(\Sigma)}(A_0, \mathring{C}^\pm) > \kappa. \quad (51)$$

Assume by contradiction that there exists a sequence $(\varepsilon_i)$ of positive numbers with $\varepsilon_i \to 0$ as $i \to \infty$, and a sequence $(s_i) \subseteq \mathbb{R}$ such that

$$\|\alpha(A(s_i))\|_{L^2(\Sigma)} < 3\varepsilon_i, \quad (52)$$

but $s_i \notin (s_0 - \delta, s_0 + \delta)$. Assume first that the sequence $(s_i)$ is unbounded. Hence we can choose a subsequence (without changing notation) such that $s_i$ converges to $-\infty$ or to $+\infty$. It follows that (for one sign or $-$)

$$A(s_i) \xrightarrow{L^2(\Sigma)} A^\pm \quad \text{as} \quad i \to \pm \infty.$$

But then (51) and (52) cannot be satisfied simultaneously. This contradiction shows that the sequence $(s_i)$ has an accumulation point $s_* \notin (s_0 - \delta, s_0 + \delta)$. So there exists a subsequence $(s_i)$ with $\lim_{i \to \infty} s_i = s_*$. Because the gradient flow line $s \mapsto A(s)$ is continuous as a map $\mathbb{R} \to L^p(\Sigma)$ it follows that $\lim_{i \to \infty} A(s_i) = A(s_*)$ in $L^p(\Sigma)$. From (52) and the $L^2$ local slice Theorem 2.1 we hence infer that

$$0 = \lim_{i \to \infty} \alpha(A(s_i)) = \alpha(A(s_*)).$$

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and therefore $A(s_*)$ is gauge equivalent to $A_0$. So $\mathcal{V}M^V(A(s_*)) = \mathcal{V}M^V(A_0)$. As $\mathcal{V}M^V(A(s))$ is strictly monotone decreasing in $s$ it follows that $s_* = s_0$, which contradicts $s_* \notin (s_0 - \delta, s_0 + \delta)$. Hence the assumption was wrong and the claim follows. \hfill \Box

**Proposition 7.5 (Model perturbation).** Let $A$, $A_0$, $s_0$ be as in Proposition 7.4. Assume $\eta \in L^{-\delta, q}$ satisfies $D_A^* \eta = 0$ and $\eta_0 := \eta(s_0) \neq 0$. Then there exists a constant $\varepsilon > 0$ and a gauge-invariant smooth map $\mathcal{V}_0 : A(P) \to \mathbb{R}$ such that the following three properties are satisfied,

(i) $\text{supp} \, \mathcal{V}_0 \subseteq \{A \in A(P) \mid \|\alpha(A)\|_{L^2(\Sigma)} \leq 2\varepsilon\}$,

(ii) $\langle \nabla \mathcal{V}_0(A_0), \eta \rangle_{\Sigma} = \|\eta_0\|_{L^2(\Sigma)}^2$,

(iii) $\langle \nabla \mathcal{V}_0(A), \eta \rangle_{\mathbb{R} \times \Sigma} > 0$.

Before turning to the proof of the proposition we remark the following. For fixed $s \in \mathbb{R}$ let $\alpha'(s)$ denote the differential of the map $A \mapsto \alpha(A)$ at the point $A(s)$ in direction of $\eta(s)$. Note that $\alpha'(s_0) = \eta_0$, as follows from Proposition Appendix A.2. Then by continuous differentiability of the path $A : \mathbb{R} \to L^2(\Sigma)$ and continuity of the map $\eta : \mathbb{R} \to L^2(\Sigma)$ there exists a constant $\delta > 0$ with the following significance. For all $s \in (s_0 - \delta, s_0 + \delta)$ we have that

(A) $\|\eta(s)\|_{L^2(\Sigma)} \leq 2\|\eta_0\|_{L^2(\Sigma)}$,

(B) $\langle \alpha'(s), \eta_0 \rangle \geq \frac{1}{2}\|\eta_0\|_{L^2(\Sigma)}^2 > 0$,

(C) and with $\mu := \|\partial_s^* A(s_0)\|_{L^2(\Sigma)} > 0$ that

$$\frac{1}{2} \mu \leq \frac{\|\alpha(A(s))\|_{L^2(\Sigma)}}{|s - s_0|} \leq \frac{3}{2} \mu.$$  

**Proof:** (Proposition 7.5) Let $\varepsilon > 0$ be such that the condition

$$\|\alpha(A(s))\|_{L^2(\Sigma)} < 3\varepsilon \quad \Rightarrow \quad s \in (s_0 - \delta, s_0 + \delta) \quad (53)$$

is satisfied for all $s \in \mathbb{R}$. The existence of such an $\varepsilon$ follows from Proposition 7.4. Now let $\rho : \mathbb{R} \to [0, 1]$ be the smooth cut-off function which was part of the data for the construction of the perturbations $\mathcal{V}_i$ in Section 2.2. Define $\rho_\varepsilon(r) := \rho(\varepsilon^{-2} r)$. Note that $\|\rho\|_{L^\infty(\mathbb{R})} < \varepsilon^{-2}$. Define

$$\mathcal{V}_0(A) := \rho_\varepsilon(\|\alpha\|_{L^2(\Sigma)}^2)(\alpha, \eta_0)$$

with $\alpha = \alpha(A)$ as above. The perturbation $\mathcal{V}_0$ clearly satisfies condition (i). Furthermore, it follows from Proposition Appendix A.2 (with $\rho_\varepsilon(0) = 1$, $\rho_\varepsilon'(0) = 0$, and $\alpha(s_0) = 0$) that

$$d\mathcal{V}_0(A_0)\eta_0 = \langle \eta_0 - d_{A_0} T_{A_0, \alpha(s_0)} \eta_0, \eta_0 \rangle = \|\eta_0\|_{L^2(\Sigma)}^2,$$
so that condition (ii) is satisfied. It remains to show property (iii). We fix constants \( \sigma_1, \sigma_2, s_1, s_2 \) with \( \sigma_1 < s_1 < s_0 < s_2 < \sigma_2 \) as follows. Let \( s_2 \) be such that \( \| \alpha(A(s_2)) \|_{L^2(\Sigma)} = \varepsilon \) and \( \| \alpha(A(s)) \|_{L^2(\Sigma)} < \varepsilon \) for all \( s \in (s_0, s_2) \), and similarly for \( s_1 \). Let \( \sigma_2 \) be such that \( \| \alpha(A(\sigma_2)) \|_{L^2(\Sigma)} = 2\varepsilon \) and \( \| \alpha(A(s)) \|_{L^2(\Sigma)} > 2\varepsilon \) for all \( s \in (\sigma_2, s_0 + \delta) \), and similarly for \( \sigma_1 \). By assumption (i) and condition (53) it then follows that

\[
\langle \nabla \nu_0(A), \eta \rangle_{R \times \Sigma} = \int_{s_0 - \delta}^{s_0 + \delta} d\nu_0(A(s)) \eta(s) \, ds
\]

\[
= \int_{s_0 - \delta}^{s_0 + \delta} \rho_\varepsilon(\| \alpha \|_{L^2(\Sigma)}^2) \langle \alpha', \eta_0 \rangle \, ds + 2 \int_{s_0 - \delta}^{s_0 + \delta} \rho_\varepsilon(\| \alpha \|_{L^2(\Sigma)}^2) \langle \alpha, \alpha' \rangle \langle \alpha, \eta_0 \rangle \, ds.
\]

We estimate the last two terms separately. For the first one we obtain

\[
\int_{s_0 - \delta}^{s_0 + \delta} \rho_\varepsilon(\| \alpha \|_{L^2(\Sigma)}^2) \langle \alpha', \eta_0 \rangle \, ds
\]

\[
\geq \int_{s_1}^{s_2} 1 \cdot \langle \alpha', \eta_0 \rangle \, ds
\]

\[
\geq \frac{1}{2} (s_2 - s_1) \| \eta_0 \|_{L^2(\Sigma)}^2
\]

\[
\geq \frac{1}{3\mu} (\| \alpha(A(s_1)) \|_{L^2(\Sigma)} + \| \alpha(A(s_2)) \|_{L^2(\Sigma)} ) \| \eta_0 \|_{L^2(\Sigma)}^2
\]

\[
= \frac{2}{3\mu} \| \eta_0 \|_{L^2(\Sigma)}^2 \varepsilon.
\]

The first and second inequality follows from property (B), while the third one is by property (C). We define functions \( f, g : \mathbb{R} \to \mathbb{R} \) by

\[
f(s) := \langle \alpha(s), \alpha'(s) \rangle \quad \text{and} \quad g(s) := \langle \alpha(s), \eta_0 \rangle.
\]

As \( \alpha(s_0) = 0 \) it follows that \( f(s_0) = g(s_0) = 0 \). By Proposition Appendix A.2 we have that \( \alpha(s_0) = A(s_0) \) and \( \alpha'(s_0) = \eta_0 \). Using Proposition 7.3 it follows that

\[
f'(s_0) = \langle \dot{\alpha}(s_0), \alpha'(s_0) \rangle + \langle \alpha(s_0), \partial_s(\alpha')(s_0) \rangle = \langle \dot{A}(s_0), \eta_0 \rangle = 0,
\]

and similarly that

\[
g'(s_0) = \langle \dot{\alpha}(s_0), \eta_0 \rangle = \langle \dot{A}(s_0), \eta_0 \rangle = 0.
\]

Hence there exists a constant \( C \) such that for all \( s \in (s_0 - \delta, s_0 + \delta) \)

\[
|f(s)| \leq C(s - s_0)^2 \quad \text{and} \quad |g(s)| \leq C(s - s_0)^2.
\]
The second term is now estimated as follows.

\[
2 \int_{s_0 - \delta}^{s_0 + \delta} \rho'_e(\|\alpha\|_{L^2(\Sigma)}^2) \langle \alpha, \alpha' \rangle \langle \alpha, \eta_0 \rangle \, ds \\
= 2 \int_{\sigma_1}^{\sigma_2} \rho'_e(\|\alpha\|_{L^2(\Sigma)}^2) \langle \alpha, \alpha' \rangle \langle \alpha, \eta_0 \rangle \, ds \\
\geq -2 \int_{\sigma_1}^{\sigma_2} \rho'_e \|\alpha\|_{L^\infty(\mathbb{R})} \langle \alpha, \alpha' \rangle \cdot \|\langle \alpha, \eta_0 \rangle \rangle \, ds \\
\geq -2 \varepsilon^{-2} C^2 \int_{\sigma_1}^{\sigma_2} (s - s_0)^4 \, ds \\
= \frac{2}{5} \varepsilon^{-2} C^2 (|\sigma_1 - s_0|^5 + |\sigma_2 - s_0|^5) \\
\geq -\frac{2}{5} \varepsilon^{-2} C^2 \left( \frac{2}{\mu} \right)^5 (\|\alpha(A(\sigma_1))\|_{L^2(\Sigma)}^5 + \|\alpha(A(\sigma_2))\|_{L^2(\Sigma)}^5) \\
= -\frac{128}{5\mu^5} C^2 \varepsilon^3.
\]

The last inequality follows from property (C). Combining these estimates we find that

\[
\langle \nabla V_0(A), \eta \rangle_{\mathcal{R} \times \Sigma} \geq \frac{2}{3\mu} \|\eta_0\|_{L^2(\Sigma)}^2 \varepsilon - \frac{128}{5\mu^5} C^2 \varepsilon^3
\]

Choosing \( \varepsilon > 0 \) still smaller if necessary (which does not affect the argumentation so far), the last expression becomes strictly positive. This shows property (iii) and completes the proof. \( \square \)

8. Yang-Mills Morse homology

8.1. Morse-Bott theory

We briefly recall Frauenfelder’s cascade construction of Morse homology for Morse functions with degenerate critical points satisfying the Morse-Bott condition (cf. [11, Appendix C]). Let \((M, g)\) be a Riemannian (Banach) manifold. A smooth function \( f : M \to \mathbb{R} \) is called Morse-Bott if the set \( \text{crit}(f) \subseteq M \) of its critical points is a finite-dimensional submanifold of \( M \), and for each \( x \in \text{crit}(f) \) the Morse-Bott condition \( T_x \text{crit}(f) = \ker \text{Hess}_x f \) is satisfied. As an additional datum, we fix a Morse function \( h : \text{crit}(f) \to \mathbb{R} \) which satisfies the Morse-Smale condition, i.e. the stable and unstable manifolds \( W^s_x(y) \) and \( W^u_x(y) \) of any two critical points \( x, y \in \text{crit}(h) \) intersect transversally. We assign to a critical point \( x \in \text{crit}(h) \subseteq \text{crit}(f) \) the index

\[ \text{Ind}(x) := \text{ind}_f(x) + \text{ind}_h(x). \]

**Definition 8.1.** Let \( x^-, x^+ \in \text{crit}(h) \) and \( m \in \mathbb{N} \). A flow line from \( x^- \) to \( x^+ \) with \( m \) cascades is a tuple \( (x, T) := (x_1, \ldots, x_m, t_1, \ldots, t_{m-1}) \) with \( x_j \in C^\infty(\mathbb{R}, M) \) and \( t_j \in \mathbb{R}^+ \) such that the following conditions are satisfied.
(i) Each $x_j$ is a nonconstant solution of the gradient flow equation $\partial_s x_j + \nabla f(x_j) = 0$.

(ii) For each $1 \leq j \leq m - 1$ there exists a solution $y_j \in C^\infty(\mathbb{R}, \text{crit}(f))$ of the gradient flow equation $\partial_s y_j + \nabla h(y_j) = 0$ such that $\lim_{s \to -\infty} x_j(s) = y_j(0)$ and $\lim_{s \to -\infty} x_{j+1}(s) = y_j(t_j)$.

(iii) There exist points $p^- \in W_u^h(x^-) \subseteq \text{crit}(f)$ and $p^+ \in W_s^h(x^+) \subseteq \text{crit}(f)$ such that $\lim_{s \to -\infty} x_1(s) = p^-$ and $\lim_{s \to \infty} x_m(s) = p^+$.

A flow line with $m = 0$ cascades is an ordinary Morse flow line of $h$ on $\text{crit}(f)$ from $x^-$ to $x^+$.

Denote by $\mathcal{M}_m(x^-, x^+)$ the set of flow lines from $x^-$ to $x^+$ with $m \geq 0$ cascades (modulo the action of the group $\mathbb{R}^m$ by time-shifts on tuples $(x_1, \ldots, x_m)$). We call

$$\mathcal{M}(x^-, x^+) := \bigcup_{m \in \mathbb{N}_0} \mathcal{M}_m(x^-, x^+)$$

the set of flow lines with cascades from $x^-$ to $x^+$. In analogy to usual Morse theory (where the Morse function is required to have only isolated non-degenerate critical points), a sequence of flow lines with cascades may converge to a limit configuration which is a connected chain of such flow lines with cascades. This limiting behaviour is captured in the following definition.

**Definition 8.2.** Let $x^-, x^+ \in \text{crit}(h)$. A broken flow line with cascades from $x^-$ to $x^+$ is a tuple $v = (v_1, \ldots, v_\ell)$ where each $v_j, j = 1, \ldots, \ell$, consists of a flow line with cascades from $x^{(j-1)}$ to $x^{(j)} \in \text{crit}(h)$ such that $x^{(0)} = x^-$ and $x^{(\ell)} = x^+$.

Let $CM_*(M, f, h)$ denote the complex generated (as a $\mathbb{Z}$ module) by the critical points of $h$ and graded by the index $\text{Ind}$. On generators of $CM_*(M, f, h)$ we set

$$\partial_k x := \sum_{\text{Ind}(x') = k-1} n(x, x')x'.$$

Here $n(x, x')$ denotes the (oriented) count of elements in the zero dimensional moduli space $\mathcal{M}(x, x')$. By linear extension to $CM_*(M, f, h)$ this formally defines a boundary operator $\partial_k : CM_k(M, f, h) \to CM_{k-1}(M, f, h)$. The aim of the next section is to show that Morse-Bott theory with cascade applies to the Yang-Mills functional on the Banach manifold $\mathcal{A}(P)/\mathcal{G}_0(P)$ and gives rise to a well-defined boundary operator and Morse-Bott homology groups

$$HM_k \left( \frac{\mathcal{A}(P)}{\mathcal{G}_0(P)}, \mathcal{Y}M^V, h \right) := \frac{\text{ker} \partial_k}{\text{im} \partial_{k+1}} \quad (k \in \mathbb{N}_0).$$
8.2. Yang-Mills Morse complex

In order to keep the presentation short and avoid to discuss orientation issues we use coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ for the construction of the Yang-Mills Morse complex. For a regular value $a \geq 0$ of $\mathcal{YM}$, we define

$$\mathcal{P}(a) := \left\{ A \in A(P) \mid d_A^* F_A = 0 \text{ and } \mathcal{YM}^V(A) \leq a \right\}$$

$G_0(P)$ to be the set of based gauge equivalence classes of Yang-Mills connections of energy at most $a$. We fix a regular perturbation $V \in Y$ (cf. Definition 2.8). Let $h : \mathcal{P}(a) \to \mathbb{R}$ be a smooth Morse function. We let

$$CM_*^a(A(P), V, h)$$

denote the complex generated as a $\mathbb{Z}_2$ module by the set $\text{crit}(h) \subseteq \mathcal{P}(a)$ of critical points of $h$. For $x^- , x^+ \in \mathcal{P}(a)$ we call the set $M(x^-, x^+)$ as in (54) the moduli space of Yang-Mills gradient flow lines with cascades from $x^-$ to $x^+$.

**Lemma 8.3.** For generic, regular $V \in Y$, Morse function $h$, and all $x^- , x^+ \in \mathcal{P}(a)$, the set $M(x^-, x^+)$ is a smooth manifold (with boundary) of dimension

$$\dim M(x^-, x^+) = \text{Ind}(x^-) - \text{Ind}(x^+) - 1.$$

**Proof:** The proof that $M(x^-, x^+)$ is a smooth manifold for generic perturbations $V$ follows the standard routine by writing $M(x^-, x^+)$ as the zero set of a Fredholm section $\tilde{F}$ of a suitable Banach space bundle, and then applying the implicit function theorem. This Fredholm problem can be reduced to the one studied in Section 5. Namely, as shown in [11, Lemma C.12], the set $M(x^-, x^+)$ arises as a submanifold of the product of moduli spaces $M(C^-_i , C^+_j)$ for suitable pairs $C^-_i , C^+_j \in CR$ of critical manifolds. This way, the Fredholm property of the linearized section $d\tilde{F}$ and the formula for its index follow from Theorem 5.4. Similarly, transversality of $\tilde{F}$ is a consequence of Theorem 7.1 and does not require any new arguments. \square

For $k \in \mathbb{N}_0$ we define the Morse boundary operator

$$\partial_k : CM_*^a(A(P), V, h) \to CM_*^{a-1}(A(P), V, h)$$

to be the linear extension of the map

$$\partial_k x := \sum_{x' \in \text{crit}(h) \atop \text{Ind}(x') = k - 1} n(x, x') x',$$

where $x \in \text{crit}(h)$ is a critical point of index $\text{Ind}(x) = k$. The numbers $n(x, x')$ are given by counting modulo 2 the oriented flow lines with cascades (with respect to $\mathcal{YM}^V$ and $h$) from $x$ to $x'$, i.e.

$$n(x, x') := \# M(x^-, x^+) \pmod{2}.$$
Lemma 8.4. The numbers $n(x,x')$ are well-defined and $\partial_\ast$ has the property of a chain map, i.e., it satisfies $\partial_\ast \circ \partial_\ast = 0$.

Proof: It follows from Theorem 6.2 by repeating the arguments of [11, Theorems C.10] that for any $x^-, x^+ \in \text{crit}(h)$ the moduli space $\mathcal{M}(x^-,x^+)$ is compact up to convergence to broken flow lines with cascades. For $\text{Ind}(x') = \text{Ind}(x) - 1$ this means that it is a finite set and $n(x,x')$ is well-defined. The chain map property follows from standard arguments making use of Theorem 4.1 on exponential decay of Yang-Mills gradient flow lines.

With these preparations, we can finally proof the main result.

Proof: (Theorem 1.1) That $(CM^a_{\ast}(\mathcal{A}(P), \mathcal{V}, h), \partial_\ast)$ is a chain complex for generic, regular perturbations $\mathcal{V} \in \mathcal{Y}$ and Morse functions $h : \mathcal{P}(a) \to \mathbb{R}$ follows from Lemmata 8.3 and 8.4. Hence Yang-Mills Morse homology exists and is well-defined. From homotopy arguments standardly used in Floer theory (cf. [10, 22]) it follows that these homology groups do not depend on the choice of regular perturbation $\mathcal{V}$ or Morse function $h$. $\square$

Appendix A. Perturbations

Subsequently we list some relevant properties of the perturbations $\mathcal{V}$ introduced in Section 2.2. Let $A \in \mathcal{A}(P)$ be a regular connection and $\alpha \in L^\infty(\Sigma, T^*\Sigma \otimes \text{ad}(P))$ be a 1-form such that $d_A^*\alpha = 0$. We introduce the following operators.

$$L_{A,\alpha} : \Omega^0(\Sigma, \text{ad}(P)) \to \Omega^0(\Sigma, \text{ad}(P)), \quad L_{A,\alpha} \lambda := \Delta_A \lambda + *[\ast \alpha \wedge d_A \lambda],$$

$$R_{A,\alpha} := L_{A,\alpha}^{-1} : \Omega^0(\Sigma, \text{ad}(P)) \to \Omega^0(\Sigma, \text{ad}(P)),$$

$$M_{\alpha} : \Omega^1(\Sigma, \text{ad}(P)) \to \Omega^0(\Sigma, \text{ad}(P)), \quad M_{\alpha} := *[\alpha \wedge \xi],$$

$$T_{A,\alpha} := R_{A,\alpha} \circ M_{\alpha} : \Omega^1(\Sigma, \text{ad}(P)) \to \Omega^0(\Sigma, \text{ad}(P)).$$

Proposition Appendix A.1. The operator

$$L_{A,\alpha} : L^2(\Sigma, \text{ad}(P)) \to L^2(\Sigma, \text{ad}(P))$$

is a densely defined self-adjoint operator with domain $W^{2,2}(\Sigma, \text{ad}(P))$. Furthermore, there exists a constant $c(A)$ such that for every $\alpha$ with $\|\alpha\|_{L^2(\Sigma)} < c(A)$ the inverse $R_{A,\alpha}$ exists as a bounded operator

$$R_{A,\alpha} : L^2(\Sigma, \text{ad}(P)) \to W^{2,2}(\Sigma, \text{ad}(P)).$$

Proof: For all $\lambda, \mu \in L^2(\Sigma, T^*\Sigma \otimes \text{ad}(P))$ it follows that

$$\langle *[\alpha \wedge d_A \lambda], \mu \rangle = \langle *[\alpha \wedge d_A \lambda], *\mu \rangle = \langle d_A \lambda, [\alpha \wedge \mu] \rangle = \langle \lambda, - * d_A *[\alpha \wedge \mu] \rangle = \langle \lambda, *[\alpha \wedge d_A \mu] \rangle,$$
which implies symmetry of the operator $L_{A,\alpha}$. As the Laplace operator $\Delta_A$ is self-adjoint with domain $W^{2,2}(\Sigma)$ the same holds true for $L_{A,\alpha}$ by the Kato-Rellich theorem (cf. [20]) because the perturbation $\ast [\ast \alpha \wedge d_A \lambda]$ is of relative bound zero. Assuming bijectivity of $L_{A,\alpha}$, boundedness of the operator $R_{A,\alpha} : L^2(\Sigma, ad(P)) \to W^{2,2}(\Sigma, ad(P))$ follows from elliptic regularity. It remains to show that $L_{A,\alpha}$ is bijective. For this we first consider for $p > 2$ the bounded operator $L^p_{A,\alpha} : W^{1,p}(\Sigma, ad(P)) \to W^{-1,p}(\Sigma, ad(P))$, $\lambda \mapsto L_{A,\alpha} \lambda$. The assumption that $A$ is regular implies that $\Delta_A = L^p_{A,0}$ is injective and therefore (by symmetry) bijective. Bijectivity is preserved under small perturbations with respect to the operator norm, and thus $L^p_{A,\alpha}$ is bijective for $\|\alpha\|_{L^2(\Sigma)} \leq c(A)$ sufficiently small because

$$\| \ast [\ast \alpha \wedge d_A \lambda] \|_{W^{-1,p}(\Sigma)} \leq c\|\alpha\|_{L^2(\Sigma)} \|d_A \lambda\|_{L^p(\Sigma)} \leq c\|\alpha\|_{L^2(\Sigma)} \|\lambda\|_{W^{1,p}(\Sigma)}.$$ 

The first estimate follows from Proposition Appendix A.3. Now let $\lambda \in \ker L_{A,\alpha}$ Then by the Sobolev embedding $W^{2,2}(\Sigma) \hookrightarrow W^{1,p}(\Sigma)$ for any $p < \infty$ and injectivity of $L^p_{A,\alpha}$ it follows that $\lambda = 0$. Hence $L_{A,\alpha}$ is injective and (by self-adjointness) bijective. This completes the proof of the proposition. \hfill $\Box$

**Proposition Appendix A.2.** The map $\mathcal{V} := \mathcal{V}_x : \mathcal{A}(P) \to \mathbb{R}$ has the following properties.

(i) (We denote $A_0 := A_i$, $\eta := \eta_{ij}$, and $\rho := \rho_k$.) Its differential and $L^2$ gradient are given by

$$d\mathcal{V}(A)\xi = 2\rho'(\|\alpha\|^2\Sigma)\langle \alpha, \xi \rangle + 2\rho\langle \|\alpha\|^2\Sigma \rangle + d_{\rho(A)}T_{A_0,\alpha}\xi, \eta,$$

$$g^{-1}\nabla\mathcal{V}(A)g = 2\rho'(\|\alpha\|^2\Sigma)\langle \alpha, \eta \rangle + 2\rho\langle \|\alpha\|^2\Sigma \rangle + T_{A_0,\alpha}(\ast[\ast \alpha \wedge \ast \eta]),$$

with $\xi := g^{-1}\xi g$. Here we assume that $\xi \in \Omega^1(\Sigma, ad(P))$ satisfies $d_A^*\xi = 0$.

(ii) We have that

$$d_A \nabla \mathcal{V}(A) = \rho(\|\alpha\|^2\Sigma)(d_A \eta + d_A T_{A_0,\alpha}(\ast[\ast \alpha \wedge \ast \eta])) + d_{\rho(A)}T_{A_0,\alpha}(\ast[\ast \alpha \wedge \ast \eta]).$$

(iii) Let $\beta \in \Omega^1(\Sigma, ad(P))$ such that $d_A^*\beta = 0$ and set $\gamma := \beta - d_{\rho(A)}T_{A_0,\alpha}\beta$. Then the Hessian of $\mathcal{V}(A)$ is the map $H_A \mathcal{V} : \Omega^1(\Sigma, ad(P)) \to \Omega^1(\Sigma, ad(P))$ determined by the formula

$$g^{-1}(H_A \mathcal{V})\beta g + [g^{-1}\nabla\mathcal{V}(A)g, \lambda] =$$

$$\rho(\|\alpha\|^2\Sigma) \left( S_{A_0,\alpha,\gamma}(\ast[\ast \alpha \wedge \ast \eta]) + T_{A_0,\alpha}(\ast[\ast \gamma \wedge \ast \eta]) \right) +$$

$$2\rho'(\|\alpha\|^2\Sigma)(\langle \alpha, \gamma \rangle \eta + T_{A_0,\alpha}(\ast[\ast \alpha \wedge \ast \eta])) + \langle \eta, \gamma \rangle \alpha + (\alpha, \eta) \gamma$$

$$+ 4\rho'(\|\alpha\|^2\Sigma)(\langle \alpha, \gamma \rangle \eta + (\alpha, \eta) \gamma).$$

Here we denote

$$S_{A_0,\alpha,\gamma} := R_{A_0,\alpha} \circ M_\gamma \circ (1 - d_A \circ R_{A_0,\alpha}) : \Omega^1(\Sigma, ad(P)) \to \Omega^0(\Sigma, ad(P)).$$
Proof:

(i) Let $A(t) = A + t\xi$. Assume $A(t)$, $\alpha(t)$ and $g(t)$ satisfy condition (4) for $t \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ sufficiently small. We denote $g := g(0)$ and set $\dot{\alpha} := \frac{d}{dt}\bigg|_{t=0} \alpha(t)$ and $\lambda := g^{-1} \frac{d}{dt}\bigg|_{t=0} g(t)$. Differentiating the equation $d_{A}^{\ast}\alpha = 0$ at $t = 0$ and using that $\alpha = g^{\ast}A - A_{0}$ and $d_{A}^{\ast}\xi = 0$ yields

$$0 = d_{A}^{\ast}_{0}(g^{-1}\xi g + d_{g^{\ast}A}\lambda)$$

$$= g^{-1}(d_{[g^{-1}A]}^{\ast}\xi_{A})g + d_{A_{0}}^{\ast}d_{A_{0}}\lambda + d_{A_{0}}^{\ast}[\alpha \wedge \lambda]$$

$$= g^{-1}(d_{A - gA - A_{0}}^{\ast}\xi)g + \Delta_{A_{0}}\lambda + d_{A_{0}}^{\ast}[\alpha \wedge \lambda]$$

$$= g^{-1} * [g\circ g^{-1} \wedge \xi]g + \Delta_{A_{0}}\lambda + d_{A_{0}}^{\ast}[\alpha \wedge \lambda]$$

$$= *[\alpha \wedge g^{-1}\xi g] + \Delta_{A_{0}}\lambda + d_{A_{0}}^{\ast}[\alpha \wedge \lambda]$$

$$= M_{\alpha}\xi + L_{A_{0},\alpha}\lambda.$$ 

Hence $\lambda = -T_{A_{0},\alpha}\xi$ by definition of $T_{A_{0},\alpha}$, and

$$\dot{\alpha} = \frac{d}{dt}\bigg|_{t=0} (g^{\ast}(t)A(t) - A_{0}) = \dot{\xi} - d_{g^{\ast}A}T_{A_{0},\alpha}\dot{\xi} \quad (A.1)$$

From this we obtain

$$dV(A)\xi = \frac{d}{dt}\bigg|_{t=0} \rho(||\alpha(t)||_{L_{2}}^{2})\langle\alpha(t), \eta\rangle$$

$$= 2\rho(||\alpha||_{L_{2}}^{2})\langle\alpha, \dot{\alpha}\rangle\langle\alpha, \eta\rangle + \rho(||\alpha||_{L_{2}}^{2})\langle\dot{\alpha}, \eta\rangle$$

$$= 2\rho(||\alpha||_{L_{2}}^{2})\langle\alpha, \xi - d_{g^{\ast}A}T_{A_{0},\alpha}\dot{\xi}\rangle\langle\alpha, \eta\rangle + \rho(||\alpha||_{L_{2}}^{2})\langle\dot{\xi} - d_{g^{\ast}A}T_{A_{0},\alpha}\dot{\xi}, \eta\rangle$$

$$= 2\rho(||\alpha||_{L_{2}}^{2})\langle\alpha, \xi\rangle\langle\alpha, \eta\rangle + \rho(||\alpha||_{L_{2}}^{2})\langle\dot{\xi} - d_{g^{\ast}A}T_{A_{0},\alpha}\dot{\xi}, \eta\rangle.$$ 

In the last line we used that $d_{g^{\ast}A}\alpha = 0$. The formula for $\nabla V(A)$ follows from this by taking adjoints and using that $d_{g^{\ast}A}\eta = d_{A_{0}}^{\ast}\eta = *[\alpha \wedge \eta] = -*[\alpha \wedge \eta]$.

(ii) This follows immediately from (i).

(iii) The formula follows from differentiating the expression for $g^{-1}\nabla V(A)g$ in (i) and formula (A.1). The operator $S_{A_{0},\alpha,\gamma}$ arises from differentiating

$$\frac{d}{dt}\bigg|_{t=0} T_{A_{0},\alpha} = \frac{d}{dt}\bigg|_{t=0} \hat{L}_{A_{0},\alpha}^{-1} M_{\alpha} = -\hat{L}_{A_{0},\alpha}^{-1} \hat{L}_{A_{0},\alpha} L_{A_{0},\alpha}^{-1} M_{\alpha} + R_{A_{0},\alpha} \hat{M}_{\alpha}$$

$$= -R_{A_{0},\alpha} L_{A_{0},\alpha} R_{A_{0},\alpha} + R_{A_{0},\alpha} M_{\gamma} = R_{A_{0},\alpha} M_{\gamma}(\mathbb{I} - d_{A} \circ R_{A_{0},\alpha}),$$

using that $\hat{L}_{A_{0},\alpha} = *[\gamma \wedge d_{A_{0}} \cdot]$ and $\hat{M}_{\alpha} = M_{\gamma}$.

$\square$
Proposition Appendix A.3. For every $p > 2$ there exists a constant $c(p)$ such the estimate

$$\|uv\|_{W^{-1,p}(\Sigma)} \leq c(p)\|u\|_{L^2(\Sigma)}\|v\|_{L^p(\Sigma)}$$

is satisfied for all functions $u \in L^2(\Sigma)$ and $v \in L^p(\Sigma)$.

Proof: Let $q < 2$ denote the dual Sobolev exponent of $p$. Let $r := \frac{2p}{2+pq} < 2$ and $s := \frac{2p}{2-pq} > 2$, i.e. $\frac{1}{r} + \frac{1}{s} = 1$. Then the Sobolev embedding $W^{1,q}(\Sigma) \hookrightarrow L^s(\Sigma)$ implies the dual embedding $L^r(\Sigma) \hookrightarrow W^{-1,p}(\Sigma)$. Hence for some constant $c(p)$ it follows that

$$\|uv\|_{W^{-1,p}(\Sigma)} \leq c(p)\|uv\|_{L^r(\Sigma)},$$

and Hölder’s inequality (with exponents $\ell = \frac{2}{r} > 1$ and $\ell' = \frac{2}{s} > 1$) then implies that

$$\|uv\|_{L^r(\Sigma)} \leq \left( \int_{\Sigma} |u|^2 \right)^{\frac{r}{2}} \left( \int_{\Sigma} |v|^{r\ell'} \right)^{\frac{1}{\ell'}} = \|u\|_{L^2(\Sigma)}\|v\|_{L^p},$$

as claimed. \qed

Lemma Appendix A.4. Let $A_0 \in A(P)$ and $p > 2$. There exist constants $c(A_0)$, $c(A_0, p)$ and $\delta(A_0, p)$ such that the estimates

$$\|\alpha\|_{W^{1,p}(\Sigma)} \leq c(A_0, p)(1 + \|F_A\|_{L^p(\Sigma)}), \quad (A.2)$$
$$\|\nabla V(A)\|_{L^p(\Sigma)} \leq c(A_0)(1 + \|F_A\|_{L^1(\Sigma)}), \quad (A.3)$$
$$\|d_A\nabla V(A)\|_{L^p(\Sigma)} \leq c(A_0, p)(1 + \|F_A\|_{L^p(\Sigma)} + \|\alpha\|_{L^{2p}(\Sigma)}^2) \quad (A.4)$$

are satisfied for all $A \in A(P)$ with $\|\alpha(A)\|_{L^2(\Sigma)} < \delta(A_0, p)$.

Proof: With $\alpha$ satisfying $d_{A_0}\alpha = F_A - F_{A_0} - \frac{1}{2} [\alpha \wedge \alpha]$ and $d^*_{A_0}\alpha$, and hence

$$\Delta_{A_0}\alpha = d^*_{A_0}(F_A - F_{A_0}) - \frac{1}{2} d^*_{A_0} [\alpha \wedge \alpha], \quad (A.5)$$

elliptic regularity of the operator $\Delta_{A_0} : W^{1,p}(\Sigma) \to W^{-1,p}(\Sigma)$ yields for a constant $c(A_0, p)$ the estimate

$$\|\alpha\|_{W^{1,p}(\Sigma)} \leq c(A_0, p)(\|d^*_{A_0}F_{A_0}\|_{W^{-1,p}(\Sigma)} + \|d^*_{A_0}F_A\|_{W^{-1,p}(\Sigma)})$$
$$+ \|d_{A_0}[\alpha \wedge \alpha]\|_{W^{-1,p}(\Sigma)} + \|\alpha\|_{L^p(\Sigma)} \quad (A.6)$$
$$\leq c(A_0, p)(1 + \|F_A\|_{L^p(\Sigma)} + \|\nabla_{A_0}\alpha, \alpha\|_{W^{-1,p}(\Sigma)} + \|\alpha\|_{L^p(\Sigma)})$$
$$\leq c(A_0, p)(1 + \|F_A\|_{L^p(\Sigma)} + \|\alpha\|_{L^2(\Sigma)}\|\nabla_{A_0}\alpha\|_{L^p(\Sigma)} + \|\alpha\|_{L^p(\Sigma)}).$$
In the last step we applied Proposition Appendix A.2. The Sobolev embedding
\( W^{1,2}(\Sigma) \hookrightarrow L^p(\Sigma) \) together with the second inequality in (A.6) yields
\[
\|\alpha\|_{L^p(\Sigma)} \leq c \|\alpha\|_{W^{1,2}(\Sigma)}
\]
\[
\leq c(A_0, 2) (1 + \|F_A\|_{L^2(\Sigma)} + \|\{\nabla A_0\alpha, \alpha\}\|_{W^{-1,2}(\Sigma)} + \|\alpha\|_{L^2(\Sigma)})
\]
\[
\leq c(A_0, p) (1 + \|F_A\|_{L^2(\Sigma)} + \|\{\nabla A_0\alpha, \alpha\}\|_{W^{-1,p}(\Sigma)})
\]
\[
\leq c(A_0, p) (1 + \|F_A\|_{L^2(\Sigma)} + \|\alpha\|_{L^2(\Sigma)} \|\nabla A_0\alpha\|_{L^p(\Sigma)}).
\]
Combining (A.6) and (A.7) it hence follows that
\[
\|\alpha\|_{L^p(\Sigma)} \leq (1 + \|F_A\|_{L^p(\Sigma)} + \|\alpha\|_{L^2(\Sigma)} \|\nabla A_0\alpha\|_{L^p(\Sigma)}).
\]
Now fix the constant \( \delta(A_0, p) > 0 \) sufficiently small such that
\[
\|\alpha\|_{L^2(\Sigma)} \|\nabla A_0\alpha\|_{L^p(\Sigma)} \leq \|\alpha\|_{W^{1,p}(\Sigma)}
\]
holds for all \( \alpha \) with \( \|\alpha\|_{L^2(\Sigma)} < \delta(A_0, p) \), to conclude (A.2). To get (A.3) we note that \( \|\nabla V(\cdot)\|_{L^p(\Sigma)} \) is controlled by \( \|\alpha\|_{L^2(\Sigma)} \) as follows from Proposition Appendix A.2. Using the Sobolev embedding \( W^{1,4}(\Sigma) \hookrightarrow L^p(\Sigma) \) and (A.2), we infer estimate (A.3). To prove (A.4) let us denote \( \beta := *[\alpha \wedge \eta] \) and \( \gamma := R_{A_0,\alpha} \beta \).

From the expression for \( d_A \nabla V(A) \) as in Proposition Appendix A.2 we see that it suffices to estimate the \( L^p \) norms of the terms \( d_A\alpha \) and
\[
d_A T^*_{A_0,\alpha,\beta} = d_A[\alpha \wedge R_{A_0,\alpha,\beta}] = [d_A\alpha \wedge \gamma] - [\alpha \wedge d_A\gamma] \quad \text{(A.7)}
\]
The claimed bound for \( d_A\alpha = d_A\alpha + [\alpha \wedge \alpha] \) follows from (A.2). Furthermore, from the Sobolev embedding \( W^{2,2}(\Sigma) \hookrightarrow C^0(\Sigma) \) and Proposition Appendix A.1 we obtain
\[
\|\gamma\|_{C^0(\Sigma)} \leq c(\gamma\|_{W^2,2(\Sigma)} \leq c(\gamma\|_{L^2(\Sigma)} \leq c(A_0)\|\alpha\|_{L^2(\Sigma)}).
\]
Since \( \|\alpha\|_{L^2(\Sigma)} \leq \delta \) for some constant \( \delta \), the required estimated for the term \( [d_A\alpha \wedge \gamma] \) in (A.7) follows. Finally consider the term \( [\alpha \wedge d_A\gamma] = [\alpha \wedge d_A\gamma] + [\alpha \wedge [\alpha \wedge \gamma]] \) in (A.7). It can be bounded similarly, using again estimates (A.2) and (A.7).

**Proposition Appendix A.5.** For every \( p > 1 \) there exists a constant \( c(A_0, p) \) such that the estimate
\[
\int_{\Sigma} |\nabla A_0 d_A \nabla V(A)|^p \leq c(A_0, p) \int_{\Sigma} 1 + |\alpha|^{4p} + |F_A|^{4p} + |\nabla A_0 F_A|^{2p}
\]
holds for all \( A \in A(\mathcal{P}) \).

**Proof:** Denote \( \gamma := R_{A_0,\alpha}(*[\alpha \wedge \eta]) \). From the expression for \( d_A \nabla V(A) \) in Proposition Appendix A.2 we see that it suffices to estimate
\[
\int_{\Sigma} |\nabla A_0 d_A \alpha|^p + |\nabla A_0 (d_A \alpha \wedge \gamma)|^p + |[d_A \alpha \wedge \nabla A_0 \gamma]|^p + |[\nabla A_0 \alpha \wedge d_A \gamma]|^p
\]
\[+ |[\alpha \wedge \nabla A_0 d_A \gamma]|^p.
\]


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Applying Hölder’s inequality, it remains to estimate the terms (apart from similar ones with Sobolev exponent smaller than \(2p\))

\[
\int_\Sigma |\nabla A_0 d\alpha|^{2p}, \quad \int_\Sigma |\nabla A_0 \gamma|^{2p}, \quad \int_\Sigma |\nabla A_0 \alpha|^{2p}, \quad \int_\Sigma \nabla A_0 |dA\gamma|^{2p}.
\]  \(\text{(A.8)}\)

After rewriting \(dA\alpha = F_A - F_{A_0} - \frac{1}{2}[\alpha \wedge \alpha]\) the estimate of the first term in (A.8) reduces to that of \(\int_\Sigma |\nabla A_0 F_A|^{2p} + |\alpha|^{4p} + |\nabla A_0 \alpha|^{4p}\). Lemma Appendix A.4 now gives the bound

\[
\int_\Sigma |\nabla A_0 \alpha|^{4p} \leq c(A_0, p) \int_\Sigma 1 + |F_A|^{4p},
\]

as required. The estimate for the third term in (A.8) follows by the same argument. By elliptic regularity of the operator \(L_{A_0,\alpha}\) it follows that \(\|\gamma\|_{W^{2,2p}(\Sigma)}\) is controlled by \(\|\alpha\|_{L^{2p}(\Sigma)}\), which gives the required bound for the second and fourth term in (A.8).

Appendix B. Perturbed Yang-Mills gradient flow

Throughout this section we fix the compact interval \(I = [a,b]\). Let \(V \in Y\) be a perturbation, where \(Y\) denotes the universal space of perturbations as introduced in (6). In this section we derive a priori estimates for solutions of the perturbed Yang-Mills gradient flow equation

\[
\partial_s A + d_A^* F_A + \nabla V(A) = 0.
\]  \(\text{(B.1)}\)

Note that solutions of equation (B.1) are in particular solutions of (8) with \(\Psi = 0\). Conversely, any solution of (8) is gauge invariant under \(G - \delta^{2,p} \hat{P}\) to a solution of (B.1), so for many purposes it is sufficient to have estimates only for these.

**Proposition Appendix B.1.** Let \(A\) be a solution of (B.1) on \(I \times \Sigma\). Then there holds the estimate

\[
\|F_A(s)\|_{L^2(\Sigma)} \leq \|F_A(a)\|_{L^2(\Sigma)} + \|V\|
\]

for all \(s \in I = [a,b]\).

**Proof:** The energy \(\mathcal{YM}^V(A) = \frac{1}{2} \int_\Sigma |F_A|^2 + V(A)\) is monotone decreasing along flow lines, hence

\[
\frac{1}{2} \|F_A(s)\|_{L^2(\Sigma)}^2 \leq \mathcal{YM}^V(A(s)) + |V(A(s))| \\
\leq \mathcal{YM}^V(A(a)) + \sup_{A \in \mathcal{A}(P)} |V(A)| \\
\leq \mathcal{YM}^V(A(a)) + \|V\|
\]

where in the last line we made use of condition (i) in Section 2.2.

\[\square\]
**Proposition Appendix B.2.** For every $2 < p < 4$ there exists a constant $C(p, |I|, \|\mathcal{V}\|)$ such that

$$\| F_A \|_{L^p(I \times \Sigma)} \leq C(p, |I|, \|\mathcal{V}\|) \left(1 + 2c \mathcal{M}^\mathcal{V}(A(a))^{\frac{1}{2}}\right)$$

is satisfied for every solution $A$ of (B.1) on $I \times \Sigma$.

**Proof:** We use Hölder’s inequality with exponents $r = \frac{2}{p-2}$ and $s = \frac{2}{4-p}$ to obtain the estimate

$$\int_I \int_\Sigma |F_A|^p \leq \int_I \left( \int_\Sigma |F_A|^2 \right)^{\frac{2}{r}} \left( \int_\Sigma |F_A|^s \right)^{\frac{1}{s}} \leq \sup_{s \in I} \| F_A(s) \|_{L^2(\Sigma)}^{\frac{2}{r}} \int_I \| F_A \|_{L^\frac{2}{r}(\Sigma)}^2 \leq c(p) \| F_A \|_{L^\infty(I \times \Sigma)} \int_I \| F_A \|_{L^2(\Sigma)}^2.$$

The third line is by the Sobolev embedding $W^{1,2}(\Sigma) \hookrightarrow L^{\frac{4}{4-p}}(\Sigma)$. Now the terms in the last line admit a universal bound as follows from Proposition Appendix B.1 and the estimate

$$\int_I \| d^* F_A \|_{L^2(\Sigma)}^2 \leq \int_I \left( \int_\Sigma |d^* F_A + \nabla \mathcal{V}(A)|^2 \right)^{\frac{1}{2}} \leq 2c \mathcal{M}^\mathcal{V}(A(a)) \int_I \| \nabla \mathcal{V}(A) \|_{L^2(\Sigma)}^2 \leq 2c \mathcal{M}^\mathcal{V}(A(a)) \sup_{A \in A(P)} \| \nabla \mathcal{V}(A) \|_{L^2(\Sigma)}^2.$$

The last inequality follows from the definition of $\| \mathcal{V} \|$ and condition (ii) in Section 2.2, with

$$\sup_{A \in A(P)} \| \nabla \mathcal{V}(A) \|_{L^2(\Sigma)} = \sup_{A \in A(P)} \| \sum_{\ell=1}^\infty \lambda_\ell \mathcal{V}_\ell \|_{L^2(\Sigma)} \leq \sum_{\ell=1}^\infty C_\ell |\lambda_\ell| = \| \mathcal{V} \|.$$

Putting the single estimates together, the claim follows. \(\square\)

Let $\Delta_\Sigma = - \ast d \ast d$ denote the (positive semidefinite) Laplace-Beltrami operator on $(\Sigma, g)$ and let $L_\Sigma := \partial_s + \Delta_\Sigma$ be the corresponding heat operator. For the following calculations we also need the Bochner-Weitzenböck formula (cf. [18])

$$\Delta_A \alpha = \nabla_A^* \nabla_A \alpha + \{ F_A, \alpha \} + \{ R_\Sigma, \alpha \} \quad \text{(B.2)}$$

relating the covariant Hodge Laplacian $\Delta_A$ and the connection Laplacian $\nabla_A^* \nabla_A$ on forms in $\Omega^{\mathcal{V}}(\Sigma, \text{ad}(P))$. Here $\{ F_A, \alpha \}$, etc. denote bilinear expressions with...
fixed coefficients in $F_A$ and $\alpha$, and $R_\Sigma$ is a term involving the curvature operator of $(\Sigma, g)$. For a form $\alpha \in \Omega^k(\Sigma, \text{ad}(P))$ there holds the identity

$$\Delta_\Sigma \frac{1}{2} |\alpha|^2 = -|\nabla_A \alpha|^2 + \langle \nabla_A^* \nabla_A \alpha, \alpha \rangle.$$  \hfill (B.3)

We shall also make use of the commutator identity

$$[\nabla_A, \nabla_A^* \nabla_A] \alpha = \{\alpha, \nabla_A \alpha\},$$  \hfill (B.4)

cf. [9, p. 17].

**Proposition Appendix B.3.** Assume that $A$ solves (B.1) on $I \times \Sigma$. Consider (for $p \geq 2$) the function $u_p : I \times \Sigma \to \mathbb{R}$ defined by $u_p(s, z) := \frac{1}{p} |* F_{A(s)}(z)|^p$. Denote $u := u_2$. Then the following holds,

$$L_\Sigma u = -|d_A * F_A|^2 - \langle * F_A, * d_A \nabla V(A) \rangle,$$

$$L_\Sigma u_p = |* F_A|^{p-2} \left(-|d_A * F_A|^2 - \langle * F_A, * d_A \nabla V(A) \rangle \right)$$

$$- (p - 2) |* F_A|^{p-2} \langle * F_A, d_A * F_A \rangle \wedge \langle * F_A, d_A * F_A \rangle.$$

Moreover, the function $\langle * F_A, d_A * F_A \rangle \wedge \langle * F_A, d_A * F_A \rangle : I \times \Sigma \to \mathbb{R}$ is non-negative.

**Proof:** We calculate using (B.1),

$$\frac{d}{ds} \frac{1}{2} \langle * F_A, * F_A \rangle = \langle * F_A, * d_A \dot{A} \rangle = \langle * F_A, -\Delta_A F_A - * d_A \nabla V(A) \rangle.$$

From this it follows that

$$L_\Sigma u = (\partial_s - * d \circ d) \frac{1}{2} \langle * F_A, * F_A \rangle$$

$$= -\langle * F_A, \Delta_A F_A + * d_A \nabla V(A) \rangle - * d \langle * F_A, d_A * F_A \rangle$$

$$= -\langle * F_A, \Delta_A F_A + * d_A \nabla V(A) \rangle - \langle * F_A, * d_A * F_A \rangle - |d_A * F_A|^2$$

$$= -\langle * F_A, * d_A \nabla V(A) \rangle - |d_A * F_A|^2.$$

The formula for $u_p$ follows from that for $u$ and the further calculation

$$- * d \circ d \frac{1}{p} |* F_A|^p$$

$$= - * d |* F_A|^{p-2} \langle * F_A, d_A * F_A \rangle$$

$$= - (p - 2) |* F_A|^{p-2} \langle * F_A, d_A * F_A \rangle \wedge \langle * F_A, d_A * F_A \rangle$$

$$- * |* F_A|^{p-2} d \langle * F_A, d_A * F_A \rangle.$$

The statement on non-negativity follows by a short calculation in local coordinates. Namely, if we write $d_A * F_A = \alpha_1 dx_1 + \alpha_2 dx_2$ with respect to local orthonormal coordinates $x_1, x_2$ and maps $\alpha_i \in C^\infty(U, g)$, then it follows that

$$\langle * F_A, d_A * F_A \rangle \wedge \langle * F_A, d_A * F_A \rangle$$

$$= \langle (\langle * F_A, \alpha_1 \rangle dx_1 + \langle * F_A, \alpha_2 \rangle dx_2) \wedge (\langle * F_A, \alpha_1 \rangle dx_2 - \langle * F_A, \alpha_2 \rangle dx_1) \rangle$$

$$= \langle (\langle * F_A, \alpha_1 \rangle^2 + \langle * F_A, \alpha_2 \rangle^2) dx_1 \wedge dx_2,$$

which is a non-negative multiple of the volume form. \hfill □
Proposition Appendix B.4. Assume that $A$ solves (B.1) on $I \times \Sigma$. Consider the function $u : I \times \Sigma \to \mathbb{R}$ defined by $u(s, z) := \frac{1}{2} \| \nabla_{A(s)} F_{A(s)}(z) \|^2$. It satisfies
\[
L_{\Sigma} u = -\| \nabla_{A} F_{A} \|^2 + \langle \nabla A F_A, \{ \nabla A F_A, F_A \} \rangle + \nabla A \{ R_{\Sigma}, F_A \} + \{ \nabla \nabla (A), F_A \}
- \nabla A d_{A} \nabla (A).
\] (B.5)

Proof: We calculate
\[
\frac{d}{ds} \nabla A F_A = \nabla A d_{A} \dot{A} + \{ \dot{A}, F_A \}
= \nabla A ( - d_A d_A^* F_A - d_A \nabla (A)) + \{ d_A^* F_A + \nabla (A), F_A \}
= \nabla A ( - d_A^* \nabla A F_A + \{ F_A, F_A \} + \{ R_{\Sigma}, F_A \} - d_A \nabla (A))
+ \{ d_A^* F_A + \nabla (A), F_A \}
= -d_A^* \nabla A \nabla A F_A + \{ \nabla A F_A, F_A \} + \nabla A \{ F_A, F_A \} + \nabla A \{ R_{\Sigma}, F_A \}
- \nabla A d_{A} \nabla (A) + \{ \nabla (A), F_A \}.
\]
The third line is by (B.2), and the last line uses (B.4). Inserting this expression into (B.3) we obtain
\[
L_{\Sigma} u = -\| \nabla_{A} F_{A} \|^2 + \langle \nabla A F_A, \{ \nabla A F_A, F_A \} \rangle + \nabla A \{ R_{\Sigma}, F_A \}
+ \{ \nabla \nabla (A), F_A \} - \nabla A d_{A} \nabla (A),
\]
as claimed.

The following lemma is an adaption of a result (for four-dimensional domains) by Struwe [27, Lemma 3.3].

Lemma Appendix B.5. Let $A \in A^{1, 2}(P)$ be a fixed connection and let $p > 1$. There exists a constant $c = c(p, P)$ such that for any form $\varphi \in \Omega^{k}(\Sigma, ad(P))$ there holds
\[
\| \varphi \|^2_{L^{p}(\Sigma)} \leq c(p, P)(\| d_A \varphi \|^2_{L^{2}(\Sigma)} + \| d_A^* \varphi \|^2_{L^{2}(\Sigma)} + c(p, P)\langle \{ F_A, \varphi \}, \varphi \rangle).
\]

Proof: The proof of [27, Lemma 3.3] applies with minor modifications. The first one is the Sobolev embedding $W^{1, 2}(\Sigma) \hookrightarrow L^p(\Sigma)$ for all $p < \infty$ (instead of only $W^{1, 2} \hookrightarrow L^{4}$ in dimension 4). The second one is that at this point we do not further estimate the term $\langle \{ F_A, \varphi \}, \varphi \rangle$.

As a consequence of this lemma we obtain the following estimate for $F_A$, provided $A$ solves (B.1).

Proposition Appendix B.6. Let $p > 1$ and $I = [a, b]$. There exists a constant $c(I, p)$ such that if $A$ is a solution of (B.1) on $I \times \Sigma$, then
\[
\int_{I} \| F_{A(s)} \|^2_{L^{p}(\Sigma)} ds \leq c(I, p)(\| \nabla \|^2 + \mathcal{Y} \mathcal{M}^\nabla (A(a)) + \mathcal{Y} \mathcal{M}^\nabla (A(a))^\frac{1}{2}).
\]
**Proof:** We integrate the estimate of Lemma Appendix B.5 with \( \varphi = F_A \) and use the Bianchi identity \( d_A F_A = 0 \) to obtain

\[
c(p, P)^{-1} \int_I \|F_A\|_{L^p(I \times \Sigma)}^2 \\
\leq \int_I (\|d_A F_A\|_{L^2(\Sigma)}^2 + \|F_A\|_{L^3(\Sigma)}^3) \\
\leq \int_I (2\|d^* A F_A + \nabla \mathcal{V}(A)\|_{L^2(\Sigma)}^2 + 2\|\nabla \mathcal{V}(A)\|_{L^2(\Sigma)}^2 + \|F_A\|_{L^3(\Sigma)}^3) \\
\leq 2 \mathcal{Y} M^V(A(a)) + 2|I| \sup_{A \in A(P)} \|\nabla \mathcal{V}(A)\|_{L^2(\Sigma)}^2 + \|F_A\|_{L^3(I \times \Sigma)}^3.
\]

The last inequality follows from the definition of \( \|V\| \) and condition (ii) in Section 2.2, with

\[
\sup_{A \in A(P)} \|\nabla \mathcal{V}(A)\|_{L^2(\Sigma)} = \sup_{A \in A(P)} \left\| \sum_{\ell=1}^{\infty} \lambda_\ell V_\ell \right\|_{L^2(\Sigma)} \leq \sum_{\ell=1}^{\infty} C_\ell |\lambda_\ell| = \|V\|.
\]

An estimate for the remaining term \( \|F_A\|_{L^3(I \times \Sigma)}^3 \) is provided by Proposition Appendix B.2 with \( p = 3 \).

**Lemma Appendix B.7 (L^p curvature estimate).** Let \( 1 < p < 4 \) and \( I = [a, b] \). There exists a constant \( c(I, p, \|V\|) \) such that if \( A \) is a solution of (B.1) on \( I \times \Sigma \), then

\[
\|F_A\|_{L^p(I \times \Sigma)}^2 \leq c(I, p, \|V\|) \left( 1 + \mathcal{Y} M^V(A(a)) \right)^{1+\frac{1}{p}}.
\]

**Proof:** Hölder’s inequality yields for \( p < 4 \) the estimate

\[
\|F_A\|_{L^p(I \times \Sigma)}^p \leq \|F_A\|_{L^\infty(I \times \Sigma)}^{p-2} \|F_A\|_{L^2(I \times \Sigma)}^2 \|F_A\|_{L^2(I \times \Sigma)}^{4-p}.
\]

Bounds for the last two factors are provided by Propositions Appendix B.1 and Appendix B.6. Putting those estimates together, the claim follows.

**Proposition Appendix B.8.** Let \( p > 1 \) and \( I = [a, b] \). There exists a constant \( c(p) \) such that if \( A \) is a solution of (B.1) on \( I \times \Sigma \), then

\[
\int_I \|d_{A(s)}^* F_A(s)\|_{L^p(\Sigma)}^2 \, ds \\
\leq c(p) \int_I \|d_{A(s)}^* d_{A(s)} F_A(s)\|_{L^2(\Sigma)}^2 \, ds + c(p) \int_I \int_{\Sigma} |F_A(s)| \cdot |d_{A(s)}^* F_A(s)|^2 \, ds.
\]
Proof: We integrate the estimate of Lemma Appendix B.5 with \( \varphi = d_A^* F_A \) over the interval \( I \) and use \( d_A^* d_A^* F_A = 0 \) to obtain

\[
\int_I \|d_A^* F_A\|^2_{L^2(\Sigma)} \leq c(p) \int_I \|d_A d_A^* F_A\|^2_{L^2(\Sigma)} + c(p) \int_I \int_{\Sigma} \{F_A, d_A^* F_A\}, d_A^* F_A) ,
\]

and hence the claim follows. \( \Box \)

Proposition Appendix B.9. Let \( I = [a, b] \). Suppose \( A \) is a solution of (B.1) on \( I \times \Sigma \). Then the function \( R(s) := \frac{1}{2}\|d_A^* F_A(s)\|^2_{L^2(\Sigma)} \) satisfies the estimate

\[
\sup_{a \leq s \leq b} R(s) \leq R(a) + \int_I \|d_A \nabla V(A)\|^2_{L^2(\Sigma)}
+ \int_I \left( |\langle d_A^* F_A, \{d_A^* F_A, F_A\} \rangle| + |\langle d_A^* F_A, \{\nabla V(A), F_A\} \rangle| \right),
\]

where \( \langle \cdot, \cdot \rangle \) indicates a certain bilinear expression with smooth time-independent coefficients.

Proof: From equation (B.1) it follows for every \( a \leq s \leq b \) that

\[
\frac{d}{ds} R(s) = \langle d_A^* F_A, d_A^* d_A \dot{A} - *([\dot{A} \wedge *F_A]) \rangle
- \langle d_A^* F_A, d_A^* d_A^* d_A^* F_A + d_A^* d_A^* \nabla V(A) - *([d_A^* F_A + \nabla V(A)] \wedge *F_A) \rangle
\]

\[
= -\|d_A d_A^* F_A\|^2_{L^2(\Sigma)} - \langle d_A d_A^* F_A, d_A \nabla V(A) \rangle
+ \langle d_A^* F_A, *([d_A^* F_A + \nabla V(A)] \wedge *F_A) \rangle
\]

\[
\leq \|d_A \nabla V(A)\|^2_{L^2(\Sigma)} + |\langle d_A^* F_A, \{d_A^* F_A, F_A\} \rangle| + |\langle d_A^* F_A, \{\nabla V(A), F_A\} \rangle|.
\]

To obtain the last step, we applied the Cauchy-Schwarz inequality to the term \( \langle d_A d_A^* F_A, d_A \nabla V(A) \rangle \). Now we integrate this inequality over the interval \( [a, s] \subseteq I \) and take the supremum over \( s \in I \) to conclude the result. \( \Box \)

Proposition Appendix B.10. Let \( I = [a, b] \) and \( I' = [a_1, b] \), where \( a_1 \in (a, b) \). There exists a constant \( c(I, I') \) such that if \( A \) is a solution of (B.1) on \( I \times \Sigma \), then

\[
\sup_{s \in I'} \|d_A^* F_A(s)\|^2_{L^2(\Sigma)} \leq c(I, I') \left( 1 + \|V\|^2 + \gamma M (A(a)) \right) + \int_{I \times \Sigma} |F_A| \cdot |d_A^* F_A|^2.
\]

Proof: By Fubini’s theorem we can find \( s_0 \in (a, a_1) \) such that

\[
\|d_A^* F_A(s_0)\|^2_{L^2(\Sigma)} \leq 2(a_1 - a)^{-1} \int_a^{a_1} \|d_A^* F_A(s)\|^2_{L^2(\Sigma)} ds
\]

\[
\leq c(a_1 - a)^{-1} \left( \gamma M^2 (A(a)) + |I| \cdot \|\nabla V\|^2 \right).
\]

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Hölder’s inequality yields for a constant $C$ to obtain
\[
\sup_{s \in I} \frac{1}{2} \| d_{A(s)}^* F_A(s) \|_{L^2(\Sigma)}^2 \leq c(a_1 - a)^{-1} (\mathcal{Y}\mathcal{M}^V(A(a)) + |I| \cdot \| V \|^2)
\]
\[
+ \int_I \| d_A \nabla V(A) \|_{L^2(\Sigma)}^2 \geq \int_I \left( \| d_{A}^* F_{A} \|_{L^2(\Sigma)}^2 + \int_I \| d_{A}^* F_{A} \{ \nabla V(A), F_A \} \right).
\]
(B.6)

To estimate the first term in the second line we combine (A.2) and (A.4) and use the Sobolev embedding $W^{1,2}(\Sigma) \hookrightarrow L^p(\Sigma)$. This yields for any $\epsilon > 0$ and $p = 2 + \epsilon$
\[
\| d_A \nabla V(A) \|_{L^2(\Sigma)}^2 \leq c \| d_A \nabla V(A) \|_{L^{2+\epsilon}(\Sigma)} \leq c(1 + \| F_A \|_{L^{2+\epsilon}(\Sigma)}^2).
\] (B.7)

By Hölder’s inequality it then follows that
\[
\int_I \| d_A \nabla V(A) \|_{L^2(\Sigma)}^2 \leq c \left| I \right| + \int_I \| F_A \|_{L^2(\Sigma)}^5 + \int_I \| F_A \|_{L^{2+\epsilon}(\Sigma)}^3 (1 + \| V \|_{L^2(\Sigma)}^2).
\]
The first term on the right-hand side can further be estimated, using monotone decay of the action $\mathcal{Y}\mathcal{M}^V(A(s))$ in $s$ and the estimate $\sup_{A \in A(P)} | V(A) | \leq \| V \|$, to get
\[
\int_I \| F_A \|_{L^2(\Sigma)}^5 \leq c \left| I \right| (\mathcal{Y}\mathcal{M}^V(A(a))^\frac{2}{p} + \| V \|_{L^2(\Sigma)}^\frac{2}{p}).
\]
Choosing $\epsilon < \frac{1}{2}$, we can use Lemma (Appendix B.7) to bound the second term on the right-hand side in the required way. Proposition Appendix B.12 gives for the last term in (B.6) the estimate
\[
\int_I \| (d_{A}^* F_{A} \{ \nabla V(A), F_A \}) \| \leq c \int_I | F_A |^2 \| d_A \nabla V(A) \| \leq c(I) \left( 1 + \mathcal{Y}\mathcal{M}^V(A(a))^2 \right).
\]
Putting all the estimates together, the claim follows.

\[\square\]

**Lemma Appendix B.11 ($L^p$ gradient estimate).** Let $1 < p < 4$, $I = [a, b]$ and $I' = [a_1, b_1]$, where $a_1 \in (a, b)$. For every solution $A$ of (B.1) on $I \times \Sigma$ there holds the gradient bound
\[
\| d_{A}^* F_{A} \|_{L^p(I' \times \Sigma)} \leq C,
\]
for a constant $C = C(I, I', p, \mathcal{Y}\mathcal{M}^V(A(a)), \| V \|)$.

**Proof:** Hölder’s inequality yields for $p < 4$ the estimate
\[
\| d_{A}^* F_{A} \|_{L^p(I \times \Sigma)} \leq \| d_{A}^* F_{A} \|_{L^\infty(I, L^2(\Sigma))} \| d_{A}^* F_{A} \|_{L^2(I, L^{\frac{4}{p-2}}(\Sigma))}^2.
\]

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For both factors on the right-hand side we have established in Propositions Appendix B.8 and Appendix B.10 bounds only involving $I, I', p, \mathcal{Y}c\mathcal{M}(A(a)), \|V\|$, and the terms

$$
\int_I\int_{\Sigma}|F_A| \cdot |d_A F_A|^2 \quad \text{and} \quad \int_I\|d_A F_A\|_{L^2(\Sigma)}^2.
$$

For these the claimed estimate by a constant $C(I, I', p, \mathcal{Y}c\mathcal{M}(A(a)), \|V\|)$ follows from Propositions Appendix B.13 and Appendix B.14.

The following Propositions Appendix B.12, Appendix B.13, Appendix B.14, and Appendix B.15 are auxiliary results needed in the proofs of Proposition Appendix B.10 and Lemma Appendix B.11.

**Proposition Appendix B.12.** Let $A$ solve (B.1) on $I \times \Sigma$. Then the product $|F_A|^2|d_A \nabla V(A)|$ admits the estimate

$$
\int_{I \times \Sigma} |F_A|^2|d_A \nabla V(A)| \leq C(I, \mathcal{Y}c\mathcal{M}(A(a)), \|V\|)
$$

with constant $C(I, \mathcal{Y}c\mathcal{M}(A(a)), \|V\|)$ independent of $A$.

**Proof:** Similarly to (B.7) we obtain for any $p > 2$ and fixed $\varepsilon > 0$ the estimate

$$
\|d_A \nabla V(A)\|_{L^p(\Sigma)} \leq c\left(1 + \|F_A\|_{L^p(\Sigma)} + \|F_A\|_{L^{2\varepsilon}(\Sigma)}\right).
$$

From Hölder’s inequality we hence get for $p = 3$ and $0 < \varepsilon \leq 1$ that

$$
\int_{I \times \Sigma} |F_A|^2|d_A \nabla V(A)| \leq \int_I\left(\int_{\Sigma}|F_A|^3\right)^{\frac{2}{3}} \cdot \left(\int_{\Sigma}|d_A \nabla V(A)|^3\right)^{\frac{1}{3}}
$$

$$
\leq c\int_I\left(\int_{\Sigma}|F_A|^3\right)^{\frac{2}{3}} \cdot \left(1 + \left(\int_{\Sigma}|F_A|^3\right)^{\frac{1}{3}} + \left(\int_{\Sigma}|F_A|^{2\varepsilon}\right)^{\frac{3}{2\varepsilon}}\right)
$$

$$
\leq c\int_I\left(\int_{\Sigma}|F_A|^3\right)^{\frac{2}{3}} + c\int_{I \times \Sigma}|F_A|^3 + c\int_I\left(\int_{\Sigma}|F_A|^3\right)^{\frac{1}{3}}
$$

$$
+ c\int_I\left(\int_{\Sigma}|F_A|^{2\varepsilon}\right)^{\frac{3}{2\varepsilon}}
$$

$$
\leq c\left(\|F_A\|_{L^3(I \times \Sigma)}^3 + \|F_A\|_{L^{2\varepsilon}(I \times \Sigma)}^4\right).
$$

The required bound for $\|F_A\|_{L^3(I \times \Sigma)}^3$ follows from Proposition Appendix B.2. Concerning the last term, we apply Hölder’s inequality to obtain

$$
\|F_A\|_{L^{4}(I, L^6(\Sigma))}^4 \leq \|F_A\|_{L^\infty(I, L^2(\Sigma))}^2 \|F_A\|_{L^2(I, L^6(\Sigma))}^2,
$$

and then estimate both factors separately as in the proof of Lemma Appendix B.7. \qed
Proposition Appendix B.13. Let $I = [a, b]$ and $I' = [a_1, b]$, where $a_1 \in (a, b)$, and assume $A$ solves (B.1) on $I \times \Sigma$. Then the map $|F_A| \cdot |d_A^2 F_A|^2 : I \times \Sigma \to \mathbb{R}$ satisfies the estimate
\[
\int_{I' \times \Sigma} |F_A| \cdot |d_A^2 F_A|^2 \leq C(I, I', \mathcal{Y} \mathcal{M}^V(A(a)), \|V\|)
\]
with constant $C(I, I', \mathcal{Y} \mathcal{M}^V(A(a)), \|V\|)$ independent of $A$.

Proof: Consider the function $u_{0, 3} = \frac{1}{3} \ast |F_A|^3 : I \times \Sigma \to \mathbb{R}$. By Proposition Appendix B.3 it satisfies
\[
L \Sigma u_{0, 3} = -\ast |F_A| \left(\|d_A \ast F_A\|^2 + \langle \ast F_A, \ast d_A \nabla V(A) \rangle \right)
- \ast |F_A|^{-1} \langle \ast F_A, d_A \ast F_A \rangle \wedge \langle \ast F_A, d_A \ast F_A \rangle,
\]
where the term $\langle \ast F_A, d_A \ast F_A \rangle \wedge \langle \ast F_A, d_A \ast F_A \rangle$ is non-negative. Lemma Appendix C.5 thus yields the estimate
\[
\int_{I' \times \Sigma} |F_A| \cdot |d_A^2 F_A|^2 \leq c(I, I') \int_{I \times \Sigma} \frac{1}{3} |F_A|^3 + |F_A| \cdot |\langle \ast F_A, d_A \nabla V(A) \rangle|.
\]
Now apply Lemma Appendix B.7 with $p = 3$ and Proposition Appendix B.12 to obtain the result. \qed

Proposition Appendix B.14. Let $I = [a, b]$ and $I' = [a_1, b]$, where $a_1 \in (a, b)$. There exists a constant $C(I, I', \mathcal{Y} \mathcal{M}^V(A(a)), \|V\|)$ such that if $A$ is a solution of (B.1) on $I \times \Sigma$, then
\[
\int_{I' \times \Sigma} |\nabla_A^2 F_A|^2 \leq C(I, I', \mathcal{Y} \mathcal{M}^V(A(a)), \|V\|).
\]

Proof: We apply the parabolic mean value inequality (C.3) to equation (B.5). After estimating the last term in (B.5) via the Cauchy-Schwarz inequality, this yields for a constant $c(I, I')$ the bound
\[
\int_{I' \times \Sigma} |\nabla_A^2 F_A|^2 \leq c(I, I') \int_{I \times \Sigma} (|\nabla_A F_A|^2 + |F_A|^2 + |\nabla_A F_A|^2 |F_A| + |F_A| \cdot |\nabla A F_A| \cdot |\nabla V(A)| + |\nabla_A d_A \nabla V(A)|^2).
\]
We estimate each of the five terms on the right-hand side separately. The integral over $|\nabla_A F_A|^2$ is bounded in terms of $|I|$, $\|V\|$, and $\mathcal{Y} \mathcal{M}^V(A(a))$, cf. the proof of Proposition Appendix B.6 for a precise estimate. A similar bound for the integral over $|F_A|^2$ is provided by Proposition Appendix B.6. The estimate for $|\nabla A F_A|^2 |F_A|$ follows from Proposition Appendix B.13. To the term $|\nabla_A d_A \nabla V(A)|^2$ we apply Proposition Appendix B.15 with $p = 3$ to obtain a bound in terms of $\int I \|F_A\|_{L^3(\Sigma)}^4$ (which can be estimated as in the last step of
the proof of Proposition Appendix B.12) and again of \( \int_I \| \nabla_A F_A \|_{L^2(\Sigma)}^2 \). The remaining integral over \( |F_A| \cdot |\nabla_A F_A| \cdot |\nabla \nu| \) can be estimated by similar arguments, using inequality (A.3). 

\[ \text{Proposition Appendix B.15. Let } I = [a, b]. \text{ For every } p > 2 \text{ there exists a constant } c(I, p, \| \nu \|) \text{ such that} \]

\[ \int_I |\nabla_A d_A \nabla \nu(A)|^2 \leq c(I, p, \| \nu \|)(1 + \int_I \| F_A \|_{L^p(\Sigma)}^4 + \int_I \| \nabla_A F_A \|_{L^2(\Sigma)}^2) \]

holds for all \( \nu \in Y \) and continuous paths \( A : I \to \mathcal{A}(P) \) of connections.

**Proof:** Let us recall the explicit formula for \( d_A \nabla \nu(A) \) given in Proposition Appendix A.2. Consider first the term \( |\nabla_A d_A \alpha|^2 \) of this expression. Because of the identity \( d_A \alpha = F_A - F_{A_0} + \frac{1}{2}[\alpha \wedge \alpha] \) it remains to estimate the term \( |\nabla_A[\alpha \wedge \alpha]|^2 \). Let \( r := \frac{5}{2} > 1 \) and \( s \) be the Hölder conjugate exponent of \( r \). It follows that

\[ \int_{I \times \Sigma} |\nabla_A[\alpha \wedge \alpha]|^2 \leq c \int_{I \times \Sigma} |\nabla_A \alpha|^2 |\alpha|^2 \leq c \int_I \| \nabla_A \alpha \|^4_{L^2(\Sigma)} + c \int_I \| \alpha \|^4_{L^{2r}(\Sigma)} \]

\[ \leq c \int_I \| \nabla_A \alpha \|^4_{L^{2r}(\Sigma)} \leq c(I, \| \nu \|)(1 + \int_I \| F_A \|^4_{L^2(\Sigma)}). \]

The third inequality is by the Sobolev embedding \( W^{1,2r}(\Sigma) \hookrightarrow L^{2s}(\Sigma) \), while in the last inequality we made use of estimate (A.2) for \( \| \alpha \|_{W^{1,2r}(\Sigma)} \). Denote \( \beta := *[\alpha \wedge \eta] \) and \( \gamma := R_{A_0, \alpha} \beta \). The remaining term to estimate in the expression for \( \nabla_A d_A \nabla \nu(A) \) (as given by Proposition Appendix A.2) is

\[ \nabla_A d_A T^*_{A_0, \alpha} \beta = \nabla_A d_A [\alpha \wedge R_{A_0, \alpha} \beta] = \nabla_A [d_A \alpha \wedge \gamma] - \nabla_A [\alpha \wedge d_A \gamma] \]

\[ = [\nabla_A d_A \alpha \wedge \gamma] + [d_A \alpha \wedge \nabla_A \gamma] - [\nabla_A \alpha \wedge d_A \gamma] - [\alpha \wedge \nabla_A d_A \gamma]. \]

The required estimates for the last four terms are obtained similarly as before for \( \nabla_A d_A \alpha \). In particular, for the integral over \( |[\nabla_A d_A \alpha \wedge \gamma]| \) we use that \( |\gamma|_{W^{2,2}(\Sigma)} \) and hence \( |\gamma|_{C^0(\Sigma)} \) is bounded in terms of \( \| \alpha \|_{L^2(\Sigma)} \) (and hence by \( \| \nu \| \)) as follows from Proposition Appendix A.1.

\[ \text{Appendix C. Auxiliary results} \]

We give an a priori estimate for the linearized Yang-Mills gradient flow along a path \( s \mapsto A(s) \in \mathcal{A}(P) \) of connections. The linearization is given by the operator \( \mathcal{D}_A = \frac{\partial}{\partial s} + \mathcal{H}_A : \Omega^{k,p} \to \mathcal{L}^{k,p} \) with

\[ \mathcal{H}_A \left( \begin{array}{c} \alpha_0 \\ \alpha_1 \\ \psi \end{array} \right) = \left( \begin{array}{c} \Delta_A \alpha_0 + *[\nu F_A \wedge \alpha_0] + [d_A \nu F_A \wedge \varphi] - d_A \alpha_0 + \text{H}\alpha_0 \\ -d_A \varphi + d_A \alpha_0 + \text{H}\alpha_1 \\ -d_A \alpha_1 + \text{H}\psi \end{array} \right) \]
denoting the augmented Yang-Mills Hessian defined by (27) and Banach spaces \( Z_{\delta,p}^B, L_{\beta,p}^B \) as introduced in Section 5.2. Here \( \alpha = \alpha_0 + \alpha_1 \) denotes the (time-varying) Hodge decomposition of \( \alpha \) with respect to the connection \( A = A(s) \) (i.e. \( d_A^* \alpha_0 = 0 \) and \( \alpha_1 = d_A \varphi \) for some \( \varphi \in \Omega^0(\Sigma, \text{ad}(P)) \)). The map \( \omega \) has been defined as a solution of the equation

\[
\Delta_A \omega = *[d_A \ast F_A \wedge \alpha].
\]

\( \Delta_A \omega = *[d_A \ast F_A \wedge \alpha]. \) \hspace{1cm} (C.1)

**Proposition Appendix C.1 (Linear estimate).** Fix \( p > 3 \) and compact intervals \( I = [a,b] \) and \( I' = [a_1,b] \) where \( a < a_1 < b \). For any path \( A \in W^{2,p}(I \times \Sigma) \) of connections there exists a constant \( c(p,A,I,I') \) such that the estimate

\[
\|\alpha_0\|_{W^{1,p}(I \times \Sigma)} + \|\alpha_1\|_{W^{1,p}(I \times \Sigma)} + \|\psi\|_{W^{1,p}(I \times \Sigma)} \leq c(p,A,I,I') (\|D_A\xi\|_{L^p(I \times \Sigma)} + \|\alpha\|_{L^p(I \times \Sigma)} + \|\psi\|_{L^p(I \times \Sigma)})
\]

is satisfied for every \( \xi = (\alpha_0, \alpha_1, \psi) \in Z_{\delta,p}^A \).

**Proof:** Denote \( \eta = (\eta_0, \eta_1, \eta_2) := D_A \xi \). We fix a smooth reference connection \( A_0 \in A(P) \) and remark that by the assumptions \( A \in W^{2,p}(I \times \Sigma) \) and \( p > 3 \) the difference \( L_A := \Delta_A - \Delta_{A_0} \) is a first order operator with continuous coefficients. Now \( \alpha_0 \) satisfies

\[
(\frac{d}{ds} + \Delta_{A_0}) \alpha_0 = \xi_0 - L_A \alpha_0 + *[\ast F_A \wedge \alpha_0] + [d_A^* F_A \wedge \varphi] - d_A \omega + \delta \alpha_0,
\]

and the right-hand side of this equation is in \( L^p(I \times \Sigma) \). The estimate for \( \alpha_0 \) thus follows from a standard parabolic estimate for the linear heat operator \( \frac{d}{ds} + \Delta_{A_0} \). To estimate the terms \( \alpha_1 \) and \( \psi \) we define the linear operator

\[
B := \begin{pmatrix} 0 & -d_A \\ -d_A^* & 0 \end{pmatrix}
\]

acting on pairs \((\alpha_1, \psi)^T\) and denote by

\[
M := [B, \partial_s] = \begin{pmatrix} 0 & \hat{\alpha} \wedge \cdot \\ - \ast [\hat{\alpha} \wedge \cdot] & 0 \end{pmatrix}
\]

its commutator with \( \partial_s \). We set

\[
L := (-\partial_s + B)(\partial_s + B) = -\partial_s^2 + B^2 + M =: \text{diag}(L_1, L_2) + M
\]

with \( L_1 = -\partial_s^2 + d_A d_A^* \) and \( L_2 = -\partial_s^2 + d_A^* d_A \). Recall the Laplace operator \( \hat{\Delta}_A = -\partial_s^2 + \Delta_A \) on ad-valued 1-forms over \( \mathbb{R} \times \Sigma \) as introduced in Section 6 and note that \( L_2 = \Delta_A \). Similarly, \( L_1 \) acts on \( \alpha_1 = d_A \varphi \) as

\[
L_1 \alpha_1 = (-\partial_s^2 + d_A d_A^* + d_A^* d_A) \alpha_1 - d_A^* d_A \varphi = \hat{\Delta}_A \alpha_1 + [\ast F_A \wedge \alpha_1] - [d_A^* F_A \wedge \varphi].
\]
We consider $-\partial_s + B$ as a bounded operator $L^p(I \times \Sigma) \to W^{-1,p}(I \times \Sigma)$ and denote by $K$ its norm. The claimed estimate then follows from elliptic regularity of the Laplace operator $\Delta_A : W^{1,p}(I \times \Sigma) \to W^{-1,p}(I \times \Sigma)$, because

$$c^{-1} (\| \alpha_1 \|_{W^{1,p}} + \| \psi \|_{W^{1,p}})$$

\[ \leq \| \hat{\Delta}_A \alpha_1 \|_{W^{-1,p}} + \| \hat{\Delta}_A \psi \|_{W^{-1,p}} + \| \alpha_1 \|_{L^p} + \| \psi \|_{L^p} \]

\[ \leq \| L_1 \alpha_1 \|_{W^{-1,p}} + \| [\ast F_A \ast \alpha_1] - [d_A^* F_A \ast \varphi] \|_{W^{-1,p}} + \| L_2 \psi \|_{W^{-1,p}} \]

\[ + \| \alpha_1 \|_{L^p} + \| \psi \|_{L^p} \]

\[ \leq \| L(\alpha_1, \psi)^T \|_{W^{-1,p}} + \| M(\alpha_1, \psi)^T \|_{W^{-1,p}} + (1 + \| A \|_{L^\infty}) \| \alpha_1 \|_{L^p} \]

\[ + \| d_A^* F_A \|_{L^p} \| \varphi \|_{L^\infty} + \| \psi \|_{L^p} \]

\[ \leq K \| \partial_s \alpha_1 - d_A \psi + d_A \omega \|_{L^p} + \| d_A \omega \|_{L^p} + K \| \partial_s \psi - d_A^* \alpha_1 \|_{L^p} \]

\[ + (1 + \| A \|_{L^\infty}) \| \alpha_1 \|_{L^p} + \| d_A^* F_A \|_{L^p} \| \alpha_1 \|_{L^p} + \| \psi \|_{L^p}. \]

Here all norms are over the domain $I' \times \Sigma$. The fourth inequality follows from boundedness of the operator $M : L^p \to L^p$. Finally, the term $d_A \omega$ satisfies the estimate $\| d_A \omega \|_{L^p(I' \times \Sigma)} \leq c(p, A, I') \| \alpha \|_{L^p(I' \times \Sigma)}$ as follows from (C.1) by elliptic regularity. This completes the proof of the proposition. \[ \square \]

**Lemma Appendix C.2.** Let $f : (-\infty, T] \to \mathbb{R}$ be a bounded $C^2$ function with $f \geq 0$, and such that the differential inequality

$$f'' \geq c_0 f + c_1 f'$$

is satisfied for constants $c_0 > 0$ and $c_1 \in \mathbb{R}$. Then $f$ satisfies the decay estimate

$$f(s) \leq e^{-\lambda(t-s)} f(T)$$

for a constant $\lambda = \lambda(c_0, c_1) > 0$ and all $-\infty < s \leq T$.

**Proof:** Set

$$k := -\frac{c_1}{2} + \frac{1}{2} \sqrt{4c_0 + c_1^2} > 0 \quad \text{and} \quad \lambda := \frac{c_1}{2} + \frac{1}{2} \sqrt{4c_0 + c_1^2} > 0. \!
$$

Then $k\lambda = -c_0$ and $k-\lambda = -c_1$. Assume by contradiction that $f'(s_0) - \lambda f(s_0) < 0$ for some $s_0 \leq T$ and set $g(s) := e^{ks}(f'(s) - \lambda f(s))$. Then

$$g' = e^{ks}(f'' + (k-\lambda)f' - k\lambda f) = e^{ks}(f'' - c_1 f' - c_0 f) \geq 0,$$

so $g$ is monotone increasing. Therefore $g(s) \leq g(s_0)$ for all $s \leq s_0$ and hence

$$f'(s) \leq \lambda f(s) + e^{k(s-s_0)}(f'(s_0) - \lambda f(s_0)).$$

Because $f$ is bounded and $f'(s_0) - \lambda f(s_0) < 0$ by assumption, it follows that $f'(s) \to -\infty$ as $s \to -\infty$. This contradicts the boundedness of $f$ as $f(s_0) -
\[ f(s) = \int_s^0 f'(\sigma) d\sigma. \] Therefore the assumption was wrong and \( f'(s) - \lambda f(s) \geq 0 \) holds for all \(-\infty < s \leq T\). Then with \( h(s) := e^{-\lambda s} f(s) \) it follows that

\[ h' = e^{-\lambda s} (f' - \lambda f) \geq 0, \]

which implies \( f(s) \leq e^{-\lambda (T-s)} f(T) \) for all \( s \leq T \). \hfill \Box

**Proposition Appendix C.3.** For every map \( u \in L^2(\Sigma) \) there holds the estimate \( \|u\|_{W^{-1,2}(\Sigma)} \leq \|u\|_{L^2(\Sigma)} \).

**Proof:** The result follows from

\[
\|u\|_{W^{-1,2}(\Sigma)} = \sup_{\|\varphi\|_{W^{1,2}(\Sigma)}=1} \left| \int_{\Sigma} u \varphi \right| \leq \sup_{\|\varphi\|_{W^{1,2}(\Sigma)}=1} \|u\|_{L^2(\Sigma)} \|\varphi\|_{L^2(\Sigma)} \leq \|u\|_{L^2(\Sigma)} \sup_{\|\varphi\|_{W^{1,2}(\Sigma)}=1} \|\varphi\|_{W^{1,2}(\Sigma)} = \|u\|_{L^2(\Sigma)},
\]

where we used the Cauchy-Schwarz inequality in the second step. \hfill \Box

In the following two lemmata, we let \( \Delta = d^* d \) denote the (positive semidefinite) Laplace operator on functions on the Riemann surface \((\Sigma, g)\), and \( L_\Sigma := \partial_\Sigma + \Delta \) the corresponding heat operator. We also use the notation \( B_r(x) \) for the open ball of radius \( r > 0 \) around \( x \in \Sigma \), and \( P_r(x) := (-r^2, 0) \times B_r(x) \) for the corresponding parabolic cylinder.

**Lemma Appendix C.4 (Parabolic mean value inequality).** There is a constant \( c > 0 \) such that the following holds for all \( r > 0 \) smaller than the injectivity radius of \( \Sigma \). If \( a \geq 0 \) and the function \( u : \mathbb{R} \times \Sigma \rightarrow \mathbb{R} \) is of class \( C^1 \) in the \( s \) variable and of class \( C^2 \) in the spatial variables such that

\[ u \geq 0 \quad \text{and} \quad L_\Sigma u \leq au, \]

then for all \( x \in \Sigma \),

\[ u(x) \leq \frac{e^{ar^2}}{r^4} \int_{P_r(x)} u. \] \hfill (C.3)

**Proof:** For a proof we refer to [23, Lemma B.2]. \hfill \Box

**Lemma Appendix C.5.** Let \( R, r > 0 \), \( u : P_{R+r} \rightarrow \mathbb{R} \) be a \( C^2 \) function and \( f, g : P_{R+r} \rightarrow \mathbb{R} \) be continuous functions such that

\[ -L_\Sigma u \geq g - f, \quad u \geq 0, \quad f \geq 0, \quad g \geq 0. \]

Then

\[
\int_{P_{R/2}} g \leq 2 \left( 1 + \frac{r}{R} \right) \left( \int_{P_{R+r}} f + \left( \frac{4}{r^2} + \frac{1}{Rr} \right) \int_{P_{R+r}} u \right).
\]

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Proof: For a proof we refer to [23, Lemma B.5].

Lemma Appendix C.6 (Interpolation). Let $r, r', s, s' \in \mathbb{R}$. The intersection $W^{r,s;2}(I \times \Sigma) \cap W^{r',s';2}(I \times \Sigma)$ is a Banach space with norm $(\parallel \cdot \parallel_{W^{r,s;2}(I \times \Sigma)} + \parallel \cdot \parallel_{W^{r',s';2}(I \times \Sigma)})^{\frac{1}{2}}$. For any $\theta \in [0,1]$ there is a bounded embedding

$W^{r,s;2}(I \times \Sigma) \cap W^{r',s';2}(I \times \Sigma) \hookrightarrow W^{\theta r+(1-\theta)r',\theta s+(1-\theta)s';2}(I \times \Sigma)$.

Proof: For a proof we refer to [35, Lemma A.0.3].

References


