A constructive theory of uniformity and its application to integration theory

Hajime Ishihara

October 20, 2025

Abstract

1 Introduction

Errett Bishop, the founder of *neutral constructive mathematics* (or *Bishop's constructive mathematics*), wrote in his book as follows [6, Appendix A: Metrizability and Separability].

> *The situation is easily summarized: Nonmetric spaces and nonseparable metric spaces play no significant role in those parts of analysis with which this book is concerned. To illustrate this point, consider the concept of a uniform space, as developed in Probs. 17 to 21 of Chap. 4. A uniform space at first sight appears to be a natural and fruitful concept for constructive mathematics, a promising substitute for the concept of a topological space. In fact, this is not the case.*

Bishop introduced the notion of a uniform space using a *family of pseudometrics* in [6, Problem 17 of Chapter 4], and adopted a rather unsatisfactory notion of a *complete uniform space*: a uniform space is complete if it is uniformly equivalent to a complete *metric* space. Although classically,

- the topology on the space l_∞ of bounded sequences of real numbers is given by the norm

$$\|(x_n)_{n\in\mathbb{N}}\| = \sup\{|x_n| \mid n \in \mathbb{N}\};$$

- the *strong topology* on the dual space (the set of bounded linear functionals) E^* of a normed space E is given by the norm

$$\|f\| = \sup\{|f(x)| \mid x \in E, \|x\| \leq 1\},$$

constructively, they are uniform topologies but are *not* given by any family of pseudometrics. We need a *more general framework* than a family of pseudometrics for defining the notion of a uniform space.

Here, as an application of a general framework for uniform spaces, we consider integration theory. One of the motivations Lebesgue developed his integration theory was to make integration and limit commute:

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n,$$

which does not hold for the Riemann integral. The Lebesgue integral is based on the Lebesgue *measure* which is a generalisation of the notions of a length, an area and a volume. Since a measure is defined on a σ-algebra which is closed under the *complementation*, the lack of *law of excluded middle* in constructive mathematics brings us a difficulty to define an appropriate domain of a measure. Bishop overcame the difficulty by introducing the notion of a *complemented set*, and developed a constructive measure and integration theory. However, the original motivation of Lebesgue is concerned with the *topological notion* of a limit. Classicaly, the notion of a convergence induces a closure operation; hence defines a topology. As far as we are concerned with convergence theorems such as the *monotone and dominated convergence theorems* of Lebesgue, we may be able to constructively deal with them topologically without invoking the notion of a measure and the notion of a complemented set. Spitters [17] took an approach using Bishop's notion of a *uniform space*, and following Bishop's advice [6, Preface].

2 Preliminaries

2.1 Intuitionistic logic

In natural deduction, *minimal logic* is formalised by introduction (I) and elimination (E) rules corresponding to each logical connective $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication), $\neg$ (negation), $\forall$ and $\exists$ (universal and existential quantifiers) as follows.

$$\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi & \psi\end{matrix}}{\varphi\wedge\psi}\ \wedge\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}\\ \varphi\wedge\psi\end{matrix}}{\varphi}\ \wedge\mathrm{E}_r \quad \dfrac{\begin{matrix}\mathcal{D}\\ \varphi\wedge\psi\end{matrix}}{\psi}\ \wedge\mathrm{E}_l$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}}{\varphi\vee\psi}\ \vee\mathrm{I}_r \quad \dfrac{\begin{matrix}\mathcal{D}\\ \psi\end{matrix}}{\varphi\vee\psi}\ \vee\mathrm{I}_l \qquad\qquad \dfrac{\begin{matrix} & [\varphi] & [\psi]\\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3\\ \varphi\vee\psi & \chi & \chi\end{matrix}}{\chi}\ \vee\mathrm{E}$$

$$\dfrac{\begin{matrix}[\varphi]\\ \mathcal{D}\\ \psi\end{matrix}}{\varphi\to\psi}\ {\to}\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi\to\psi & \varphi\end{matrix}}{\psi}\ {\to}\mathrm{E}$$

$$\dfrac{\begin{matrix}[\varphi]\\ \mathcal{D}\\ \bot\end{matrix}}{\neg\varphi}\ \neg\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \neg\varphi & \varphi\end{matrix}}{\bot}\ \neg\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}}{\forall y\varphi[x/y]}\ \forall\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}\\ \forall x\varphi\end{matrix}}{\varphi[x/t]}\ \forall\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \varphi[x/t]\end{matrix}}{\exists x\varphi}\ \exists\mathrm{I} \qquad\qquad \dfrac{\begin{matrix} & [\varphi]\\ \mathcal{D}_1 & \mathcal{D}_2\\ \exists y\varphi[x/y] & \chi\end{matrix}}{\chi}\ \exists\mathrm{E},$$

where $\bot$ denotes an absurdity (or a contradiction); there are some variable conditions for $\forall$I, $\forall$E $\exists$I and $\exists$E.

Intuitionistic logic is obtained by adding the following left rule, called EFQ (*ex falso quodlibet*), to minimal logic, and *classical logic* is obtained by adding the right rule, called RAA (*reductio ad absurdum*) to intuitionistic logic.

$$\dfrac{\begin{matrix}\mathcal{D}\\ \bot\end{matrix}}{\varphi}\ \mathrm{EFQ} \qquad\qquad \dfrac{\begin{matrix}[\neg\varphi]\\ \mathcal{D}\\ \bot\end{matrix}}{\varphi}\ \mathrm{RAA}$$

By using RAA, we can prove double negation elimination (DNE : $\neg\neg\varphi\to\varphi$) and law of excluded middle (LEM : $\varphi\vee\neg\varphi$) as follows.

$$\dfrac{\dfrac{\dfrac{[\neg\neg\varphi]^2 \quad [\neg\varphi]^1}{\bot}\ \neg\mathrm{E}}{\varphi}\ \mathrm{RAA}^1}{\neg\neg\varphi\to\varphi}\ {\to}\mathrm{I}^2 \qquad \dfrac{\dfrac{[\neg(\varphi\vee\neg\varphi)]^2 \quad \dfrac{\dfrac{\dfrac{[\neg(\varphi\vee\neg\varphi)]^2 \quad \dfrac{[\varphi]^1}{\varphi\vee\neg\varphi}\ \vee\mathrm{I}_r}{\bot}\ \neg\mathrm{E}}{\neg\varphi}\ \neg\mathrm{I}^1}{\varphi\vee\neg\varphi}\ \vee\mathrm{I}_l}{\bot}\ \neg\mathrm{E}}{\varphi\vee\neg\varphi}\ \mathrm{RAA}^2$$

Conversely, using DNE and LEM, we can simulate RAA in minimal logic and intuitionistic logic, respectively, as follows.

$$\dfrac{\neg\neg\varphi\to\varphi \quad \dfrac{\begin{matrix}[\neg\varphi]^1\\ \mathcal{D}\\ \bot\end{matrix}}{\neg\neg\varphi}\ \neg\mathrm{I}^1}{\varphi}\ {\to}\mathrm{E} \qquad\qquad \dfrac{\varphi\vee\neg\varphi \quad [\varphi]^1 \quad \dfrac{\begin{matrix}[\neg\varphi]^1\\ \mathcal{D}\\ \bot\end{matrix}}{\varphi}\ \mathrm{EFQ}}{\varphi}\ \vee\mathrm{E}^1$$

Note that $\neg$I is a rule showing $\neg\varphi$ by deducing a contradiction from φ, and RAA is a rule showing φ by deducing a contradiction from $\neg\varphi$. For example, we learn, say in high school mathematics, a proof $\mathcal{D}_{\sqrt{2}}$ deducing a contradiction from an assumption "$\sqrt{2}$ is rational".

$$\begin{matrix}\sqrt{2}\text{ is rational}\\ \mathcal{D}_{\sqrt{2}}\\ \bot\end{matrix}$$

The following left deduction is a proof of "$\sqrt{2}$ is irrational" by using $\neg$I (without using RAA), and the right deduction is a proof of "$\sqrt{2}$ is irrational" by using RAA, if "$\sqrt{2}$ is irrational" is defined to be $\neg(\sqrt{2}$ is rational).

$$\dfrac{\begin{matrix}[\sqrt{2}\text{ is rational}]^1\\ \mathcal{D}_{\sqrt{2}}\\ \bot\end{matrix}}{\sqrt{2}\text{ is irrational}}\,\neg\mathrm{I}^1 \qquad\qquad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg(\sqrt{2}\text{ is irrational})]^2 \quad [\sqrt{2}\text{ is irrational}]^1}{\bot}\,\neg\mathrm{E}}{\sqrt{2}\text{ is rational}}\,\mathrm{RAA}^1\\ \mathcal{D}_{\sqrt{2}}\\ \bot\end{matrix}}{\sqrt{2}\text{ is irrational}}\,\mathrm{RAA}^2$$

Intuitionistic logic, just lacking RAA, is consistent with, and has less deducible formulae than classical logic used in usual mathematical practice. However, as BHK-interpretation and Curry-Howard isomorphism show, intuitionistic logic has an affinity for computation.

2.2 Constructive set theory

In general, the language of a set theory contains variables for sets and the binary predicates $=$ and $\in$. We use the following notations:

$$\begin{aligned}\forall x \in a\, \varphi(x) &\equiv \forall x(x \in a \to \varphi(x)); & \exists x \in a\, \varphi(x) &\equiv \exists x(x \in a \wedge \varphi(x));\\ a \subseteq b &\equiv \forall x(x \in a \to x \in b); & a \between b &\equiv \exists x(x \in a \wedge x \in b);\\ 0 &\equiv \emptyset; & x+1 &\equiv x \cup \{x\};\end{aligned}$$

note that $n = \{0, \dots, n-1\}$.

Definition 2.1. The axioms and rules of **iZF** are those of *intuitionistic predicate logic* with equality. In addition, **iZF** has the set theoretic axioms of the classical Zermelo-Fraenkel set theory **ZF**:

Extensionality: $\forall a \forall b[\forall x(x \in a \leftrightarrow x \in b) \to a = b]$;

Pairing: $\forall a \forall b \exists c \forall x(x \in c \leftrightarrow x = a \vee x = b)$;

Emptyset: $\exists a \forall x(x \notin a)$;

Union: $\forall a \exists b \forall x[x \in b \leftrightarrow \exists y \in a\,(x \in y)]$;

Separation:

$$\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \varphi(x)]$$

for all formula $\varphi(x)$;

Replacement:

$$\forall a[\forall x \in a \exists! y \varphi(x, y) \to \exists b \forall y(y \in b \leftrightarrow \exists x \in a\, \varphi(x, y))]$$

for all formulae $\varphi(x, y)$, where b is not free in $\varphi(x, y)$;

Powerset: $\forall a \exists b \forall x[x \in b \leftrightarrow x \subseteq a]$;

Infinity: $\exists a[0 \in a \wedge \forall x(x \in a \to x + 1 \in a)]$;

$\in$-Induction:

$$\forall a(\forall x \in a \varphi(x) \to \varphi(a)) \to \forall a \varphi(a),$$

for all formula $\varphi(a)$.

Remark 2.2. The intuitionistic Zermelo-Fraenkel set theory **IZF**, introduced by Friedman [**?**] and Myhill [**?**], adopts the following axiom of *collection* instead of the axiom of replacement.

Collection:

$$\forall a[\forall x \in a\, \exists y \varphi(x, y) \to \exists b \forall x \in a\, \exists y \in b\, \varphi(x, y)]$$

for all formulae $\varphi(x, y)$, where b is not free in $\varphi(x, y)$.

Classically, they are equivalent, but, constructively, Replacement is weaker than Collection; hence **iZF** is weaker than **IZF**.

The classical Zermelo-Fraenkel set theory **ZF** adopts, instead of $\in$-Induction, the following classically equivalent axiom of *foundation*.

Foundation:

$$\exists x \varphi(x) \to \exists x[\varphi(x) \wedge \forall y(y \in x \to \neg\varphi(y))]$$

for every formula $\varphi(x)$.

However, constructively, it implies DNE, but $\in$-Induction not; see [**?**].

Proposition 2.3. *The axiom of foundation implies* DNE.

Proof. Consider a foumula φ such that $\neg\neg\varphi$, and a set a given by

$$a = \{y \in \{0, 1\} \mid y = 1 \vee (y = 0 \wedge \varphi)\}.$$

Note that $1 \in a$ and $0 \in a$ if and only if φ. Then, since

$$\exists x(x \in a) \to \exists x[x \in a \wedge \forall y(y \in x \to \neg(y \in a))],$$

by Foundation, there exists $x \in a$ such that $\neg(y \in a)$ for all $y \in x$. If $x = 1 = \{0\}$, then $\neg(0 \in a)$; hence $\neg\varphi$, a contradiction. Therefore $x = 0$, and so φ. $\square$

The *axiom of choice* may be used freely in the practice of classical mathematics. However, constructively, it implies DNE. Here we consider the following form of the axiom of choice:

$$\forall a[\forall x \in a \exists y(y \in x) \to \exists f \in (\bigcup a)^a\, \forall x \in a\, (f(x) \in x)].$$

Proposition 2.4. *The axiom of choice implies* DNE.

Proof. For each formula φ with $\neg\neg\varphi$, define sets x_0 and x_1 as follows:

$$x_0 = \{y \in \{0,1\} \mid y = 0 \vee \varphi\}, \qquad x_1 = \{y \in \{0,1\} \mid y = 1 \vee \varphi\},$$

and let $a = \{x_0, x_1\}$. Then, since $0 \in x_0$ and $1 \in x_1$, we have

$$\forall x \in a\, \exists y (y \in x).$$

Hence, by the axiom of choice, there exists a function $f : a \to \bigcup a = \{0,1\}$ such that

$$\forall x \in a\, (f(x) \in x).$$

Note that if φ, then, since $x_0 = x_1$, we have $f(x_0) = f(x_1)$. If $f(x_0) = 0$ and $f(x_1) = 1$, then $\neg\varphi$, a contradiction. Therefore either $f(x_0) = 1$ or $f(x_1) = 0$, and so $1 \in x_0$ or $0 \in x_1$. Thus φ. $\square$

The set $\mathrm{Pow}(1)$ is considered as the set of truth values; note that $1 = \{0\} = \{\emptyset\}$. For a formula φ, a subset $[\![\varphi]\!]$ of 1 given by

$$[\![\varphi]\!] = \{u \in 1 \mid \varphi\},$$

whrere u is not free in φ, is considerd as a truth value of φ. Note that $\{0,1\} \subseteq \mathrm{Pow}(1)$, but not $\mathrm{Pow}(1) \subseteq \{0,1\}$, constructively; $\mathrm{Pow}(1) \subseteq \{0,1\}$ implies DNE (Exercise). In **iZF**, we can show that $\mathrm{Pow}(1)$ is a subobject classifier in the category of sets and functions.

In constructive set theory, *predicativity* is considered very often. The following is an example of *impredicative* definition of a set:

$$\begin{aligned} S &= \{x \in \mathbb{N} \mid \forall a \in \mathrm{Pow}(\mathbb{N})\, (x \in a \to \cdots)\} \\ &= \{x \in \mathbb{N} \mid \forall a (a \subseteq \mathbb{N} \wedge x \in a \to \cdots)\}. \end{aligned}$$

The set S is a subset of $\mathbb{N}$, that is, $S \in \mathrm{Pow}(\mathbb{N})$; the variable a ranges over $\mathrm{Pow}(\mathbb{N})$; hence we may take the set S being defined as a. A *predicative* set theory does not allow this kind of circular argument (*vicious circle*) in defining sets; does allow only constructions of sets from sets already constructed. As the above example shows, we have to give up the axioms (full) Separation and Powerset to make a set theory predicative.

The elementary constructive (and predicative) set theory **ECST** was introduced by Aczel and Rathjen and is a subsystem of **CZF**; see their book draft [3] written by extending their research report [2].

Definition 2.5. The axioms and rules of **ECST** are those of *intuitionistic predicate logic* with equality. In addition, **ECST** has the set theoretic axioms Extensionality, Pairing, Emptyset, Union, Replacement and

Bounded Separation:

$$\forall a \exists b \forall x (x \in b \leftrightarrow x \in a \wedge \varphi(x))$$

for all *bounded* formulae $\varphi(x)$, where b is not free in $\varphi(x)$; here a formula $\varphi(x)$ is *bounded*, or Δ_0, if all its quantifiers are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$.

Strong Infinity:

$$\exists a[0 \in a \wedge \forall x(x \in a \to x+1 \in a) \\ \wedge \forall y(0 \in y \wedge \forall x(x \in y \to x+1 \in y) \to a \subseteq y)].$$

Remark 2.6. With (full) Separation, Infinity implies Strong Infinity: for, since there exists a set N_0 such that $0 \in N_0 \wedge \forall x(x \in N_0 \to x+1 \in N_0)$, by Infinity, there exists a set $\mathbb{N}$ given by

$$\mathbb{N} = \{z \in N_0 \mid \forall y(0 \in y \wedge \forall x(x \in y \to x+1 \in y) \to z \in y)\},$$

by (full) Separation; hence $\mathbb{N}$ is a unique set such that

$$0 \in \mathbb{N} \wedge \forall x(x \in \mathbb{N} \to x+1 \in \mathbb{N}) \\ \wedge \forall y(0 \in y \wedge \forall x(x \in y \to x+1 \in y) \to \mathbb{N} \subseteq y).$$

For a set theory without (full) Separation, like **ECST**, we have to adopt Strong Infinity instead of Infinity.

In **ECST**, we are able to perform basic set constructions in mathematical practice. First note that a set A is *inhabited* if $\exists x(x \in A)$, or $A \between A$, and *nonempty* if $\neg\forall x\neg(x \in A)$, or $\neg(A = \emptyset)$; "A is nonempty" is the double negation of "A is inhabited". Using Pairing, the *ordered pair*

$$(x, y) = \{\{x\}, \{x, y\}\}$$

of x and y, and, using Replacement and Union, the *cartesian product* $A \times B$ of sets A and B consisting of the ordered pairs (x, y) with $x \in A$ and $y \in B$ can be introduced in **ECST**.

A *relation* R between sets A and B is a subset of $A \times B$, and we then write $x\ R\ y$ for $(x, y) \in R$; the *inverse relation* $R^{-1} \subseteq B \times A$ of R is given by $y\ R^{-1}\ x \Leftrightarrow x\ R\ y$ for all $x \in A$ and $y \in B$. We write $R' \circ R$ for the *composition* of relations $R \subseteq A \times B$ and $R' \subseteq B \times C$, and Δ_A for the *diagonal* subset of $A \times A$.

A relation $R \subseteq A \times B$ is *total* (or is a *multivalued function*) if for every $x \in A$ there exists $y \in B$ such that $x\ R\ y$; *single valued* if for every $x \in A$ there exists at most one $y \in B$ such that $x\ R\ y$. The class of total relations between sets A and B is denoted by $\mathrm{mv}(A, B)$, or more formally

$$R \in \mathrm{mv}(A, B) \Leftrightarrow R \subseteq A \times B \wedge \forall x \in A\, \exists y \in B\, (x\ R\ y).$$

A *function* from a set A into a set B is a total and single valued relation $f \subseteq A \times B$, that is, for every $x \in A$ there is exactly one $y \in B$, denoted by $f(x)$, with $x\ f\ y$; we then write $f : A \to B$. The class of functions from a set A to a set B is denoted by B^A, or more formally

$$f \in B^A \Leftrightarrow f \in \mathrm{mv}(A, B) \wedge \forall x \in A\, \forall y, z \in B\, (x\ f\ y \wedge x\ f\ z \to y = z).$$

A function $f : A \to B$ is a *surjection* if for each $y \in B$, there exists $x \in A$ such that $y = f(x)$; an *injection* if for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$; a *bijection* if it is a surjection and an injection; we write $A \cong B$ if there is a bijection between A and B. The composition $g \circ f$ of functions $f : A \to B$

and $g : B \to C$ is a function from A into C, and the diagonal subset Δ_A is a bijection between A and A, called the *identity* function on A and denoted by id_A.

Let A and B be sets. Then the *projections* $\pi_0^{A,B} : A \times B \to A$ and $\pi_1^{A,B} : A \times B \to B$, given by

$$\pi_0^{A,B} : (x, y) \mapsto x, \qquad\qquad \pi_1^{A,B} : (x, y) \mapsto y$$

for all $(x, y) \in A \times B$, are surjective whenever A and B are inhabited; the superscripts of $\pi_0^{A,B}$ and $\pi_1^{A,B}$ will be mostly omitted. The cartesian product $A \times B$ of sets A and B is the *product* of A and B in the category of sets and functions: for each set C and each pair of functions $f : C \to A$ and $g : C \to B$, there exists a unique function $\langle f, g\rangle : C \to A \times B$ such that $\pi_0 \circ \langle f, g\rangle = f$ and $\pi_1 \circ \langle f, g\rangle = g$.

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B
\end{tikzcd}$$

For functions $f : A \to C$ and $g : B \to D$, we write $f \times g$ for $\langle f \circ \pi_0^{A,B}, g \circ \pi_1^{A,B}\rangle : A \times B \to C \times D$.

In **ECST**, the structure $(\mathbb{N}, 0, S)$, where $0 = \emptyset$ and $S : \mathbb{N} \to \mathbb{N}$ is given by $S(x) = x + 1$ for all $x \in \mathbb{N}$, satisfies the *Dedekind-Peano axioms*:

1. $0 \neq S(x)$ for all $x \in \mathbb{N}$;
2. S is an injection;
3. if A is a subset of $\mathbb{N}$ such that $0 \in A$ and $S(x) \in A$ for all $x \in A$, then $A = \mathbb{N}$.

By mathematical induction which is a consequence of (3) above, we are able to show, without invoking LEM, that the equality $=$ on $\mathbb{N}$ is decidable, that is,

$$x = y \vee \neg(x = y)$$

for all $x, y \in \mathbb{N}$.

In this paper, we assume, in addition to the axioms of **ECST**, *Exponentiation Axiom* asserting that the class B^A of functions from a set A into a set B, called the *exponential* of A and B, forms a set:

Exponentiation: $\forall a \forall b \exists c \forall f (f \in c \leftrightarrow f \in b^a)$.

The exponential B^A of a set A and B is the *exponential* object of A and B in the category of sets and functions with a function $\mathrm{ev}^{A,B} : B^A \times A \to B$, given by

$$\mathrm{ev}^{A,B} : (f, x) \mapsto f(x)$$

for all $f \in B^A$ and $x \in A$; the superscripts of $\mathrm{ev}^{A,B}$ will be mostly omitted: for each set C and each function $h : C \times A \to B$, there exists a unique function $\hat{h} : C \to B^A$ such that $\mathrm{ev}^{A,B} \circ (\hat{h} \times \mathrm{id}_A) = h$.

$$\begin{tikzcd} C \times A \arrow[d, "\hat{h}\times \mathrm{id}_A"'] \arrow[dr, "h"] & \\ B^A \times A \arrow[r, "\mathrm{ev}"'] & B \end{tikzcd}$$

Note that for all sets A, B and C,

$$B^{C\times A} \cong (B^A)^C$$

with a bijection $\hat{(\text{-})} : h \mapsto \hat{h}$ between $B^{C\times A}$ and $(B^A)^C$, and

$$A^C \times B^C \cong (A \times B)^C$$

with a bijection $\langle\text{-},\text{-}\rangle : (f, g) \mapsto \langle f, g\rangle$ between $A^C \times B^C$ and $(A \times B)^C$.

In the presence of Exponentiation, functions on $\mathbb{N}$ are defined by primitive recursion.

Proposition 2.7. *Let A and B be sets, and let $f : B \to A$ and $g : B\times\mathbb{N}\times A \to A$ be functions. Then there exists a unique function $h : B \times \mathbb{N} \to A$ such that*

$$h(x, 0) = f(x), \qquad\qquad h(x, y + 1) = g(x, y, h(x, y))$$

for all $x \in B$ and $y \in \mathbb{N}$.

Proof. Note that Exponentiation implies *small iteration axiom* **SIA**, and see [3, Theorem 6.4.3]. $\square$

Therefore we can define addition, multiplication, predecessor and truncated subtraction funcitons on $\mathbb{N}$ by primitive recursion as follows.

$$\begin{array}{llll} x + 0 = x, & x + (y + 1) = (x + y) + 1; & x0 = 0, & x(y + 1) = (xy) + x; \\ \mathrm{prd}(0) = 0, & \mathrm{prd}(y + 1) = y; & x \dot{-} 0 = x, & x \dot{-} y + 1 = \mathrm{prd}(x \dot{-} y); \end{array}$$

hence the maximun function is given by $\max(x, y) = (x \dot{-} y) + y$.

A binary relation $R \subseteq A \times A$ on a set A is a *preorder* if it is reflexive and transitive; an *equivalence relation* if it is reflexive, symmetric and transitive. A *preordered set* is a pair $I = (\underline{I}, \preceq_I)$ of a set $\underline{I}$ and a preorder $\preceq_I$ on $\underline{I}$. Let $I = (\underline{I}, \preceq_I)$ and $I' = (\underline{I}', \preceq_{I'})$ be preordered sets. Then a function $f : \underline{I} \to \underline{I}'$ is *monotone* if

$$x \preceq_I y \Rightarrow f(x) \preceq_{I'} f(y)$$

for all $x, y \in \underline{I}$; we write $\hom(I, I')$ for the set of monotone functions between I and I'. The *product* of preordered sets $I = (\underline{I}, \preceq_I)$ and $I' = (\underline{I}', \preceq_{I'})$ is given by $I \times I' = (\underline{I} \times \underline{I}', \preceq_{I\times I'})$, where

$$(x, x') \preceq_{I\times I'} (y, y') \Leftrightarrow x \preceq_I y \text{ and } x' \preceq_{I'} y'$$

for all $(x, x'), (y, y') \in \underline{I} \times \underline{I}'$. We denote the constant function $i \mapsto i'$, where $i' \in \underline{I}'$, from $\underline{I}$ into $\underline{I}'$ by $\overline{i'}$; the *diagonal function* $i \mapsto (i, i)$ from $\underline{I}$ into $\underline{I} \times \underline{I}$ by δ_I; then $\overline{i'} \in \hom(I, I')$ and $\delta_I \in \hom(I, I \times I)$. A preordered set $I = (\underline{I}, \preccurlyeq_I)$ is *directed* if $\underline{I}$ is inhabited, and there exists $\mathrm{ub}_I \in \hom(I \times I, I)$, called an *upper bound function*, such that for each $x, y \in \underline{I}$, $x \preccurlyeq_I \mathrm{ub}_I(x, y)$ and $y \preccurlyeq_I \mathrm{ub}_I(x, y)$. It is straigtforward to see that the product of directed preordered sets is directed.

Let A be a set, and let R be an equivalence relation on A. Then the *quotient set*

$$A/R = \{[x]_R \mid x \in A\},$$

where $[x]_R = \{y \in A \mid x\ R\ y\}$ is an equivalence class of $x \in A$, is constructed by Replacement. However, we have to be careful with using quotient sets, because to pick up each representative of equivalence classes by a function $f : A/R \to A$, we have to invoke *the axiom of choice* which is not acceptable in constructive set theory. Therefore, we cannot uniformly refer structure and data of each representative, constructively.

2.3 Setoids

A *setoid* (or *Bishop set*) X is a pair $(\underline{X}, =_X)$ of a set $\underline{X}$ and an equivalence relation $=_X$ on $\underline{X}$; it is *discrete* if $x =_X y$ or $\neg(x =_X y)$ for all $x, y \in \underline{X}$; *stable* if $\neg\neg(x =_X y)$ implies $x =_X y$ for all $x, y \in \underline{X}$. If X is discrete, then it is stable. We may consider a set $\underline{X}$ together with the equality $=$ on sets as a setoid $(\underline{X}, =)$.

Let $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_Y)$ be setoids. Then a function $f : \underline{X} \to \underline{Y}$ is a *setoid mapping* (or simply, *mapping*) of X into Y if

$$x =_X y \Rightarrow f(x) =_Y f(y)$$

for all $x, y \in \underline{X}$, and we then write $f : X \to Y$; for each pair of functions $f, g : \underline{X} \to \underline{Y}$, f is a setoid mapping if and only if g is a setoid mapping, whenever $f = g$. Two setoid mappings $f, g : X \to Y$ are *identical*, denoted by $f \sim g$, if

$$x =_X y \Rightarrow f(x) =_Y g(y)$$

for all $x, y \in \underline{X}$, or equivalently $f(x) =_Y g(x)$ for all $x \in \underline{X}$; hence, if $f = g$, then $f \sim g$. A setoid mapping $f : X \to Y$ is a *setoid surjection* if for every $y \in \underline{Y}$ there exists $x \in \underline{X}$ such that $y =_Y f(x)$; a *setoid injection* if

$$f(x) =_Y f(y) \Rightarrow x =_X y$$

for all $x, y \in \underline{X}$; a *setoid bijection* if it is a surjection and an injection; we write $X \cong Y$ if there is a setoid bijection between X and Y.

The composition $g \circ f$ of setoid mappings $f : X \to Y$ and $g : Y \to Z$ is a setoid mapping of X into Z, and for all setoid mappings $f' : X \to Y$ and $g' : Y \to Z$, if $f \sim f'$ and $g \sim g'$, then $g \circ f \sim g' \circ f'$; the identity function $\mathrm{id}_{\underline{X}} : \underline{X} \to \underline{X}$ is a setoid bijection, denoted by id_X; for all setoid mappings $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$,

$$h \circ (g \circ f) \sim (h \circ g) \circ f \qquad \text{and} \qquad \mathrm{id}_Y \circ f \sim f \circ \mathrm{id}_X \sim f.$$

It is straightforward to show in **ECST**, by virtue of the following lemma, that for each setoid mapping $f : X \to Y$, f is a setoid surjection if and only if f is an epimorphism (that is, for each pair of setoid mapping $g, h : Y \to Z$, if $g \circ f \sim h \circ f$, then $g \sim h$), and f is a setoid injection if and only if f is a monomorphism (that is, for each pair of setoid mapping $g, h : Z \to X$, if $f \circ g \sim f \circ h$, then $g \sim h$), in the category of setoids and setoid mappings.

Lemma 2.8. *Let X and Y be setoids, and let $f : X \to Y$ be a setoid mapping. Then f is a setoid surjection whenever f is an epimorphism in the category of setois and setoid mappings.*

Proof. Let $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_Y)$, and suppose that for all $g, h : Y \to Z$, if $g \circ f \sim h \circ f$, then $g \sim h$. For each $y \in \underline{Y}$, let $c_y = [\![\exists x \in \underline{X}\, (y =_Y f(x))]\!]$. Then, since $\forall y \in \underline{Y}\, \exists! z(z = c_y)$,

$$C = \{z \mid \exists y \in \underline{Y}\, (z = c_y)\}$$

is a set by Replacement. Consider a setoid $Z = (\underline{Z}, =)$ where $\underline{Z} = C \cup \{1\}$ and $=$ is the equality $=$ on sets, and define $g, h : \underline{Y} \to \underline{Z}$ by $g(y) = c_y$ and $h(y) = 1$ for all $y \in \underline{Y}$. Then $g, h : \underline{Y} \to \underline{Z}$ are setoid mappings, and

$$(g \circ f)(x) = g(f(x)) = c_{f(x)} = 1 = h(f(x)) = (h \circ f)(x)$$

for all $x \in \underline{X}$; hence $g \circ f \sim h \circ f$. Therefore $g \sim h$, and so $c_y = g(y) = h(y) = 1$ for all $y \in \underline{Y}$. Thus $\forall y \in \underline{Y}\, \exists x \in \underline{X}\, (y =_Y f(x))$, that is, f is surjective. $\square$

Remark 2.9. For each pair of setoid mappings $f : X \to Y$ and $g : Y \to Z$, if f and g are setoid surjections, then so is $g \circ f$; if $g \circ f$ is a setoid surjection, then so is g; if f and g are setoid injections, then so is $g \circ f$; if $g \circ f$ is a setoid injection, then so is f.

A *subsetoid* S of a setoid $X = (\underline{X}, =_X)$ is a pair $(\underline{S}, \iota)$ of a set $\underline{S}$ and a function $\iota : \underline{S} \to \underline{X}$; in which case, $(\underline{S}, =_S)$ is a setoid, where $=_S$ is an equivalence relation on $\underline{S}$ given by

$$x =_S y \Leftrightarrow \iota(x) =_X \iota(y)$$

for all $x, y \in \underline{S}$, and $\iota : S \to X$ is a setoid injection. We may consider a subset $\underline{S}$ of $\underline{X}$ together with the inclusion function $\iota : \underline{S} \to \underline{X}$ as a subsetoid $S = (\underline{S}, \iota)$ of X.

The *cartesian product* of setoids $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_X)$, denoted by $X \times Y$, is a pair of the set $\underline{X} \times \underline{Y}$ and an equivalence relation $=_{X\times Y}$ on $\underline{X} \times \underline{Y}$ given by

$$(x, y) =_{X\times Y} (x', y') \Leftrightarrow x =_X x' \text{ and } y =_Y y'$$

for all $(x, y), (x', y') \in \underline{X} \times \underline{Y}$. It is straightforward to see that the projections $\underline{\pi_0^{X,Y}} : \underline{X} \times \underline{Y} \to \underline{X}$ and $\underline{\pi_1^{X,Y}} : \underline{X} \times \underline{Y} \to \underline{Y}$ are setoid mappings $\pi_0^{X,Y} : X \times Y \to X$ and $\pi_1^{X,Y} : X \times Y \to Y$, respectively. The product $X \times Y$ is the *product* of X and Y in the category of setoids and setoid mappings: for each setoid Z and each pair of setoid mappings $f : Z \to X$ and $g : Z \to Y$, there exists a unique setoid mapping $\langle f, g\rangle : Z \to X \times Y$ such that $\pi_0^{X,Y} \circ \langle f, g\rangle \sim f$ and $\pi_1^{X,Y} \circ \langle f, g\rangle \sim g$.

Let $X = (\underline{X}, =_X)$ be a setoid. Then an *apartness* $\#_X$ on X is a binary relation on $\underline{X}$ which is *irreflexive*: $x =_X y \Rightarrow \neg(x \#_X y)$ for all $x, y \in \underline{X}$, *symmetric*: $x \#_X y \Rightarrow y \#_X x$ for all $x, y \in \underline{X}$, and *cotransitive*:

$$x \#_X y \Rightarrow x \#_X z \text{ or } z \#_X y$$

for all $x, y, z \in \underline{X}$; it is *tight* if $\neg(x \#_X y) \Rightarrow x =_X y$ for all $x, y \in \underline{X}$. Note that an apartness $\#_X$ is a binary relation on the setoid X in the sense that

$$x \#_X y,\ x =_X x' \text{ and } y =_X y' \Rightarrow x' \#_X y'$$

for all $x, x', y, y' \in \underline{X}$: for if $x \mathrel{\#_X} y$, $x =_X x'$ and $y =_X y'$, then, since $\neg(x \mathrel{\#_X} x')$ and $\neg(y' \mathrel{\#_X} y)$ by irreflexivity and symmetry, we have $x' \mathrel{\#_X} y$, by cotransitivity; hence $x' \mathrel{\#_X} y'$, by cotransitivity. If X is a stable setoid, then the *denial inequality* $\neq_X$, given by

$$x \neq_X y \Leftrightarrow \neg(x =_X y)$$

for all $x, y \in \underline{X}$, is a tight apartness.

Let $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_X)$ be setoids with apartesses $\#_X$ and $\#_Y$, respectively. Then a setoid mapping $f : X \to Y$ is *strongly extensional* if

$$f(x) \mathrel{\#_Y} f(y) \Rightarrow x \mathrel{\#_X} y$$

for all $x, y \in \underline{X}$; an apartness $\#_{X\times Y}$ on the product $X \times Y$ is given by

$$(x, y) \mathrel{\#_{X\times Y}} (x', y') \Leftrightarrow x \mathrel{\#_X} x' \text{ or } y \mathrel{\#_Y} y'$$

for all $(x, y), (x', y') \in \underline{X} \times \underline{Y}$. It is straightforward to see that the projections $\pi_0^{X,Y} : X \times Y \to X$ and $\pi_1^{X,Y} : X \times Y \to Y$ are strongly extensional.

A *partial order* $\leq_X$ on X is a binary relation on $\underline{X}$ which is *reflexive*:

$$x =_X y \Rightarrow x \leq_X y$$

for all $x, y \in \underline{X}$, *antisymmetric*: $x \leq_X y$ and $y \leq_X x \Rightarrow x =_X y$ for all $x, y \in \underline{X}$ and *transitive*: $x \leq_X y$ and $y \leq_X z \Rightarrow x \leq_X z$ for all $x, y, z \in \underline{X}$; it is *total* (or *linear*) order if $x \leq_X y$ or $y \leq_X x$ for all $x, y \in \underline{X}$; *quasi-total* (or *quasi-linear*) order if $\neg(x \leq_X y) \Rightarrow y \leq_X x$ for all $x, y \in \underline{X}$; if $\leq_X$ is total, then it is quasi-total.

A (join) *semilattice* is a setoid $X = (\underline{X}, =_X)$ equipped with a setoid mapping $(x, y) \mapsto x \vee y$ of $X \times X$ into X, called a *join*, such that

$$x \vee (y \vee z) =_X (x \vee y) \vee z, \qquad x \vee y =_X y \vee x, \qquad x \vee x =_X x$$

for all $x, y, z \in \underline{X}$. Let $X = (\underline{X}, =_X)$ be a semilattice. Then the (canonical) partial order $\leq_X$ on X is given by

$$x \leq_X y \Leftrightarrow x \vee y =_X y$$

for all $x, y \in \underline{X}$, and $x \vee y$ is the least upper bound of $\{x, y\}$ for all $x, y \in \underline{X}$.

For an apartness $\#_X$ and a quasi-total order $\leq_X$ on X, a binary relation $<_X$ on X is given by

$$x <_X y \Leftrightarrow x \leq_X y \text{ and } x \#_X y$$

for all $x, y \in \underline{X}$.

Proposition 2.10. *Let $X = (\underline{X}, =_X)$ be a semilattice with an apartness $\#_X$ such that the canonical partial order $\leq_X$ is quasi-total. Then*

1. *if $x <_X y$, then $\neg(y <_X x)$;*
2. *$x \mathrel{\#_X} y$ if and only if $x <_X y$ or $y <_X x$;*
3. *if $x \leq_X y$, then $\neg(y <_X x)$, and if $\neg(y <_X x)$, then $x \leq_X y$ whenever $\#_X$ is tight*

for all $x, y \in \underline{X}$. Forthermore, suppose that the join $\vee : X \times X \to X$ is strongly extensional. Then

4. *if $x <_X y$ and $y <_X z$, then $x <_X z$;*

5. *if $x <_X y$, then $x <_X z$ or $z <_X y$*

for all $x, y, z \in \underline{X}$.

Proof. (1): Consider $x, y \in \underline{X}$ with $x <_X y$. If $y <_X x$, then, since $x \leq_X y$ and $y \leq_X x$, we have $x =_X y$ which contradicts $x \#_X y$. Hence $\neg(y <_X x)$.

(2): Consider $x, y \in \underline{X}$. Then it is trivial that $x <_X y$ or $y <_X x$ implies $x \#_X y$. Assume that $x \#_X y$. Then either $x \#_X (x \vee y)$ or $(x \vee y) \#_X y$. In the first case, since $\neg(y \leq_X x)$ by irreflexivity, we have $x \leq_X y$, by quasi-totality; hence $x <_X y$. In the second case, similarly, we have $y <_X x$.

(3): Consider $x, y \in \underline{X}$ with $x \leq_X y$. If $y <_X x$, then, since $y \leq_X x$ and $y \#_X x$, we have $x =_X y$, a contradiction. Hnece $\neg(y <_X x)$.

Suppose that $\#_X$ is tight, and consider $x, y \in \underline{X}$ such that $\neg(y <_X x)$ and $x \vee y \#_X y$. Then, since $\neg(x \leq_X y)$ by irreflexivity, we have $y \leq_X x$, by quasi-totality; hence $x \#_X y$ as $x \vee y =_X x$. Therefore $y <_X x$, a contradiction, and so $x \vee y =_X y$, by tightness. Thus $x \leq_X y$.

(4): Consider $x, y, z \in \underline{X}$ such that $x <_X y$ and $y <_X z$. Then, since $x \leq_X y$ and $y \leq_X z$, we have $x \leq_X z$. Since $x \vee y =_X y$, $z \vee y =_X z$ and $y \#_X z$, we have $(x \vee y) \#_X (z \vee y)$; hence either $x \#_X z$ or $y \#_X y$. Therefore, since the former must be the case, we have $x <_X z$.

(5): Consider $x, y, z \in \underline{X}$ with $x <_X y$. Then, since $x \#_X y$, either $x \#_X z$ or $z \#_X y$. In the first case, either $x <_X z$ or $z <_X x$, by (2); since $z <_X y$ in the latter case by (4), we have $x <_X z$ or $z <_X y$. In the second case, similarly, we have $x <_X z$ or $z <_X y$. ☐

2.4 Number systems and their algebraic structures

A *ring* is a setoid $X = (\underline{X}, =_X)$ equipped with a setoid mapping $(x, y) \mapsto x + y$ of $X \times X$ into X, called *addition*, a setoid mapping $x \mapsto -x$ of X into X, called *inverse*, a setoid mapping $(x, y) \mapsto xy$ of $X \times X$ into X, called *multiplication*, and an element 0 of $\underline{X}$, called the *zero element*, such that

$$(x + y) + z =_X x + (y + z), \quad x + y =_X y + x, \quad x + 0 =_X x, \quad x + (-x) =_X 0,$$
$$(xy)z =_X x(yz), \quad x(y + z) =_X xy + xz, \quad (y + z)x =_X yx + zx$$

for all $x, y, z \in \underline{X}$; it is *unitary* if there exists an element 1 of $\underline{X}$, called the *unity element*, such that $1x =_X x1 =_X x$ for all $x \in \underline{X}$; *commutative* if $xy =_X yx$ for all $x, y \in \underline{X}$.

The setoid of *integers* is given as follows. Let $\underline{\mathbb{Z}} = \mathbb{N} \times \mathbb{N}$, and let $=_\mathbb{Z}$ be an equivalence relation on $\underline{\mathbb{Z}}$ given by

$$a =_\mathbb{Z} b \Leftrightarrow m + n' = m' + n$$

for all $a = (m, n), b = (m', n') \in \underline{\mathbb{Z}}$. Then $\mathbb{Z} = (\underline{\mathbb{Z}}, =_\mathbb{Z})$ is a discrete setoid of integers. Define the addition $(a, b) \mapsto a + b$, the inverse $a \mapsto -a$, the multiplication $(a, b) \mapsto ab$ and the join $(a, b) \mapsto \max_\mathbb{Z}(a, b)$ by

$$a + b = (m + m', n + n'), \qquad -a = (n, m),$$
$$ab = (mm' + nn', mn' + nm'), \quad \max_\mathbb{Z}(a, b) = (\max(m + n', m' + n), n + n'),$$

respectively, for all $a = (m, n), b = (m', n') \in \underline{\mathbb{Z}}$. Then they are setoid mappings, and $\mathbb{Z}$ is a unitary commutative ring with the zero element $0 = (0, 0)$ and the unity element $1 = (1, 0)$, and $\mathbb{Z}$ is a semilattice such that the canonical partial order $\leq_{\mathbb{Z}}$ is total, and

1. $\max_{\mathbb{Z}}(a, b) + c =_{\mathbb{Z}} \max_{\mathbb{Z}}(a + c, b + c)$;
2. if $0 \leq_{\mathbb{Z}} c$, then $\max_{\mathbb{Z}}(a, b)c =_{\mathbb{Z}} \max_{\mathbb{Z}}(ac, bc)$

for all $a, b, c \in \underline{\mathbb{Z}}$. The set $\mathbb{N}$ of natural numbers is a subsetoid of $\mathbb{Z}$ with a function $\iota_{\mathbb{N}} : \mathbb{N} \to \underline{\mathbb{Z}}$ given by $\iota_{\mathbb{N}} : n \mapsto (n, 0)$ for all $n \in \mathbb{N}$, and then

$$m = n \Leftrightarrow \iota_{\mathbb{N}}(m) =_{\mathbb{Z}} \iota_{\mathbb{N}}(n), \qquad \iota_{\mathbb{N}}(m + n) =_{\mathbb{Z}} \iota_{\mathbb{N}}(m) + \iota_{\mathbb{N}}(n),$$
$$\iota_{\mathbb{N}}(m \cdot n) =_{\mathbb{Z}} \iota_{\mathbb{N}}(m)\iota_{\mathbb{N}}(n), \qquad \iota_{\mathbb{N}}(\max(m, n)) =_{\mathbb{Z}} \max_{\mathbb{Z}}(\iota_{\mathbb{N}}(m), \iota_{\mathbb{N}}(n))$$

for all $m, n \in \mathbb{N}$.

A *field* is a unitary commutative ring $X = (\underline{X}, =_X)$ with the unity element 1 and a tight apartness $\#_X$ on X such that for each $x \in \underline{X}$, if $x \#_X 0$, then there exists $y \in \underline{X}$ such that $xy =_X 1$. An *ordered field* is a field $X = (\underline{X}, =_X)$ equipped with a quasi-total order $\leq_X$ such that

1. if $x <_X y$, then $x + z <_X y + z$;
2. if $x <_X y$ and $0 <_X z$, then $xz <_X yz$

for all $x, y, z \in \underline{X}$; it is *Archimedean* if for all $x, y \in \underline{X}$ with $0 <_X x$ and $0 <_X y$, there exists $n \in \mathbb{N}$ such that $x <_X ny$.

The setoid of *rationals* is given as follows. Let $\underline{\mathbb{Q}} = \underline{\mathbb{Z}} \times \{a \in \mathbb{Z} \mid \neg(a =_{\mathbb{Z}} 0)\}$, and let $=_{\mathbb{Q}}$ be an equivalence relation on $\underline{\mathbb{Q}}$ given by

$$p =_{\mathbb{Q}} q \Leftrightarrow b'a =_{\mathbb{Z}} ba'$$

for all $p = (a, b), q = (a', b') \in \underline{\mathbb{Q}}$. Then $\mathbb{Q} = (\underline{\mathbb{Q}}, =_{\mathbb{Q}})$ is a discrete setoid of rationals. Define the addition $(p, q) \mapsto p + q$, the inverse $p \mapsto -p$, the multiplication $(p, q) \mapsto pq$ and the join $(p, q) \mapsto \max_{\mathbb{Q}}(p, q)$ by

$$p + q = (b'a + ba', bb'), \qquad -p = (-a, b),$$
$$pq = (aa', bb'), \qquad \max_{\mathbb{Q}}(p, q) = (\max_{\mathbb{Z}}(b'a, ba'), bb'),$$

respectively, for all $p = (a, b), q = (a', b') \in \underline{\mathbb{Q}}$. Then they are setoid mappings, and $\mathbb{Q}$ is a unitary commutative ring with the zero element $0 = (0, 1)$ and the unity element $1 = (1, 1)$, and $\mathbb{Q}$ is a semilattice such that the canonical partial order $\leq_{\mathbb{Q}}$ is total, and

1. $\max_{\mathbb{Q}}(p, q) + r =_{\mathbb{Q}} \max_{\mathbb{Q}}(p + r, q + r)$;
2. if $0 \leq_{\mathbb{Q}} r$, then $\max_{\mathbb{Q}}(p, q)r =_{\mathbb{Z}} \max_{\mathbb{Q}}(pr, qr)$

for all $p, q, r \in \underline{\mathbb{Z}}$. The setoid $\mathbb{Z}$ of integers is a subsetoid of $\mathbb{Q}$ with a function $\iota_{\mathbb{Z}} : \underline{\mathbb{Z}} \to \underline{\mathbb{Q}}$ given by $\iota_{\mathbb{Z}} : a \mapsto (a, 1)$ for all $a \in \underline{Z}$, and then

$$a =_{\mathbb{Z}} b \Leftrightarrow \iota_{\mathbb{Z}}(a) =_{\mathbb{Q}} \iota_{\mathbb{Z}}(b), \quad \iota_{\mathbb{Z}}(a + b) =_{\mathbb{Q}} \iota_{\mathbb{Z}}(a) + \iota_{\mathbb{Z}}(b), \quad \iota_{\mathbb{Z}}(-a) =_{\mathbb{Q}} -\iota_{\mathbb{Z}}(a),$$
$$\iota_{\mathbb{Z}}(ab) =_{\mathbb{Q}} \iota_{\mathbb{Z}}(a)\iota_{\mathbb{Z}}(b), \quad \iota_{\mathbb{Z}}(\max_{\mathbb{Z}}(a, b)) =_{\mathbb{Q}} \max_{\mathbb{Q}}(\iota_{\mathbb{Z}}(a), \iota_{\mathbb{Z}}(b))$$

for all $a, b \in \underline{\mathbb{Z}}$. The denial inequality $\neq_{\mathbb{Q}}$ is a tight apartness on $\mathbb{Q}$ such that

1. if $p \neq_{\mathbb{Q}} q$, then $p + r \neq_{\mathbb{Q}} q + r$;
2. if $p \neq_{\mathbb{Q}} q$ and $r \neq_{\mathbb{Q}} 0$, then $pr \neq_{\mathbb{Q}} qr$

for all $p, q, r \in \underline{\mathbb{Q}}$. The setoid $\mathbb{Q}$ is an Archimedean ordered field.

3 Uniform spaces

3.1 Fundamental definitions

Definition 3.1. A *uniform structure* on a setoid $X = (\underline{X}, =_X)$ is a triple consisting of a directed preordered set $I_X = (\underline{I}_X, \preccurlyeq_{I_X})$, a function $\rho_X \in \hom(I_X, I_X)$, and a relation $\Vdash_X$ between $\underline{X} \times \underline{X}$ and $\underline{I}_X$ such that

1. for all $x, y \in \underline{X}$, $x =_X y$ if and only if $(x, y) \Vdash_X a$ for all $a \in \underline{I}_X$;
2. for all $a \in \underline{I}_X$ and $x, y, x', y' \in \underline{X}$, if $x =_X x'$, $y =_X y'$ and $(x, y) \Vdash_X a$, then $(x', y') \Vdash_X a$;
3. for all $a \in \underline{I}_X$ and $x, y \in \underline{X}$, if $(x, y) \Vdash_X a$, then $(y, x) \Vdash_X a$;
4. for all $a, b \in \underline{I}_X$ and $x, y \in \underline{X}$, if $a \preccurlyeq_{I_X} b$ and $(x, y) \Vdash_X b$, then $(x, y) \Vdash_X a$;
5. for all $a \in \underline{I}_X$ and $x, y, z \in \underline{X}$, if $(x, y) \Vdash_X \rho_X(a)$ and $(y, z) \Vdash_X \rho_X(a)$, then $(x, z) \Vdash_X a$.

A *uniform space* is a setoid equipped with a uniform structure.

Example 3.2. Let $\underline{X}$ be a set, and let $d : \underline{X} \times \underline{X} \to \mathbb{R}$ be a pseudometric on $\underline{X}$. Then a binary relation $=_X$ on $\underline{X}$, given by

$$x =_X y \Leftrightarrow d(x, y) = 0$$

for all $x, y \in \underline{X}$, is an equivalence relation; hence $X = (\underline{X}, =_X)$ is a setoid.

Let ρ_X and $\Vdash_X$ be a monotone function from $\mathbb{N}$ into $\mathbb{N}$ and a relation between $\underline{X} \times \underline{X}$ and $\mathbb{N}$ defined by

$$\rho_X(n) = n + 1,$$
$$(x, y) \Vdash_X n \Leftrightarrow d(x, y) \leq 2^{-n}$$

for all $n \in \mathbb{N}$ and $x, y \in \underline{X}$, respectively. Then $(\mathbb{N}, \rho_X, \Vdash_X)$ is a uniform structure on the setoid X.

Lemma 3.3. *Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on a setoid X. Then for each $n \in \mathbb{N}$,*

1. *for all $a \in \underline{I}_X$ and $x, y \in \underline{X}$, if $(x, y) \Vdash_X \rho_X^n(a)$, then $(x, y) \Vdash_X a$;*
2. *for all $a \in \underline{I}_X$ and $x_0, \dots x_{n+1} \in \underline{X}$, if $(x_k, x_{k+1}) \Vdash_X \rho_X^n(a)$ for all $k \in \mathbb{N}$ with $0 \leq k \leq n$, then $(x_0, x_{n+1}) \Vdash_X a$.*

Proof. (1): We proceed by induction on n. For $n = 0$, it is trivial. Given $a \in \underline{I}_X$ and $x, y \in \underline{X}$, if $(x, y) \Vdash_X \rho_X^{n+1}(a)$, then, since $(y, y) \Vdash_X \rho_X^{n+1}(a)$, we have $(x, y) \Vdash_X \rho_X^n(a)$; hence $(x, y) \Vdash_X a$, by the induction hypothesis.

(2): We proceed by induction on n. For $n = 0$ and $x_0, x_1 \in \underline{X}$, it is trivial. Given $a \in \underline{I}_X$ and $x_0, \dots, x_{n+1}, x_{n+2} \in \underline{X}$, if $(x_k, x_{k+1}) \Vdash_X \rho_X^{n+1}(a)$ for all $k \in \mathbb{N}$ with $0 \leq k \leq n+1$, then, since $(x_k, x_{k+1}) \Vdash_X \rho_X^n(\rho_X(a))$ for all $k \in \mathbb{N}$ with $0 \leq k \leq n$ and $(x_{n+1}, x_{n+2}) \Vdash_X \rho_X^{n+1}(a)$, we have $(x_0, x_{n+1}) \Vdash_X \rho_X(a)$ and $(x_{n+1}, x_{n+2}) \Vdash_X \rho_X(a)$, by the induction hypothesis and (1); hence $(x_0, x_{n+2}) \Vdash_X a$. $\square$

Definition 3.4. Let X be a setoid, and let $J = (\underline{J}, \preccurlyeq_J)$ be a directed preordered set. Then a function $\boldsymbol{x} : j \mapsto x_j$ from $\underline{J}$ into $\underline{X}$ is called a *net* (or *Moore-Smith sequence*) in X on J, and is denoted by $(x_j)_{j\in\underline{J}}$; a net $(x_n)_{n\in\mathbb{N}}$ on the linearly ordered set $(\mathbb{N}, \leq)$ is called a *sequence* in X; we write $\underline{X}^J$ for the set $\underline{X}^{\underline{J}}$ of nets in X on J.

Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on X. Then a net $\boldsymbol{x} = (x_j)_{j\in\underline{J}} \in \underline{X}^J$ *converges* to an element x of $\underline{X}$ in X if there exists $\beta \in \hom(I_X, J)$, called a *modulus* (of convergence), such that for each $a \in \underline{I}_X$,

$$(x_j, x) \Vdash_X a$$

for all $j \in \underline{J}$ with $\beta(a) \preccurlyeq_J j$. We then write $\boldsymbol{x} \to x$, and x is called a *limit* of $\boldsymbol{x}$.

A net $(x_j)_{j\in\underline{J}} \in \underline{X}^J$ is a *Cauchy net* in X if there exists $\alpha \in \hom(I_X, J)$, called a (Cauchy) *modulus*, such that for each $a \in \underline{I}_X$,

$$(x_j, x_{j'}) \Vdash_X a$$

for all $j, j' \in \underline{J}$ with $\alpha(a) \preccurlyeq_J j, j'$. For each $x \in \underline{X}$, the constant function $j \mapsto x$ from $\underline{J}$ into $\underline{X}$, denoted by $(x)_{j\in\underline{J}}$, is a Cauchy net in X with any modulus $\alpha \in \hom(I_X, J)$.

Lemma 3.5. *Let $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_Y)$ be a uniform space, and let J be a directed preordered set. Then for each pair of nets $\boldsymbol{x} = (x_j)_{j\in\underline{J}}, \boldsymbol{y} = (y_j)_{j\in\underline{J}} \in \underline{X}^J$ such that $x_j =_X y_j$ for all $j \in \underline{J}$ and each pair of setoid mappings $f, g : X \to Y$ with $f \sim g$,*

1. *for all $x, x' \in \underline{X}$, if $\boldsymbol{x} \to x$ and $\boldsymbol{y} \to x'$, then $x =_X x'$;*

2. *for all $y, y' \in \underline{Y}$, if $f \circ \boldsymbol{x} \to y$ and $g \circ \boldsymbol{y} \to y'$, then $y =_Y y'$.*

Proof. Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on X, and let $\boldsymbol{x} = (x_j)_{j\in\underline{J}}, \boldsymbol{y} = (y_j)_{j\in\underline{J}} \in \underline{X}^J$ be such that $x_j =_X y_j$ for all $j \in \underline{J}$.

(1): For each pair of elements $x, x' \in \underline{X}$, if $\boldsymbol{x} \to x$ and $\boldsymbol{y} \to x'$ with moduli $\beta \in \hom(I_X, J)$ and $\beta' \in \hom(I_X, J)$, respectively, then for all $a \in \underline{I}_X$, since $(x_j, x) \Vdash_X \rho_X(a)$ and $(y_j, x') \Vdash_X \rho_X(a)$ for $j = \mathrm{ub}_J(\beta'(\rho_X(a)), \beta(\rho_X(a)))$, we have $(x, x') \Vdash_X a$ as $x_j =_X y_j$; hence $x =_X y$.

(2) For each pair of setoid mappings $f, g : X \to Y$ of X into $Y = (\underline{Y}, =_Y)$ and each pair of elements $y, y' \in \underline{Y}$ such that $f \sim g$, $f \circ \boldsymbol{x} \to y$ and $g \circ \boldsymbol{y} \to y'$, since $f(x_j) =_Y g(y_j)$ for all $j \in \underline{J}$, we have $y =_Y y'$, by (1). □

Lemma 3.6. *Every convergent net in a uniform space is a Cauchy net.*

Proof. Let X be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$ and let $J = (\underline{J}, \preccurlyeq_J)$ be a directed preordered set. Consider $\boldsymbol{x} = (x_j)_{j\in\underline{J}} \in \underline{X}^J$ and $x \in \underline{X}$ such that $\boldsymbol{x} \to x$ with a modulus $\beta \in \hom(I_X, J)$. Then for each $a \in \underline{I}_X$, since $(x_j, x) \Vdash_X \rho_X(a)$ and $(x_{j'}, x) \Vdash_X \rho_X(a)$ for all $j, j' \in \underline{J}$ with $\beta(\rho_X(a)) \preccurlyeq_J j, j'$, we have $(x_j, x_{j'}) \Vdash_X a$ for all $j, j' \in \underline{J}$ with $\beta(\rho_X(a)) \preccurlyeq_J j, j'$. Therefore $\boldsymbol{x}$ is a Cauchy net with a modulus $\beta \circ \rho_X \in \hom(I_X, J)$. □

Definition 3.7. Let X and Y be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respectively. Then a function $f : \underline{X} \to \underline{Y}$ is *uniformly continuous* if there exists $\gamma \in \hom(I_Y, I_X)$, called a *modulus* (of uniform continuity), such that for each $b \in \underline{I_Y}$,

$$(x, y) \Vdash_X \gamma(b) \Rightarrow (f(x), f(y)) \Vdash_Y b$$

for all $x, y \in \underline{X}$.

A uniformly continuous mapping $f : X \to Y$ is a *uniform isomorphism* if there exists a uniformly continuous mapping $g : Y \to X$, called an *inverse* of f, such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$; X and Y are *uniformly equivalent* if there exists a uniform isomorphism between X and Y; we then write $X \simeq Y$.

Lemma 3.8. *Let X and Y be uniform spaces. Then every uniformly continuous function $f : \underline{X} \to \underline{Y}$ is a setoid mapping.*

Proof. Let $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$ be uniform structures on $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_Y)$, respectively, and let $\gamma \in \hom(I_Y, I_X)$ be a modulus of uniform continuity of f, and consider $x, y \in \underline{X}$ with $x =_X y$. Then for each $b \in \underline{I_Y}$, since $(x, y) \Vdash_X \gamma(b)$, we have $(f(x), f(y)) \Vdash_Y b$; hence $f(x) =_Y f(y)$. □

Remark 3.9. For each pair of setoid mappings $f, g : X \to Y$ between uniform spaces X and Y with $f \sim g$, f is uniformly continuous if and only if g is uniformly continuous. Let X, Y and Z be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$, $(I_Y, \rho_Y, \Vdash_Y)$ and $(I_Z, \rho_Z, \Vdash_Z)$, respectively. Then the composition $g \circ f : X \to Z$ of uniformly continuous mappings $f : X \to Y$ and $g : Y \to Z$ with moduli $\gamma^f \in \hom(I_Y, I_X)$ and $\gamma^g \in \hom(I_Z, I_Y)$, respectively, is uniformly continuous with a modulus $\gamma^f \circ \gamma^g \in \hom(I_Z, I_X)$, and the identity mapping $\mathrm{id}_X : X \to X$ is a uniform isomorphism.

Lemma 3.10. *Let X and Y be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respectively, let J be a directed preordered set. Then for each uniformly continuous function $f : \underline{X} \to \underline{Y}$ with a modulus $\gamma \in \hom(I_Y, I_X)$ and each net $\boldsymbol{x} \in \underline{X}^J$,*

1. *for all $x \in \underline{X}$, if $\boldsymbol{x} \to x \in \underline{X}$ in X, then $f \circ \boldsymbol{x} \to f(x)$ in Y;*
2. *if $\boldsymbol{x}$ is a Cauchy net in X with a modulus $\alpha \in \hom(I_X, J)$, then $f \circ \boldsymbol{x}$ is a Cauchy net in Y with a modulus $\alpha \circ \gamma \in \hom(I_Y, J)$.*

Proof. Let $J = (\underline{J}, \preccurlyeq_J)$, and consider a net $\boldsymbol{x} = (x_j)_{j \in \underline{J}} \in \underline{X}^J$.

(1): For each element $x \in \underline{X}$, if $\boldsymbol{x} \to x \in \underline{X}$ with a modulus $\beta \in \hom(I_X, J)$, then for all $b \in \underline{I_Y}$ and $j \in \underline{J}$ with $\beta(\gamma(b)) \preccurlyeq_J j$, since $(x_j, x) \Vdash_X \gamma(b)$, we have $(f(x_j), f(x)) \Vdash_Y b$; hence $f \circ \boldsymbol{x} \to f(x)$ with a modulus $\beta \circ \gamma \in \hom(I_Y, J)$.

(2): If $\boldsymbol{x}$ is a Cauchy net with a modulus $\alpha \in \hom(I_X, J)$, then for all $b \in \underline{I_Y}$ and $j, j' \in \underline{J}$ with $\alpha(\gamma(b)) \preccurlyeq_J j, j'$, since $(x_j, x_{j'}) \Vdash_X \gamma(b)$, we have $(f(x_j), f(x_{j'})) \Vdash_Y b$; hence $f \circ \boldsymbol{x}$ is a Cauchy net in Y with a modulus $\alpha \circ \gamma \in \hom(I_Y, J)$. □

Definition 3.11. Let X be a uniform space. Then a uniform space S is a *uniform subspace* of X if there exists a uniformly continuous setoid injection $\iota : S \to X$.

Proposition 3.12. *Let X be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, let $S = (\underline{S}, \iota)$ be a subsetoid of X, and define a relation $\Vdash_S$ between $\underline{S} \times \underline{S}$ and $\underline{I}_X$ by*

$$(x, y) \Vdash_S a \Leftrightarrow (\iota(x), \iota(y)) \Vdash_X a$$

for all $a \in \underline{I}_X$ and $x, y \in \underline{S}$. Then $(I_X, \rho_X, \Vdash_S)$ is a uniform structure on S, and $\iota_S : S \to X$ is a uniformly continuous setoid injection; the uniform space S is called a uniform subspace induced by a subsetoid $S = (\underline{S}, \iota)$ of X.

Proof. It is straightforward to see that $(I_X, \rho_X, \Vdash_S)$ is a uniform structure on S, and $\iota : S \to X$ is a uniformly continuous setoid injection with a modulus $\mathrm{id}_{\underline{I}_X} \in \hom(I_X, I_X)$. $\square$

Lemma 3.13. *Let X be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let S be the uniform subspace induced by a subsetoid $S = (\underline{S}, \iota)$ of X. Then for each uniform space Z,*

1. *for all function $f : \underline{Z} \to \underline{S}$, f is uniformly continuous if and only if $\iota \circ f$ is uniformly continuous,*

and for each directed preordered set J,

2. *for all $\boldsymbol{x} \in \underline{S}^J$ and $x \in \underline{S}$, $\boldsymbol{x} \to x$ in S if and only if $\iota \circ \boldsymbol{x} \to \iota(x)$ in X;*

3. *for all $\boldsymbol{x} \in \underline{S}^J$ and $\alpha \in \hom(I_X, J)$, $\boldsymbol{x}$ is a Cauchy net in S with a modulus α if and only if $\iota \circ \boldsymbol{x}$ is a Cauchy net in X with a modulus α.*

Proof. Straightforward. $\square$

Proposition 3.14. *Let X and Y be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respectively, and define $\rho_{X\times Y} \in \hom(I_X \times I_Y, I_X \times I_Y)$ and a relation $\Vdash_{X\times Y}$ between $(\underline{X} \times \underline{Y}) \times (\underline{X} \times \underline{Y})$ and $\underline{I}_X \times \underline{I}_Y$ by*

$$\rho_{X\times Y} = \rho_X \times \rho_Y,$$
$$((x, y), (x', y')) \Vdash_{X\times Y} (a, b) \Leftrightarrow (x, x') \Vdash_X a \text{ and } (y, y') \Vdash_Y b$$

for all $(a, b) \in \underline{I}_X \times \underline{I}_Y$ and $(x, y), (x', y') \in \underline{X} \times \underline{Y}$. Then $(I_X \times I_Y, \rho_{X\times Y}, \Vdash_{X\times Y})$ is a uniform structure on the product setoid $X \times Y$ A uniform space $X \times Y$ with the uniform structure is called the product *of uniform spaces X and Y.*

Proof. (1): For all $(x, y), (x', y') \in \underline{X} \times \underline{Y}$, if $(x, y) =_{X\times Y} (x', y')$, then, since $x =_X x'$ and $y =_Y y'$, we have $(x, x') \Vdash_X a$ and $(y, y') \Vdash_Y b$ for all $(a, b) \in \underline{I}_X \times \underline{I}_Y$; hence $((x, y), (x', y')) \Vdash_{X\times Y} (a, b)$ for all $(a, b) \in \underline{I}_X \times \underline{I}_Y$. Conversely, if $((x, y), (x', y')) \Vdash_{X\times Y} (a, b)$ for all $(a, b) \in \underline{I}_X \times \underline{I}_Y$, then, since $(x, x') \Vdash_X a$ and $(y, y') \Vdash_Y b$ for all $a \in \underline{I}_X$ and $b \in \underline{I}_Y$, we have $x =_X x'$ and $y =_Y y'$; hence $(x, y) =_{X\times Y} (x', y')$.

(2) and (3): Straightforward.

(4): Consider $(a, b), (a', b') \in \underline{I}_X \times \underline{I}_Y$ and $(x, y), (x', y') \in \underline{X} \times \underline{Y}$ such that

$$(a, b) \preccurlyeq_{I_X\times I_Y} (a', b') \qquad \text{and} \qquad ((x, y), (x', y')) \Vdash_{X\times Y} (a', b').$$

Then $(x, x') \Vdash_X a'$ and $(y, y') \Vdash_Y b'$. Since $a \preccurlyeq_{I_X} a'$ and $b \preccurlyeq_{I_Y} b'$, we have $(x, x') \Vdash_X a$ and $(y, y') \Vdash_Y b$; hence $((x, y), (x', y')) \Vdash_{X\times Y} (a, b)$.

(5): Consider $(a,b) \in \underline{I_X} \times \underline{I_Y}$ and $(x,y),(x',y'),(x'',y'') \in \underline{X} \times \underline{Y}$ such that $((x,y),(x',y')) \Vdash_{X\times Y} \rho_{X\times Y}(a,b)$ and $((x',y'),(x'',y'')) \Vdash_{X\times Y} \rho_{X\times Y}(a,b)$. Then, since $(x,x') \Vdash_X \rho_X(a)$, $(y,y') \Vdash_Y \rho_Y(b)$, $(x',x'') \Vdash_X \rho_X(a)$ and $(y',y'') \Vdash_Y \rho_Y(b)$, we have $(x,x'') \Vdash_X a$ and $(y,y'') \Vdash_Y b$; hence

$$((x,y),(x'',y'')) \Vdash_{X\times Y} (a,b). \qquad \square$$

Theorem 3.15. *Let X and Y be uniform spaces. Then the projections $\pi_0 : X \times Y \to X$ and $\pi_1 : X \times Y \to Y$ are uniformly continuous, and for each uniform space Z and each pair of uniformly continuous mappings $f : Z \to X$ and $g : Z \to Y$, there exists a unique uniformly continuous mapping $\langle f, g\rangle : Z \to X \times Y$ such that $\pi_0 \circ \langle f, g\rangle \sim f$ and $\pi_1 \circ \langle f, g\rangle \sim g$.*

Proof. Let $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$ be uniform structure on $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_Y)$, respectively, and define $\gamma^{\pi_0} \in \hom(I_X, I_X \times I_Y)$ by

$$\gamma^{\pi_0} : a \mapsto (a, b_0)$$

for all $a \in \underline{I_X}$, where b_0 is an inhabitant of $\underline{I_Y}$. Then for all $a \in \underline{I_X}$ and $(x,y),(x',y') \in \underline{X} \times \underline{Y}$, if $((x,y),(x',y')) \Vdash_{X\times Y} \gamma^{\pi_0}(a)$, then, since $(x,x') \Vdash_X a$, we have $(\pi_0(x,y), \pi_0(x',y')) \Vdash_X a$. Hence $\pi_0 : X \times Y \to X$ is a uniformly continuous mapping with a modulus γ^{π_0}. Similarly, $\pi_1 : X \times Y \to Y$ is a uniformly continuous mapping with a modulus $\gamma^{\pi_1} \in \hom(I_Y, I_X \times I_Y)$ given by $\gamma^{\pi_1} : b \mapsto (a_0, b)$ for all $b \in \underline{I_Y}$, where a_0 is an inhabitant of $\underline{I_X}$.

Consider a uniform space $Z = (\underline{Z}, =_Z)$ with a uniform structure $(I_Z, \rho_Z, \Vdash_Z)$ and uniformly continuous mappings $f : Z \to X$ and $g : Z \to Y$. Then it suffices to show that the unique function $\langle f, g\rangle : \underline{Z} \to \underline{X} \times \underline{Y}$ such that $\pi_0 \circ \langle f, g\rangle = f$ and $\pi_1 \circ \langle f, g\rangle = g$, is uniformly continuous. To this end, assume that f and g are uniformly continuous with moduli $\gamma^f \in \hom(I_X, I_Z)$ and $\gamma^g \in \hom(I_Y, I_Z)$, respectively, and define $\gamma^{\langle f,g\rangle} \in \hom(I_X \times I_Y, I_Z)$ by

$$\gamma^{\langle f,g\rangle} = \mathrm{ub}_{I_Z} \circ (\gamma^f \times \gamma^g).$$

Then for all $(a,b) \in \underline{I_X} \times \underline{I_Y}$ and $z, z' \in \underline{Z}$ with

$$(z,z') \Vdash_Z \gamma^{\langle f,g\rangle}(a,b),$$

since $\gamma^f(a) \preccurlyeq_{I_Z} \gamma^{\langle f,g\rangle}(a,b)$ and $\gamma^g(b) \preccurlyeq_{I_Z} \gamma^{\langle f,g\rangle}(a,b)$, we have $(z,z') \Vdash_Z \gamma^f(a)$ and $(z,z') \Vdash_Z \gamma^g(b)$. Therefore

$$(f(z), f(z')) \Vdash_X a \qquad \text{and} \qquad (g(z), g(z')) \Vdash_Y b,$$

and so $(\langle f, g\rangle(z), \langle f, g\rangle(z')) \Vdash_{X\times Y} (a,b)$. Thus $\langle f, g\rangle : Z \to X \times Y$ is uniformly continuous with a modulus $\gamma^{\langle f,g\rangle} \in \hom(I_X \times I_Y, I_Z)$. $\square$

Corollary 3.16. *Let X, X', Y and Y' be uniform spaces. Then each pair of uniformly continuous mappings $f : X \to X'$ and $g : Y \to Y'$, $f \times g : X \times Y \to X' \times Y'$ is a uniformly continuous mapping such that $\pi_0^{X',Y'} \circ (f \times g) \sim f \circ \pi_0^{X,Y}$ and $\pi_1^{X',Y'} \circ (f \times g) \sim g \circ \pi_1^{X,Y}$.*

Proof. Straightforward by Theorem 3.15. $\square$

Remark 3.17. ♣ Theorem 3.15 shows that the product uniform space $X \times Y$ is the *product* of X and Y in the category of uniform spaces and uniformly continuous mappings. For each pair of uniform isomorphisms $f : X \to Y$ between uniform spaces X and Y and $g : X' \to Y'$ between uniform spaces X' and Y', since $f \times g : X \times X' \to Y \times Y'$ is a uniform isomorphism, we have $X \times Y \simeq X' \times Y'$ whenever $X \simeq X'$ and $Y \simeq Y'$.

3.2 Completeness

Lemma 3.18. *Let $X = (\underline{X}, =_X)$ be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let $J = (\underline{J}, \preccurlyeq_J)$ be a directed preordered set. Then a binary relation $=_{X^J}$ on $\underline{X}^J$ given by*

$$\boldsymbol{x} =_{X^J} \boldsymbol{y} \Leftrightarrow \forall a \in \underline{I}_X\, \exists j \in \underline{J}\, \forall i \in \underline{J}\, (j \preccurlyeq_J i \Rightarrow (x_i, y_i) \Vdash_X a)$$

for all $\boldsymbol{x} = (x_i)_{i \in \underline{J}}, \boldsymbol{y} = (y_i)_{i \in \underline{J}} \in \underline{X}^J$, is an equivalence relation on $\underline{X}^J$; hence $X^J = (\underline{X}^J, =_{X^J})$ is a setoid.

Proof. Consider $\boldsymbol{x} = (x_i)_{i \in \underline{J}} \in \underline{X}^J$. Then for each $a \in \underline{I}_X$, since $(x_i, x_i) \Vdash_X a$ for all $i \in \underline{J}$, we have $\boldsymbol{x} =_{X^J} \boldsymbol{x}$.

Consider $\boldsymbol{x} = (x_i)_{i \in \underline{J}}, \boldsymbol{y} = (y_i)_{i \in \underline{J}} \in \underline{X}^J$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{y}$. Then for each $a \in \underline{I}_X$, there exists $j \in \underline{J}$ such that for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, $(x_i, y_i) \Vdash_X a$; hence $(y_i, x_i) \Vdash_X a$. Therefore $\boldsymbol{y} =_{X^J} \boldsymbol{x}$.

Consider $\boldsymbol{x} = (x_i)_{i \in \underline{J}}, \boldsymbol{y} = (y_i)_{i \in \underline{J}}, \boldsymbol{z} = (z_i)_{i \in \underline{J}} \in \underline{X}^J$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{y}$ and $\boldsymbol{y} =_{X^J} \boldsymbol{z}$. Then given an $a \in \underline{I}_X$, there exists $j \in \underline{J}$ such that $(x_i, y_i) \Vdash_X \rho_X(a)$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, and there exists $j' \in \underline{J}$ such that $(y_i, z_i) \Vdash_X \rho_X(a)$ for all $i \in \underline{J}$ with $j' \preccurlyeq_J i$. Therefore, since J is directed, there exists $j'' \in \underline{J}$ such that $j \preccurlyeq_J j''$ and $j' \preccurlyeq_J j''$, and so for each $i \in \underline{J}$, if $j'' \preccurlyeq_J i$, then, since $j \preccurlyeq_J i$ and $j' \preccurlyeq_J i$, we have $(x_i, y_i) \Vdash_X \rho_X(a)$ and $(y_i, z_i) \Vdash_X \rho_X(a)$; hence $(x_i, z_i) \Vdash_X a$. Thus $\boldsymbol{x} =_{X^J} \boldsymbol{z}$. □

Proposition 3.19. *Let $X = (\underline{X}, =_X)$ be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let $J = (\underline{J}, \preccurlyeq_J)$ be a directed preordered set. Then a relation $\Vdash_{X^J}$ between $\underline{X}^J \times \underline{X}^J$ and $\underline{I}_X$ given by*

$$\begin{aligned}(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a \Leftrightarrow \exists \boldsymbol{x}', \boldsymbol{y}' \in \underline{X}^J\, [&\boldsymbol{x} =_{X^J} \boldsymbol{x}' \wedge \boldsymbol{y} =_{X^J} \boldsymbol{y}' \\ &\wedge \exists j \in \underline{J}\, \forall i \in \underline{J}\, (j \preccurlyeq_J i \Rightarrow (x'_i, y'_i) \Vdash_X a)]\end{aligned}$$

for all $a \in \underline{I}_X$ and $\boldsymbol{x}, \boldsymbol{y} \in \underline{X}^J$ where $\boldsymbol{x}' = (x'_i)_{i \in \underline{J}}$ and $\boldsymbol{y}' = (y'_i)_{i \in \underline{J}}$, gives a uniform structure $(I_X, \rho_X^2, \Vdash_{X^J})$ on $X^J = (\underline{X}^J, =_{X^J})$; hence X^J is a uniform space.

Proof. (1): Consider $\boldsymbol{x} = (x_i)_{i \in \underline{J}}, \boldsymbol{y} = (y_i)_{i \in \underline{J}}. \in \underline{X}^J$. If $\boldsymbol{x} =_{X^J} \boldsymbol{y}$, then for each $a \in \underline{I}_X$, since $\boldsymbol{x} =_{X^J} \boldsymbol{x}$, $\boldsymbol{y} =_{X^J} \boldsymbol{y}$ and there exists $j \in \underline{J}$ such that $(x_i, y_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, we have $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a$. Conversely, suppose that $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a$ for all $a \in \underline{I}_X$. Then given an $a \in \underline{I}_X$, there exist $\boldsymbol{x}' = (x'_i)_{i \in \underline{J}}, \boldsymbol{y}' = (y'_i)_{i \in \underline{J}} \in \underline{X}^J$ and $j' \in \underline{J}$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{x}'$, $\boldsymbol{y} =_{X^J} \boldsymbol{y}'$ and $(x'_i, y'_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j' \preccurlyeq_J i$. Therefore there exist $j'', j''' \in \underline{J}$ such that $(x'_i, x_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j'' \preccurlyeq_J i$ and $(y'_i, y_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j''' \preccurlyeq_J i$. Since J is directed, there exists $j \in \underline{J}$ such that $j' \preccurlyeq_J j$, $j'' \preccurlyeq_J j$ and $j''' \preccurlyeq_J j$; hence $(x_i, y_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, by Lemma 3.3 (2). Therefore $\boldsymbol{x} =_{X^J} \boldsymbol{y}$.

(2): Consider $a \in \underline{I}_X$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}', \boldsymbol{y}' \in \underline{X}^J$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{x}'$, $\boldsymbol{y} =_{X^J} \boldsymbol{y}'$ and $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a$. Then there exist $\boldsymbol{x}'' = (x''_i)_{i \in \underline{J}}, \boldsymbol{y}'' = (y''_i)_{i \in \underline{J}} \in \underline{X}^J$ and $j \in \underline{J}$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{x}''$, $\boldsymbol{y} =_{X^J} \boldsymbol{y}''$ and $(x''_i, y''_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$. Therefore, since $\boldsymbol{x}' =_{X^J} \boldsymbol{x}''$ and $\boldsymbol{y}' =_{X^J} \boldsymbol{y}''$, we have $(\boldsymbol{x}', \boldsymbol{y}') \Vdash_{X^J} a$.

(3): Consider $a \in \underline{I}_X$ and $\boldsymbol{x}, \boldsymbol{y} \in \underline{X}^J$ such that $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a$. Then there exist $\boldsymbol{x}' = (x'_i)_{i \in \underline{J}}, \boldsymbol{y}' = (y'_i)_{i \in \underline{J}} \in \underline{X}^J$ and $j \in \underline{J}$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{x}'$, $\boldsymbol{y} =_{X^J} \boldsymbol{y}'$ and $(x'_i, y'_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$; hence $(y'_i, x'_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$. Therefore $(\boldsymbol{y}, \boldsymbol{x}) \Vdash_{X^J} a$.

(4): Consider $a, b \in \underline{I}_X$ and $\boldsymbol{x}, \boldsymbol{y} \in \underline{X}^J$ such that $a \preccurlyeq_{I_X} b$ and $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} b$. Then there exist $\boldsymbol{x}' = (x'_i)_{i \in \underline{J}}, \boldsymbol{y}' = (y'_i)_{i \in \underline{J}} \in \underline{X}^J$ and $j \in \underline{J}$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{x}'$, $\boldsymbol{y} =_{X^J} \boldsymbol{y}'$ and $(x'_i, y'_i) \Vdash_X b$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$; hence $(y'_i, x'_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$. Therefore $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a$.

(5): Consider $a \in \underline{I}_X$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \underline{X}^J$ such that $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} \rho_X^2(a)$ and $(\boldsymbol{y}, \boldsymbol{z}) \Vdash_{X^J} \rho_X^2(a)$. Then there exist $\boldsymbol{x}' = (x'_i)_{i \in \underline{J}}, \boldsymbol{y}' = (y'_i)_{i \in \underline{J}} \in \underline{X}^J$ and $j' \in \underline{J}$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{x}'$, $\boldsymbol{y} =_{X^J} \boldsymbol{y}'$ and $(x'_i, y'_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j' \preccurlyeq_J i$, and there exist $\boldsymbol{y}'' = (y''_i)_{i \in \underline{J}}, \boldsymbol{z}' = (z'_i)_{i \in \underline{J}} \in \underline{X}^J$ and $j'' \in \underline{J}$ such that $\boldsymbol{y} =_{X^J} \boldsymbol{y}''$, $\boldsymbol{z} =_{X^J} \boldsymbol{z}'$ and $(y''_i, z'_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j'' \preccurlyeq_J i$. Since $\boldsymbol{y}' =_{X^J} \boldsymbol{y}''$, there exists $j''' \in \underline{J}$ such that $(y'_i, y''_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j''' \preccurlyeq_J i$. Therefore, since J is directed, there exists $j \in \underline{J}$ such that $j' \preccurlyeq_J j$, $j'' \preccurlyeq_J j$ and $j''' \preccurlyeq_J j$, and so for each $i \in \underline{J}$, if $j \preccurlyeq_J i$, then $(x'_i, z'_i) \Vdash_X a$, by Lemma 3.3 (2). Thus $(\boldsymbol{x}, \boldsymbol{z}) \Vdash_{X^J} a$. □

Lemma 3.20. *Let $X = (\underline{X}, =_X)$ be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let $J = (\underline{J}, \preccurlyeq_J)$ be a directed preordered set. Then for all $a \in \underline{I}_X$ and $\boldsymbol{x} = (x_i)_{i \in \underline{J}}, \boldsymbol{y} = (y_i)_{i \in \underline{J}} \in \underline{X}^J$,*

1. *if $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} \rho_X^2(a)$, then there exists $j \in \underline{J}$ such that $(x_i, y_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$;*
2. *if there exists $j \in \underline{J}$ such that $(x_i, y_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, then $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a$.*

Proof. (1): Consider $a \in \underline{I}_X$ and $\boldsymbol{x} = (x_i)_{i \in \underline{J}}, \boldsymbol{y} = (y_i)_{i \in \underline{J}} \in \underline{X}^J$, and suppose that $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} \rho_X^2(a)$. Then there exist $\boldsymbol{x}' = (x'_i)_{i \in \underline{J}}, \boldsymbol{y}' = (y'_i)_{i \in \underline{J}} \in \underline{X}^J$ and $j' \in \underline{J}$ such that $\boldsymbol{x} =_{X^J} \boldsymbol{x}'$, $\boldsymbol{y} =_{X^J} \boldsymbol{y}'$ and $(x'_i, y'_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j' \preccurlyeq_J i$. Since $\boldsymbol{x} =_{X^J} \boldsymbol{x}'$ and $\boldsymbol{y} =_{X^J} \boldsymbol{y}'$, there exist $j'', j''' \in \underline{J}$ such that $(x_i, x'_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j'' \preccurlyeq_J i$, and $(y_i, y'_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j''' \preccurlyeq_J i$. Therefore, since J is directed, there exists $j \in \underline{J}$ such that $j' \preccurlyeq_J j$, $j'' \preccurlyeq_J j$ and $j''' \preccurlyeq_J j$, and so

$$(x'_i, y'_i) \Vdash_X \rho_X^2(a), \qquad (x'_i, x_i) \Vdash_X \rho_X^2(a), \qquad (y'_i, y_i) \Vdash_X \rho_X^2(a)$$

for all $i \in \underline{J}$ with $j \preccurlyeq_J i$. Thus $(x_i, y_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, by Lemma 3.3 (2).

(2): Consider $a \in \underline{I}_X$ and $\boldsymbol{x} = (x_i)_{i \in \underline{J}}, \boldsymbol{y} = (y_i)_{i \in \underline{J}} \in \underline{X}^J$, and suppose that there exists $j \in \underline{J}$ such that $(x_i, y_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$. Then, since $\boldsymbol{x} =_{X^J} \boldsymbol{x}$ and $\boldsymbol{y} =_{X^J} \boldsymbol{y}$, we have $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a$. □

Definition 3.21. Let X and Y be setoids, and let J and J' be directed preordered sets. Then a function $\eta_X^J : \underline{X} \to \underline{X}^J$ is defined by

$$\eta_X^J : x \mapsto (x)_{j \in \underline{J}}$$

for all $x \in \underline{X}$; for each function $f : \underline{X} \to \underline{Y}$, a function $f^J : \underline{X}^J \to \underline{Y}^J$ is defined by

$$f^J : \boldsymbol{x} \mapsto f \circ \boldsymbol{x}$$

for all $\boldsymbol{x} \in \underline{X}^J$; for each $\sigma \in \hom(J', J)$, a function $\sigma_X : \underline{X}^J \to \underline{X}^{J'}$ is defined by

$$\sigma_X : \boldsymbol{x} \mapsto \boldsymbol{x} \circ \sigma$$

for all $\boldsymbol{x} \in \underline{X}^J$; in which case

$$f^J \circ \eta_X^J = \eta_Y^J \circ f \qquad \text{and} \qquad \sigma_X \circ \eta_X^J = \eta_X^{J'}.$$

Lemma 3.22. *Let X be a uniform space, and let J be a directed preordered set. Then $\eta_X^J : \underline{X} \to \underline{X}^J$ is a uniformly continuous setoid injection such that*

$$\eta_X^J \circ \boldsymbol{x} \to \boldsymbol{x}$$

in X^J for all Cauchy net $\boldsymbol{x} \in \underline{X}^J$; hence X is a uniform subspace of X^J.

Proof. Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on X. For all $a \in \underline{I}_X$ and $x, y \in \underline{X}$, if $(x, y) \Vdash_X a$, then, since $(x, y) \Vdash_X a$ for all $i \in \underline{J}$ with $j_0 \preccurlyeq_J i$ where j_0 is an inhabitant of $\underline{J}$, we have $(\eta_X^J(x), \eta_X^J(y)) \Vdash_{X^J} a$, by Lemma 3.20 (2). Therefore $\eta_X^J : \underline{X} \to \underline{X}^J$ is uniformly continuous with a modulus $\mathrm{id}_{\underline{I}_X} \in \hom(I_X, I_X)$.

Consider $x, y \in \underline{X}$ with $\eta_X^J(x) =_{X^J} \eta_X^J(y)$. Then for all $a \in \underline{I}_X$, since there exists $j \in \underline{J}$ such that $(x, y) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, we have $(x, y) \Vdash_X a$; hence $x =_X y$. Therefore $\eta_X^J : \underline{X} \to \underline{X}^J$ is a setoid injection; see Lemma 3.8.

Consider a Cauchy net $\boldsymbol{x} = (x_j)_{j \in \underline{J}} \in \underline{X}^J$ with a modulus $\alpha \in \hom(I_X, J)$. Then for each $a \in \underline{I}_X$, since $(x_j, x_{j'}) \Vdash_X a$ for all $j, j' \in \underline{J}$ with $\alpha(a) \preccurlyeq_J j, j'$, we have $(\eta_X^J(x_j), \boldsymbol{x}) \Vdash_{X^J} a$ for all $j \in \underline{J}$ with $\alpha(a) \preccurlyeq_J j$, by Lemma 3.20 (2). Therefore $\eta_X^J \circ \boldsymbol{x}$ converges to $\boldsymbol{x}$ in X^J with a modulus α. □

Lemma 3.23. *Let X and Y be uniform spaces, and let J be a directed preordered set. Then for each uniformly continuous function $f : \underline{X} \to \underline{Y}$, $f^J : \underline{X}^J \to \underline{Y}^J$ is uniformly continuous.*

Proof. Let $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$ be uniform structures on X and Y, respcetively, let $J = (\underline{J}, \preccurlyeq_J)$, and suppose that $f : X \to Y$ is uniformly continuous with a modulus $\gamma \in \hom(I_Y, I_X)$. Then for all $b \in \underline{I}_Y$ and $\boldsymbol{x} = (x_j)_{j \in \underline{J}}, \boldsymbol{y} = (y_j)_{j \in \underline{J}} \in \underline{X}^J$, if

$$(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} \rho_X^2(\gamma(b)),$$

then, since there exists $j \in \underline{J}$ such that $(x_i, y_i) \Vdash_X \gamma(b)$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, by Lemma 3.20 (1) , we have $(f(x_i), f(y_i)) \Vdash_Y b$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$; hence $(f \circ \boldsymbol{x}, f \circ \boldsymbol{y}) \Vdash_{Y^J} b$, or

$$(f^J(\boldsymbol{x}), f^J(\boldsymbol{y})) \Vdash_{Y^J} b,$$

by Lemma 3.20 (2) . Therefore $f^J : \underline{X}^J \to \underline{Y}^J$ is uniformly continuous with a modulus $\rho_X^2 \circ \gamma \in \hom(I_Y, I_X)$. □

Lemma 3.24. *Let X be a uniform space, and let $J = (\underline{J}, \preccurlyeq_J)$ and $J' = (\underline{J}', \preccurlyeq_{J'})$ be directed preordered sets. Then for each $\sigma \in \hom(J', J)$, $\sigma_X : \underline{X}^J \to \underline{X}^{J'}$ is uniformly continuous whenever for each $j \in \underline{J}$ there exists $j' \in \underline{J}'$ such that $j \preccurlyeq_J \sigma(j')$.*

Proof. Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on X, let $\sigma \in \hom(J', J)$ be such that for each $j \in \underline{J}$ there exists $j' \in \underline{J}'$ with $j \preccurlyeq_J \sigma(j')$, and consider $a \in \underline{I}_X$ and $\boldsymbol{x} = (x_j)_{j\in\underline{J}}, \boldsymbol{y} = (y_j)_{j\in\underline{J}} \in \underline{X}^J$ with

$$(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} \rho_X^2(a).$$

Then there exists $j \in \underline{J}$ such that $(x_i, y_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, by Lemma 3.20 (1) ; hence there exists $j' \in \underline{J}'$ such that $j \preccurlyeq_J \sigma(j')$. Therefore for each $i' \in \underline{J}'$, if $j' \preccurlyeq_J i'$, then, since $j \preccurlyeq_J \sigma(j') \preccurlyeq_J \sigma(i')$, we ahve $(x_{\sigma(i')}, y_{\sigma(i')}) \Vdash_X a$; hence $(\boldsymbol{x} \circ \sigma, \boldsymbol{y} \circ \sigma) \Vdash_{X^{J'}} a$, or

$$(\sigma_X(\boldsymbol{x}), \sigma_X(\boldsymbol{y})) \Vdash_{X^{J'}} a,$$

by Lemma 3.20 (2) . Thus $\sigma_X : \underline{X}^J \to \underline{X}^{J'}$ is uniformly continuous with a modulus $\rho_X^2 \in \hom(I_X, I_X)$. □

Recall that for all setoids $X = (\underline{X}, =_X)$, $Y = (\underline{Y}, =_Y)$ and all directed preordered sets J and J',

$$\underline{X}^{J\times J'} \cong (\underline{X}^{J'})^J$$

with the bijection $\hat{(\text{-})} : \underline{X}^{J\times J'} \to (\underline{X}^{J'})^J$, and

$$\underline{X}^J \times \underline{Y}^J \cong (\underline{X} \times \underline{Y})^J$$

with the bijection $\langle\text{-},\text{-}\rangle : \underline{X}^J \times \underline{Y}^J \to (\underline{X} \times \underline{Y})^J$;

$$\hat{(\text{-})} \circ \eta_X^{J\times J'} = \eta_{X^{J'}}^J \circ \eta_X^{J'} \qquad \text{and} \qquad \langle\text{-},\text{-}\rangle \circ (\eta_X^J \times \eta_Y^J) = \eta_{X\times Y}^J.$$

Proposition 3.25. *Let $X = (\underline{X}, =_X)$ be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let J and J' be directed preordered sets. Then $\hat{(\text{-})} : \underline{X}^{J\times J'} \to (\underline{X}^{J'})^J$ is a uniformly continuous mapping such that for each $a \in \underline{I}_X$,*

$$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \Vdash_{(X^{J'})^J} \rho_X^8(a) \Rightarrow (\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^{J\times J}} a$$

for all pair of Cauchy nets $\boldsymbol{x}, \boldsymbol{y} \in \underline{X}^{J\times J'}$.

Proof. Let $J = (\underline{J}, \preccurlyeq_J)$ and $J' = (\underline{J}', \preccurlyeq_{J'})$, and consider $a \in \underline{I}_X$ and $\boldsymbol{x} = (x_{j,j'})_{(j,j')\in\underline{J}\times\underline{J}'}, \boldsymbol{y} = (y_{j,j'})_{(j,j')\in\underline{J}\times\underline{J}'} \in \underline{X}^{J\times J'}$ with

$$(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^{J\times J'}} \rho_X^2(a).$$

Then there exists $(j, j') \in \underline{J}\times\underline{J}'$ such that $(x_{i,i'}, y_{i,i'}) \Vdash_X a$ for all $(i, i') \in \underline{J}\times\underline{J}'$ with $(j, j') \preccurlyeq_{J\times J'} (i, i')$, by Lemma 3.20 (1). Consider $i \in \underline{J}$ with $j \preccurlyeq_J i$. Then for all $i' \in \underline{J}'$ with $j' \preccurlyeq_{J'} i'$, since $(j, j') \preccurlyeq_{J\times J'} (i, i')$, we have $(x_{i,i'}, y_{i,i'}) \Vdash_X a$; hence $(\hat{\boldsymbol{x}}(i), \hat{\boldsymbol{y}}(i)) \Vdash_{X^{J'}} a$, by Lemma 3.20 (2). Therefore

$$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \Vdash_{(X^{J'})^J} a,$$

by Lemma 3.20 (2). Thus $\hat{(\text{-})} : \underline{X}^{J\times J'} \to (\underline{X}^{J'})^J$ is a uniformly continuous mapping with a modulus $\rho_X^2 \in \hom(I_X, I_X)$; see Lemma 3.8.

Consider $a \in \underline{I}_X$ and Cauchy nets $\boldsymbol{x} = (x_{j,j'})_{(j,j') \in \underline{J} \times \underline{J'}}, \boldsymbol{y} = (y_{j,j'})_{(j,j') \in \underline{J} \times \underline{J'}} \in \underline{X}^{J \times J'}$ with moduli $\alpha \in \hom(I_X, J \times J')$ and $\alpha' \in \hom(I_X, J \times J')$, respcetively, with

$$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \Vdash_{X^{J'J}} \rho_X^8(a),$$

and note that $\rho_{X^{J'}} = \rho_X^2$; see Proposition 3.19. Then, since $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \Vdash_{X^{J'J}} \rho^2_{X^{J'}}(\rho_X^4(a))$, there exits $j \in \underline{J}$ such that $((x_{i,j'})_{j' \in \underline{J'}}, (y_{i,j'})_{j' \in \underline{J'}}) \Vdash_{X^{J'}} \rho_X^4(a)$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$. Choose $k \in \underline{J}$ such that $j \preccurlyeq_J k$, $\pi_0(\alpha(\rho_X^2(a))) \preccurlyeq_J k$ and $\pi_0(\alpha'(\rho_X^2(a))) \preccurlyeq_J k$. Then, since $((x_{k,j'})_{j' \in \underline{J'}}, (y_{k,j'})_{j' \in \underline{J'}}) \Vdash_{X^{J'}} \rho_X^4(a)$, there exists $j' \in \underline{J'}$ such that $(x_{k,i'}, y_{k,i'}) \Vdash_X \rho_X^2(a)$ for all $i' \in \underline{J}$ with $j' \preccurlyeq_J i'$. Choose $k' \in \underline{J'}$ such that $j' \preccurlyeq_{J'} k'$, $\pi_1(\alpha(\rho_X^2(a))) \preccurlyeq_{J'} k'$ and $\pi_1(\alpha'(\rho_X^2(a))) \preccurlyeq_{J'} k'$. Then for all $(i, i') \in \underline{J} \times \underline{J'}$ with $(k, k') \preccurlyeq_{J \times J'} (i, i')$, since $(x_{k,k'}, y_{k,k'}) \Vdash_X \rho_X^2(a)$, $\alpha(\rho_X^2(a)) \preccurlyeq_{J \times J'} (k, k')$ and $\alpha'(\rho_X^2(a)) \preccurlyeq_{J \times J'} (k, k')$, we have $(x_{i,i'}, x_{k,k'}) \Vdash_X \rho_X^2(a)$ and $(y_{i,i'}, y_{k,k'}) \Vdash_X \rho_X^2(a)$; hence $(x_{i,i'}, y_{i,i'}) \Vdash_X a$, by Lemma 3.3 (2). Therefore

$$(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^{J \times J'}} a,$$

by Lemma 3.20 (2). □

Proposition 3.26. *Let X and Y be uniform spaces, and let J be a directed preordered set. Then $\langle \text{-}, \text{-} \rangle : \underline{X}^J \times \underline{Y}^J \to (\underline{X} \times \underline{Y})^{\underline{J}}$ is a uniform isomorphism; hence*

$$X^J \times Y^J \simeq (X \times Y)^J.$$

Proof. Let $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$ be uniform structures on X and Y, respectively, let $J = (\underline{J}, \preccurlyeq_J)$, and consider $(a, b) \in \underline{I}_X \times \underline{I}_Y$ and $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}') \in \underline{X}^J \times \underline{Y}^J$ with

$$((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')) \Vdash_{X^J \times Y^J} (\rho_X^2 \times \rho_Y^2)((a, b)),$$

where $\boldsymbol{x} = (x_j)_{j \in \underline{J}}, \boldsymbol{x}' = (x'_j)_{j \in \underline{J}} \in \underline{X}^J$ and $\boldsymbol{y} = (y_j)_{j \in \underline{J}}, \boldsymbol{y}' = (y'_j)_{j \in \underline{J}} \in \underline{Y}^J$. Then, since $(\boldsymbol{x}, \boldsymbol{x}') \Vdash_{X^J} \rho_X^2(a)$ and $(\boldsymbol{y}, \boldsymbol{y}') \Vdash_{Y^J} \rho_Y^2(b)$, there exist $j, j' \in \underline{J}$ such that $(x_i, x'_i) \Vdash_X a$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$ and $(y_{i'}, y'_{i'}) \Vdash_Y b$ for all $i' \in \underline{J}$ with $j' \preccurlyeq_J i'$, by Lemma 3.20 (1). Therefore for each $i \in \underline{J}$, if $\mathrm{ub}_J(j, j') \preccurlyeq_J i$, then, since $(x_i, x'_i) \Vdash_X a$ and $(y_i, y'_i) \Vdash_Y b$, we have $((x_i, y_i), (x'_i, y'_i)) \Vdash_{X \times Y} (a, b)$; hence

$$(\langle \boldsymbol{x}, \boldsymbol{y} \rangle, \langle \boldsymbol{x}', \boldsymbol{y}' \rangle) \Vdash_{(X \times Y)^J} (a, b).$$

Thus $\langle \text{-}, \text{-} \rangle : \underline{X}^J \times \underline{Y}^J \to (\underline{X} \times \underline{Y})^{\underline{J}}$ is a uniformly continuous mapping with a modulus $\rho_X^2 \times \rho_Y^2 \in \hom(I_X \times I_Y, I_X \times I_Y)$; see Lemma 3.8.

Let $\pi_0 : X \times Y \to X$ and $\pi_1 : X \times Y \to Y$ be the projections. Then, since π_0 and π_1 are uniformly continuous, by Theorem 3.15, $\pi_0^J : (X \times Y)^J \to X^J$ and $\pi_1^J : (X \times Y)^J \to Y^J$ are uniformly continuous, by Lemma 3.23; hence $\langle \pi_0^J, \pi_1^J \rangle : (X \times Y)^J \to X^J \times Y^J$ is uniformly continuous, by Theorem 3.15. Furthermore, since

$$\begin{aligned} \langle \pi_0^J, \pi_1^J \rangle (\langle \boldsymbol{x}, \boldsymbol{y} \rangle) &= (\pi_0^J(\langle \boldsymbol{x}, \boldsymbol{y} \rangle), \pi_1^J(\langle \boldsymbol{x}, \boldsymbol{y} \rangle)) \\ &= (\pi_0 \circ \langle \boldsymbol{x}, \boldsymbol{y} \rangle, \pi_1 \circ \langle \boldsymbol{x}, \boldsymbol{y} \rangle) = (\boldsymbol{x}, \boldsymbol{y}) \end{aligned}$$

for all $\boldsymbol{x} \in \underline{X}^J$ and $\boldsymbol{y} \in \underline{Y}^J$, $\langle \pi_0^J, \pi_1^J \rangle : (X \times Y)^J \to X^J \times Y^J$ is an inverse of $\langle \text{-}, \text{-} \rangle : X^J \times Y^J \to (X \times Y)^J$. □

Definition 3.27. For each uniform space $X = (\underline{X}, =_X)$ with a uniform structure $(I_X, \rho_X, \Vdash_X)$ and each directed preordered set $J = (\underline{J}, \preccurlyeq_J)$, let $\underline{\tilde{X}}^J$ be a subset of $\underline{X}^J \times \hom(I_X, J)$ given by

$$\underline{\tilde{X}}^J = \{(\boldsymbol{x}, \alpha) \in \underline{X}^J \times \hom(I_X, J) \mid \boldsymbol{x} \text{ is a Cauchy net with a modulus } \alpha\},$$

an let $\tilde{\iota}^J_X : \underline{\tilde{X}}^J \to \underline{X}^J$ be a function given by

$$\tilde{\iota}^J_X : (\boldsymbol{x}, \alpha) \mapsto \boldsymbol{x}$$

for all $(\boldsymbol{x}, \alpha) \in \underline{\tilde{X}}^J$. Then a uniform space $\tilde{X}^J$ is a uniform subspace induced by a subsetoid $\tilde{X}^J = (\underline{\tilde{X}}^J, \tilde{\iota}^J_X)$ of X^J; see Proposition 3.12. Note that for all $(\boldsymbol{x}, \alpha), (\boldsymbol{y}, \alpha') \in \underline{\tilde{X}}^J$ and $a \in \underline{I}_X$,

$$(\boldsymbol{x}, \alpha) =_{\tilde{X}^J} (\boldsymbol{y}, \alpha') \Leftrightarrow \boldsymbol{x} =_{X^J} \boldsymbol{y}, \quad ((\boldsymbol{x}, \alpha), (\boldsymbol{y}, \alpha')) \Vdash_{\tilde{X}^J} a \Leftrightarrow (\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a.$$

Lemma 3.28. *Let X be a uniform space, and let J be a directed preordered set. Then there exists a uniformly continuous setoid injection $\tilde{\eta}^J_X : X \to \tilde{X}^J$ which makes the following diagram commute, that is, $\tilde{\iota}^J_X \circ \tilde{\eta}^J_X \sim \eta^J_X$; hence X is a subspace of $\tilde{X}^J$.*

$$\begin{tikzcd}
X \arrow[r, "\tilde{\eta}^J_X"] \arrow[dr, "\eta^J_X"'] & \tilde{X}^J \arrow[d, "\tilde{\iota}^J_X"] \\
& X^J
\end{tikzcd}$$

Proof. Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on $X = (\underline{X}, =_X)$. Then for each $x \in \underline{X}$, $\eta^J_X(x)$ is a Cauchy net with a modulus $\overline{j_0} \in \hom(I_X, J)$, where j_0 is an inhabitant of $\underline{J}$. Define a function $\tilde{\eta}^J_X : \underline{X} \to \underline{\tilde{X}}^J$ by

$$\tilde{\eta}^J_X : x \mapsto (\eta^J_X(x), \overline{j_0})$$

for all $x \in \underline{X}$. Then, since $\tilde{\iota}^J_X \circ \tilde{\eta}^J_X = \eta^J_X$, we have $\tilde{\iota}^J_X \circ \tilde{\eta}^J_X \sim \eta^J_X$, and, since $\eta^J_X : X \to X^J$ is uniformly continuous, by Lemma 3.22, $\tilde{\eta}^J_X : X \to \tilde{X}^J$ is a uniformly continuous mapping, by Lemma 3.13 (1) and Lemma 3.8. Since $\tilde{\iota}^J_X \circ \tilde{\eta}^J_X$ is a setoid injection, so is $\tilde{\eta}^J_X$; see Remark 2.9. $\square$

Lemma 3.29. *Let X and Y be uniform spaces, and let J be a directed preordered set. Then for each pair of uniformly continuous mappings $f, g : \tilde{X}^J \to Y$, if $f \circ \tilde{\eta}^J_X \sim g \circ \tilde{\eta}^J_X$, then $f \sim g$; that is, $\tilde{\eta}^J_X$ is an epimorphism in the category of uniform spaces and uniformly continuous mappings.*

Proof. Consider uniformly continuous mappings $f, g : \tilde{X}^J \to Y$ with $f \circ \tilde{\eta}^J_X \sim g \circ \tilde{\eta}^J_X$. Then for each $(\boldsymbol{x}, \alpha) \in \underline{\tilde{X}}^J$, since $\eta^J_X \circ \boldsymbol{x} \to \boldsymbol{x}$ in $\underline{X}^J$, by Lemma 3.22, we have $\eta^J_X \circ \tilde{\iota}^J_X(\boldsymbol{x}, \alpha) \to \tilde{\iota}^J_X(\boldsymbol{x}, \alpha)$; hence $\tilde{\iota}^J_X \circ \tilde{\eta}^J_X \circ \tilde{\iota}^J_X(\boldsymbol{x}, \alpha) \to \tilde{\iota}^J_X(\boldsymbol{x}, \alpha)$, by Lemma 3.28. Therefore $\tilde{\eta}^J_X \circ \tilde{\iota}^J_X(\boldsymbol{x}, \alpha) \to (\boldsymbol{x}, \alpha)$ in $\tilde{X}^J$, by Lemma 3.13 (2), and so $f \circ \tilde{\eta}^J_X \circ \tilde{\iota}^J_X(\boldsymbol{x}, \alpha) \to f(\boldsymbol{x}, \alpha)$ and $g \circ \tilde{\eta}^J_X \circ \tilde{\iota}^J_X(\boldsymbol{x}, \alpha) \to g(\boldsymbol{x}, \alpha)$ in Y, by Lemma 3.10 (1). Thus $f(\boldsymbol{x}, \alpha) =_Y g(\boldsymbol{x}, \alpha)$, by Lemma 3.5 (2), and so, since $(\boldsymbol{x}, \alpha) \in \underline{\tilde{X}}^J$ is arbitrary, we have $f \sim g$. $\square$

Lemma 3.30. *Let $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_Y)$ be uniform spaces with uniformly structures $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respcetively, and let J and J' be directed preordered sets. Then for each pair of functions $F : \underline{X}^J \to \underline{Y}^{J'}$ and $G : \tilde{\underline{X}}^J \to \mathrm{hom}(I_Y, J')$ such that for all $(\boldsymbol{x}, \alpha) \in \tilde{\underline{X}}^J$, $F(\boldsymbol{x})$ is a Cauchy net with a modulus $G(\boldsymbol{x}, \alpha)$, there exists a function $\tilde{F} : \tilde{\underline{X}}^J \to \tilde{\underline{Y}}^{J'}$ such that $F \circ \tilde{\iota}_X^J \sim \tilde{\iota}_Y^{J'} \circ \tilde{F}$. Furthermore, if F is uniformly continuous, then $\tilde{F}$ is uniformly continuous.*

Proof. Define a function $\tilde{F} : \tilde{\underline{X}}^J \to \tilde{\underline{Y}}^{J'}$ by

$$\tilde{F} : (\boldsymbol{x}, \alpha) \mapsto (F(\boldsymbol{x}), G(\boldsymbol{x}, \alpha))$$

for all $(\boldsymbol{x}, \alpha) \in \tilde{\underline{X}}^J$. Then, since $\tilde{\iota}_Y^{J'} \circ \tilde{F} = F \circ \tilde{\iota}_X^J$, we have $\tilde{\iota}_Y^{J'} \circ \tilde{F} \sim F \circ \tilde{\iota}_X^J$. If $F : \underline{X}^J \to \underline{Y}^{J'}$ is uniformly continuous, then, since $F \circ \tilde{\iota}_X^J : \tilde{X}^J \to Y^J$ is uniformly continuous, by Proposition 3.12, $\tilde{\iota}_Y^{J'} \circ \tilde{F} : \tilde{X}^J \to Y^{J'}$ is uniformly continuous; hence $\tilde{F} : \tilde{\underline{X}}^J \to \tilde{\underline{Y}}^{J'}$ is uniformly continuous, by Lemma 3.13 (1). $\square$

Proposition 3.31. *Let X and Y be uniform spaces, and let J be a directed preordered set. Then for each uniformly continuous function $f : X \to Y$, there exists a uniformly continuous mapping $\tilde{f}^J : \tilde{X}^J \to \tilde{Y}^J$ which makes the following diagram commute, that is, $f^J \circ \tilde{\iota}_X^J \sim \tilde{\iota}_Y^J \circ \tilde{f}^J$ and $\tilde{f}^J \circ \tilde{\eta}_X^J \sim \tilde{\eta}_Y^J \circ f$.*

$$\begin{tikzcd}
X \arrow[r, "\tilde{\eta}_X^J"] \arrow[d, "f"'] & \tilde{X}^J \arrow[r, "\tilde{\iota}_X^J"] \arrow[d, "\tilde{f}^J"] & X^J \arrow[d, "f^J"] \\
Y \arrow[r, "\tilde{\eta}_Y^J"'] & \tilde{Y}^J \arrow[r, "\tilde{\iota}_Y^J"'] & Y^J
\end{tikzcd}$$

Proof. Let $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$ be uniform structures on X and Y, respcetively, let $J = (\underline{J}, \preccurlyeq_J)$, and suppose that $f : X \to Y$ is uniformly continuous with a modulus $\gamma \in \mathrm{hom}(I_Y, I_X)$. Then for each Cauchy net $\boldsymbol{x} \in \underline{X}^J$ with a modulus $\alpha \in \mathrm{hom}(I_X, J)$, $f^J(\boldsymbol{x})$, or $f \circ \boldsymbol{x}$, is a Cauchy net in Y with a modulus $\alpha \circ \gamma \in \mathrm{hom}(I_Y, J)$, by Lemma 3.10 (2). Apply Lemma 3.30 with $F : \boldsymbol{x} \mapsto f^J(\boldsymbol{x})$ and $G : (\boldsymbol{x}, \alpha) \mapsto \alpha \circ \gamma$. Then there exists a uniformly continuous mapping $\tilde{f}^J : \tilde{X}^J \to \tilde{Y}^J$ such that $\tilde{\iota}_Y^J \circ \tilde{f}^J \sim f^J \circ \tilde{\iota}_X^J$. Since $f^J \circ \eta_X^J = \eta_Y^J \circ f$, we have

$$\tilde{\iota}_Y^J \circ \tilde{f}^J \circ \tilde{\eta}_X^J \sim f^J \circ \tilde{\iota}_X^J \circ \tilde{\eta}_X^J \sim f^J \circ \eta_X^J \sim \eta_Y^J \circ f \sim \tilde{\iota}_Y^J \circ \tilde{\eta}_Y^J \circ f;$$

hence $\tilde{f}^J \circ \tilde{\eta}_X^J \sim \tilde{\eta}_Y^J \circ f$, as $\tilde{\iota}_Y^J$ is a monomorphism. $\square$

Proposition 3.32. *Let X be a uniform space, and let $J = (\underline{J}, \preccurlyeq_J)$ and $J' = (\underline{J}', \preccurlyeq_{J'})$ be directed preordered sets. Then for each pair of $\sigma \in \mathrm{hom}(J', J)$ and $\tau \in \mathrm{hom}(J, J')$ such that $j \preccurlyeq_J \sigma(\tau(j))$ for all $j \in \underline{J}$, there exists a uniformly continuous mapping $\tilde{\sigma}_X : \tilde{X}^J \to \tilde{X}^{J'}$ which makes the following diagram commute, that is, $\sigma_X \circ \tilde{\iota}_X^J \sim \tilde{\iota}_X^{J'} \circ \tilde{\sigma}_X$ and $\tilde{\sigma}_X \circ \tilde{\eta}_X^J \sim \tilde{\eta}_X^{J'}$.*

$$\begin{tikzcd}
X \arrow[r, "\tilde{\eta}^J_X"] \arrow[dr, "\tilde{\eta}^{J'}_X"'] & \tilde{X}^J \arrow[r, "\tilde{\iota}^J_X"] \arrow[d, "\tilde{\sigma}_X"] & X^J \arrow[d, "\sigma_X"] \\
& \tilde{X}^{J'} \arrow[r, "\tilde{\iota}^{J'}_X"'] & X^{J'}
\end{tikzcd}$$

Proof. Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on $X = (\underline{X}, =_X)$, and consider $\sigma \in \hom(J', J)$ and $\tau \in \hom(J, J')$ such that $j \preccurlyeq_J \sigma(\tau(j))$ for all $j \in \underline{J}$. Then for each Cauchy net $\boldsymbol{x} \in \underline{X}^J$ with a modulus $\alpha \in \hom(I_X, J)$, $\sigma_X(\boldsymbol{x})$, or $\boldsymbol{x} \circ \sigma$, is a Cauchy net on J' with a modulus $\tau \circ \alpha \in \hom(I_X, J')$: in fact, for all $a \in \underline{I}_X$ and $i', j' \in \underline{J}'$, if $\tau(\alpha(a)) \preccurlyeq_{J'} i', j'$, then, since $\alpha(a) \preccurlyeq_J \sigma(\tau(\alpha(a))) \preccurlyeq_J \sigma(i'), \sigma(j')$, we have $(\boldsymbol{x}(\sigma(i')), \boldsymbol{x}(\sigma(j'))) \Vdash_X a$, or $((\boldsymbol{x} \circ \sigma)(i'), (\boldsymbol{x} \circ \sigma)(j')) \Vdash_X a$. Apply Lemma 3.30 with $F : \boldsymbol{x} \mapsto \sigma^J(\boldsymbol{x})$ and $G : (\boldsymbol{x}, \alpha) \mapsto \tau \circ \alpha$. Then there exists a uniformly continuous mapping $\tilde{\sigma}_X : \tilde{X}^J \to \tilde{X}^{J'}$ such that $\sigma_X \circ \tilde{\iota}^J_X \sim \tilde{\iota}^{J'}_X \circ \tilde{\sigma}_X$. Since $\sigma_X \circ \eta^J_X = \eta^{J'}_X$, we have

$$\tilde{\iota}^{J'}_X \circ \tilde{\sigma}_X \circ \tilde{\eta}^J_X \sim \sigma_X \circ \tilde{\iota}^J_X \circ \tilde{\eta}^J_X \sim \sigma_X \circ \eta^J_X \sim \eta^{J'}_X \sim \tilde{\iota}^{J'}_X \circ \tilde{\eta}^{J'}_X;$$

hence $\tilde{\sigma}_X \circ \tilde{\eta}^J_X \sim \tilde{\eta}^{J'}_X$, as $\tilde{\iota}^{J'}_X$ is a monomorphism. $\square$

Theorem 3.33. *Let X and Y be uniform spaces, and let J be a directed preordered set. Then there exists a uniform isomorphism $\widetilde{\langle \text{-}, \text{-} \rangle} : \underline{\tilde{X}}^J \times \underline{\tilde{Y}}^J \to (\underline{\widetilde{X \times Y}})^J$ which makes the following diagram commute, that is, $\widetilde{\langle \text{-}, \text{-} \rangle} \circ (\tilde{\eta}^J_X \times \tilde{\eta}^J_Y) \sim \tilde{\eta}^J_{X \times Y}$ and $\langle \text{-}, \text{-} \rangle \circ (\tilde{\iota}^J_X \times \tilde{\iota}^J_Y) \sim \tilde{\iota}^J_{X \times Y} \circ \widetilde{\langle \text{-}, \text{-} \rangle}$; hence*

$$\tilde{X}^J \times \tilde{Y}^J \simeq (\widetilde{X \times Y})^J.$$

$$\begin{tikzcd}
X \times Y \arrow[r, "\tilde{\eta}^J_X \times \tilde{\eta}^J_Y"] \arrow[dr, "\tilde{\eta}^J_{X \times Y}"'] & \tilde{X}^J \times \tilde{Y}^J \arrow[r, "\tilde{\iota}^J_X \times \tilde{\iota}^J_Y"] \arrow[d, "\widetilde{\langle \text{-}, \text{-} \rangle}"] & X^J \times Y^J \arrow[d, "\langle \text{-}, \text{-} \rangle"] \\
& \widetilde{X \times Y}^J \arrow[r, "\tilde{\iota}^J_{X \times Y}"'] & (X \times Y)^J
\end{tikzcd}$$

Proof. Let $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$ be uniform structures on X and Y, respectively, and let $J = (\underline{J}, \preccurlyeq_J)$ with an upper bound function $\mathrm{ub}_J \in \hom(J \times J, J)$. Then for each pair of Cauchy nets $\boldsymbol{x} = (x_j)_{j \in \underline{J}} \in \underline{X}^J$ and $\boldsymbol{y} = (y_j)_{j \in \underline{J}} \in \underline{Y}^J$ with moduli $\alpha \in \hom(I_X, J)$ and $\alpha' \in \hom(I_Y, J)$, respectively, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = ((x_j, y_j))_{j \in \underline{J}} \in \underline{X \times Y}^J$ is a Cauchy net with a modulus $\mathrm{ub}_J \circ (\alpha \times \alpha') \in \hom(I_X \times I_Y, J)$: in fact, for all $(a, b) \in \underline{I}_X \times \underline{I}_Y$ and $i, j \in \underline{J}$, if $\mathrm{ub}_J(\alpha(a), \alpha'(b)) \preccurlyeq_J i, j$, then, since $\alpha(a) \preccurlyeq_J i, j$ and $\alpha'(b) \preccurlyeq_J i, j$, we have $(x_i, x_j) \Vdash_X a$ and $(y_i, y_j) \Vdash_Y b$; hence $((x_i, y_i), (x_j, y_j)) \Vdash_{X \times Y} (a, b)$. Define a function $\widetilde{\langle \text{-}, \text{-} \rangle} : \underline{\tilde{X}}^J \times \underline{\tilde{Y}}^J \to (\underline{\widetilde{X \times Y}})^J$ by

$$\widetilde{\langle \text{-}, \text{-} \rangle} : ((\boldsymbol{x}, \alpha), (\boldsymbol{y}, \alpha')) \mapsto (\langle \boldsymbol{x}, \boldsymbol{y} \rangle, \mathrm{ub}_J \circ (\alpha \times \alpha'))$$

for all $(\boldsymbol{x}, \alpha) \in \underline{\tilde{X}}^J$ and $(\boldsymbol{y}, \alpha') \in \underline{\tilde{Y}}^J$. Then, since $\langle \text{-}, \text{-} \rangle \circ (\tilde{\iota}^J_X \times \tilde{\iota}^J_Y) = \tilde{\iota}^J_{X \times Y} \circ \widetilde{\langle \text{-}, \text{-} \rangle}$, we have $\langle \text{-}, \text{-} \rangle \circ (\tilde{\iota}^J_X \times \tilde{\iota}^J_Y) \sim \tilde{\iota}^J_{X \times Y} \circ \widetilde{\langle \text{-}, \text{-} \rangle}$, and, since $\langle \text{-}, \text{-} \rangle \circ (\tilde{\iota}^J_X \times \tilde{\iota}^J_Y)$ is uniformly continuous, by Proposition 3.12 and Proposition 3.26, $\tilde{\iota}^J_{X \times Y} \circ \widetilde{\langle \text{-}, \text{-} \rangle} : \tilde{X}^J \times \tilde{Y}^J \to (X \times Y)^J$ is uniformly continuous; hence $\widetilde{\langle \text{-}, \text{-} \rangle} : \underline{\tilde{X}}^J \times \underline{\tilde{Y}}^J \to (\underline{\widetilde{X \times Y}})^J$ is a uniformly continuous mapping, by Lemma 3.13 (1) and Lemma 3.8. Since $\langle \text{-}, \text{-} \rangle \circ (\eta^J_X \times \eta^J_Y) = \eta^J_{X \times Y}$, we have

$$
\begin{aligned}
\tilde{\iota}^J_{X \times Y} \circ \widetilde{\langle \text{-}, \text{-} \rangle} \circ (\tilde{\eta}^J_X \times \tilde{\eta}^J_Y) &\sim \langle \text{-}, \text{-} \rangle \circ (\tilde{\iota}^J_X \times \tilde{\iota}^J_Y) \circ (\tilde{\eta}^J_X \times \tilde{\eta}^J_Y) \\
&\sim \langle \text{-}, \text{-} \rangle \circ (\eta^J_X \times \eta^J_Y) \sim \eta^J_{X \times Y} \sim \tilde{\iota}^J_{X \times Y} \circ \tilde{\eta}^J_{X \times Y};
\end{aligned}
$$

hence $\widetilde{\langle \text{-}, \text{-} \rangle} \circ (\tilde{\eta}^J_X \times \tilde{\eta}^J_Y) \sim \tilde{\eta}^J_{X \times Y}$, as $\tilde{\iota}^J_{X \times Y}$ is a monomorphism.

For each Cauchy net $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = ((x_j, y_j))_{j \in \underline{J}} \in \underline{X \times Y}^J$ with a modulus $\alpha \in \hom(I_X \times I_Y, J)$, $\boldsymbol{x} = (x_j)_{j \in \underline{J}} \in \underline{X}^J$ and $\boldsymbol{y} = (y_j)_{j \in \underline{J}} \in \underline{Y}^J$ are Cauchy nets with moduli $\alpha \circ (\mathrm{id}_{\underline{I}_X} \times \overline{b_0}) \in \hom(I_X, J)$ and $\alpha \circ (\overline{a_0} \times \mathrm{id}_{\underline{I}_Y}) \in \hom(I_Y, J)$ where a_0 and b_0 are inhabitants of $\underline{I}_X$ and $\underline{I}_Y$, respectively: in fact, for all $a \in \underline{I}_X$ and $i, j \in \underline{J}$, if $\alpha(a, b_0) \preccurlyeq_J i, j$, then, since $((x_i, y_i), (x_j, y_j)) \Vdash_{X \times Y} (a, b_0)$, we have $(x_i, x_j) \Vdash_X a$; similarly, for all $b \in \underline{I}_Y$ and $i, j \in \underline{J}$, if $\alpha(a_0, b) \preccurlyeq_J i, j$, then $(x_i, x_j) \Vdash_X b$. Let $\pi_0 : X \times Y \to X$ and $\pi_1 : X \times Y \to Y$ be the projections, and define a function $\widetilde{\langle \pi^J_0, \pi^J_1 \rangle} : (\underline{\widetilde{X \times Y}})^J \to \underline{\tilde{X}}^J \times \underline{\tilde{Y}}^J$ by

$$
\widetilde{\langle \pi^J_0, \pi^J_1 \rangle} : (\langle \boldsymbol{x}, \boldsymbol{y} \rangle, \alpha) \mapsto ((\boldsymbol{x}, \alpha \circ (\mathrm{id}_{\underline{I}_X} \times \overline{b_0})), (\boldsymbol{y}, \alpha \circ (\overline{a_0} \times \mathrm{id}_{\underline{I}_Y})))
$$

for all $(\langle \boldsymbol{x}, \boldsymbol{y} \rangle, \alpha) \in \underline{\widetilde{X \times Y}}^J$; see the proof of Proposition 3.26. Then, since $\tilde{\iota}^J_{X \times Y} \circ \widetilde{\langle \pi^J_0, \pi^J_1 \rangle} = \langle \pi^J_0, \pi^J_1 \rangle \circ (\tilde{\iota}^J_X \times \tilde{\iota}^J_Y)$, we have $\tilde{\iota}^J_{X \times Y} \circ \widetilde{\langle \pi^J_0, \pi^J_1 \rangle} \sim \langle \pi^J_0, \pi^J_1 \rangle \circ (\tilde{\iota}^J_X \times \tilde{\iota}^J_Y)$, and, since $\langle \pi^J_0, \pi^J_1 \rangle \circ (\tilde{\iota}^J_X \times \tilde{\iota}^J_Y)$ is uniformly continuous, by Proposition 3.12 and Proposition 3.26, $\tilde{\iota}^J_{X \times Y} \circ \widetilde{\langle \pi^J_0, \pi^J_1 \rangle} : (X \times Y)^J \to \tilde{X}^J \times \tilde{Y}^J$ is uniformly continuous; hence $\widetilde{\langle \pi^J_0, \pi^J_1 \rangle} : (\underline{\widetilde{X \times Y}})^J \to \underline{\tilde{X}}^J \times \underline{\tilde{Y}}^J$ is a uniformly continuous mapping, by Lemma 3.13 (1) and Lemma 3.8.

♣

$\widetilde{\langle \pi^J_0, \pi^J_1 \rangle} : (\widetilde{X \times Y})^J \to \tilde{X}^J \times \tilde{Y}^J$ is an inverse of $\widetilde{\langle \text{-}, \text{-} \rangle} : \tilde{X}^J \times \tilde{Y}^J \to (\widetilde{X \times Y})^J$. □

Definition 3.34. Let $X = (\underline{X}, =_X)$ be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$. Then a *regular net* in X is a Cauchy net on the directed preordered set I_X with a modulus $\mathrm{id}_{\underline{I}_X} \in \hom(I_X, I_X)$.

Let $\underline{\tilde{X}}$ be a subset of $\underline{X}^{I_X}$ given by

$$
\underline{\tilde{X}} = \{\boldsymbol{x} \in \underline{X}^{I_X} \mid \boldsymbol{x} \text{ is a regular net}\},
$$

and let $\tilde{\iota}_X : \underline{\tilde{X}} \to \underline{\tilde{X}}^{I_X}$ be a function given by

$$
\tilde{\iota}_X : \boldsymbol{x} \mapsto (\boldsymbol{x}, \mathrm{id}_{\underline{I}_X})
$$

for all $\boldsymbol{x} \in \underline{\tilde{X}}$. Then a uniform space $\tilde{X}$, called a *completion* of X, is a uniform subspace induced by a subsetoid $\tilde{X} = (\underline{\tilde{X}}, \tilde{\iota}_X)$ of $\tilde{X}^{I_X}$; see Proposition 3.12. Note that for all $\boldsymbol{x}, \boldsymbol{y} \in \underline{\tilde{X}}$ and $a \in \underline{I}_X$,

$$
\boldsymbol{x} =_{\tilde{X}} \boldsymbol{y} \Leftrightarrow \boldsymbol{x} =_{X^{I_X}} \boldsymbol{y}, \qquad (\boldsymbol{x}, \boldsymbol{y})) \Vdash_{\tilde{X}} a \Leftrightarrow (\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^{I_X}} a.
$$

Since $\eta_X^{I_X}(x) \in \underline{\tilde{X}}$ for all $x \in \underline{X}$, we denote $\eta_X^{I_X} : X \to X^{I_X}$ by $\eta_X : X \to \tilde{X}$; note that, since $\tilde{\iota}_X^{I_X} \circ \tilde{\iota}_X \circ \eta_X^{I_X} \sim \eta_X^{I_X}$, we have $\tilde{\iota}_X^{I_X} \circ \tilde{\iota}_X \circ \eta_X \sim \tilde{\iota}_X^{I_X} \circ \tilde{\eta}_X^{I_X}$, by Lemma 3.22; hence $\tilde{\iota}_X \circ \eta_X \sim \tilde{\eta}_X^{I_X}$ as $\tilde{\iota}_X^{I_X}$ is a monomorphism.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\tilde{\eta}_X^{I_X}"'] & \tilde{X} \arrow[d, "\tilde{\iota}_X"] \\ & \tilde{X}^{I_X} \end{tikzcd}$$

Lemma 3.35. *Let X be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let J be a directed preordered set. Then*

1. *if $\boldsymbol{x} \in \underline{X}^J$ is a Cauchy net with a modulus $\alpha \in \hom(I_X, J)$, then $\boldsymbol{x} \circ \alpha \in \underline{X}^{I_X}$ is a regular net;*

2. *if $\boldsymbol{x}, \boldsymbol{y} \in \underline{X}^J$ are Cauchy nets with moduli $\alpha, \alpha' \in \hom(I_X, J)$, respectively, then for each $a \in \underline{I}_X$,*
$$(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} \rho_X^4(a) \Rightarrow (\boldsymbol{x} \circ \alpha, \boldsymbol{y} \circ \alpha') \Vdash_{X^{I_X}} a;$$

3. *if $\boldsymbol{x}, \boldsymbol{y} \in \underline{X}^J$ are Cauchy nets with moduli $\alpha, \alpha' \in \hom(I_X, J)$, respectively, then for each $a \in \underline{I}_X$,*
$$(\boldsymbol{x} \circ \alpha, \boldsymbol{y} \circ \alpha') \Vdash_{X^{I_X}} \rho_X^4(a) \Rightarrow (\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a;$$

4. *if $\boldsymbol{x}, \boldsymbol{y} \in \underline{X}^J$ are Cauchy nets with moduli $\alpha, \alpha' \in \hom(I_X, J)$, respectively, then*
$$\boldsymbol{x} =_{X^J} \boldsymbol{y} \Leftrightarrow \boldsymbol{x} \circ \alpha =_{X^{I_X}} \boldsymbol{y} \circ \alpha';$$

5. *if $\boldsymbol{x} \in \underline{X}^{I_X}$ is a Cauchy net with a modulus $\alpha \in \hom(I_X, I_X)$, then*
$$\boldsymbol{x} \circ \alpha =_{X^{I_X}} \boldsymbol{x}.$$

Proof. Let $I_X = (\underline{I}_X, \preccurlyeq_{I_X})$ with an upper bound function $\mathrm{ub}_{I_X} \in \hom(I_X \times I_X, I_X)$, and let $J = (\underline{J}, \preccurlyeq_J)$.

(1): Consider a Cauchy net $\boldsymbol{x} = (x_j)_{j \in \underline{J}} \in \underline{X}^J$ with a modulus $\alpha \in \hom(I_X, J)$ and $a \in \underline{I}_X$. Then for all $c, c' \in \underline{I}_X$ with $a \preccurlyeq_{I_X} c, c'$, since $\alpha(a) \preccurlyeq_J \alpha(c), \alpha(c')$, we have $(x_{\alpha(c)}, x_{\alpha(c')}) \Vdash_X a$. Therefore $\boldsymbol{x} \circ \alpha \in \underline{X}^{I_X}$ is a regular net.

(2): Consider Cauchy nets $\boldsymbol{x} = (x_j)_{j \in \underline{J}}, \boldsymbol{y} = (y_j)_{j \in \underline{J}} \in \underline{X}^J$ with moduli $\alpha, \alpha' \in \hom(I_X, J)$, respectively, and $a \in \underline{I}_X$ such that $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} \rho_X^4(a)$. Then there exists $j \in \underline{J}$ such that $(x_i, y_i) \Vdash_X \rho_X^2(a)$ for all $i \in \underline{J}$ with $j \preccurlyeq_J i$, by Lemma 3.20 (1). Choose $i \in \underline{J}$ such that $j \preccurlyeq_J i$, $\alpha(\rho_X^2(a)) \preccurlyeq_J i$ and $\alpha'(\rho_X^2(a)) \preccurlyeq_J i$. Then for all $c \in \underline{I}_X$ with $\rho_X^2(a) \preccurlyeq_{I_X} c$, since $\alpha(\rho_X^2(a)) \preccurlyeq_J \alpha(c), i$ and $\alpha'(\rho_X^2(a)) \preccurlyeq_J \alpha(c), i$, we have $(x_i, y_i) \Vdash_X \rho_X^2(a)$, $(x_{\alpha(c)}, x_i) \Vdash_X \rho_X^2(a)$ and $(y_{\alpha'(c)}, y_i) \Vdash_X \rho_X^2(a)$; hence we have $(x_{\alpha(c)}, y_{\alpha'(c)}) \Vdash_X a$, by Lemma 3.3 (2) . Therefore $(\boldsymbol{x} \circ \alpha, \boldsymbol{y} \circ \alpha') \Vdash_{X^{I_X}} a$, by Lemma 3.20 (2).

(3): Consider Cauchy nets $\boldsymbol{x} = (x_j)_{j \in \underline{J}}, \boldsymbol{y} = (y_j)_{j \in \underline{J}} \in \underline{X}^J$ with moduli $\alpha, \alpha' \in \hom(I_X, J)$, respectively, and $a \in \underline{I}_X$ such that $(\boldsymbol{x} \circ \alpha, \boldsymbol{y} \circ \alpha') \Vdash_{X^{I_X}}$

$\rho^4_X(a)$. Then there exists $b \in \underline{I_X}$ such that $(x_{\alpha(c)}, y_{\alpha'(c)}) \Vdash_X \rho^2_X(a)$ for all $c \in \underline{I_X}$ with $b \preccurlyeq_{I_X} c$, by Lemma 3.20 (1). Choose $c \in \underline{I_X}$ such that $b \preccurlyeq_{I_X} c$, and $\rho^2_X(a) \preccurlyeq_{I_X} c$. Then for all $i \in \underline{J}$ with $\mathrm{ub}_J(\alpha(\rho^2_X(a)), \alpha'(\rho^2_X(a)) \preccurlyeq_J i$, since $\alpha(\rho^2_X(a)) \preccurlyeq_J \alpha(c), i$ and $\alpha'(\rho^2_X(a)) \preccurlyeq_J \alpha'(c), i$, we have $(x_{\alpha(c)}, y_{\alpha'(c)}) \Vdash_X \rho^2_X(a)$, $(x_{\alpha(c)}, x_i) \Vdash_X \rho^2_X(a)$ and $(y_{\alpha'(c)}, y_i) \Vdash_X \rho^2_X(a)$; hence we have $(x_i, y_i) \Vdash_X a$, by Lemma 3.3 (2) . Therefore $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a$, by Lemma 3.20 (2).

(4): It follows from (2) and (3).

(5): Consider a Cauchy net $\boldsymbol{x} = (x_j)_{j \in \underline{J}} \in \underline{X}^J$ with a modulus $\alpha \in \mathrm{hom}(I_X, J)$ and $a \in \underline{I_X}$. Then for all $c \in \underline{I_X}$ with $\mathrm{ub}_{I_X}(a, \alpha(a)) \preccurlyeq_{I_X} c$, since $\alpha(a) \preccurlyeq_J c$ and $\alpha(a) \preccurlyeq_J \alpha(c)$, we have $(x_{\alpha(c)}, x_c) \Vdash_X a$. Therefore, since $a \in \underline{I_X}$ is arbitrary, we have $\boldsymbol{x} \circ \alpha =_{X^{I_X}} \boldsymbol{x}$. ☐

Proposition 3.36. *Let X be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let J be a directed preordered set. Then there exists a uniformly continuous setoid injection $\nu^J_X : \tilde{X}^J \to \tilde{X}$ which makes the following diagram commute, that is, $\nu^J_X \circ \tilde{\eta}^J_X \sim \eta_X$; hence $\tilde{X}^J$ is a subspace of $\tilde{X}$. Especially, $\nu^{I_X}_X : \tilde{X}^{I_X} \to \tilde{X}$ is a uniform isomorphism with an inverse $\tilde{\imath}_X : \tilde{X} \to \tilde{X}^{I_X}$; hence*

$$\tilde{X} \simeq \tilde{X}^{I_X}.$$

$$\begin{tikzcd}
X \arrow[r, "\tilde{\eta}^J_X"] \arrow[dr, "\eta_X"'] & \tilde{X}^J \arrow[d, "\nu^J_X"] \\
& \tilde{X}
\end{tikzcd}$$

Proof. Note that for each $(\boldsymbol{x}, \alpha) \in \underline{\tilde{X}}^J$, $\boldsymbol{x} \circ \alpha \in \underline{\tilde{X}}$, by Lemma 3.35 (1), and define a function $\nu^J_X : \underline{\tilde{X}}^J \to \underline{\tilde{X}}$ by

$$\nu^J_X : (\boldsymbol{x}, \alpha) \mapsto \boldsymbol{x} \circ \alpha$$

for all $(\boldsymbol{x}, \alpha) \in \underline{\tilde{X}}^J$. For all $(\boldsymbol{x}, \alpha), (\boldsymbol{y}, \alpha') \in \underline{\tilde{X}}^J$ and $a \in \underline{I_X}$, if $((\boldsymbol{x}, \alpha), (\boldsymbol{y}, \alpha')) \Vdash_{\tilde{X}^J} \rho^4_X(a)$, then, since $(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} \rho^4_X(a)$, we have $(\boldsymbol{x} \circ \alpha, \boldsymbol{y} \circ \alpha') \Vdash_{X^{I_X}} a$, by Lemma 3.35 (2); hence $(\boldsymbol{x} \circ \alpha, \boldsymbol{y} \circ \alpha') \Vdash_{\tilde{X}} a$. Therefore $\nu^J_X : \underline{\tilde{X}}^J \to \underline{\tilde{X}}$ is uniformly continuous with a modulus $\rho^4_X \in \mathrm{hom}(I_X, I_X)$. It is straightforward to see that $\nu^J_X : \tilde{X}^J \to \tilde{X}$ is a setoid injection, by Lemma 3.35 (4), and, since $\nu^J_X \circ \tilde{\eta}^J_X = \eta_X$, we have $\nu^J_X \circ \tilde{\eta}^J_X \sim \eta_X$.

For each $(\boldsymbol{x}, \alpha) \in \underline{\tilde{X}}^{I_X}$, since $\boldsymbol{x} \circ \alpha =_{X^{I_X}} \boldsymbol{x}$, we have

$$\begin{aligned}
(\tilde{\imath}_X \circ \nu^{I_X}_X)(\boldsymbol{x}, \alpha) &=_{\tilde{X}^{I_X}} \tilde{\imath}_X(\nu^{I_X}_X(\boldsymbol{x}, \alpha)) =_{\tilde{X}^{I_X}} \tilde{\imath}_X(\boldsymbol{x} \circ \alpha) \\
&=_{\tilde{X}^{I_X}} (\boldsymbol{x} \circ \alpha, \mathrm{id}_{\underline{I_X}}) =_{\tilde{X}^{I_X}} (\boldsymbol{x}, \alpha),
\end{aligned}$$

and for each $\boldsymbol{x} \in \underline{\tilde{X}}$, since $\boldsymbol{x} \circ \mathrm{id}_{\underline{I_X}} = \boldsymbol{x}$, we have

$$(\nu^{I_X}_X \circ \tilde{\imath}_X)(\boldsymbol{x}) =_{\tilde{X}} \nu^{I_X}_X(\tilde{\imath}_X(\boldsymbol{x})) =_{\tilde{X}} \nu^{I_X}_X(\boldsymbol{x}, \mathrm{id}_{\underline{I_X}}) =_{\tilde{X}} \boldsymbol{x} \circ \mathrm{id}_{\underline{I_X}} =_{\tilde{X}} \boldsymbol{x}.$$

Therefore $\tilde{\imath}_X : \tilde{X} \to \tilde{X}^{I_X}$ is an inverse of $\nu^J_X : \tilde{X}^J \to \tilde{X}$. ☐

Lemma 3.37. *Let X and Y be uniform spaces. Then for each pair of uniformly continuous mappings $f, g : \tilde{X} \to Y$, if $f \circ \eta_X \sim g \circ \eta_X$, then $f \sim g$; that is, η_X is an epimorphism in the category of uniform spaces and uniformly continuous mappings.*

Proof. Consider uniformly continuous mappings $f, g : \tilde{X} \to Y$ with $f \circ \eta_X \sim g \circ \eta_X$. Then, since $f \circ \nu_X^{I_X} \circ \tilde{\eta}_X^{I_X} \sim g \circ \nu_X^{I_X} \circ \tilde{\eta}_X^{I_X}$, we have $f \circ \nu_X^{I_X} \sim g \circ \nu_X^{I_X}$, by Lemma 3.29; hence $f \sim f \circ \nu_X^{I_X} \circ \tilde{\iota}_X \sim g \circ \nu_X^{I_X} \circ \tilde{\iota}_X \sim g$, by Proposition 3.36. $\square$

Proposition 3.38. *Let X be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let J be a directed preordered set. Then $\nu_X^{I_X \times J} : \tilde{X}^{I_X \times J} \to \tilde{X}$ and $\nu_X^{J \times I_X} : \tilde{X}^{J \times I_X} \to \tilde{X}$ are uniform isomorphisms; hence*

$$\tilde{X}^{I_X \times J} \simeq \tilde{X} \qquad \textit{and} \qquad \tilde{X}^{J \times I_X} \simeq \tilde{X}.$$

Proof. Let $\pi_0 : \underline{I}_X \times \underline{J} \to \underline{I}_X$ be the projection, and let $\tau_0 \in \hom(I_X, I_X \times J)$ be given by

$$\tau_0 : a \mapsto (a, j_0)$$

for all $a \in \underline{I}_X$, where j_0 is an inhabitant of $\underline{J}$. Then $\pi_0 \in \hom(I_X \times J, I_X)$ and $a \preccurlyeq_{I_X} \pi_0(\tau_0(a))$ for all $a \in \underline{I}_X$; hence there exist uniformly continuous mapping $(\widetilde{\pi_0})_X : \tilde{X}^{I_X} \to \tilde{X}^{I_X \times J}$ such that $(\widetilde{\pi_0})_X \circ \tilde{\eta}_X^{I_X} \sim \eta_X^{I_X \times J}$, by Proposition 3.32. Therefore, since $\tilde{\iota}_X \circ \eta_X \sim \tilde{\eta}_X^{I_X}$ and $\nu_X^{I_X \times J} \circ \eta_X^{I_X \times J} \sim \eta_X$, by Proposition 3.36, we have $\nu_X^{I_X \times J} \circ (\widetilde{\pi_0})_X \circ \tilde{\iota}_X \circ \eta_X \sim \eta_X$ and $(\widetilde{\pi_0})_X \circ \tilde{\iota}_X \circ \nu_X^{I_X \times J} \circ \eta_X^{I_X \times J} \sim \eta_X^{I_X \times J}$, and so $\nu_X^{I_X \times J} \circ (\widetilde{\pi_0})_X \circ \tilde{\iota}_X \sim \mathrm{id}_{\tilde{X}}$ and $(\widetilde{\pi_0})_X \circ \tilde{\iota}_X \circ \nu_X^{I_X \times J} \sim \mathrm{id}_{\tilde{X}^{I_X \times J}}$, by Lemma **??** (**??**) and (**??**). Thus $(\widetilde{\pi_0})_X \circ \tilde{\iota}_X : \tilde{X} \to \tilde{X}^{I_X \times J}$ is an inverse of $\nu_X^{I_X \times J} : \tilde{X}^{I_X \times J} \to \tilde{X}$.

Similarly, for the projection $\pi_1 : \underline{J} \times \underline{I}_X \to \underline{I}_X$ and $\tau_1 \in \hom(I_X, J \times I_X)$ given by $\tau_1 : a \mapsto (j_0, a)$ for all $a \in \underline{I}_X$, $(\widetilde{\pi_1})_X \circ \tilde{\iota}_X : \tilde{X} \to \tilde{X}^{J \times I_X}$ is an inverse of $\nu_X^{J \times I_X} : \tilde{X}^{J \times I_X} \to \tilde{X}$. $\square$

Proposition 3.39. *Let X be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let J be a directed preordered set. Then there exists a uniformly continuous mapping $\theta_X^J : \tilde{\tilde{X}}^J \to \tilde{X}^{J \times I_X}$ which makes the following diagram commute, that is, $\theta_X^J \circ \eta_{\tilde{X}}^J \circ \eta_X \sim \eta_X^{J \times I_X}$.*

$$\begin{tikzcd}
X \arrow[r, "\tilde{\eta}_X^{J \times I_X}"] \arrow[d, "\eta_X"'] & \tilde{X}^{J \times I_X} \\
\tilde{X} \arrow[r, "\tilde{\eta}_{\tilde{X}}^J"'] & \tilde{\tilde{X}}^J \arrow[u, "\theta_X^J"']
\end{tikzcd}$$

Proof. Let $X = (\underline{X}, =_X)$ be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let $J = (\underline{J}, \preccurlyeq_J)$ be a directed preordered set.

We show that for each $\boldsymbol{x} \in \underline{X}^{J \times I_X}$, if $\hat{\boldsymbol{x}} \in \underline{\tilde{X}}^J$ is a Cauchy net with a modulus $\alpha \in \hom(I_X, J)$, then $\boldsymbol{x}$ is a Cauchy net with a modulus $\langle \alpha \circ \rho_X^4, \rho_X^2 \rangle \in \hom(I_X, J \times I_X)$. To this end, consider $a \in \underline{I}_X$ and $(j, c), (j', c') \in \underline{J} \times \underline{I}_X$ with $(\alpha(\rho_X(a)), \rho_X^2(a)) \preccurlyeq_{J \times I_X} (j, c), (j', c')$. Then, since $\alpha(\rho_X^4(a)) \preccurlyeq_J j, j'$, we have $((x_{j,b})_{b \in \underline{I}_X}, (x_{j',b})_{b \in \underline{I}_X}) \Vdash_{\tilde{X}} \rho_X^4(a)$; hence there exists $b \in \underline{I}_X$

such that $(x_{j,b'}, x_{j',b'})) \Vdash_X \rho_X^2(a)$ for all $b' \in \underline{I}_X$ with $b \preccurlyeq_{I_X} b'$. Set $b' = \mathrm{ub}_{I_X}(b, \rho_X^2(a))$. Then, since $\rho_X^2(a) \preccurlyeq_{I_X} c, c', b'$, we have $(x_{j,c}, x_{j,b'}) \Vdash_X \rho_X^2(a)$ and $(x_{j',b'}, x_{j,c'}) \Vdash_X \rho_X^2(a)$; hence $(x_{j,c}, x_{j',c'}) \Vdash_X a$, by Lemma 3.3 (2). $\square$

Definition 3.40. A uniform space X is *complete* if $\eta_X : X \to \tilde{X}$ is a uniform isomorphism.

Proposition 3.41. *Every Cauchy net in a complete uniform space X converges.*

Proof. Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on X, let $g : \underline{\tilde{X}} \to \underline{X}$ be a uniformly continuous inverse of $\eta_X : X \to \tilde{X}$, and consider a Cauchy net $\boldsymbol{x}$ in X on a directed preordered set J with a modulus $\alpha \in \hom(I_X, J)$, that is, $(\boldsymbol{x}, \alpha) \in \underline{\tilde{X}}^J$. Then, since $\eta_X^J \circ \boldsymbol{x} \to \boldsymbol{x}$ in X^J, by Lemma 3.22, we have $\tilde{\iota}_X^J \circ \tilde{\eta}_X^J \circ \boldsymbol{x} \to \tilde{\iota}_X^J(\boldsymbol{x}, \alpha)$ in X^J, by Lemma 3.28; hence $\tilde{\eta}_X^J \circ \boldsymbol{x} \to (\boldsymbol{x}, \alpha)$ in $\tilde{X}^J$, by Lemma 3.13 (2). Therefore, since $\nu_X^J \circ \tilde{\eta}_X^J \circ \boldsymbol{x} \to \nu_X^J(\boldsymbol{x}, \alpha)$ in $\tilde{X}$, by Lemma 3.10 (1), we have $\eta_X \circ \boldsymbol{x} \to \nu_X^J(\boldsymbol{x}, \alpha)$ in $\tilde{X}$, and so $g \circ \eta_X \circ \boldsymbol{x} \to g(\nu_X^J(\boldsymbol{x}, \alpha))$ in X, by Lemma 3.10 (1). Thus $\boldsymbol{x} \to g(\nu_X^J(\boldsymbol{x}, \alpha))$ in X. $\square$

Theorem 3.42. *The completion $\tilde{X}$ of a uniform space X is complete.*

Proof. Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on X, and note that $\nu_{\tilde{X}}^{I_X} : \tilde{\tilde{X}}^{I_X} \to \tilde{\tilde{X}}$ is a uniform isomorphism, by Proposition 3.36. Define a uniformly continuous mapping $g : \tilde{\tilde{X}} \to \tilde{X}$ by

$$g = \nu_X^{I_X \times I_X} \circ \theta_X^{I_X} \circ (\nu_{\tilde{X}}^{I_X})^{-1}.$$

Then, since $g \circ \eta_{\tilde{X}} \circ \eta_X \sim \nu_X^{I_X \times I_X} \circ \theta_X^{I_X} \circ \eta_{\tilde{X}}^{I_X} \circ \eta_X \sim \nu_X^{I_X \times I_X} \circ \eta_X^{I_X \times I_X} \sim \eta_X$ and $\eta_{\tilde{X}} \circ g \circ \eta_{\tilde{X}} \circ \eta_X \sim \eta_{\tilde{X}} \circ \eta_X$, we have $g \circ \eta_{\tilde{X}} \sim \mathrm{id}_{\tilde{X}}$ and $\eta_{\tilde{X}} \circ g \circ \eta_{\tilde{X}} \sim \eta_{\tilde{X}}$, by Lemma 3.37; hence $\eta_{\tilde{X}} \circ g \sim \mathrm{id}_{\tilde{\tilde{X}}}$, by Lemma 3.37. Therefore $\eta_{\tilde{X}} : \tilde{X} \to \tilde{\tilde{X}}$ is a uniform isomorphism with an inverse $g : \tilde{\tilde{X}} \to \tilde{X}$.

$$\begin{tikzcd}
\tilde{X} \arrow[r, leftrightarrow, "\nu_X^{I_X \times I_X}"'] \arrow[rrr, bend left, "\eta_{\tilde{X}}"] & \tilde{X}^{I_X \times I_X} & \tilde{\tilde{X}}^{I_X} \arrow[l, "\theta_X^{I_X}"'] \arrow[r, leftrightarrow, "\nu_{\tilde{X}}^{I_X}"'] & \tilde{\tilde{X}} \\
 & X \arrow[ul, "\eta_X"] \arrow[u, "\eta_X^{I_X \times I_X}"] \arrow[r, "\eta_X"] & \tilde{X} \arrow[u, "\eta_{\tilde{X}}^{I_X}"] \arrow[ur, "\eta_{\tilde{X}}"'] &
\end{tikzcd}$$

$\square$

Theorem 3.43. *Let X and Y be uniform spaces. Then*

$$\tilde{X} \times \tilde{Y} \simeq \widetilde{X \times Y},$$

and $X \times Y$ is complete whenever so are X and Y.

Proof. Let $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$ be uniform structures on X and Y, respectively. Then, since $\nu_X^{I_X\times I_Y} : \tilde{X}^{I_X\times I_Y} \to \tilde{X}$ and $\nu_Y^{I_X\times I_Y} : \tilde{Y}^{I_X\times I_Y} \to \tilde{Y}$ are uniform isomorphisms, by Proposition 3.38, we have $\nu_X^{I_X\times I_Y} \times \nu_Y^{I_X\times I_Y} : \tilde{Y}^{I_X\times I_Y} \times \tilde{X}^{I_X\times I_Y} \to \tilde{X} \times \tilde{Y}$ is a uniform isomorphism. Therefore, since $\nu_{X\times Y}^{I_X\times I_Y} : \widetilde{X\times Y}^{I_X\times I_Y} \to \widetilde{X\times Y}$ and $\widetilde{\langle \text{-},\text{-}\rangle} : \tilde{X}^{I_X\times I_Y} \times \tilde{Y}^{I_X\times I_Y} \to \widetilde{X\times Y}^{I_X\times I_Y}$ are uniform isomorphisms, by Proposition 3.36 and Theorem 3.33, a mapping $g : \tilde{X} \times \tilde{Y} \to \widetilde{X\times Y}$ given by

$$g = \nu_{X\times Y}^{I_X\times I_Y} \circ \widetilde{\langle \text{-},\text{-}\rangle} \circ (\nu_X^{I_X\times I_Y} \times \nu_Y^{I_X\times I_Y})^{-1}$$

is a uniform isomorphism. If X and Y are complete, then, since $\eta_X : X \to \tilde{X}$ and $\eta_Y : Y \to \tilde{Y}$ are uniform isomorphisms, $\eta_X \times \eta_Y : X \times Y \to \tilde{X} \times \tilde{Y}$ is a uniform isomorphism; hence so is $\eta_{X\times Y} \sim g \circ (\eta_X \times \eta_Y) : X \times Y \to \widetilde{X\times Y}$.

$$\begin{tikzcd}
 & \tilde{X}^{I_X\times I_Y} \times \tilde{Y}^{I_X\times I_Y} \arrow[rr, leftrightarrow, "\widetilde{\langle \text{-},\text{-}\rangle}"] & & \widetilde{X\times Y}^{I_X\times I_Y} \arrow[dr, leftrightarrow, "\nu_{X\times Y}^{I_X\times I_Y}"] & \\
\tilde{X} \times \tilde{Y} \arrow[ur, leftrightarrow, "\nu_X^{I_X\times I_Y} \times \nu_Y^{I_X\times I_Y}"] & & & & \widetilde{X\times Y} \\
 & & X \times Y \arrow[ull, "\eta_X \times \eta_Y"] \arrow[uul, "\eta_X^{I_X\times I_Y} \times \eta_Y^{I_X\times I_Y}"'] \arrow[uur, "\eta_{X\times Y}^{I_X\times I_Y}"] \arrow[urr, "\eta_{X\times Y}"'] & &
\end{tikzcd}$$

□

Definition 3.44. Let X and Y be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respectively. Then a function $f : \underline{X} \to \underline{Y}$ is *locally uniformly continuous* if there exists a function $\boldsymbol{z} \mapsto \gamma_{\boldsymbol{z}}$ from $\underline{\tilde{X}}$ into $\mathrm{hom}(I_Y, I_X)$, called a *family of local moduli*, such that for each $b \in \underline{I_Y}$,

$$(\boldsymbol{z}, \eta_X(x)) \Vdash_{\tilde{X}} \gamma_{\boldsymbol{z}}(b) \text{ and } (\boldsymbol{z}, \eta_X(y)) \Vdash_{\tilde{X}} \gamma_{\boldsymbol{z}}(b) \ \Rightarrow (f(y), f(z)) \Vdash_Y b$$

for all $\boldsymbol{z} \in \underline{\tilde{X}}$ and $x, y \in \underline{X}$

Lemma 3.45. *Let X and Y be uniform spaces. Then every locally uniformly continuous function $f : \underline{X} \to \underline{Y}$ is a setoid mapping.*

Proof. Let $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$ be uniform structures on X and Y, respectively, and suppose that f is locally uniformly continuous with a family $\boldsymbol{z} \mapsto \gamma_{\boldsymbol{z}}$ of local moduli. For all $x, y \in \underline{X}$, if $x =_X y$, then, for all $b \in \underline{I_Y}$, since $(x, x) \Vdash_X \gamma_{\eta_X(x)}(b)$ and $(x, y) \Vdash_X \gamma_{\eta_X(x)}(b)$, we have $(\eta_X(x), \eta_X(x)) \Vdash_{\tilde{X}} \gamma_{\eta_X(x)}(b)$ and $(\eta_X(x), \eta_X(y)) \Vdash_{\tilde{X}} \gamma_{\eta_X(x)}(b)$; hence $(f(x), f(y)) \Vdash_Y b$. Therefore $f(x) =_Y f(y)$. □

Lemma 3.46. *Let X and Y be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X$ $)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respectively, and let J be a directed preordered set. Then for each locally uniformly continuous function $f : \underline{X} \to \underline{Y}$ with a family of local moduli $\boldsymbol{z} \mapsto \gamma_{\boldsymbol{z}}$,*

1. *for all $\boldsymbol{x} \in \underline{X}^J$ and $x \in \underline{X}$, if $\boldsymbol{x} \to x$, then $f \circ \boldsymbol{x} \to f(x)$;*

2. *for all $\boldsymbol{x} \in \underline{X}^J$ and $\alpha \in \hom(I_X, J)$, if $\boldsymbol{x}$ is a Cauchy net with a modulus α, then $f \circ \boldsymbol{x}$ is a Cauchy net with a modulus $\alpha \circ \gamma_{(\boldsymbol{x}\circ\alpha)} \in \hom(I_Y, J)$.*

Proof. (1): Consider $\boldsymbol{x} = (x_j)_{j\in\underline{J}} \in \underline{X}^J$ and $x \in \underline{X}$ such that $\boldsymbol{x} \to x$ with a modulus $\beta \in \hom(I_X, J)$. Then for all $b \in \underline{I}_Y$ and $j \in \underline{J}$ with $\beta(\gamma_{\eta_X(x)}(b)) \preccurlyeq_J j$, since $(x, x_j) \Vdash_X \gamma_{\eta_X(x)}(b)$ and $(x, x) \Vdash_X \gamma_{\eta_X(x)}(b)$, we have $(\eta_X(x), \eta_X(x_j)) \Vdash_{\tilde{X}} \gamma_{\eta_X(x)}(b)$ and $(\eta_X(x), \eta_X(x)) \Vdash_{\tilde{X}} \gamma_{\eta_X(x)}(b)$; hence $(f(x_j), f(x)) \Vdash_Y b$. Therefore $f \circ \boldsymbol{x} \to f(x)$ with a modulus $\beta \circ \gamma_{\eta_X(x)} \in \hom(I_Y, J)$.

(2): Consider a Cauchy net $\boldsymbol{x} = (x_j)_{j\in\underline{J}} \in \underline{X}^J$ with a modulus $\alpha \in \hom(I_X, J)$. Then for all $b \in \underline{I}_Y$ and $j, j' \in \underline{J}$ with $\alpha(\gamma_{(\boldsymbol{x}\circ\alpha)}(b)) \preccurlyeq_J j, j'$, since $(x_{\alpha(c)}, x_j) \Vdash_X \gamma_{(\boldsymbol{x}\circ\alpha)}(b)$ and $(x_{\alpha(c)}, x_{j'}) \Vdash_X \gamma_{(\boldsymbol{x}\circ\alpha)}(b)$ for all $c \in \underline{I}_X$ with $\gamma_{(\boldsymbol{x}\circ\alpha)}(b) \preccurlyeq_{I_X} c$, we have $(f \circ \alpha, \eta_X(x_j)) \Vdash_{\tilde{X}} \gamma_{(\boldsymbol{x}\circ\alpha)}(b)$ and $(f \circ \alpha, \eta_X(x_{j'})) \Vdash_{\tilde{X}} \gamma_{(\boldsymbol{x}\circ\alpha)}(b)$; hence $(f(x_j), f(x_{j'})) \Vdash_Y b$. Therefore $f \circ \boldsymbol{x}$ is a Cauchy net in Y with a modulus $\alpha \circ \gamma_{(\boldsymbol{x}\circ\alpha)} \in \hom(I_Y, J)$. $\square$

Remark 3.47. For each pair of setoid mappings $f, g : X \to Y$ between uniform spaces X and Y with $f \sim g$, f is locally uniformly continuous if and only if g is locally uniformly continuous. Let $X = (\underline{X}, =_X)$, $Y = (\underline{Y}, =_Y)$ and $Z = (\underline{Z}, =_Z)$ be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$, $(I_Y, \rho_Y, \Vdash_Y)$ and $(I_Z, \rho_Z, \Vdash_Z)$, respectively. Then the composition $g \circ f : \underline{X} \to \underline{Z}$ of locally uniformly continuous functions $f : \underline{X} \to \underline{Y}$ and $g : \underline{Y} \to \underline{Z}$ with families $\boldsymbol{z} \mapsto \gamma^f_{\boldsymbol{z}}$ and $\boldsymbol{z}' \mapsto \gamma^g_{\boldsymbol{z}'}$ of local moduli, respcetively, is locally uniformly continuous with a family $\boldsymbol{z} \mapsto \gamma^f_{\boldsymbol{z}} \circ \gamma^g_{(f\circ\boldsymbol{z}\circ\gamma^f_{\boldsymbol{z}})}$ of local moduli. Every uniformly continuous function $f : \underline{X} \to \underline{Y}$ with a modulus $\gamma \in \hom(I_Y, I_X)$ is locally uniformly continuous with the constant function $\boldsymbol{z} \mapsto \rho_X \circ \gamma$ as a family of local moduli.

Lemma 3.48. *Let X and Y be uniform spaces, and let J be a directed preordered set. Then*

1. *for each pair of locally uniformly continuous mappings $f, g : \tilde{X}^J \to Y$, if $f \circ \tilde{\eta}^J_X \sim g \circ \tilde{\eta}^J_X$, then $f \sim g$;*

2. *for each pair of locally uniformly continuous mappings $f, g : \tilde{X} \to Y$, if $f \circ \eta_X \sim g \circ \eta_X$, then $f \sim g$.*

Proof. $\square$

Proposition 3.49. *Let X and Y be uniform spaces, and let J be a directed preordered set. Then for each locally uniformly continuous mapping $f : X \to Y$, there exists a locally uniformly continuous mapping $\tilde{f}^J : \tilde{X}^J \to \tilde{Y}^J$ which makes the following diagram commute, that is, $f^J \circ \tilde{\iota}^J_X \sim \tilde{\iota}^J_Y \circ \tilde{f}^J$ and $\tilde{f}^J \circ \tilde{\eta}^J_X \sim \tilde{\eta}^J_Y \circ f$. Furthermore, such an $\tilde{f}^J$ is unique in the sense that for each locally uniformly continuous mapping $h : \tilde{X}^J \to \tilde{Y}^J$, if $h \circ \tilde{\eta}^J_X \sim \tilde{\eta}^J_Y \circ f$, then $h \sim \tilde{f}^J$.*

$$
\begin{tikzcd}
X \arrow[r, "\tilde{\eta}^J_X"] \arrow[d, "f"'] & \tilde{X}^J \arrow[r, "\tilde{\iota}^J_X"] \arrow[d, "\tilde{f}^J"] & X^J \arrow[d, "f^J"] \\
Y \arrow[r, "\tilde{\eta}^J_Y"'] & \tilde{Y}^J \arrow[r, "\tilde{\iota}^J_Y"'] & Y^J
\end{tikzcd}
$$

Proof. □

Theorem 3.50. *Let X and Y be uniform spaces. Then for each uniformly continuous (respectively, locally uniformly continuous) function $f : X \to Y$, there exists a uniformly continuous (respectively, locally uniformly continuous) function $\tilde{f} : \underline{\tilde{X}} \to \underline{\tilde{Y}}$ which makes the following diagram commute, that is, $\tilde{f} \circ \eta_X \sim \eta_Y \circ f$. Furthermore, such an $\tilde{f}$ is unique in the sense that for each locally uniformly continuous mapping $h : \tilde{X} \to \tilde{Y}$, if $h \circ \eta_X \sim \eta_Y \circ f$, then $h \sim \tilde{f}$.*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "\tilde{f}"] \\ Y \arrow[r, "\eta_Y"'] & \tilde{Y} \end{tikzcd}$$

Proof. □

Remark 3.51. For uniform spaces X and Y, it is straightforward to see that if $X \simeq Y$, then $\tilde{X} \simeq \tilde{Y}$; for uniformly continuous mappings $f, g : X \to Y$, if $f \sim g$, then $\tilde{f} \sim \tilde{g}$; furthermore $\widetilde{\mathrm{id}}_X \sim \mathrm{id}_{\tilde{X}}$, and $\widetilde{(f \circ g)} \sim \tilde{f} \circ \tilde{g}$ for uniformly continuous mappings $f : X \to Y$ and $g : Y \to Z$ among uniform spaces X, Y and Z.

4 Topological vector spaces and lattices

4.1 Real numbers

Proposition 4.1. *Let $\rho_{\mathbb{Q}}$ and $\Vdash_{\mathbb{Q}}$ be a monotone function from $\mathbb{N}$ into $\mathbb{N}$ and a relation between $\underline{\mathbb{Q}} \times \underline{\mathbb{Q}}$ and $\mathbb{N}$ defined by*

$$\rho_{\mathbb{Q}}(n) = n + 1,$$
$$(p, q) \Vdash_{\mathbb{Q}} n \Leftrightarrow |p - q| \leq_{\mathbb{Q}} 2^{-n}$$

for all $n \in \mathbb{N}$ and $p, q \in \underline{\mathbb{Q}}$, respectively. Then $(\mathbb{N}, \rho_{\mathbb{Q}}, \Vdash_{\mathbb{Q}})$ is a uniform structure on the setoid $\mathbb{Q} = (\underline{\mathbb{Q}}, =_{\mathbb{Q}})$ such that the addition $(p, q) \mapsto p+q$, the inverse $p \mapsto -p$ and $(p, q) \mapsto \max_{\mathbb{Q}}(p, q)$ are uniformly continuous, and the multiplication $(p, q) \mapsto pq$ is locally uniformly continuous.

Proof. □

The uniform space $\mathbb{R}$ of *real numbers* is the completion $\tilde{\mathbb{Q}}$ of the uniform space $\mathbb{Q}$.

Lemma 4.2. *The setoid $\mathbb{R}$ is stable.*

Proof. Consider $s = (p_n)_{n \in \mathbb{N}}, t = (q_n)_{n \in \mathbb{N}}$ with $\neg\neg(s =_{\mathbb{R}} t)$. Given an $n \in \mathbb{N}$, assume that $|p_m - q_m| >_{\mathbb{Q}} 2^{-n}$ for some $m \in \mathbb{N}$ with $n + 2 \leq m$. Then for all $m' \in \mathbb{N}$ with $n+2 \leq m'$, since $|p_m - p_{m'}| \leq_{\mathbb{Q}} 2^{-(n+2)}$ and $|q_m - q_{m'}| \leq_{\mathbb{Q}} 2^{-(n+2)}$, we have

$$\begin{aligned} |p_{m'} - q_{m'}| &\geq_{\mathbb{Q}} |p_m - q_m| - |p_m - p_{m'}| - |q_m - q_{m'}| \\ &>_{\mathbb{Q}} 2^{-n} - 2^{-(n+2)} - 2^{-(n+2)} =_{\mathbb{Q}} 2^{-(n+1)}. \end{aligned}$$

If $s =_{\mathbb{R}} t$, then there exists $n' \in \mathbb{N}$ such that $|p_{m'} - q_{m'}| \leq_{\mathbb{Q}} 2^{-(n+1)}$ for all $m' \in \mathbb{N}$ with $n' \leq m'$; hence $|p_{m'} - q_{m'}| \leq_{\mathbb{Q}} 2^{-(n+1)}$ for $m' = \max(n+2, n')$, a contradiction. Therefore $\neg(s =_{\mathbb{R}} t)$, a contradiction, and so $|p_m - q_m| \leq_{\mathbb{Q}} 2^{-n}$ for all $m \in \mathbb{N}$ with $n + 2 \leq m$. Thus $s =_{\mathbb{R}} t$. □

Proposition 4.3. *The setoid* $\mathbb{R}$ *is a unitary commutative ring with the zero element* $0 = \eta_{\mathbb{Q}}(0)$ *and the unity element* $1 = \eta_{\mathbb{Q}}(1)$*, and a semilattice such that*

1. $\max_{\mathbb{R}}(s,t) + r =_{\mathbb{R}} \max_{\mathbb{R}}(s + r, t + r)$;
2. *if* $0 \leq_{\mathbb{R}} r$*, then* $\max_{\mathbb{R}}(s,t)r =_{\mathbb{R}} \max_{\mathbb{R}}(sr, tr)$

for all $r, s, t \in \underline{\mathbb{R}}$.

Proof. □

The uniform space $\mathbb{Q}$ of rationals is a uniform subspace of $\mathbb{R}$ with a function $\eta_{\mathbb{Q}} : \underline{\mathbb{Q}} \to \underline{\mathbb{R}}$, and then

$$p =_{\mathbb{Q}} q \Leftrightarrow \eta_{\mathbb{Q}}(p) =_{\mathbb{R}} \eta_{\mathbb{Q}}(q), \quad \eta_{\mathbb{Q}}(p+q) =_{\mathbb{R}} \eta_{\mathbb{Q}}(p) + \eta_{\mathbb{Q}}(q), \quad \eta_{\mathbb{Q}}(-p) =_{\mathbb{R}} -\eta_{\mathbb{Q}}(p),$$
$$\eta_{\mathbb{Q}}(pq) =_{\mathbb{R}} \eta_{\mathbb{Q}}(p)\eta_{\mathbb{Q}}(q), \quad \eta_{\mathbb{Q}}(\max_{\mathbb{Q}}\{p,q\}) =_{\mathbb{R}} \max_{\mathbb{R}}\{\eta_{\mathbb{Q}}(p), \eta_{\mathbb{Q}}(q)\}$$

for all $p, q \in \underline{\mathbb{Q}}$.

Lemma 4.4. *The canonical partial order* $\leq_{\mathbb{R}}$ *on* $\mathbb{R}$ *is quasi-total.*

Proof. Consider $s = (p_n)_{n \in \mathbb{N}}, t = (q_n)_{n \in \mathbb{N}}$ with $\neg(s \leq_{\mathbb{R}} t)$, and note that, since

$$|\max\{p_n, q_n\} - \max\{p_m, q_m\}| \leq_{\mathbb{Q}} \max\{|p_n - p_m|, |q_n - q_m|\}$$

for all $n, m \in \mathbb{N}$, we have $(\max\{p_n, q_n\})_{n \in \mathbb{N}} \in \mathbb{R}$, and, since $\eta_{\mathbb{Q}}(p_m) \to s$ and $\eta_{\mathbb{Q}}(q_m) \to t$, we have $\eta_{\mathbb{Q}}(\max\{p_m, q_m\}) =_{\mathbb{R}} \max\{\eta_{\mathbb{Q}}(p_m), \eta_{\mathbb{Q}}(q_m)\}) \to \max\{s, t\}$; hence $\max\{s,t\} =_{\mathbb{R}} (\max\{p_n, q_n\})_{n \in \mathbb{N}}$. Given an $n \in \mathbb{N}$, assume that $q_m - p_m >_{\mathbb{Q}} 2^{-n}$ for some $m \in \mathbb{N}$ with $n + 2 \leq m$. Then for all $m' \in \mathbb{N}$ with $n + 2 \leq m'$, since $|p_m - p_{m'}| \leq_{\mathbb{Q}} 2^{-(n+2)}$ and $|q_m - q_{m'}| \leq_{\mathbb{Q}} 2^{-(n+2)}$, we have

$$\begin{aligned} p_{m'} - q_{m'} &\leq_{\mathbb{Q}} -(q_m - p_m) + |p_m - p_{m'}| + |q_m - q_{m'}| \\ &<_{\mathbb{Q}} -2^{-n} + 2^{-(n+2)} + 2^{-(n+2)} =_{\mathbb{Q}} -2^{-(n+1)}; \end{aligned}$$

hence

$$0 \leq_{\mathbb{Q}} \max\{p_{m'}, q_{m'}\} - q_{m'} =_{\mathbb{Q}} \max\{p_{m'} - q_{m'}, 0\} \leq_{\mathbb{Q}} \max\{0, -2^{-(n+1)}\} =_{\mathbb{Q}} 0.$$

Therefore $\max\{p_{m'}, q_{m'}\} =_{\mathbb{Q}} q_{m'}$ for all $m' \in \mathbb{N}$ with $n + 2 \leq m'$, and so $\max\{s,t\} =_{\mathbb{R}} t$, or $s \leq_{\mathbb{R}} t$, a contradiction. Thus for all $m \in \mathbb{N}$ with $n + 2 \leq m$, $q_m - p_m \leq_{\mathbb{Q}} 2^{-n}$, and so

$$\begin{aligned} |\max\{p_m, q_m\} - p_m| &=_{\mathbb{Q}} \max\{p_m, q_m\} - p_m =_{\mathbb{Q}} \max\{0, q_m - p_m\} \\ &\leq_{\mathbb{Q}} \max\{0, 2^{-(n+1)}\} =_{\mathbb{Q}} 2^{-(n+1)} <_{\mathbb{Q}} 2^{-n}. \end{aligned}$$

This entails that $\max\{s,t\} =_{\mathbb{R}} s$, or $t \leq_{\mathbb{R}} s$. □

Lemma 4.5. *Let $\#_{\mathbb{R}}$ be a binary relation on $\underline{\mathbb{R}}$ given by*

$$s \#_{\mathbb{R}} t \Leftrightarrow \exists n \in \mathbb{N} \exists m \in \mathbb{N} \forall m' \in \mathbb{N} \, (m \leq m' \Rightarrow |p_{m'} - q_{m'}| >_{\mathbb{Q}} 2^{-n})$$

for all $s = (p_n)_{n\in\mathbb{N}}, t = (q_n)_{n\in\mathbb{N}} \in \underline{\mathbb{R}}$. Then $\#_{\mathbb{R}}$ is a tight apartness on $\mathbb{R}$ such that

1. *if $(s + r) \#_{\mathbb{R}} (t + r)$, then $s \#_{\mathbb{R}} t$;*
2. *if $sr \#_{\mathbb{R}} tr$, then $s \#_{\mathbb{R}} t$;*
3. *if $\max\{s, r\} \#_{\mathbb{R}} \max\{t, r\}$, then $s \#_{\mathbb{R}} t$*

for all $r, s, t \in \underline{\mathbb{R}}$.

Proof. □

Lemma 4.6. *The join $\max_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is strongly extensional.*

Proof. □

Proposition 4.7. *The following hold.*

1. *if $s <_{\mathbb{R}} t$, then $\neg(t <_{\mathbb{R}} s)$;*
2. *if $s <_{\mathbb{R}} t$ and $t <_{\mathbb{R}} r$, then $s <_{\mathbb{R}} r$;*
3. *if $s <_{\mathbb{R}} t$, then $s <_{\mathbb{R}} r$ or $r <_{\mathbb{R}} t$;*
4. *$s \#_{\mathbb{R}} t$ if and only if $s <_{\mathbb{R}} t$ or $t <_{\mathbb{R}} s$;*
5. *$s \leq_{\mathbb{R}} t$ if and only if $\neg(t <_{\mathbb{R}} s)$*

for all $s, t, r \in \underline{\mathbb{R}}$.

Proof. □

Theorem 4.8. *The setoid $\mathbb{R}$ is an Archimedean ordered field.*

Proof. □

♣ subspace $[0, 1]$

4.2 Topological vector spaces

Definition 4.9. A *vector space* (over $\mathbb{R}$) is a setoid $X = (\underline{X}, =_X)$ equipped with a setoid mapping $(x, y) \mapsto x + y$ of $X \times X$ into X, called *addition*, a setoid mapping $x \mapsto -x$ of X into X, called *inverse*, a setoid mapping $(s, x) \mapsto sx$ of $\mathbb{R} \times X$ into X, called *scalar multiplication* and an element 0 of $\underline{X}$, called the *zero element*, such that

$$\begin{aligned} (x + y) + z &=_X x + (y + z), & x + y &=_X y + x, \\ x + 0 &=_X x, & x + (-x) &=_X 0, \\ s(x + y) &=_X sx + sy, & (s + t)x &=_X sx + tx, \\ s(tx) &=_X (st)x, & 1x &=_X x \end{aligned}$$

for all $x, y, z \in \underline{X}$ and $s, t \in \underline{\mathbb{R}}$.

Example 4.10. Let $\underline{F}[0,1]$ be the set of setoid mappings of $[0,1]$ into $\mathbb{R}$. Then the setoid $F[0,1] = (\underline{F}[0,1], \sim)$ is a vector space equipped with addition, inverse, scalar multiplication and zero element given by

$$(f+g)(x) = f(x) + g(x), \qquad (-f)(x) = -f(x),$$
$$(sf)(x) = sf(x), \qquad 0(x) = 0$$

for all $f, g \in \underline{F}[0,1]$, $s \in \mathbb{R}$ and $x \in [0,1]$.

Definition 4.11. Let $X = (\underline{X}, =_X)$ is a vector space. Then a *linear functional* on X is a setoid mapping $f : X \to \mathbb{R}$ such that

$$f(x+y) =_{\mathbb{R}} f(x) + f(y) \qquad \text{and} \qquad f(sx) =_{\mathbb{R}} sf(x)$$

for all $x, y \in \underline{X}$ and $s \in \underline{\mathbb{R}}$.

Definition 4.12. A *topological vector space* is a vector space $X = (\underline{X}, =_X)$ equipped with a uniform structure $(I_X, \rho_X, \Vdash_X)$ such that

1. the addition $+ : X \times X \to X$ is uniformly continuous;
2. there exists a function $\xi^X : \underline{I}_X \times \underline{X} \to \mathbb{N}$ such that for each $a \in \underline{I}_X$,
$$(0, sx) \Vdash_X a$$
for all $x \in \underline{X}$ and $s \in \underline{\mathbb{R}}$ with $|s| \leq_{\mathbb{R}} 2^{-\xi^X(a,x)}$;
3. for each $a \in \underline{I}_X$,
$$(0, x) \Vdash_X a \Rightarrow (0, sx) \Vdash_X a$$
for all $x \in \underline{X}$ and $s \in \underline{\mathbb{R}}$ with $|s| \leq_{\mathbb{R}} 1$.

Remark 4.13. ♣ radial (absorbing)
♣ circled

Lemma 4.14. *Let X be a topological vector space. Then the inverse $x \mapsto -x$ is uniformly continuous, and the scalar multiplication $(s, x) \mapsto sx$ of $\mathbb{R} \times X$ is locally uniformly continuous.*

Proof. □

Theorem 4.15. *If X is a topological vector space, then so is its completion $\tilde{X}$.*

Proof. □

4.3 Topological vector lattices

Definition 4.16. A *vector lattice* is a vector space $X = (\underline{X}, =_X)$ such that X is a semilattice, and

1. $(x+z) \vee (y+z) =_X (x \vee y) + z$,
2. if $0 \leq_{\mathbb{R}} s$, then $s(x \vee y) =_X (sx) \vee (sy)$

for all $x, y, z \in \underline{X}$ and $s \in \underline{\mathbb{R}}$.

Example 4.17. Let $\underline{C}[0,1]$ be a set given by

$$\underline{C}[0,1] = \{(f,\gamma) \in \underline{F}[0,1] \times \hom(\mathbb{N},\mathbb{N}) \mid \\ f \text{ is uniformly continuous with a modulus } \gamma\ \},$$

and let $=_{C[0,1]}$ be an equivalence relation on $\underline{C}[0,1]$ given by

$$(f,\gamma^f) =_{C[0,1]} (g,\gamma^g) \Leftrightarrow f =_{F[0,1]} g.$$

Then the setoid $C[0,1] = (\underline{C}[0,1], =_{C[0,1]})$, is a vector lattice equipped with addition, inverse, scalar multiplication and zero element given in Example 4.10 with appropriate moduli, and join given by

$$(f,\gamma^f) \vee (g,\gamma^g) = (\max_{\mathbb{R}} \circ (f \times g), \max_{\mathbb{R}} \circ (\gamma^f \times \gamma^g) \circ \rho_{\mathbb{R}})$$

for all $(f,\gamma^f),(g,\gamma^g) \in \underline{C}[0,1]$.

Lemma 4.18. *Let $X = (\underline{X}, =_X)$ be a vector lattice. Then*

1. *if $x \leq_X y$, then $x + z \leq_X y + z$;*
2. *if $x \leq_X y$ and $0 \leq_{\mathbb{R}} s$, then $sx \leq_X sy$*

for all $x,y,z \in \underline{X}$ and $s \in \underline{\mathbb{R}}$.

Proof. (1): For all $x,y,z \in \underline{X}$, if $x \leq_X y$, then, since $x \vee y =_X y$, we have $(x+z) \vee (y+z) =_X x \vee y + z =_X y + z$; hence $x + z \leq_X y + z$.

(2): For all $x,y \in \underline{X}$ and $s \in \underline{\mathbb{R}}$, if $x \leq_X y$ and $0 \leq_{\mathbb{R}} s$, then, since $x \vee y =_X y$, we have $(sx) \vee (sy) =_X s(x \vee y) =_X sy$; hence $sx \leq_X sy$. $\square$

Proposition 4.19. *Let $X = (\underline{X}, =_X)$ be a vector lattice, and let $(x,y) \mapsto x \wedge y$ be a setoid mapping of $X \times X$ into X, called a* meet, *given by*

$$x \wedge y = -(-x \vee -y)$$

for all $x,y \in \underline{X}$. Then

$$x \wedge (y \wedge z) =_X (x \wedge y) \wedge z, \quad x \wedge y =_X y \wedge x, \quad x \wedge x =_X x,$$
$$x \vee (x \wedge y) =_X x, \quad x \wedge (x \vee y) =_X x$$

for all $x,y,z \in \underline{X}$. Furthermore, $x \leq_X y \Leftrightarrow x \wedge y =_X x$ and $x \wedge y$ is the greatest lower bound of $\{x,y\}$ for all $x,y \in \underline{X}$.

Proof. It is straightforward to see the first three equations. To see the rest, consider $x,y \in \underline{X}$. Then, since $-x \leq_X -x \vee -y$, we have $x \wedge y \leq_X x$, by Lemma 4.18 (1); hence $x \vee (x \wedge y) \leq_X x$. Therefore, since $x \leq_X x \vee (x \wedge y)$, we have $x \vee (x \wedge y) =_X x$. Since $x \leq_X x \vee y$, we have $-(x \vee y) \leq_X -x$, by Lemma 4.18 (1); hence $-x \vee -(x \vee y) \leq_X -x$. Therefore $x \leq_X x \wedge (x \vee y)$, by Lemma 4.18 (1), and so, since $-x \leq_X -x \vee -(x \vee y)$, we have $x \wedge (x \vee y) \leq_X x$, by Lemma 4.18 (1). Thus $x \wedge (x \vee y) =_X x$.

If $x \leq_X y$, then, since $x \vee y =_X y$, we have $x \wedge y =_X x \wedge (x \vee y) =_X x$; conversely, if $x \wedge y =_X x$, then, since $x \vee y =_X (x \wedge y) \vee y =_X y$, we have $x \leq_X y$. Since $(x \wedge y) \vee x =_X x$ and $(x \wedge y) \vee y =_X y$, we have $x \wedge y \leq_X x$ and $x \wedge y \leq_X y$. For each $z \in \underline{X}$, if $z \leq_X x$ and $z \leq_X y$, then, since $z \wedge x =_X z$ and $z \wedge y =_X z$, we have $z \wedge (x \wedge y) =_X (z \wedge x) \wedge y =_X z \wedge y =_X z$; hence $z \leq_X x \wedge y$. $\square$

Lemma 4.20. *Let $X = (\underline{X}, =_X)$ be a vector lattice. Then*

1. $x + y =_X x \vee y + x \wedge y$;
2. $(x + z) \wedge (y + z) =_X (x \wedge y) + z$;
3. *if* $0 \leq s$*, then* $s(x \wedge y) =_X sx \wedge sy$

for all $x, y, z \in \underline{X}$ *and* $s \in \underline{\mathbb{R}}$.

Proof. (1): For all $x, y \in \underline{X}$, since

$$-x \vee -y + (x + y) =_X (-x + x + y) \vee (-y + x + y) =_X y \vee x =_X x \vee y,$$

we have $x + y =_X x \vee y - (-x \vee -y) =_X x \vee y + x \wedge y$, by Lemma 4.18 (1).

(2): For all $x, y, z \in \underline{X}$, we have

$$(x+z) \wedge (y+z) =_X -(-x-z) \vee (-y-z) =_X -((-x \vee -y) - z) =_X (x \wedge y) + z.$$

(3): For all $x, y \in \underline{X}$ and $s \in \underline{\mathbb{R}}$, if $0 \leq s$, then

$$s(x \wedge y) =_X -s(-x \vee -y) =_X -(-sx \vee -sy) =_X sx \wedge sy. \qquad \square$$

Proposition 4.21. *Every vector lattice $X = (\underline{X}, =_X)$ is a distributive lattice, that is, $x \vee (y \wedge z) =_X (x \vee y) \wedge (x \vee z)$ or $x \wedge (y \vee z) =_X (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in \underline{X}$.*

Proof. Let $x, y, z \in \underline{X}$. Then, since $x \leq_X x \vee z$ and $y \wedge z \leq_X z \leq_X x \vee z$, we have $x \vee (y \wedge z) \leq_X x \vee z$; similarly, we have $x \vee (y \wedge z) \leq_X y \vee z$. Hence $x \vee (y \wedge z) \leq_X (x \vee z) \wedge (y \vee z)$. Let $w = (x \vee y) \wedge (x \vee z)$. Then, since $w \leq_X x \vee y$, we have $w + x \wedge y \leq_X x \vee y + x \wedge y =_X x + y$, by Lemma 4.20 (1); similarly, we have $w + x \wedge z \leq_X x + z$. Hence

$$\begin{aligned} w + (x \wedge y \wedge z) &=_X w + (x \wedge y \wedge z \wedge z) =_X w + (x \wedge z) \wedge (y \wedge z) \\ &\leq_X (w + x \wedge y) \wedge (w + x \wedge z) \leq_X (x + y) \wedge (x + z) \\ &=_X x + (y \wedge z), \end{aligned}$$

by Lemma 4.20 (2). Therefore, since, by Lemma 4.20 (1),

$$x + (y \wedge z) =_X x \vee (y \wedge z) + (x \wedge y \wedge z),$$

we have $w \leq_X x + (y \wedge z) - (x \wedge y \wedge z) =_X x \vee (y \wedge z)$. Thus

$$x \vee (y \wedge z) =_X (x \vee y) \wedge (x \vee z).$$

$\square$

Definition 4.22. Let $X = (\underline{X}, =_X)$ be a vector lattice. Then the subset

$$\underline{C}_X = \{x \in \underline{X} \mid 0 \leq_X x\}$$

is called a *positive cone* of $\underline{X}$; note that $C_X = (\underline{C}_X, \leq_X)$ with $\mathrm{ub}_{C_X} = \vee \in \hom(\underline{C}_X \times \underline{C}_X, \underline{C}_X)$ is a directed preordered set.

A linear functional f on X is *positive* if $0 \leq_{\mathbb{R}} f(x)$ for all $x \in \underline{C}_X$.

Example 4.23. Let $R : \underline{C}[0,1] \to \mathbb{R}$ be a function given by

$$R(f,\gamma) = \int f,$$

where $\int$ is the Riemann integral, for all $(f,\gamma) \in \underline{C}[0,1]$. Then R is a positive linear functional.

Definition 4.24. Let $X = (\underline{X}, =_X)$ be a vector lattice, and let $(\text{-})^+ : X \to X$, $(\text{-})^- : X \to X$ and $|\text{-}| : X \to X$ be setoid mappings given by

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0, \qquad |x| = x \vee (-x),$$

respectively, for all $x \in \underline{X}$; note that $x^+, x^- \in \underline{C}_X$. Two elements $x, y \in \underline{X}$ are *disjoint* if $|x| \wedge |y| =_X 0$.

Lemma 4.25. *Let $X = (\underline{X}, =_X)$ be a vector lattice. Then*

1. $x =_X x^+ - x^-$;
2. x^+ *and* x^- *are disjoint;*
3. *for each pair of disjoint elements $u, v \in \underline{C}_X$, if $x =_X u - v$ then $u =_X x^+$ and $v =_X x^-$;*
4. $|-x| =_X |x| =_X x^+ + x^- \in \underline{C}_X$;
5. *for each $s \in \mathbb{R}$, if $0 \leq_{\mathbb{R}} s$, then $(sx)^+ =_X sx^+$, $(sx)^- =_X sx^-$ and $|sx| =_X s|x|$;*
6. *x and y are disjoint if and only if $|x| \vee |y| =_X |x| + |y|$;*
7. *if x and y are disjoint, then $(x+y)^+ =_X x^+ + y^+$, $(x+y)^- =_X x^- + y^-$ and $|x+y| =_X |x| + |y|$*

for all $x, y, z \in \underline{X}$.

Proof. Let $x, y, z \in \underline{X}$. Then

(1): $x =_X x + 0 =_X x \vee 0 + x \wedge 0 =_X x \vee 0 - (-x) \vee 0 =_X x^+ - x^-$, by Lemma 4.20 (1).

(2):

$$\begin{aligned} x^+ \wedge x^- &=_X (x^+ - x^-) \wedge 0 + x^- =_X x \wedge 0 + x^- =_X -(-x \vee 0) + x^- \\ &=_X -x^- + x^- =_X 0, \end{aligned}$$

by Lemma 4.20 (2) and (1).

(3): Consider $u, v \in \underline{C}_X$ such that $u \wedge v =_X 0$ and $x =_X u - v$. Then, since $u =_X x + v$ and $v =_X u - x$, we have $x \leq_X u$ and $-x \leq_X v$; hence $x^+ \leq u$ and $x^- \leq v$. Note that, since $u - v =_X x =_X x^+ - x^-$, we have $u - x^+ =_X v - x^-$. Then, since $u - x^+ \leq_X u$ and $v - x^- \leq_X v$, we have

$$\begin{aligned} 0 &\leq_X u - x^+ =_X (u - x^+) \wedge (u - x^+) =_X (u - x^+) \wedge (v - x^-) \\ &\leq_X u \wedge v =_X 0; \end{aligned}$$

hence $u =_X x^+$. Similarly, we have $v =_X x^-$.

(4):

$$\begin{aligned}|-x| &=_X -x \vee -(-x) =_X -x \vee x =_X x + (-2x) \vee 0 =_X x + 2(-x \vee 0)\\ &=_X x + 2x^- =_X (x^+ - x^-) + 2x^- =_X x^+ + x^- \in \underline{C}_X,\end{aligned}$$

by (1).

(5): for each $s \in \underline{\mathbb{R}}$, if $0 \leq_{\mathbb{R}} s$, then $(sx)^+ =_X (sx) \vee 0 =_X s(x \vee 0) =_X sx^+$ and, similary, $(sx)^- =_X sx^-$; hence

$$|sx| =_X (sx)^+ + (sx)^- =_X sx^+ + sx^- =_X s(x^+ + x^-) =_X s|x|,$$

by (4).

(6): $|x|+|y| =_X |x| \vee |y| + |x| \wedge |y|$, by Lemma 4.20 (1); hence $|x| \wedge_X |y| =_X 0$ if and only if $|x| + |y| =_X |x| \vee |y|$.

(7): Assume that x and y are disjoint, and note that, since

$$0 \leq_X x^+ \wedge y^+, x^- \wedge y^- \leq_X |x| \wedge |y| =_X 0,$$

x^+ and y^+, and x^- and y^- are disjoint, respectively. Then, since, by (4) and (6),

$$\begin{aligned}(x^+ + y^+) \vee (x^- + y^-) &=_X (x^+ \vee y^+) \vee (x^- \vee y^-) =_X (x^+ \vee x^-) \vee (y^+ \vee y^-)\\ &=_X (x^+ + x^-) \vee (y^+ + y^-) =_X |x| \vee |y|\\ &=_X |x| + |y| =_X (x^+ + x^-) + (y^+ + y^-)\\ &=_X (x^+ + y^+) + (x^- + y^-),\end{aligned}$$

$x^+ + y^+$ and $x^- + y^-$ are disjoint, by (6). Therefore, since

$$x + y =_X (x^+ - x^-) + (y^+ - y^-) =_X (x^+ + y^+) - (x^- + y^-),$$

we have $(x + y)^+ =_X x^+ + y^+$ and $(x + y)^- =_X x^- + y^-$, by (3), and so

$$\begin{aligned}|x + y| &=_X (x + y)^+ + (x + y)^- =_X (x^+ + y^+) + (x^- + y^-)\\ &=_X (x^+ + x^-) + (y^+ + y^-) =_X |x| + |y|,\end{aligned}$$

by (4). □

Lemma 4.26. *Let $X = (\underline{X}, =_X)$ be a vector lattice. Then*

1. $|x + y| \leq_X |x| + |y|$;
2. $|sx| \leq_X |s||x|$;
3. $|x \vee z - y \vee z| \leq_X |x - y|$;
4. $|x \wedge z - y \wedge z| \leq_X |x - y|$;
5. $(|x| + |y|) \wedge u \leq_X |x| \wedge u + |y| \wedge u$;
6. $|sx| \wedge u \leq_X (|s| + 1)(|x| \wedge u)$;
7. $|x \wedge |y| - x \wedge |z|| \leq_X |y - z| \wedge |x|$

for all $x, y, z \in \underline{X}$, $s \in \underline{\mathbb{R}}$ and $u \in \underline{C}_X$.

Proof. Consider $x, y, z \in \underline{X}$, $s \in \underline{\mathbb{R}}$ and $u \in \underline{C}_X$. Then

(1): Since $x, -x \leq_X |x|$ and $y, -y \leq_X |y|$, we have

$$x + y \leq_X |x| + |y| \qquad \text{and} \qquad -(x+y) =_X -x - y \leq_X |x| + |y|;$$

hence $|x + y| \leq_X |x| + |y|$.

(2): For each $s \in \underline{\mathbb{R}}$, we write s^+ and s^- for $\max_{\mathbb{R}}(s, 0)$ and $\max_{\mathbb{R}}(-s, 0)$, respectively. Then

$$\begin{aligned} |sx| &=_X |(s^+ - s^-)x| =_X |s^+x - s^-x| \leq_X |s^+x| + |-s^-x| \\ &=_X |s^+x| + |s^-x| =_X s^+|x| + s^-|x| =_X (s^+ + s^-)|x| \\ &=_X |s||x|, \end{aligned}$$

by (1) and Lemma 4.25 (4) and (5).

(3): Since, by (1),

$$x \leq_X |x - y| + y \leq_X |x - y| + y \vee z \quad \text{and} \quad z \leq_X y \vee z \leq_X |x - y| + y \vee z,$$

we have $x \vee z \leq_X |x - y| + y \vee z$; hence $x \vee z - y \vee z \leq_X |x - y|$. Similarly, we have $y \vee z - x \vee z \leq_X |x - y|$. Therefore $|x \vee z - y \vee z| \leq_X |x - y|$.

(4): Since by (1), $x \wedge z \leq_X x \leq_X |x - y| + y$ and $x \wedge z \leq_X z \leq_X |x - y| + z$, we have $x \wedge z \leq_X (|x - y| + y) \wedge (|x - y| + z) =_X |x - y| + y \wedge z$, by Lemma 4.20 (2); hence $x \wedge z - y \wedge z \leq_X |x - y|$. Similarly, we have $y \wedge z - x \wedge z \leq_X |x - y|$. Therefore $|x \wedge z - y \wedge z| \leq_X |x - y|$.

(5): Since $(|x| + |y|) \wedge u \leq_X |x| + |y|$ and $(|x| + |y|) \wedge u \leq_X u \leq_X u + |y|$, we have $(|x| + |y|) \wedge u \leq_X (|x| + |y|) \wedge (u + |y|) =_X (|x| \wedge u) + |y|$, by Proposition 4.19 and Lemma 4.20 (2). Therefore, since $(|x| + |y|) \wedge u \leq_X u \leq_X (|x| \wedge u) + u$, we have

$$(|x| + |y|) \wedge u \leq_X (|x| \wedge u + |y|) \wedge (|x| \wedge u + u) =_X |x| \wedge u + |y| \wedge u,$$

by Proposition 4.19 and Lemma 4.20 (2).

(6): Since, by (2), $|sx| \wedge u \leq_X |sx| \leq_X |s||x| \leq_X |s||x| + |x| =_X (|s| + 1)|x|$ and $|sx| \wedge u \leq_X u \leq_X |s|u + u =_X (|s| + 1)u$, we have

$$|sx| \wedge u \leq_X (|s| + 1)|x| \wedge (|s| + 1)u =_X (|s| + 1)(|x| \wedge u),$$

by Proposition 4.19 and Lemma 4.20 (3).

(7): Since $x \wedge |y| \leq_X x \leq_X |x| + x$ and $x \wedge |y| \leq_X x \leq_X |x| \leq_x |x| + |z|$, we have $x \wedge |y| \leq (|x| + x) \wedge (|x| + |z|) =_X |x| + x \wedge |z|$ by Proposition 4.19 and Lemma 4.20 (2); hence $x \wedge |y| - x \wedge |z| \leq_X |x|$. Similarly, we have $x \wedge |z| - x \wedge |y| \leq_X |x|$, and so $|x \wedge |y| - x \wedge |z|| \leq_X |x|$, by Proposition 4.19. Since, by (1), $|y| \leq_X |y - z| + |z|$ and $|z| \leq_X |y - z| + |y|$, we have $|y| - |z| \leq_X |y - z|$ and $|z| - |y| \leq_X |y - z|$; hence $|x \wedge |y| - x \wedge |z|| \leq_X ||y| - |z|| \leq_X |y - z|$. Therefore $|x \wedge |y| - x \wedge |z|| \leq_X |y - z| \wedge |x|$, by Proposition 4.19. ☐

Definition 4.27. A vector lattice $X = (\underline{X}, =_X)$ is *Archimedean* if for each $x \in \underline{X}$, $x \leq_X 0$ whenever there exists $y \in \underline{X}$ such that $x \leq_X 2^{-n}y$ for all $n \in \mathbb{N}$.

Lemma 4.28. *Let $X = (\underline{X}, =_X)$ be an Archimedean vector lattice. Then*

1. *if $0 \leq_X x$, then $\max_{\mathbb{R}}(s, t)x =_X sx \vee tx$;*

2. *if* $0 \leq_X x$*, then* $\min_{\mathbb{R}}(s,t)x =_X sx \wedge tx$*;*

3. $|sx| =_X |s||x|$

for all $s, t \in \underline{\mathbb{R}}$ *and* $x \in \underline{X}$*.*

Proof. (1): Consider $s, t \in \underline{\mathbb{R}}$ and $x \in \underline{X}$ with $0 \leq_X x$. Then, since $s, t \leq_{\mathbb{R}} \max_{\mathbb{R}}(s,t)$, we have $sx, tx \leq_X \max_{\mathbb{R}}(s,t)x$, by Lemma 4.18 (2); hence $sx \vee tx \leq_X \max_{\mathbb{R}}(s,t)x$. For each $n \in \mathbb{N}$, either $\max_{\mathbb{R}}(s,t) \leq_{\mathbb{R}} s + 2^{-n}$ or $\max_{\mathbb{R}}(s,t) \leq_{\mathbb{R}} t + 2^{-n}$, by Lemma ♣ ; in the first case, we have

$$\max_{\mathbb{R}}(s,t)x \leq_X (s + 2^{-n})x \leq_X sx + 2^{-n}x \leq_X sx \vee tx + 2^{-n}x;$$

in the second case, similarly, we have $\max_{\mathbb{R}}(s,t)x \leq_X sx \vee tx + 2^{-n}x$. Therefore $\max_{\mathbb{R}}(s,t)x - sx \vee tx \leq_X 2^{-n}x$ for all $n \in \mathbb{N}$, and so $\max_{\mathbb{R}}(s,t)x - sx \vee tx \leq_X 0$, that is, $\max_{\mathbb{R}}(s,t)x \leq_X sx \vee tx$. Thus $\max_{\mathbb{R}}(s,t)x =_X sx \vee tx$.

(2): For all $s, t \in \underline{\mathbb{R}}$ and $x \in \underline{X}$ with $0 \leq_X x$, since $\max_{\mathbb{R}}(-s,-t)x =_X (-sx) \vee (-tx)$, by (1), we have

$$\min_{\mathbb{R}}(s,t)x =_X -(\max_{\mathbb{R}}(-s,-t)x) =_X -((-sx) \vee (-tx)) =_X sx \wedge tx.$$

(3): For each $s \in \underline{\mathbb{R}}$, we write s^+ and s^- for $\max_{\mathbb{R}}(s,0)$ and $\max_{\mathbb{R}}(-s,0)$, respectively. Then for all $s \in \underline{\mathbb{R}}$ and $x \in \underline{X}$, since $\min_{\mathbb{R}}(s^+, s^-) =_{\mathbb{R}} 0$, we have $s^+|x| \wedge s^-|x| =_X \min_{\mathbb{R}}(s^+, s^-)|x| =_X 0$, by (2); hence $s^+|x| =_X |s^+x|$ and $s^-|x| =_X |s^-x| =_X |-s^-x|$ are disjoint, by Lemma 4.25 (4) and (5). Therefore

$$\begin{aligned}|s||x| &=_X (s^+ + s^-)|x| =_X s^+|x| + s^-|x| =_X |s^+x| + |-s^-x| \\ &=_X |s^+x - s^-x| =_X |(s^+ - s^-)x| =_X |sx|,\end{aligned}$$

by Lemma 4.25 (7). □

Definition 4.29. A *topological vector lattice* is a vector lattice $X = (\underline{X}, =_X)$ equipped with a uniform structure $(I_X, \rho_X, \Vdash_X)$ such that

1. X is a topological vector space with the uniform structure;

2. the join $\vee : X \times X \to X$ is uniformly continuous;

3. for each $a \in \underline{I}_X$,

$$(0, y) \Vdash_X a \Rightarrow (0, x) \Vdash_X a$$

   for all $x, y \in \underline{C}_X$ with $x \leq_X y$.

Remark 4.30. ♣ the positive cone $\underline{C}_X$ is normal

Lemma 4.31. *Every topological vector lattice is Archimedean.*

Proof. Let $X = (\underline{X}, =_X)$ be a topological vector lattice equipped with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and consider $x, y \in \underline{X}$ such that $x \leq_X 2^{-n}y$ for all $n \in \mathbb{N}$. Then $x^+ =_X (2^{-n}y)^+ \leq_X 2^{-n}y^+$ for for all $n \in \mathbb{N}$. Therefore for each $a \in \underline{I}_X$, since $x^+ \leq_X 2^{-\xi^X(a,y^+)}y^+$ and $(0, 2^{-\xi^X(a,y^+)}y^+) \Vdash_X a$, we have $(0, x^+) \Vdash_X a$. Thus $x^+ =_X 0$, and so $x \leq_X 0$. □

Theorem 4.32. *If* X *is a topological vector lattice, then so is its completion* $\tilde{X}$*.*

Proof. □

5 Integration theory

An *abstract integration space* is a vector lattice $X = (\underline{X}, =_X)$ equipped with a positive linear functional E on X.

Example 5.1. $(C[0,1], R)$ is an abstract integration space; see Example 4.17 and Example 4.23.

In the following, we fix an abstract integration space (X, E).

5.1 Integrable functions

Lemma 5.2. *Let $=_L$ be a binary relation on $\underline{X}$ given by*

$$x =_L y \Leftrightarrow \forall n \in \mathbb{N}\,(E(|x-y|) \leq_{\mathbb{R}} 2^{-n})$$

for all $x, y \in \underline{X}$. Then $=_L$ is an equivalence relation on $\underline{X}$; hence $L = (\underline{X}, =_L)$ is a setoid.

Proof. It is trivial that for all $x, y \in \underline{X}$, $x =_L x$ and if $x =_L y$ then $y =_L x$. For all $x, y, z \in \underline{X}$, if $x =_L y$ and $y =_L z$, then, since $E(|x-y|) \leq_{\mathbb{R}} 2^{-(n+1)}$ and $E(|x-y|) \leq_{\mathbb{R}} 2^{-(n+1)}$ for all $n \in \mathbb{N}$, we have

$$E(|x-z|) \leq_{\mathbb{R}} E(|x-y|) + E(|y-z|) \leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n}$$

for all $n \in \mathbb{N}$; hence $x =_L z$. □

Proposition 5.3. *Let ρ_L and $\Vdash_L$ be a monotone function from $\mathbb{N}$ into $\mathbb{N}$ and a relation between $\underline{X} \times \underline{X}$ and $\mathbb{N}$ defined by*

$$\rho_L(n) = n + 1 \qquad \textit{and} \qquad (x, y) \Vdash_L n \Leftrightarrow E(|x-y|) \leq_{\mathbb{R}} 2^{-n},$$

respectively, for all $n \in \mathbb{N}$ and $(x, y) \in \underline{X} \times \underline{X}$. Then $(\mathbb{N}, \rho_L, \Vdash_L)$ is a uniform structure on the setoid $L = (\underline{X}, =_L)$.

Proof. (1): For all $x, y \in \underline{X}$, if $x =_L y$, then, since $E(|x-y|) \leq_{\mathbb{R}} 2^{-n}$ for all $n \in \mathbb{N}$, we have $(x, y) \Vdash_L n$ for all $n \in \mathbb{N}$; conversely, if $(x, y) \Vdash_L n$ for all $n \in \mathbb{N}$, then $E(|x-y|) \leq_{\mathbb{R}} 2^{-n}$ for all $n \in \mathbb{N}$; hence $x =_L y$.

(2) For all $n \in \mathbb{N}$ and $x, x', y, y' \in \underline{X}$, if $x =_L x'$, $y =_L y'$ and $(x, y) \Vdash_L n$, then, since $E(|x-x'|) \leq_{\mathbb{R}} 2^{-(m+1)}$ and $E(|y-y'|) \leq_{\mathbb{R}} 2^{-(m+1)}$ for all $m \in \mathbb{N}$, we have

$$\begin{aligned} E(|x'-y'|) &\leq_{\mathbb{R}} E(|x'-x|) + E(|x-y|) + E(|y-y'|) \\ &\leq_{\mathbb{R}} 2^{-(m+1)} + 2^{-n} + 2^{-(m+1)} =_{\mathbb{R}} 2^{-m} + 2^{-n} \end{aligned}$$

for all $m \in \mathbb{N}$; hence $E(|x'-y'|) \leq_{\mathbb{R}} 2^{-n}$. Therefore $(x', y') \Vdash_L n$.

(3): Straightforward.

(4): For all $m, n \in \mathbb{N}$ and $x, y \in \underline{X}$, if $m \leq n$ and $(x, y) \Vdash_L n$, then

$$E(|x-y|) \leq_{\mathbb{R}} 2^{-n} \leq_{\mathbb{R}} 2^{-m};$$

hence $(x, y) \Vdash_L m$.

(5): For all $n \in \mathbb{N}$ and $x, y, z \in \underline{X}$, if $(x, y) \Vdash_L \rho_L(n)$ and $(y, z) \Vdash_L \rho_L(n)$, then, since $E(|x-y|) \leq_{\mathbb{R}} 2^{-(n+1)}$ and $E(|y-z|) \leq_{\mathbb{R}} 2^{-(n+1)}$, we have

$$E(|x-z|) \leq_{\mathbb{R}} E(|x-y|) + E(|y-z|) \leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n};$$

hence $(x, z) \Vdash_L n$. □

Proposition 5.4. *L is a topological vector lattice.*

Proof. Let $\gamma^+ \in \hom(\mathbb{N}, \mathbb{N}\times\mathbb{N})$ be given by $\gamma^+(n) = (n+1, n+1)$ for all $n \in \mathbb{N}$, and consider $n \in \mathbb{N}$ and $(x, y), (x', y') \in \underline{L} \times \underline{L}$ such that

$$((x, y), (x', y')) \Vdash_{L\times L} \gamma^+(n).$$

Then, since $(x, x') \Vdash_L n + 1$ and $(y, y') \Vdash_L n + 1$, we have

$$\begin{aligned} E(|(x + y) - (x' + y')|) &\leq_{\mathbb{R}} E(|(x - x') + (y - y')|) \\ &\leq_{\mathbb{R}} E(|x - x'| + |y - y'|) \\ &=_{\mathbb{R}} E(|x - x'|) + E(|y - y'|) \\ &\leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n}, \end{aligned}$$

by Lemma 4.26 (1); hence $(x + x', y + y') \Vdash_L n$. Therefore $+ : L \times L \to L$ is uniformly continuous with a modulus γ^+.

Let $\gamma^\vee \in \hom(\mathbb{N}, \mathbb{N} \times \mathbb{N})$ be given by $\gamma^\vee(n) = (n + 1, n + 1)$ for all $n \in \mathbb{N}$, and consider $n \in \mathbb{N}$ and $(x, y), (x', y') \in \underline{L} \times \underline{L}$ such that

$$((x, y), (x', y')) \Vdash_{L\times L} \gamma^\vee(n).$$

Then, since $(x, x') \Vdash_L n + 1$ and $(y, y') \Vdash_L n + 1$, we have

$$\begin{aligned} E(|x \vee y - x' \vee y'|) &=_{\mathbb{R}} E(|(x \vee y - x' \vee y) + (x' \vee y - x' \vee y')|) \\ &\leq_{\mathbb{R}} E(|x \vee y - x' \vee y| + |x' \vee y - x' \vee y'|) \\ &\leq_{\mathbb{R}} E(|x \vee y - x' \vee y|) + E(|x' \vee y - x' \vee y'|) \\ &\leq_{\mathbb{R}} E(|x - x'|) + E(|y - y'|) \\ &\leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n}, \end{aligned}$$

by Lemma 4.26 (1) and (3); hence $(x+x', y+y') \Vdash_L n$. Therefore $\vee : L\times L \to L$ is uniformly continuous with a modulus $\gamma^\vee$.

□

We write $\mathfrak{L}$ for the completion $\tilde{L}$ of the topological vector lattice L, and call an element of $\mathfrak{L}$ an *integrable function* over the abstract integration space (X, E).

Proposition 5.5. *There exists a uniformly continuous mapping $\int : \mathfrak{L} \to \mathbb{R}$ such that*

1. $\int \eta_L(x) =_{\mathbb{R}} E(x)$,
2. $\int (f + g) =_{\mathbb{R}} \int f + \int g$,
3. $\int (sf) =_{\mathbb{R}} s \int f$,
4. *if* $0 \leq_{\mathfrak{L}} f$, *then* $0 \leq_{\mathbb{R}} \int f$

for all $x \in \underline{L}$, $f, g \in \underline{\mathfrak{L}}$ and $s \in \underline{\mathbb{R}}$. For $f \in \underline{\mathfrak{L}}$, $\int f$ is called the integral *of f.*

Proof. Consider $n \in \mathbb{N}$ and $x, y \in \underline{L}$ with $(x, y) \Vdash_L n$, and note that, since $x - y \leq_X |x - y|$ and $y - x \leq_X |x - y|$, we have $E(x) - E(y) \leq_{\mathbb{R}} E(|x - y|)$ and $E(y) - E(x) \leq_{\mathbb{R}} E(|x - y|)$; hence $|E(x) - E(y)| \leq_{\mathbb{R}} E(|x - y|)$. Then, since

$$|E(x) - E(y)| \leq_{\mathbb{R}} E(|x - y|) \leq_{\mathbb{R}} 2^{-n},$$

we have ♣ $(E(x), E(y)) \Vdash_{\mathbb{R}} n$. Therefore $E : L \to \mathbb{R}$ is a uniformly continuous mapping, and so there exists a uniformly continuous mapping $\tilde{E} : \mathfrak{L} \to \tilde{\mathbb{R}}$ such that $\eta_{\mathbb{R}} \circ E = \tilde{E} \circ \eta_L$, by Theorem 3.50. Since $\mathbb{R}$ is complete, there exists a uniformly continuous mapping $\varepsilon_{\mathbb{R}} : \tilde{\mathbb{R}} \to \mathbb{R}$ such that $\varepsilon_{\mathbb{R}} \circ \eta_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$. Let $\int = \varepsilon_{\mathbb{R}} \circ \tilde{E}$, and note that

$$\int \eta_L(x) =_{\mathbb{R}} \varepsilon_{\mathbb{R}}(\tilde{E}(\eta_L(x))) =_{\mathbb{R}} \varepsilon_{\mathbb{R}}(\eta_{\mathbb{R}}(E(x))) =_{\mathbb{R}} E(x)$$

for all $x \in \underline{L}$. Then, since

$$\begin{aligned}\int (\eta_L(x) + \eta_L(y)) &=_{\mathbb{R}} \int \eta_L(x + y) =_{\mathbb{R}} E(x + y) \\ &=_{\mathbb{R}} E(x) + E(y) =_{\mathbb{R}} \int \eta_L(x) + \int \eta_L(y)\end{aligned}$$

for all $x, y \in \underline{L}$, we have $\int (f + g) =_{\mathbb{R}} \int f + \int g$ for all $f, g \in \mathfrak{L}$. Similarly, since

$$\int s\eta_L(x) =_{\mathbb{R}} \int \eta_L(sx) =_{\mathbb{R}} E(sx) =_{\mathbb{R}} sE(x) =_{\mathbb{R}} s \int \eta_L(x)$$

for all $x \in \underline{L}$ and $s \in \underline{\mathbb{R}}$, we have $\int sf =_{\mathbb{R}} s \int f$ for all $f \in \mathfrak{L}$ and $s \in \underline{\mathbb{R}}$.

Consider $f = (x_n)_{n \in \mathbb{N}} \in \mathfrak{L}$ with $0 \leq_{\mathfrak{L}} f$. Then, since $(\text{-})^+ : \mathfrak{L} \to \mathfrak{L}$ is uniformly continuous and $\eta_L(x_n) \to f$ by Corollary **??**, we have $(\eta_L(x_n))^+ \to f^+$, and, since $f^+ =_{\mathfrak{L}} f$ and $(\eta_L(x_n))^+ =_{\mathfrak{L}} \eta_L(x_n^+)$ for all $n \in \mathbb{N}$, we have $\eta_L(x_n^+) \to f$. Therefore, since $\int : \mathfrak{L} \to \mathfrak{L}$ is uniformly continuous and

$$0 \leq_{\mathbb{R}} E(x_n^+) =_{\mathbb{R}} \int \eta_L(x_n^+)$$

for all $n \in \mathbb{N}$, we have $0 \leq_{\mathbb{R}} \int f$. ☐

Proposition 5.6. *Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence of integrable functions, that is, $f_n \leq_{\mathfrak{L}} f_{n+1}$ for all $n \in \mathbb{N}$. If $(\int f_n)_{n \in \mathbb{N}}$ converges in $\mathbb{R}$, then $(f_n)_{n \in \mathbb{N}}$ converges in $\mathfrak{L}$.*

Proof. Suppose that $(\int f_n)_{n \in \mathbb{N}}$ converges in $\mathbb{R}$. Then $(\int f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, by Proposition **??**; hence there exists a Cauchy modulus $\alpha \in \hom(\mathbb{N}, \mathbb{N})$ such that for each $n \in \mathbb{N}$, $|\int f_m - \int f_{m'}| \leq 2^{-n}$ for all $m, m' \in \mathbb{N}$ with $\alpha(n) \leq m, m'$.

Given an $n \in \mathbb{N}$, consider $m, m' \in \mathbb{N}$ such that $\alpha(\rho_{\mathfrak{L}}(n)+1) \leq m \leq m'$. Then, since $f_m \leq_{\mathfrak{L}} f_{m'}$, we have $\int f_m \leq_{\mathbb{R}} \int f_{m'}$; hence $\int f_{m'} - \int f_m \leq 2^{-(\rho_{\mathfrak{L}}(n)+1)}$. Let $f_m = (x_{n'})_{n' \in \mathbb{N}}$ and $f_{m'} = (y_{n'})_{n' \in \mathbb{N}}$. Then, since $\eta_L(x_{n'}) \to f_m$, $\eta_L(y_{n'}) \to f_{m'}$ and $\vee : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ is uniformly continuous, we have

$$\eta_L(x_{n'} \vee y_{n'}) =_{\mathfrak{L}} \eta_L(x_{n'}) \vee \eta_L(y_{n'}) \to f_m \vee f_{m'} =_{\mathfrak{L}} f_{m'};$$

hence $E(x_{n'}) =_{\mathbb{R}} \int \eta_L(x_{n'}) \to \int f_m$ and $E(x_{n'} \vee y_{n'}) =_{\mathbb{R}} \int \eta_L(x_{n'} \vee y_{n'}) \to \int f_{m'}$. Choose $n' \in \mathbb{N}$ such that

$$(f_m, \eta_L(x_{n'})) \Vdash_{\mathfrak{L}} \rho^2_{\mathfrak{L}}(n), \qquad (f_{m'}, \eta_L(x_{n'} \vee y_{n'})) \Vdash_{\mathfrak{L}} \rho^2_{\mathfrak{L}}(n),$$
$$|E(x_{n'}) - \int f_m| \leq_R 2^{-(\rho^2_{\mathfrak{L}}(n)+2)}, \quad |E(x_{n'} \vee y_{n'}) - \int f_{m'}| \leq_R 2^{-(\rho^2_{\mathfrak{L}}(n)+2)}.$$

Then

$$\begin{aligned}
E(|x_{n'} \vee y_{n'} - x_{n'}|) &=_{\mathbb{R}} E(x_{n'} \vee y_{n'} - x_{n'}) =_{\mathbb{R}} E(x_{n'} \vee y_{n'}) - E(x_{n'}) \\
&=_{\mathbb{R}} (E(x_{n'} \vee y_{n'}) - \int f_{m'}) + (\int f_{m'} - \int f_m) \\
&\qquad\qquad\qquad + (\int f_m - E(x_{n'})) \\
&\leq_{\mathbb{R}} |E(x_{n'} \vee y_{n'}) - \int f_{m'}| + (\int f_{m'} - \int f_m) \\
&\qquad\qquad\qquad + |\int f_m - E(x_{n'})| \\
&\leq_{\mathbb{R}} 2^{-(\rho^2_{\mathfrak{L}}(n)+2)} + 2^{-(\rho^2_{\mathfrak{L}}(n)+1)} + 2^{-(\rho^2_{\mathfrak{L}}(n)+2)} \\
&=_{\mathbb{R}} 2^{-(\rho^2_{\mathfrak{L}}(n))};
\end{aligned}$$

hence $(x_{n'} \vee y_{n'}, x_{n'}) \Vdash_L \rho^2_{\mathfrak{L}}(n)$. Therefore $(\eta_L(x_{n'} \vee y_{n'}), \eta_L(x_{n'})) \Vdash_{\mathfrak{L}} \rho^2_{\mathfrak{L}}(n)$, by Lemma **??**, and so $(f_m, f_{m'}) \Vdash_{\mathfrak{L}} n$. Thus $(f_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathfrak{L}$ with a modulus $n \mapsto \alpha(\rho_{\mathfrak{L}}(n) + 1)$, and so converges. □

5.2 Measurable functions

Lemma 5.7. *Let $=_M$ be a binary relation on $\underline{X}$ given by*

$$x =_M y \Leftrightarrow \forall u \in \underline{C}_X \, \forall n \in \mathbb{N} \, (E(|x - y| \wedge u) \leq_{\mathbb{R}} 2^{-n})$$

for all $x, y \in \underline{X}$. Then $=_M$ is an equivalence relation on $\underline{X}$; hence $M = (\underline{X}, =_M)$ is a setoid.

Proof. It is trivial that for all $x, y \in \underline{X}$, $x =_M x$ and if $x =_M y$ then $y =_M x$. For all $x, y, z \in \underline{X}$, if $x =_M y$ and $y =_M z$, then, since $E(|x-y| \wedge u) \leq_{\mathbb{R}} 2^{-(n+1)}$ and $E(|x - y| \wedge u) \leq_{\mathbb{R}} 2^{-(n+1)}$ for all $u \in \underline{C}_X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
E(|x - z| \wedge u) &=_{\mathbb{R}} E(|(x - y) + (y - z)| \wedge u) \\
&\leq_{\mathbb{R}} E((|x - y| + |y - z|) \wedge u) \\
&\leq_{\mathbb{R}} E(|x - y| \wedge u + |y - z| \wedge u) \\
&=_{\mathbb{R}} E(|x - y| \wedge u) + E(|y - z| \wedge u) \\
&\leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n}
\end{aligned}$$

for all $u \in \underline{C}_X$ and $n \in \mathbb{N}$, by Lemma 4.26 (1) and (5); hence $x =_M z$. □

Proposition 5.8. *Let ρ_M and $\Vdash_M$ be a monotone function from $C_X \times \mathbb{N}$ into $C_X \times \mathbb{N}$, and a relation between $\underline{X} \times \underline{X}$ and $\underline{C}_X \times \mathbb{N}$, respectively, defined by*

$$\rho_M((u, n)) = (u, n + 1) \quad \text{and} \quad (x, y) \Vdash_M (u, n) \Leftrightarrow E(|x - y| \wedge u) \leq_{\mathbb{R}} 2^{-n},$$

respectively, for all $(u, n) \in \underline{C}_X \times \mathbb{N}$ and $(x, y) \in \underline{X} \times \underline{X}$. Then $(C_X \times \mathbb{N}, \rho_M, \Vdash_M)$ is a uniform structure on the setoid $M = (\underline{X}, =_M)$.

Proof. (1): For all $x, y \in \underline{X}$, if $x =_M y$, then, since $E(|x-y| \wedge u) \leq_{\mathbb{R}} 2^{-n}$ for all $u \in \underline{C}_X$ and $n \in \mathbb{N}$, we have $(x, y) \Vdash_M (u, n)$ for all $(u, n) \in \underline{C}_X \times \mathbb{N}$; conversely, if $(x, y) \Vdash_M (u, n)$ for all $(u, n) \in \underline{C}_X \times \mathbb{N}$, then, since $E(|x-y| \wedge u) \leq_{\mathbb{R}} 2^{-n}$ for all $u \in \underline{C}_X$ and $n \in \mathbb{N}$, we have $x =_M y$.

(2): For all $(u, n) \in \underline{C}_X \times \mathbb{N}$ and $x, x', y, y' \in \underline{X}$, if $x =_M x'$, $y =_M y'$ and $(x, y) \Vdash_M (u, n)$, then, since $E(|x - x'| \wedge u) \leq_{\mathbb{R}} 2^{-(m+1)}$ and $E(|y - y'| \wedge u) \leq_{\mathbb{R}} 2^{-(m+1)}$ for all $m \in \mathbb{N}$, we have

$$\begin{aligned}
E(|x' - y'| \wedge u) &=_{\mathbb{R}} E(|(x' - x) + (x - y')| \wedge u) \\
&\leq_{\mathbb{R}} E((|x' - x| + |x - y'|) \wedge u) \\
&\leq_{\mathbb{R}} E(|x' - x| \wedge u + |x - y'| \wedge u) \\
&=_{\mathbb{R}} E(|x' - x| \wedge u + |(x - y) + (y - y')| \wedge u) \\
&\leq_{\mathbb{R}} E(|x' - x| \wedge u + (|x - y| + |y - y'|) \wedge u) \\
&\leq_{\mathbb{R}} E(|x' - x| \wedge u + |x - y| \wedge u + |y - y'| \wedge u) \\
&=_{\mathbb{R}} E(|x' - x| \wedge u) + E(|x - y| \wedge u) + E(|y - y'| \wedge u) \\
&\leq_{\mathbb{R}} 2^{-(m+1)} + 2^{-n} + 2^{-(m+1)} =_{\mathbb{R}} 2^{-m} + 2^{-n}
\end{aligned}$$

for all $m \in \mathbb{N}$, by Lemma 4.26 (1) and (5); hence $E(|x' - y'| \wedge u) \leq_{\mathbb{R}} 2^{-n}$. Therefore $(x', y') \Vdash_M (u, n)$.

(3): Straightforward.

(4): For all $(u, n), (v, m) \in \underline{C}_X \times \mathbb{N}$ and $x, y \in \underline{X}$, if $(u, n) \preccurlyeq_{C_X \times \mathbb{N}} (v, m)$ and $(x, y) \Vdash_M (v, m)$, then, since $u \leq_X v$ and $n \leq m$, we have

$$E(|x - y| \wedge u) \leq_{\mathbb{R}} E(|x - y| \wedge v) \leq_{\mathbb{R}} 2^{-m} \leq_{\mathbb{R}} 2^{-n};$$

hence $(x, y) \Vdash_M (u, n)$.

(5): For all $(u, n) \in \underline{C}_X \times \mathbb{N}$ and $x, y, z \in \underline{X}$, if $(x, y) \Vdash_M \rho_M((u, n))$ and $(y, z) \Vdash_M \rho_M((u, n))$, then, since $E(|x-y| \wedge u) \leq_{\mathbb{R}} 2^{-(n+1)}$ and $E(|y-z| \wedge u) \leq_{\mathbb{R}} 2^{-(n+1)}$, we have

$$\begin{aligned}
E(|x - z| \wedge u) &\leq_{\mathbb{R}} E(|(x - y) + (y - z)| \wedge u) \\
&\leq_{\mathbb{R}} E((|x - y| + |y - z|) \wedge u) \\
&\leq_{\mathbb{R}} E(|x - y| \wedge u + |y - z| \wedge u) \\
&\leq_{\mathbb{R}} E(|x - y| \wedge u) + E(|y - z| \wedge u) \\
&\leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n},
\end{aligned}$$

by Lemma 4.26 (1) and (5); hence $(x, z) \Vdash_M (u, n)$. □

Proposition 5.9. *M is a topological vector lattice.*

Proof. Let $\gamma^+ \in \hom(C_X \times \mathbb{N}, (C_X \times \mathbb{N}) \times (C_X \times \mathbb{N}))$ be given by

$$\gamma^+((u, n)) = ((u, n + 1), (u, n + 1))$$

for all $(u, n) \in \underline{C}_X \times \mathbb{N}$, and consider $(u, n) \in \underline{C}_X \times \mathbb{N}$ and $(x, y), (x', y') \in \underline{M} \times \underline{M}$ such that $((x, y), (x', y')) \Vdash_{M \times M} \gamma^+((u, n))$. Then, since $(x, x') \Vdash_M (u, n + 1)$

and $(y, y') \Vdash_M (u, n+1)$, we have

$$\begin{aligned} E(|(x+y)-(x'+y')| \wedge u) &\leq_{\mathbb{R}} E(|(x-x')+(y-y')| \wedge u) \\ &\leq_{\mathbb{R}} E((|x-x'|+|y-y'|) \wedge u) \\ &\leq_{\mathbb{R}} E(|x-x'| \wedge u + |y-y'| \wedge u)) \\ &=_{\mathbb{R}} E(|x-x'| \wedge u) + E(|y-y'| \wedge u) \\ &\leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n}, \end{aligned}$$

by Lemma 4.26 (5); hence $(x+x', y+y') \Vdash_M (u, n)$. Therefore $+ : M \times M \to M$ is uniformly continuous with a modulus γ^{+}.

Let $\gamma^{\vee} \in \hom(C_X \times \mathbb{N}, (C_X \times \mathbb{N}) \times (C_X \times \mathbb{N}))$ be given by

$$\gamma^{\vee}((u, n)) = ((u, n+1), (u, n+1))$$

for all $(u, n) \in \underline{C}_X \times \mathbb{N}$, and consider $(u, n) \in \underline{C}_X \times \mathbb{N}$ and $(x, y), (x', y') \in \underline{M} \times \underline{M}$ such that $((x, y), (x', y')) \Vdash_{M \times M} \gamma^{\vee}(n)$. Then, since $(x, x') \Vdash_M (u, n+1)$ and $(y, y') \Vdash_M (u, n+1)$, we have

$$\begin{aligned} E(|x \vee y - x' \vee y'| \wedge u) &=_{\mathbb{R}} E(|(x \vee y - x' \vee y) + (x' \vee y - x' \vee y')| \wedge u) \\ &\leq_{\mathbb{R}} E((|x \vee y - x' \vee y| + |x' \vee y - x' \vee y'|) \wedge u) \\ &\leq_{\mathbb{R}} E(|x \vee y - x' \vee y| \wedge u + |x' \vee y - x' \vee y'| \wedge u) \\ &\leq_{\mathbb{R}} E(|x \vee y - x' \vee y| \wedge u) + E(|x' \vee y - x' \vee y'| \wedge u) \\ &\leq_{\mathbb{R}} E(|x-x'| \wedge u) + E(|y-y'| \wedge u) \\ &\leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n}, \end{aligned}$$

by Lemma 4.26 (1), (3) and (5); hence $(x+x', y+y') \Vdash_M (u, n)$. Therefore $\vee : M \times M \to M$ is uniformly continuous with a modulus $\gamma^{\vee}$.

□

We write $\mathfrak{M}$ for the completion $\tilde{M}$ of the topological vector lattice M, and call an element of $\mathfrak{M}$ a *measurable function* over the abstract integration space (X, E).

5.3 Convergence theorems

Lemma 5.10. *The function* $\mathrm{id}_{\underline{X}} : \underline{X} \to \underline{X}$ *is a uniformly continuous setoid injection of* L *into* M.

Proof. Let $\gamma \in \hom(C_X \times \mathbb{N}, \mathbb{N})$ be given by

$$\gamma((u, n)) = n$$

for all $(u, n) \in \underline{C}_X \times \mathbb{N}$. Then for all $(u, n) \in \underline{C}_X \times \mathbb{N}$ and $x, y \in \underline{L}$, if $(x, y) \Vdash_L \gamma((u, n))$, then, since $E(|x-y|) \leq_{\mathbb{R}} 2^{-n}$, we have

$$E(|\mathrm{id}_{\underline{X}}(x) - \mathrm{id}_{\underline{X}}(y)| \wedge u) =_{\mathbb{R}} E(|x-y| \wedge u) \leq_{\mathbb{R}} E(|x-y|) \leq_{\mathbb{R}} 2^{-n};$$

hence $(\mathrm{id}_{\underline{X}}(x), \mathrm{id}_{\underline{X}}(y)) \Vdash_M (u, n)$. Therefore $\mathrm{id}_{\underline{X}} : L \to M$ is a uniformly continuous mapping, by Lemma ♣ 3.8.

For all $x, y \in \underline{L}$, if $\mathrm{id}_{\underline{X}}(x) =_M \mathrm{id}_{\underline{X}}(y)$, then, since $E(|x-y| \wedge u) =_{\mathbb{R}} E(|\mathrm{id}_{\underline{X}}(x) - \mathrm{id}_{\underline{X}}(y)| \wedge u) \leq_{\mathbb{R}} 2^{-n}$ for all $u \in \underline{C}_X$ and $n \in \mathbb{N}$, we have

$$E(|x-y|) =_{\mathbb{R}} E(|x-y| \wedge |x-y|) \leq_{\mathbb{R}} 2^{-n}$$

for all $n \in \mathbb{N}$; hence $x =_L y$. Thus $\mathrm{id}_{\underline{X}} : \underline{L} \to \underline{M}$ is a setoid injection. □

Proposition 5.11. *There exists a uniformly continuous embedding $\lambda : \mathfrak{L} \to \mathfrak{M}$ such that $\eta_M \circ \mathrm{id}_{\underline{X}} = \lambda \circ \eta_L$.*

Proof. Since, by Lemma 5.10, $\mathrm{id}_{\underline{X}} : L \to M$ is uniformly continuous, there exists a unique uniformly continuous mapping $\lambda : \mathfrak{L} \to \mathfrak{M}$ such that $\eta_M \circ \mathrm{id}_{\underline{X}} = \lambda \circ \eta_L$.

To see that λ preserves all vector lattice operations, we only show that

$$\lambda(f+g) =_{\mathfrak{M}} \lambda(f) + \lambda(g)$$

for all $f, g \in \mathfrak{L}$. Other operations are similar. Let $F, G : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{M}$ be setoid mappings given by $F(f,g) = \lambda(f+g)$ and $G(f,g) = \lambda(f) + \lambda(g)$ for all $f, g \in \mathfrak{L}$, respectively. Note that F and G are uniformly continuous. Then, since

$$\begin{aligned} F(\eta_L(x), \eta_L(y)) &=_{\mathfrak{M}} \lambda(\eta_L(x) + \eta_L(y)) =_{\mathfrak{M}} \lambda(\eta_L(x+y)) =_{\mathfrak{M}} \eta_M(\mathrm{id}_{\underline{X}}(x+y)) \\ &=_{\mathfrak{M}} \eta_M(\mathrm{id}_{\underline{X}}(x) + \mathrm{id}_{\underline{X}}(y)) =_{\mathfrak{M}} \eta_M(\mathrm{id}_{\underline{X}}(x)) + \eta_M(\mathrm{id}_{\underline{X}}(y)) \\ &=_{\mathfrak{M}} \lambda(\eta_L(x)) + \lambda(\eta_L(y)) =_{\mathfrak{M}} G(\eta_L(x), \eta_L(y)) \end{aligned}$$

for all $x, y \in \underline{L}$, we have $F = G$ by the uniqueness of such uniformly continuous mapping. Therefore $\lambda(f+g) =_{\mathfrak{M}} \lambda(f) + \lambda(g)$ for all $f, g \in \mathfrak{L}$.

To show that λ is injective, consider $f = (x_m)_{m \in \mathbb{N}} \in \mathfrak{L}$ with $\lambda(f) =_{\mathfrak{M}} \eta_M(0)$. Then, since $\eta_L(x_m) \to f$ in $\mathfrak{L}$, we have $\lambda(\eta_L(x_m)) \to \eta_M(0)$ in $\mathfrak{M}$. Given an $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $(\lambda(\eta_L(x_{m'})), \eta_M(0)) \Vdash_{\mathfrak{M}} \rho_M^2((|x_{n+1}|, n+1))$, that is,

$$(\eta_M(\mathrm{id}_{\underline{X}}(x_{m'})), \eta_M(0)) \Vdash_{\mathfrak{M}} \rho_M^2((|x_{n+1}|, n+1))$$

for all $m' \in \mathbb{N}$ with $m \leq m'$; hence $(\mathrm{id}_{\underline{X}}(x_{m'}), 0)) \Vdash_M (|x_{n+1}|, n+1)$ for all $m' \in \mathbb{N}$ with $m \leq m'$, by Lemma **??**. Therefore for all $m' \in \mathbb{N}$ with $\max\{m, n+1\} \leq m'$, since $(x_{m'}, x_{n+1}) \Vdash_L n+1$, we have

$$\begin{aligned} E(|x_{m'}|) &\leq_{\mathbb{R}} E(|x_{m'}| \wedge |x_{m'}|) \leq_{\mathbb{R}} E(|x_{n+1} + (x_{m'} - x_{n+1})| \wedge |x_{m'}|) \\ &\leq_{\mathbb{R}} E((|x_{n+1}| + |x_{m'} - x_{n+1}|) \wedge |x_{m'}|) \\ &\leq_{\mathbb{R}} E(|x_{n+1}| \wedge |x_{m'}| + |x_{m'} - x_{n+1}| \wedge |x_{m'}|) \\ &\leq_{\mathbb{R}} E(|x_{m'}| \wedge |x_{n+1}| + |x_{m'} - x_{n+1}|) \\ &\leq_{\mathbb{R}} E(|x_{m'}| \wedge |x_{n+1}|) + E(|x_{m'} - x_{n+1}|) \\ &\leq_{\mathbb{R}} 2^{-(n+1)} + 2^{-(n+1)} =_{\mathbb{R}} 2^{-n}, \end{aligned}$$

by Lemma 4.26 (1) and (5), and so $(x_{m'}, 0) \Vdash_L n$. Thus $f =_{\mathfrak{L}} 0$. □

Lemma 5.12. *The function $(x, y) \mapsto x \wedge |y|$ from $\underline{X} \times \underline{X}$ into $\underline{X}$ is a locally uniformly continuous mapping of $L \times M$ into L.*

Proof. □

Proposition 5.13. *For each $g \in \mathfrak{L}$, there exists a uniformly continuous mapping $\mu_g : \mathfrak{M} \to \mathfrak{L}$ such that*

$$\mu_g(\lambda(f)) =_{\mathfrak{L}} g \wedge |f| \qquad \textit{and} \qquad \lambda(\mu_g(h)) =_{\mathfrak{M}} \lambda(g) \wedge |h|$$

for all $f \in \mathfrak{L}$ and $h \in \mathfrak{M}$.

Proof. □

Theorem 5.14. *Let f be a measurable function. If there exists an integrable function g such that $|f| \leq_{\mathfrak{M}} \lambda(g)$, then there exists an integrable function $f_{\mathfrak{L}}$ such that $f =_{\mathfrak{M}} \lambda(f_{\mathfrak{L}})$.*

Proof. Suppose that $|f| \leq_{\mathfrak{M}} \lambda(g)$ for some integrable function g. Then, since $f^+ \leq_{\mathfrak{M}} |f| \leq_{\mathfrak{M}} \lambda(g)$ and $f^+ \leq_{\mathfrak{M}} |f| \leq_{\mathfrak{M}} \lambda(g)$, we have

$$\lambda(\mu_g(f^+)) =_{\mathfrak{M}} \lambda(g) \wedge |f^+| =_{\mathfrak{M}} \lambda(g) \wedge f^+ =_{\mathfrak{M}} f^+$$

and

$$\lambda(\mu_g(f^-)) =_{\mathfrak{M}} \lambda(g) \wedge |f^-| =_{\mathfrak{M}} \lambda(g) \wedge f^- =_{\mathfrak{M}} f^-.$$

Therefore, setting $f_{\mathfrak{L}} = \mu_g(f^+) - \mu_g(f^-)$, we have

$$\lambda(f_{\mathfrak{L}}) =_{\mathfrak{M}} \lambda(\mu_g(f^+) - \mu_g(f^-)) =_{\mathfrak{M}} \lambda(\mu_g(f^+)) - \lambda(\mu_g(f^-)) =_{\mathfrak{M}} f^+ - f^- =_{\mathfrak{M}} f.$$

□

Definition 5.15. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions, and let f be an integrable function. Then

1. $(f_n)_{n\in\mathbb{N}}$ *converges in norm* to f if $f_n \to f$ in $\mathfrak{L}$;
2. $(f_n)_{n\in\mathbb{N}}$ *converges in measure* to f if $\lambda(f_n) \to \lambda(f)$ in $\mathfrak{M}$.

Lemma 5.16. *Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of integrable functions converging in measure to an integrable function f. Then $f_n \leq_{\mathfrak{L}} f$ for all $n \in \mathbb{N}$.*

Proof. Given an $n \in \mathbb{N}$, since $f_n \wedge f_m =_{\mathfrak{L}} f_n$ for all $m \in \mathbb{N}$ with $n \leq m$, we have $(f_n \wedge f_m, f_n) \Vdash_{\mathfrak{L}} n$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ with $n \leq m$; hence $(f_n \wedge f_m)_{m\in\in\mathbb{N}}$ converges to f_n. Therefore

$$\lambda(f_n) \wedge \lambda(f_m) =_{\mathfrak{M}} \lambda(f_n \wedge f_m) \to \lambda(f_n)$$

in $\mathfrak{M}$. On the other hand, since $\lambda(f_m) \to \lambda(f)$ in $\mathfrak{M}$, we have

$$\lambda(f_n) \wedge \lambda(f_m) \to \lambda(f_n) \wedge \lambda(f).$$

Thus $\lambda(f_n \wedge f) =_{\mathfrak{M}} \lambda(f_n) \wedge \lambda(f) =_{\mathfrak{M}} \lambda(f_n)$, and so, since $\lambda : \mathfrak{L} \to \mathfrak{M}$ is injective, we have $f_n \wedge f =_{\mathfrak{L}} f_n$; hence $f_n \leq_{\mathfrak{L}} f$. □

Theorem 5.17 (Lebesgue's Monotone Convergence Theorem)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of integrable functions. Then the following are equivalent.*

1. *$(f_n)_{n\in\mathbb{N}}$ converges in measure to some integrable function f,*
2. *$(f_n)_{n\in\mathbb{N}}$ converges in norm to some integrable function f,*

3. $(\int f_n)_{n\in\mathbb{N}}$ *converges; in which case*

$$\int f_n \to \int f.$$

Proof. (1) $\Rightarrow$ (2): Note that $f_n \leq_{\mathfrak{L}} f$ for all $n \in \mathbb{N}$, by Lemma 5.16. Then, by replacing f_n and f by $f_n - f_0$ and $f - f_0$, respectively, we may assume without loss of generality that $0 \leq_{\mathfrak{L}} f_n$ for all $n \in \mathbb{N}$ and $0 \leq_{\mathfrak{L}} f$. Since $\lambda(f_n) \to \lambda(f)$ in $\mathfrak{M}$,

$$f_n =_{\mathfrak{L}} f \wedge f_n =_{\mathfrak{L}} f \wedge |f_n| =_{\mathfrak{L}} \mu_f(\lambda(f_n)) \to \mu_f(\lambda(f)) =_{\mathfrak{L}} f \wedge |f| =_{\mathfrak{L}} f$$

in $\mathfrak{L}$.

(2) $\Rightarrow$ (3): Since $\int : \mathfrak{L} \to \mathbb{R}$, we have $\int f_n \to \int f$.

(3) $\Rightarrow$ (1): Suppose that $(\int f_n)_{n\in\mathbb{N}}$ converses. Then $(f_n)_{n\in\mathbb{N}}$ converges in $\mathfrak{L}$ to some $f \in \mathfrak{L}$, by Proposition 5.6. Therefore, since $\lambda : \mathfrak{L} \to \mathfrak{M}$ is uniformly continuous, we have $\lambda(f_n) \to \lambda(f)$ in $\mathfrak{M}$. □

Theorem 5.18 (Fatou's Lemma)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions converging in measure to an integrable function f such that $0 \leq_{\mathfrak{L}} f_n$ and $\int f_n \leq B$ for all $n \in \mathbb{N}$. Then $\int f \leq B$.*

Proof. Since $\lambda(f_n) \to \lambda(f)$ in $\mathfrak{M}$,

$$f \wedge f_n =_{\mathfrak{L}} f \wedge |f_n| =_{\mathfrak{L}} \mu_f(\lambda(f_n)) \to \mu_f(\lambda(f)) =_{\mathfrak{L}} f \wedge |f| =_{\mathfrak{L}} f$$

in $\mathfrak{L}$; hence $\int f \wedge f_n \to \int f$. Since $\int f \wedge f_n \leq \int f_n \leq B$ for all $n \in \mathbb{N}$, we have $\int f \leq B$. □

Lemma 5.19. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions converging in measure to an integrable function f, and let g be an integrable function such that $|f_n| \leq_{\mathfrak{L}} g$ for all $n \in \mathbb{N}$. Then $|f| \leq_{\mathfrak{L}} g$.*

Proof. Since $\lambda(f_n) \to \lambda(f)$ in $\mathfrak{M}$ and $|f_n| =_{\mathfrak{L}} g \wedge |f_n|$ for all $n \in \mathbb{N}$,

$$|f_n| =_{\mathfrak{L}} g \wedge |f_n| =_{\mathfrak{L}} \mu_g(\lambda(f_n)) \to \mu_g(\lambda(f)) =_{\mathfrak{L}} g \wedge |f|$$

in $\mathfrak{L}$; hence $|f| \wedge |f_n| \to |f| \wedge (g \wedge |f|) =_{\mathfrak{L}} g \wedge |f|$ in $\mathfrak{L}$. On the other hand,

$$|f| \wedge |f_n| =_{\mathfrak{L}} \mu_{|f|}(\lambda(f_n)) \to \mu_{|f|}(\lambda(f)) =_{\mathfrak{L}} |f| \wedge |f| =_{\mathfrak{L}} |f|$$

in $\mathfrak{L}$. Therefore $g \wedge |f| =_{\mathfrak{L}} |f|$, and so $|f| \leq_{\mathfrak{L}} g$. □

Theorem 5.20 (Lebesgue's Dominated Convergence Theorem)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions converging in measure to an integrable function f, and let g be an integrable function such that $|f_n| \leq_{\mathfrak{L}} g$ for all $n \in \mathbb{N}$. Then $(f_n)_{n\in\mathbb{N}}$ converges in norm to f.*

Proof. Note that $|f| \leq_{\mathfrak{L}} g$, by Lemma 5.19. Then, since $\lambda(f_n) \to \lambda(f)$ in $\mathfrak{M}$, $\lambda(f_n^+) =_{\mathfrak{M}} \lambda(f_n)^+ \to \lambda(f)^+ =_{\mathfrak{M}} \lambda(f^+)$ in $\mathfrak{M}$; hence

$$g \wedge f_n^+ =_{\mathfrak{L}} g \wedge |f_n^+| =_{\mathfrak{L}} \mu_g(\lambda(f_n^+)) \to \mu_g(\lambda(f^+)) =_{\mathfrak{L}} g \wedge |f^+| =_{\mathfrak{L}} g \wedge f^+$$

in $\mathfrak{L}$. Therefore, since $f_n^+ \leq_{\mathfrak{L}} |f_n| \leq_{\mathfrak{L}} g$ for all $n \in \mathbb{N}$ and $f^+ \leq_{\mathfrak{L}} |f| \leq_{\mathfrak{L}} g$,

$$f_n^+ \to f^+$$

in $\mathfrak{L}$. Similarly, $f_n^- \to f^-$ in $\mathfrak{L}$. Thus $f_n =_{\mathfrak{L}} f_n^+ - f_n^- \to f^+ - f^- =_{\mathfrak{L}} f$ in $\mathfrak{L}$. □

Acknowledgment

The author hearty thanks Helmut Schwichtenberg for having been encouraging him, always with helpful comments and suggestions, to make an early rough draft in 2004 into two unpublished papers in 2020 and the papers into the present form. He thanks Tatsuji Kawai for comments and suggestions on the draft, Samuele Maschio for stimulus discussions and Andrej Bauer for discussions leading to substantial improvements. He also thanks the Japan Society for the Promotion of Science (JSPS), Core-to-Core Program (A. Advanced Research Networks), and JSPS KAKENHI Grant Numbers JP21KK0045 and JP22K03400 for supporting the research.

References

[1] Peter Aczel, *The type theoretic interpretation of constructive set theory*, In: A. Macintyre, L. Pacholski, J. Paris, eds., *Logic Colloquium '77*, North-Holland, Amsterdam, 1978, 55–66.

[2] Peter Aczel and Michael Rathjen, *Notes on constructive set theory*, Report No. 40, Institut Mittag-Leffler, The Royal Swedish Academy of Sciences, 2001.

[3] Peter Aczel and Michael Rathjen, *CST Book draft*, August 19, 2010, https://www1.maths.leeds.ac.uk/˜rathjen/book.pdf.

[4] Marian Alexandru Baroni, *On the order dual of a Riesz space*, Discrete mathematics and theoretical computer science, 109–117, Lecture Notes in Comput. Sci., 2731, Springer, Berlin, 2003.

[5] Josef Berger, Hajime Ishihara, Erik Palmgren and Peter Schuster, *A predicative completion of a uniform space*, Ann. Pure Appl. Logic **163** (2012), no. 8, 975–980.

[6] Errett Bishop, *Foundations of constructive analysis*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1967.

[7] Errett Bishop and Douglas Bridges, *Constructive analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 279, Springer-Verlag, Berlin, 1985.

[8] Errett Bishop and Henry Cheng, *Constructive measure theory*, Memoirs of the American Mathematical Society, No. 116, American Mathematical Society, Providence, R.I., 1972.

[9] Douglas S. Bridges and Luminiţa Simona Vîţă, *Techniques of constructive analysis*, Universitext, Springer, New York, 2006.

[10] Douglas S. Bridges and Luminiţa Simona Vîţă, *Apartness and uniformity, A constructive development*, Theory and Applications of Computability, Springer, Heidelberg, 2011.

[11] Thierry Coquand and Erik Palmgren, *Metric Boolean algebras and constructive measure theory*, Arch. Math. Logic **41** (2002), no. 7, 687–704.

[12] Laura Crosilla, Hajime Ishihara and Peter Schuster, *On constructing completions*, J. Symbolic Logic **70** (2005), no. 3, 969–978.

[13] Giovanni Curi, *On the collection of points of a formal space*, Ann. Pure Appl. Logic **137** (2006), no. 1-3, 126–146.

[14] Tatsuji Kawai, *Localic completion of uniform spaces*, Log. Methods Comput. Sci. **13** (2017), no. 3, Paper No. 22, 39 pp.

[15] John L. Kelley, *General topology*, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27, Springer-Verlag, New York-Berlin, 1975.

[16] Giovanni Sambin, *Positive topology: A new practice in constructive mathematics*, Oxford University Press, forthcoming.

[17] Bas Spitters, *Constructive algebraic integration theory*, Ann. Pure Appl. Logic **137** (2006), no. 1-3, 380–390.

[18] Anne S. Troelstra and Dirk van Dalen, *Constructivism in mathematics. Vol. I. An introduction*, Studies in Logic and the Foundations of Mathematics, 121, North-Holland Publishing Co., Amsterdam, 1988.

[19] Anne S. Troelstra and Dirk van Dalen, *Constructivism in mathematics. Vol. II. An introduction*, Studies in Logic and the Foundations of Mathematics, 123. North-Holland Publishing Co., Amsterdam, 1988.

Hajime Ishihara
Faculty of Science, Toho University
2-2-1 Miyama, Funabashi-shi, Chiba 274-8510, Japan
E-mail: `hajime.ishihara@ext.toho-u.ac.jp`