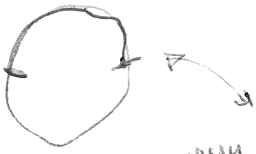
Chapter 6 | Outlook

Other pbs / generalisations

(1) Boundary Condition: Homogeneous Dirichlet: "$u = 0$ on $\partial\Omega$"

Γ_1: Dirichlet

in-hom-Dir "$u = \varphi$ on $\partial\Omega$"



Neumann: $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$

Mixed bdry cond

(or.. $\frac{\partial u}{\partial n} = \vec{\varphi}$ on $\partial\Omega$)

Γ_2: Neumann

Robin (3rd) $au + b\frac{\partial u}{\partial n} = \varphi$ on $\partial\Omega$)

(Steklov ?).

Generally: $B(x, u, Du) = 0$ on $\partial\Omega$ (1st order PDE on $\partial\Omega$ - or ΨDE..)

(2) In $-\Delta u = h$, $h \in L^2(\Omega)$ (and generalisations), could study case when $h \in H^{-1}(\Omega)$ $(= [H_0^1(\Omega)]')$

(ref. PDE2 ; [E] p. 320)

(3) $-\Delta + q \rightsquigarrow -\sum_{i,j} \partial_i (a_{ij}(x) \partial_j) + q(x)$, $q(x) \in L^{N/2}(\Omega)$

$A(x) = a_{ij}(x)$ unif. elliptic, $L^\infty(\Omega)$, measurable / cont.

($+ \bar{b} \cdot \nabla$ if not variational / symmetric)

(4) p-Laplace etc : ("non-Newtonian fluid mechanics")

If, in the variational formulation one replaces

$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx$ (Dirichlet integral; corresponds to kinetic energy / dispersion relation $\frac{p^2}{2m}$; $m = 1$; $p \rightsquigarrow -i\nabla$)

with

$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx$ (not same p !)

16 then $I'(u)$ gets (instead of $-\Delta u = 0$), a quasi-linear eq

$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ $\quad -\operatorname{div}(A(x,u,\nabla u) \nabla u) = 0$

[BS] pp. 86 - 93

(5) More general non-linearity than $f(u)$:

Instead of $F(u) = \int_{\mathbb{R}^N} F(x,u(x))\,dx$

could have $G(u) = \int_{\mathbb{R}^N} (K * u)(x)\,dx$

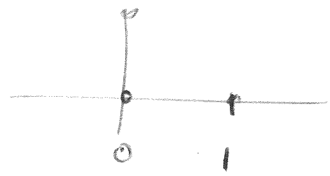
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \dots \quad ??$$

Ex: Choquard - Pekar!

---

(6) Other pb.'s: $N=1$: ODE
$$\begin{cases} u''(x) = f(x,u(x),u''(x)) \\ u(0) = u(1) = 0 \end{cases}$$

or periodic (unknown period!)



(some T!)
$$\begin{cases} u''(x) = f(x,u(x),u''(x)) \\ u(x) = u(x+T) \end{cases}$$

---

(7) Study classical solutions ($u \in C^2(\Omega) \cap C^0(\bar{\Omega})$) instead of weak solutions – most often in Hölder spaces $C^{k,\alpha}(\Omega)$ (i.e. $C^{2,\alpha}(\Omega)$; $C^{\alpha}(\Omega)$).

Nonlinear FA + various methods (Fixpoint, monotone iteration etc) can be made to work here

IFT (see website for literature)

Compactness properties are provided using Arzelà - Ascoli.

(8) Plateau's Problem

$\boxed{\text{Minimal surfaces}}$ Given a curve in $\mathbb{R}^3$, a surface which is a critical (!) pt. for the surface functional (see Struwe pp. 19–25)
(also other geometrical functionals, f.ex. Wilmore)
FA-setup ~~...~~ often more complicated
– needed "tools" the same (generalised sol.'s/objects, compactness-properties etc.)
— (huge topic, (prescribed mean curvature) huge history).

(9) $\boxed{\text{Hamiltonian systems}}$

(*) $\dot{x} = J \nabla H(x)$ $J = \begin{pmatrix} 0 & -I_n \\ -I_n & 0 \end{pmatrix}$, $H: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$
Goal: Understand 'global' structure of set of trajectories of (*) & their asymptotic behavior. Complex! So do study of: stat. pts, periodic orbits, invariant tori etc. (H: convex very often)

(10) $\boxed{\text{Duality Method}}$ (Clarke + Ekeland)
Via Legendre transform (also, Legendre-Fenchel Transform).

(more later maybe...) (see next page)

(11) $\boxed{\text{Regularity}}$ Weak sol. is in H^1 – then prove is H^2; prove $W^{2,p}$ & $C^{2,\alpha}$-reg. & estimates (see website for ref.'s)

(12) $\boxed{\text{Systems / Vector valued case}}$ (already mentioned)
Also, when value taken in submanifold, f.ex. on $S^n \subseteq \mathbb{R}^n$ ("Harmonic maps").

(13) "Any" functional inequality: Just as

$$\lambda_1(-\Delta) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}$$

i.e. ~~...~~ related to the ~~...~~ Poincaré inequality, any "functional inequality" is nothing ~~...~~ but a pb. in Calc. of Var.

16)

More on:

[Hamilton systems] [Thm] $H \in C^1(\mathbb{R}^{2N})$, strictly convex, non-negative, coercive, $H(0)=0$.

$\forall \alpha > 0$ $\exists$ periodic $x \in C^1(\mathbb{R};\mathbb{R}^{2N})$ solving $\circledast$ with $H(x(t)) = \alpha$ $\forall t$. The period T is not specified.

Idea: $\circledast$ is E-L eq. of $E(x) = \frac{1}{2}\int_0^1 \langle x, J\dot{x}\rangle dt$

on $C_\alpha = \{x \in C^1(\mathbb{R};\mathbb{R}^{2N}) \mid x(t+1) = x(t), \int_0^1 H(x(t))\,dt = \alpha\}$.

(pf: $x \in C_\alpha$ crit. of E
$\Rightarrow$ $\dot{x} = T J \nabla H(x)$, some T - rescaling time by factor T gives T-periodic sol. on energy surface $H = \alpha$. (see Struwe, Mawhin-Willem)

(More: Struwe p. 62 [!])

~~[illegible]~~ : ~~[illegible]~~

Themes / techniques not touched upon:

(1) Inverse & implicit Function Th's in Banach spaces ("nonlin. FA") (including "Lyapunov-Schmidt reduction")

(2) Degree-theory / Leray-Schauder top. degree. (incl. Brouwer degree)

(3) Bifurcation theory (also related to (1))

(4) Linking ("generalisation" of MPT / SPT).

For ref's, see website.

---

(5) "Lack of compactness" (see next pages)

[45] [B]

Outlook

Lack of compactness (& how to possibly restore it...)

Recall: For pb's of the form: Div. BVP for

$$-\Delta u + q(x) = f(u) + h(x) \text{ in } \Omega$$

with $\Omega \subseteq \mathbb{R}^N$ open & bounded,

the compact embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$

$\forall\, p < p^* = \frac{2N}{N-2}$

plays crucial role.

Do not have, f.ex., for $\Omega \equiv \mathbb{R}^N$, or if

$f(t) = |t|^{2^*-2}t$ is critical

(i) For $\Omega = \mathbb{R}^N$: The invariance of $\mathbb{R}^N$ wrt. translations

means that, f.ex., $\{u_k(x)\}_{k\in\mathbb{N}}$, $u(x+ky) = u_k(x)$ (*)

$y \in \mathbb{R}^N$, $u \in H^1(\mathbb{R}^N)$

is bounded in $H^1(\mathbb{R}^N)$ but has no convergent subseq

in any $L^p(\mathbb{R}^N)$

(ii) 1. Idea: Remove invariance by studying subspace not invariant by transl.

Ex: $H_r = \{u \in H^1(\mathbb{R}^N) \mid u \text{ is radial}\}$

$D_r = \{u \in D^{1,2}(\mathbb{R}^N) \mid u \text{ is radial}\}$ with

$D^{1,2}(\mathbb{R}^N) = \left\{u \in L^{2^*}(\mathbb{R}^N) \mid \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^N)\right\} \supseteq C_0^\infty(\mathbb{R}^N)$

and $\langle u,v\rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx$ is scalarproduct;

$H^1(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N)$ $\left(u(x) = \frac{1}{(1+|x|)^{N/2}} \notin L^2(\mathbb{R}^N). \right)$

$D^{1,2}(\mathbb{R}^N) \longrightarrow L^{2^*}(\mathbb{R}^N)$ (not compact)

Note: $H_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in [2, 2^*]$

[Lemma] This embedding is compact if $p \in (2, 2^*)$.

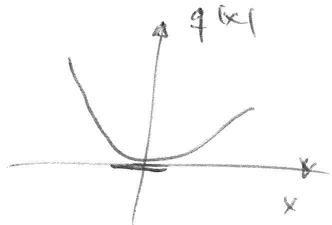
(Further tools: Schwarz / spherical symmetrization - see L.-L.)

[Lemma] $$\int_{\mathbb{R}^N} |\nabla u^*|^2 \le \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$$

$$|u^*|_p = |u|_p \quad \forall p > 1.$$

(Now Nehari, etc. can be used, see [BS])

(ii) 2. Idea: Restore compactness via $q(x)$:



$(q)_1$ $\quad \inf_{x \in \mathbb{R}^N} q(x) > 0, \quad \lim_{|x| \to +\infty} q(x) = +\infty$

(then maybe $\int_{\mathbb{R}^N} q(x) u^2 = +\infty$ for $u \in H^1(\mathbb{R}^N)$)

Let $$X = \left\{ u \in H^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} q(x) u^2 \, dx < +\infty \right\}.$$

$$X \hookrightarrow H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad p \in [2, 2^*]$$

([Lemma] This embedding is compact if $p \in [2, 2^*)$)

(Now Nehari etc can be used, see [BS])

(iii) 3. Idea:

$(q)_2$ $\quad q$ cont. & $\quad 0 < \delta = \inf_{\mathbb{R}^N} q < \sup_{\mathbb{R}^N} q = \alpha < +\infty$

(now, X above not compactly embedded)

(Nehari works! - almost) not ~~restore~~ compactness

- but carefully classify the possible behaviours

of minimizing seq.'s, and rule out some

possibilities (s. ex. like $(*)$ above)

(simple example of refined theory, called "Concentration - Compactness" - by P-L-Lions)

goal: To ensure strong enough convergence (in some L^p-sense, of minimum seq. (bdd in $H^1(\mathbb{R}^N)$ say) so weak-H^1-conv.)

(ex [BS] early p. 119)

(2) Critical exponent: $f(t) = |t|^{2^*-2} t$

ex: $-\Delta u = |u|^{2^*-2} u \qquad u \in D^{1,2}(\mathbb{R}^N)$

now pb. also invariant by scaling.

$J(v_\lambda) = J(v) \quad ; \quad v_\lambda(x) = \lambda^{\frac{N-2}{2}} v(\lambda x)$

$v \in C_0^\infty(\mathbb{R}^N)$

(**) $v_\lambda \longrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$ (spreading / concentration) for $\lambda \to 0$ and $\lambda \to +\infty$.

(Invariance by translation removed via D_r as before)
Then minimise over L^{2^*}-sphere - and a hard analysis to rule out the spreading / concentration in (**)