Yet another predicative completion of a uniform space

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Abstract

We give a predicative completion of a uniform space with pseudometrics by means of the notion of a net. Since the notion of a net is a generalization of the notion of a sequence, the completion is a generalization of, and parallel to the usual completion of a metric space. The completion is given in the elementary constructive set theory **ECST** with Exponentiation Axiom **Exp**. Since **ECST** + **Exp** is a subsystem of the constructive and predicative Zermelo–Fraenkel set theory **CZF**, we do not make use of Fullness, an axiom of **CZF**.

1 Introduction

Bishop introduced a constructive notion of a uniform space with a set of pseudometrics, and construct a completion of a uniform space by means of the set of Cauchy filters in [5, Problems 4.17 and 4.19]; see also [6, Problems 4.22 and 4.24]. Note that the construction of the set of Cauchy filters is problematic from a predicative point of view. Bishop also gave a construction of the product of uniform spaces; see [5, Problem 4.21] and also [6, Problem 4.26].

On the other hand, Bridges and Vîţă [8] employed a set of entourages (with an extra condition) to define a uniform space. Berger et al. [4] pointed out that the notion of a uniform space with entourages has an advantage over the one with pseudometrics, and gave a predicative completion of a uniform space given by entourages (without any extra condition) in **CZF**, the constructive and predicative Zermelo–Fraenkel set theory founded by Aczel [1].

However, the notion of a uniform space with pseudometrics has many applications in other areas of constructive mathematics such as locally convex spaces [7, 5.4]; see also [11].

In this paper, we give a predicative completion of a uniform space with pseudometrics by means of the notion of a net or Moore–Smith sequence (see [13, Chapter 2]) in the elementary constructive set theory **ECST** with Exponentiation Axiom **Exp**. Since the notion of a net is a generalization of the notion of a sequence, the completion is a generalization of, and parallel to the usual completion of a metric space; see [5, 4.3] and [6, 4.3].

The elementary constructive set theory **ECST** was introduced by Aczel and Rathjen and is a subsystem of **CZF**; see their book draft [3] written by extending their research report [2]. The set theory **ECST** uses intuitionistic logic and has the set theoretic axioms: Extensionality, Pairing, Union, Replacement and

Bounded Separation:

$$\forall a \exists b \forall x (x \in b \leftrightarrow x \in a \wedge \varphi(x))$$

for every *bounded* formula $\varphi(x)$; here a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$;

Strong Infinity:

$$\begin{aligned} &\exists a[0 \in a \wedge \forall x(x \in a \to x+1 \in a) \\ &\qquad\qquad \wedge \forall y(0 \in y \wedge \forall x(x \in y \to x+1 \in y) \to a \subseteq y)], \end{aligned}$$

where $x+1$ is $x \cup \{x\}$, and 0 is the empty set $\emptyset$.

In **ECST**, we are able to perform basic set constructions in mathematical practice such as finite cartesian products and infinite disjoint unions. A relation $r \subseteq a \times b$ between sets a and b is *total* (or is a *multivalued function*) if for every $x \in a$ there exists $y \in b$ such that $(x, y) \in r$. The class of total relations between a and b is denoted by $\mathrm{mv}(a, b)$, or more formally

$$r \in \mathrm{mv}(a, b) \Leftrightarrow r \subseteq a \times b \wedge \forall x \in a \exists y \in b((x, y) \in r).$$

A *function* from a to b is a total relation $f \subseteq a \times b$ such that for every $x \in a$ there is exactly one $y \in b$ with $(x, y) \in f$. The class of functions from a to b is denoted by b^a, or more formally

$$f \in b^a \Leftrightarrow f \in \mathrm{mv}(a, b) \wedge \forall x \in a \forall y, z \in b((x, y) \in f \wedge (x, z) \in f \to y = z).$$

The set theory **CZF** is obtained from **ECST** by replacing Replacement by Strong Collection and adding Subset Collection and Set Induction; see [2, 3] for details. In **ECST**, Subset Collection implies

Fullness: $\forall a \forall b \exists c(c \subseteq \mathrm{mv}(a, b) \wedge \forall r \in \mathrm{mv}(a, b) \exists s \in c(s \subseteq r))$,

and Fullness and Strong Collection imply Subset Collection. The notable consequence of Fullness is that b^a forms a set, that is, Exponentiation Axiom:

Exp: $\forall a \forall b \exists c \forall f(f \in c \leftrightarrow f \in b^a)$.

Note that the construction of a completion in [4] made use of Fullness. Since the completion in this paper is a subset of the set of functions, we do not make use of Fullness, but **Exp** which is a strictly weaker axiom than Fullness in **ECST**. Note that we can construct infinite cartesian products, the natural numbers (and their operations) and the rationals (and their operations) in **ECST** + **Exp**. Although the Dedekind reals was constructed in [2] with Fullness (see also [9]), we adapt the Cauchy reals within **ECST** + **Exp**, which is constructed as the completion of the rationals in this paper.

In Section 2, we give our construction of a completion of a uniform space by means of the notion of a net. Since the completion is similar to the usual completion of a metric space, if the reader is familiar with it, then it is easy to follow. In Section 3, we introduce the notion of a uniformly (respectively, locally uniformly) continuous mapping, and show a universal property, which a completion must have, for uniformly (respectively, locally uniformly) continuous mappings. In Section 4, we give the product of uniform spaces following Bishop, and show that the product of complete uniform spaces is complete. Finally, we show that constructions of the completion and the product commute and, as a corollary, we give a stronger version of the universal property which is crucial in application [11].

There are other constructive treatments of a uniform space and its completion; for example, see [10] for uniform space in formal topology, and [12] for localic completion of a uniform space.

2 A completion of a uniform space

Definition 1. A *pseudometric* d on a set X is a mapping $d : X \times X \to \mathbf{R}$ such that

1. $d(x, x) = 0$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$

for each $x, y, z \in X$.

Definition 2. A (metrically) *uniform space* is a pair (X, D) of a set X and a family $D = \{d_i \mid i \in I\}$ of pseudometrics indexed by an inhabited set I such that

$$\forall i \in I(d_i(x, y) = 0) \Rightarrow x = y$$

for each $x, y \in X$. If I is a singleton, then (X, D) is called a *metric space*.

Definition 3. For a set S, we write S^* for the set of finite sequences of S with the following notations:

1. $|\sigma|$ denotes the *length* of $\sigma \in S^*$;
2. ϵ denotes the *empty sequence* with $|\epsilon| = 0$;
3. $\sigma(l)$ denotes the *l-th element* of $\sigma \in S^*$, where $l < |\sigma|$;
4. $s \in \sigma$ denotes that $s = \sigma(l)$ for some $l < |\sigma|$;
5. $\sigma * \tau$ denotes the *concatenation* of $\sigma \in S^*$ and $\tau \in S^*$;
6. s^n denotes the the *constant sequence* $\langle s, \ldots, s\rangle$ of the length n.

We define a binary relation $\preceq_S$ on S^* by

$$\sigma \preceq_S \tau \Leftrightarrow |\sigma| \leq |\tau| \wedge \forall s \in S(s \in \sigma \to s \in \tau)$$

for each $\sigma, \tau \in S^*$. If S is inhabited by $s_0 \in S$, then for each n, we write σ^{+n} for the sequence $\sigma * s_0^n$; note that $\sigma^{+n} \preceq_S \tau^{+n}$ whenever $\sigma \preceq_S \tau$.

Lemma 4. *Let S be a set. Then $(S^*, \preceq_S)$ is a directed preordered set.*

Proof. It is obvious that $\preceq_S$ is a preorder on S^*, and for each $\sigma, \tau \in S^*$, we have $\sigma \preceq_S \sigma * \tau$ and $\tau \preceq_S \sigma * \tau$. $\square$

Remark 5. If S is a singleton $\{s\}$, then $(S^*, \preceq_S)$ is order isomorphic to $(\mathbf{N}, \leq)$ by the mapping $\sigma \mapsto |\sigma|$ and its inverse $n \mapsto s^n$.

Definition 6. Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$, and for each $\sigma \in I^*$, let d_σ be a pseudometric on X given by

$$d_\sigma(x, y) = \max\{d_i(x, y) \mid i \in \sigma\}$$

for each $x, y \in X$; if $\sigma = \epsilon$, then let $d_\sigma(x, y) = 0$. Let $(\Lambda, \preccurlyeq)$ be a directed preordered set. Then a map $\lambda \mapsto x_\lambda$ of Λ into X is called a *net* (or *Moore-Smith sequence*) on $(\Lambda, \preccurlyeq)$ in X, and is denoted by $(x_\lambda)_{\lambda \in \Lambda}$, or simply (x_λ). A net (x_λ) *converges* to an element x of X with a *modulus* $\beta : I^* \to \Lambda$ if

$$\beta(\sigma) \preccurlyeq \lambda \Rightarrow d_\sigma(x_\lambda, x) \leq 2^{-|\sigma|}$$

for each $\sigma \in I^*$ and $\lambda \in \Lambda$. We then write $x_\lambda \to x$, and x is called a *limit* of (x_λ). A net (x_λ) is a *Cauchy net* with a *modulus* $\alpha : I^* \to \Lambda$ if

$$\alpha(\sigma) \preccurlyeq \mu, \nu \Rightarrow d_\sigma(x_\mu, x_\nu) \leq 2^{-|\sigma|}$$

for each $\sigma \in I^*$ and $\mu, \nu \in \Lambda$. A uniform space (X, D) is *complete* if every Cauchy net converges.

Lemma 7. *Let (X, D) be a uniform space. If a net (x_λ) in X converges to elements x and y of X, then $x = y$.*

Proof. Let $D = \{d_i \mid i \in I\}$, and suppose that a net (x_λ) on $(\Lambda, \preccurlyeq)$ converges to $x \in X$ with a modulus $\alpha : I^* \to \Lambda$ and to $y \in X$ with a modulus $\beta : I^* \to \Lambda$. Then for each $i \in I$ and n, there exists $\lambda \in \Lambda$ such that $\alpha(i^n) \preccurlyeq \lambda$ and $\beta(i^n) \preccurlyeq \lambda$, and hence

$$d_i(x, y) = d_{i^n}(x, y) \leq d_{i^n}(x, x_\lambda) + d_{i^n}(x_\lambda, y) \leq 2^{-n} + 2^{-n}.$$

Therefore, letting $n \to \infty$, we have $d_i(x, y) = 0$ for each $i \in I$, and so $x = y$. $\square$

Definition 8. Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$. A *regular* net in X is a Cauchy net on $(I^*, \preceq_I)$ with the modulus id_{I^*}. We write $\tilde{X}$ for the set of all regular nets in X.

Lemma 9. *Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$, and let $i \in I$. Then the limit*

$$\tilde{d}_i(\boldsymbol{x}, \boldsymbol{y}) = \lim_{n \to \infty} d_i(x_{i^n}, y_{i^n})$$

exists for each $\boldsymbol{x} = (x_\rho), \boldsymbol{y} = (y_\rho) \in \tilde{X}$, and $\tilde{d}_i$ is a pseudometric on $\tilde{X}$.

Proof. Let $\boldsymbol{x} = (x_\rho), \boldsymbol{y} = (y_\rho) \in \tilde{X}$. Then we show that $(d_i(x_{i^n}, y_{i^n}))_n$ is a Cauchy sequence in $\mathbf{R}$ with a modulus $n \mapsto n+1$. In fact, for each $m, m' \geq n+1$, since

$$d_{i^{n+1}}(x_{i^m}, x_{i^{m'}}) \leq 2^{-(n+1)} \quad \text{and} \quad d_{i^{n+1}}(y_{i^m}, y_{i^{m'}}) \leq 2^{-(n+1)}$$

we have

$$\begin{aligned} d_i(x_{i^m}, y_{i^m}) - d_i(x_{i^{m'}}, y_{i^{m'}}) &\leq d_i(x_{i^m}, x_{i^{m'}}) + d_i(x_{i^{m'}}, y_{i^{m'}}) \\ &\quad + d_i(y_{i^{m'}}, y_{i^m}) - d_i(x_{i^{m'}}, y_{i^{m'}}) \\ &= d_{i^{n+1}}(x_{i^m}, x_{i^{m'}}) + d_{i^{n+1}}(y_{i^{m'}}, y_{i^m}) \\ &\leq 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}. \end{aligned}$$

It is obvious that $\tilde{d}_i(\boldsymbol{x},\boldsymbol{x}) = 0$ and $\tilde{d}_i(\boldsymbol{x},\boldsymbol{y}) = \tilde{d}_i(\boldsymbol{y},\boldsymbol{x})$ for each $\boldsymbol{x},\boldsymbol{y} \in \tilde{X}$. For the triangle inequality, we have

$$\begin{aligned}\tilde{d}_i(\boldsymbol{x},\boldsymbol{y}) &= \lim_{n\to\infty} d_i(x_{i^n}, y_{i^n}) \leq \lim_{n\to\infty} d_i(x_{i^n}, z_{i^n}) + \lim_{n\to\infty} d_i(z_{i^n}, y_{i^n}) \\ &= \tilde{d}_i(\boldsymbol{x},\boldsymbol{z}) + \tilde{d}_i(\boldsymbol{z},\boldsymbol{y})\end{aligned}$$

for each $\boldsymbol{x} = (x_\rho), \boldsymbol{y} = (y_\rho), \boldsymbol{z} = (z_\rho) \in \tilde{X}$. □

Lemma 10. *Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$. Define the* inclusion map ι_X *of X into $\tilde{X}$ by*

$$(\iota_X(x))(\sigma) = x$$

for each $x \in X$ and $\sigma \in I^*$. Then

$$d_\sigma(x,y) = \tilde{d}_\sigma(\iota_X(x), \iota_X(y))$$

for each $\sigma \in I^*$ and $x, y \in X$.

Proof. Straightforward. □

Lemma 11. *Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$, and let $\boldsymbol{x} = (x_\rho) \in \tilde{X}$. Then*

$$\tilde{d}_\sigma(\boldsymbol{x}, \iota_X(x_\tau)) \leq 2^{-|\tau|}$$

for each $\sigma, \tau \in I^*$ with $\sigma \preceq_I \tau$.

Proof. Consider $\sigma, \tau \in I^*$ with $\sigma \preceq_I \tau$. Then for each $i \in \sigma$ and n, since $\boldsymbol{x}$ is regular, we have $d_{i^n}(x_{i^n}, x_{i^n*\tau}) \leq 2^{-n}$ and $d_\tau(x_{i^n*\tau}, x_\tau) \leq 2^{-|\tau|}$, and hence

$$\begin{aligned}d_i(x_{i^n}, x_\tau) &\leq d_i(x_{i^n}, x_{i^n*\tau}) + d_i(x_{i^n*\tau}, x_\tau) \\ &\leq d_{i^n}(x_{i^n}, x_{i^n*\tau}) + d_\tau(x_{i^n*\tau}, x_\tau) \leq 2^{-n} + 2^{-|\tau|}.\end{aligned}$$

Therefore, letting $n \to \infty$, we have $\tilde{d}_i(\boldsymbol{x}, \iota_X(x_\tau)) \leq 2^{-|\tau|}$ for each $i \in \sigma$, and so $\tilde{d}_\sigma(\boldsymbol{x}, \iota_X(x_\tau)) \leq 2^{-|\tau|}$. □

Definition 12. The *completion* of a uniform space (X, D) with $D = \{d_i \mid i \in I\}$ is the uniform space $(\tilde{X}, \tilde{D})$ with $\tilde{D} = \{\tilde{d}_i \mid i \in I\}$ and with the equality $=_{\tilde{X}}$ given by

$$\boldsymbol{x} =_{\tilde{X}} \boldsymbol{y} \Leftrightarrow \forall i \in I(\tilde{d}_i(\boldsymbol{x},\boldsymbol{y}) = 0)$$

for each $\boldsymbol{x},\boldsymbol{y} \in \tilde{X}$.

Theorem 13. *The completion $(\tilde{X}, \tilde{D})$ of a uniform space (X, D) is complete.*

Proof. Let $D = \{d_i \mid i \in I\}$, and suppose that $(\boldsymbol{x}_\lambda)_{\lambda\in\Lambda} = ((x_{\lambda,\rho})_{\rho\in I^*})_{\lambda\in\Lambda}$ is a Cauchy net on $(\Lambda \preccurlyeq)$ in $\tilde{X}$ with a modulus $\alpha : I^* \to \Lambda$. For each $\sigma \in I^*$, define a net $\boldsymbol{y} = (y_\rho)$ on $(I^*, \preceq_I)$ in X by

$$y_\rho = x_{\alpha(\rho^{+2}),\rho^{+2}}$$

for each $\rho \in I^*$. We show that $\boldsymbol{y}$ is a regular net. To this end, consider $\sigma, \tau, \upsilon \in I^*$ with $\sigma \preceq_I \tau, \upsilon$. Then there exists $\lambda \in \Lambda$ such that $\alpha(\tau^{+2}) \preccurlyeq \lambda$ and $\alpha(\upsilon^{+2}) \preccurlyeq \lambda$, and, since $\sigma \preceq_I \tau^{+2}, \upsilon^{+2}$, we have

$$\begin{aligned}
d_\sigma(y_\tau, y_\upsilon) &= \tilde{d}_\sigma(\iota_X(x_{\alpha(\tau^{+2}),\tau^{+2}}), \iota_X(x_{\alpha(\upsilon^{+2}),\upsilon^{+2}})) \\
&\leq \tilde{d}_\sigma(\iota_X(x_{\alpha(\tau^{+2}),\tau^{+2}}), \boldsymbol{x}_{\alpha(\tau^{+2})}) + \tilde{d}_\sigma(\boldsymbol{x}_{\alpha(\tau^{+2})}, \boldsymbol{x}_\lambda) + \tilde{d}_\sigma(\boldsymbol{x}_\lambda, \boldsymbol{x}_{\alpha(\upsilon^{+2})}) \\
&\quad + \tilde{d}_\sigma(\boldsymbol{x}_{\alpha(\upsilon^{+2})}, \iota_X(x_{\alpha(\upsilon^{+2}),\upsilon^{+2}})) \\
&\leq 2^{-|\tau^{+2}|} + \tilde{d}_{\tau^{+2}}(\boldsymbol{x}_{\alpha(\tau^{+2})}, \boldsymbol{x}_\lambda) + \tilde{d}_{\upsilon^{+2}}(\boldsymbol{x}_\lambda, \boldsymbol{x}_{\alpha(\upsilon^{+2})}) + 2^{-|\upsilon^{+2}|} \\
&\leq 2^{-|\tau^{+2}|} + 2^{-|\tau^{+2}|} + 2^{-|\upsilon^{+2}|} + 2^{-|\upsilon^{+2}|} \\
&\leq 2^{-(|\tau|+1)} + 2^{-(|\upsilon|+1)} \leq 2^{-|\sigma|},
\end{aligned}$$

by Lemma 11. Therefore $\boldsymbol{y}$ is regular. Define $\beta : I^* \to \Lambda$ by

$$\beta(\sigma) = \alpha(\sigma^{+3})$$

for each $\sigma \in I^*$. If $\beta(\sigma) \preccurlyeq \lambda$, then

$$\begin{aligned}
\tilde{d}_\sigma(\boldsymbol{x}_\lambda, \boldsymbol{y}) &\leq \tilde{d}_\sigma(\boldsymbol{x}_\lambda, \boldsymbol{x}_{\alpha(\sigma^{+3})}) + \tilde{d}_\sigma(\boldsymbol{x}_{\alpha(\sigma^{+3})}, \iota_X(x_{\alpha(\sigma^{+3}),\sigma^{+3}})) + \tilde{d}_\sigma(\iota_X(y_{\sigma^{+1}}), \boldsymbol{y}) \\
&\leq 2^{-|\sigma^{+3}|} + 2^{-|\sigma^{+3}|} + 2^{-|\sigma^{+1}|} < 2^{-|\sigma|},
\end{aligned}$$

by Lemma 11. Therefore $(\boldsymbol{x}_\lambda)$ converges to $\boldsymbol{y}$ with the modulus β. □

3 A universal property

Definition 14. Let (X, D) and (Y, D') be uniform spaces with $D = \{d_i \mid i \in I\}$ and $D' = \{d'_j \mid j \in J\}$. Then a mapping $f : X \to Y$ is *uniformly continuous* with a monotone *modulus* $\alpha : J^* \to I^*$ (that is, $\sigma' \preceq_J \tau'$ implies $\alpha(\sigma') \preceq_I \alpha(\tau')$ for each $\sigma', \tau' \in J^*$) if

$$d_{\alpha(\sigma')}(x, y) \leq 2^{-|\alpha(\sigma')|} \Rightarrow d'_{\sigma'}(f(x), f(y)) \leq 2^{-|\sigma'|}$$

for each $\sigma' \in J^*$ and $x, y \in X$. A uniformly continuous mapping $f : X \to Y$ is a *uniform isomorphism* if it has a uniformly continuous inverse, and (X, D) and (Y, D') are *uniformly equivalent* if there exists a uniform isomorphism between X and Y.

A mapping $f : X \to Y$ is *locally uniformly continuous* if for each $\boldsymbol{x} \in \tilde{X}$ there exists a monotone *modulus* $\alpha : J^* \to I^*$ such that

$$y, z \in U_{\alpha(\sigma')}(\boldsymbol{x}) \Rightarrow d'_{\sigma'}(f(y), f(z)) \leq 2^{-|\sigma'|}$$

for each $\sigma' \in J^*$ and $y, z \in X$, where

$$U_\sigma(\boldsymbol{x}) = \{z \in X \mid \tilde{d}_\sigma(\boldsymbol{x}, \iota_X(z)) \leq 2^{-|\sigma|}\}$$

for each $\sigma \in I^*$.

Remark 15. For each $\rho' \in J^*$, the set $\{\sigma' \in J^* \mid \sigma' \preceq_J \rho'\}$ is finitely enumerable; for $\{\sigma' \in J^* \mid \sigma' \preceq_J \rho'\} = \{\rho' \circ \pi \mid \pi \in |\rho'|^n, n \leq |\rho'|\}$, where $|\rho'|^n$ is the finite set of functions from $\{0, \ldots, n-1\}$ into $\{0, \ldots, |\rho'| - 1\}$. Therefore we may replace any modulus $\alpha : J^* \to I^*$ of uniform continuity by a *monotone* modulus $\alpha' : J^* \to I^*$ given by

$$\alpha'(\rho') = \text{the concatenation of } \{\alpha(\sigma') \mid \sigma' \preceq_J \rho'\}$$

for each $\rho' \in J^*$.

Remark 16. We introduced here the notion of a locally uniformly continuous mapping, since it is convenient to deal with it in constructing the completions of some important spaces. For example, the scalar multiplication $(a, x) \mapsto ax$ of a normed space is not uniformly continuous, but locally uniformly continuous.

Remark 17. A uniform space (X, D) with a countable set $D = \{d_n \mid n \in \mathbf{N}\}$ of pseudometrics is uniformly equivalent to a metric space (X, d) with a metric d defined by

$$d(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}$$

for each $x, y \in X$. In fact, it is straightforward to show that $\mathrm{id}_X : (X, D) \to (X, d)$ and $\mathrm{id}_X : (X, d) \to (X, D)$ are uniformly continuous; see [5, Problem 4.17] and [6, Problem 4.22].

Lemma 18. *Let (X, D) and (Y, D') be uniform spaces. Then every uniformly continuous mapping $f : X \to Y$ is locally uniformly continuous.*

Proof. Let $D = \{d_i \mid i \in I\}$ and $D' = \{d'_j \mid j \in J\}$, and let $f : X \to Y$ be a uniformly continuous mapping with a modulus $\alpha : J^* \to I^*$. Consider $\boldsymbol{x} \in \tilde{X}$. For each $\sigma' \in J^*$ and $x, y \in X$, if $x, y \in U_{\alpha(\sigma')^{+1}}(\boldsymbol{x})$, then, since

$$\begin{aligned} d_{\alpha(\sigma')}(x, y) &= \tilde{d}_{\alpha(\sigma')}(\iota_X(x), \boldsymbol{x}) + \tilde{d}_{\alpha(\sigma')}(\boldsymbol{x}, \iota_X(y)) \\ &\leq \tilde{d}_{\alpha(\sigma')^{+1}}(\iota_X(x), \boldsymbol{x}) + \tilde{d}_{\alpha(\sigma')^{+1}}(\boldsymbol{x}, \iota_X(y)) \\ &\leq 2^{-|\alpha(\sigma')^{+1}|} + 2^{-|\alpha(\sigma')^{+1}|} = 2^{-|\alpha(\sigma')|}, \end{aligned}$$

we have $d'_{\sigma'}(f(x), f(y)) \leq 2^{-|\sigma'|}$. Therefore f is locally uniformly continuous at $\boldsymbol{x}$ with a modulus $\sigma' \mapsto \alpha(\sigma')^{+1}$. $\square$

Lemma 19. *Let (X, D) and (Y, D') be uniform spaces with $D = \{d_i \mid i \in I\}$ and $D' = \{d'_j \mid j \in J\}$. Then a locally uniformly continuous mapping $f : (X, D) \to (Y, D')$ is pointwise continuous, in the sense that for each $x \in X$ there exists $\alpha : J^* \to I^*$ such that*

$$d_{\alpha(\sigma')}(x, y) \leq 2^{-|\alpha(\sigma')|} \Rightarrow d'_{\sigma'}(f(x), f(y)) \leq 2^{-|\sigma'|}$$

for each $y \in X$ and $\sigma' \in J^*$. Especially,

$$x_\lambda \to x \Rightarrow f(x_\lambda) \to f(x)$$

for each net (x_λ) in X.

Proof. Let $x \in X$, and let $\alpha : J^* \to I^*$ be a modulus of local uniform continuity at $\iota_X(x)$. For each $y \in X$ and $\sigma' \in J^*$, if $d_{\alpha(\sigma')}(x, y) \leq 2^{-|\alpha(\sigma')|}$, then $x, y \in U_{\alpha(\sigma')}(\iota_X(x))$, and hence

$$d'_{\sigma'}(f(x), f(y)) \leq 2^{-|\sigma'|}.$$

Suppose that a net (x_λ) on $(\Lambda, \preccurlyeq)$ in X converges to a limit $x \in X$ with a modulus $\beta : I^* \to \Lambda$. Then for each $\sigma' \in J^*$ and $\lambda \in \Lambda$ with $\beta(\alpha(\sigma')) \preccurlyeq \lambda$, since $d_{\alpha(\sigma')}(x_\lambda, x) \leq 2^{-|\alpha(\sigma')|}$, we have $x_\lambda, x \in U_{\alpha(\sigma')}(\iota_X(x))$, and hence $d'_{\sigma'}(f(x_\lambda), f(x)) \leq 2^{-|\sigma'|}$. Therefore the net $(f(x_\lambda))_{\lambda\in\Lambda}$ converges to $f(x)$ with a modulus $\beta \circ \alpha$. □

Lemma 20. *Let (X, D), (Y, D') and (Z, D'') be uniform spaces. Then the composition $g \circ f$ of uniformly (respectively, locally uniformly) continuous mappings $f : (X, D) \to (Y, D')$ and $g : (Y, D') \to (Z, D'')$ is uniformly (respectively, locally uniformly) continuous.*

Proof. Let $D = \{d_i \mid i \in I\}$, $D' = \{d'_j \mid j \in J\}$ and $D'' = \{d''_k \mid k \in K\}$. Since it is straightforward for uniformly continuous mappings, we only show for locally uniformly continuous mappings.

Consider $\boldsymbol{x} = (x_\rho)_{\rho\in I^*} \in \tilde{X}$, and let $\alpha : J^* \to I^*$ be a modulus of local uniform continuity for f at $\boldsymbol{x}$. Then for each $\sigma' \in J^*$ and $\tau \in I^*$ with $\alpha(\sigma') \preceq_I \tau$, since $\tilde{d}_{\alpha(\sigma')}(\boldsymbol{x}, \iota_X(x_\tau)) \leq 2^{-|\alpha(\sigma')|}$, by Lemma 11, we have $x_\tau \in U_{\alpha(\sigma')}(\boldsymbol{x})$. Let $\boldsymbol{y} = (y_{\rho'})_{\rho'\in J^*}$ be a net in Y given by

$$y_{\rho'} = f(x_{\alpha(\rho')})$$

for each $\rho' \in J^*$. Then for each $\sigma', \tau', \upsilon' \in J^*$ with $\sigma' \preceq_J \tau', \upsilon'$, since $\alpha(\sigma') \preceq_I \alpha(\tau'), \alpha(\upsilon')$, we have $x_{\alpha(\tau')}, x_{\alpha(\upsilon')} \in U_{\alpha(\sigma')}(\boldsymbol{x})$, and hence

$$d'_{\sigma'}(y_{\tau'}, y_{\upsilon'}) = d'_{\sigma'}(f(x_{\alpha(\tau')}), f(x_{\alpha(\upsilon')})) \leq 2^{-|\sigma'|}.$$

Therefore $\boldsymbol{y}$ is a regular net in Y. Let $\beta : K^* \to J^*$ be a modulus of local uniform continuity for g at $\boldsymbol{y}$. Consider $\sigma'' \in K^*$ and $x \in X$ with $x \in U_{\alpha(\beta(\sigma''))}(\boldsymbol{x})$. Then for each n, since $x_{\alpha(\beta(\sigma'')^{+n})} \in U_{\alpha(\beta(\sigma''))}(\boldsymbol{x})$, we have $d'_{\beta(\sigma'')}(f(x_{\alpha(\beta(\sigma'')^{+n})}), f(x)) \leq 2^{-|\beta(\sigma'')|}$, and hence

$$\begin{aligned}\tilde{d}'_{\beta(\sigma'')}(\boldsymbol{y}, f(x)) &\leq \tilde{d}'_{\beta(\sigma'')}(\boldsymbol{y}, y_{\beta(\sigma'')^{+n}}) + \tilde{d}'_{\beta(\sigma'')}(f(x_{\alpha(\beta(\sigma'')^{+n})}), f(x)) \\ &\leq 2^{-|\beta(\sigma'')^{+n}|} + 2^{-|\beta(\sigma'')|} \leq 2^{-n} + 2^{-|\beta(\sigma'')|},\end{aligned}$$

by Lemma 11. Therefore, letting $n \to \infty$, we have $\tilde{d}'_{\beta(\sigma'')}(\boldsymbol{y}, f(x)) \leq 2^{-|\beta(\sigma'')|}$, and so $f(x) \in U'_{\beta(\sigma'')}(\boldsymbol{y})$, where $U'_{\sigma'}(\boldsymbol{y}) = \{z \in Y \mid \tilde{d}'_{\sigma'}(\boldsymbol{y}, \iota_Y(z)) \leq 2^{-|\sigma'|}\}$ for each $\sigma' \in J^*$. Thus for each $\sigma'' \in K^*$ and $x, y \in X$, if $x, y \in U_{\alpha(\beta(\sigma''))}(\boldsymbol{x})$, then $f(x), f(y) \in U'_{\beta(\sigma'')}(\boldsymbol{y})$, and hence $d''_{\sigma''}(g(f(x)), g(f(y))) \leq 2^{-|\sigma''|}$; and so $g \circ f$ is locally uniformly continuous at $\boldsymbol{x}$ with a modulus $\alpha \circ \beta : K^* \to I^*$. □

Theorem 21. *Let (X, D) be a uniform spaces, and let (Y, D') be a complete uniform space. Then for each uniformly (respectively, locally uniformly) continuous mapping $f : (X, D) \to (Y, D')$, there exists a unique uniformly (respectively, locally uniformly) continuous mapping $\tilde{f} : (\tilde{X}, \tilde{D}) \to (Y, D')$ which makes the following diagram commute.*

$$\begin{tikzcd} (\tilde{X}, \tilde{D}) \arrow[r, "\tilde{f}"] & (Y, D') \\ (X, D) \arrow[u, "\iota_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. Let $D = \{d_i \mid i \in I\}$ and $D' = \{d'_j \mid j \in J\}$. Since it is similar and easier for uniformly continuous f, we only show for locally uniformly continuous f.

We first construct $\tilde{f}(\boldsymbol{x}) \in Y$ for each $\boldsymbol{x} \in \tilde{X}$. To this end, consider $\boldsymbol{x} \in \tilde{X}$ with $\boldsymbol{x} = (x_\rho)_{\rho \in I^*}$, and note that for each $\sigma, \tau \in I^*$, if $\sigma \preceq_I \tau$, then, since $\tilde{d}_\sigma(\boldsymbol{x}, \iota_X(x_\tau)) \leq 2^{-|\tau|} \leq 2^{-|\sigma|}$, by Lemma 11, we have $x_\tau \in U_\sigma(\boldsymbol{x})$. Let $\alpha : J^* \to I^*$ be a modulus of local uniform continuity for f at $\boldsymbol{x}$. Then for each $\sigma' \in J^*$ and $\tau, \upsilon \in I^*$ with $\alpha(\sigma') \preceq_I \tau, \upsilon$, since $x_\tau, x_\upsilon \in U_{\alpha(\sigma')}(\boldsymbol{x})$, we have $d'_{\sigma'}(f(x_\tau), f(x_\upsilon)) \leq 2^{-|\sigma'|}$. Therefore $(f(x_\rho))$ is a Cauchy net on $(I^*, \preceq_I)$ in Y with modulus α, and so converges to the unique limit $\tilde{f}(\boldsymbol{x})$.

To show that $\tilde{f} : \tilde{X} \to Y$ is locally uniformly continuous, consider $(\boldsymbol{x}_\rho)_{\rho \in I^*} \in \tilde{\tilde{X}}$. We now construct a modulus of local uniform continuity for $\tilde{f}$ at $(\boldsymbol{x}_\rho)$. Since $(\boldsymbol{x}_\rho)$ is Cauchy and $\tilde{X}$ is complete, $(\boldsymbol{x}_\rho)$ converges to $\boldsymbol{x} \in \tilde{X}$ with a modulus $\beta : I^* \to I^*$. Note that for each $\boldsymbol{y} \in \tilde{X}$ and $i \in I$, since

$$\begin{aligned}
\tilde{d}_i(\boldsymbol{x}_{i^n}, \boldsymbol{y}) &\leq \tilde{d}_{i^n}(\boldsymbol{x}_{i^n}, \boldsymbol{x}_{i^n * \beta(i^n)}) + \tilde{d}_{i^n}(\boldsymbol{x}_{i^n * \beta(i^n)}, \boldsymbol{x}) + \tilde{d}_i(\boldsymbol{x}, \boldsymbol{y}) \\
&\leq 2^{-n} + 2^{-n} + \tilde{d}_{i^n}(\boldsymbol{x}, \boldsymbol{y})
\end{aligned}$$

and similarly $\tilde{d}_i(\boldsymbol{x}, \boldsymbol{y}) \leq 2^{-n} + 2^{-n} + \tilde{d}_i(\boldsymbol{x}_{i^n}, \boldsymbol{y})$ for each n, we have

$$\tilde{\tilde{d}}_i((\boldsymbol{x}_\rho), \iota_{\tilde{X}}(\boldsymbol{y})) = \lim_{n \to \infty} \tilde{d}_i(\boldsymbol{x}_{i^n}, \boldsymbol{y}) = \tilde{d}_i(\boldsymbol{x}, \boldsymbol{y}).$$

Let $\alpha : J^* \to I^*$ be a modulus of local uniform continuity for f at $\boldsymbol{x}$, and define $\tilde{\alpha} : J^* \to I^*$ by

$$\tilde{\alpha}(\sigma') = \alpha(\sigma')^{+1}$$

for each $\sigma' \in J^*$. Then for each $\boldsymbol{y} = (y_\rho) \in \tilde{U}_{\sigma^{+1}}((\boldsymbol{x}_\rho))$, where

$$\tilde{U}_\sigma((\boldsymbol{x}_\rho)) = \{\boldsymbol{y} \in \tilde{X} \mid \tilde{\tilde{d}}_\sigma((\boldsymbol{x}_\rho), \boldsymbol{y}) \leq 2^{-|\sigma|}\},$$

and each $\sigma' \in J^*$ and $\tau \in I^*$ with $\tilde{\alpha}(\sigma') \preceq_I \tau$, since

$$\begin{aligned}
\tilde{d}_{\alpha(\sigma')}(\boldsymbol{x}, \iota(y_\tau)) \leq \tilde{d}_{\tilde{\alpha}(\sigma')}(\boldsymbol{x}, \iota(y_\tau)) &\leq \tilde{d}_{\tilde{\alpha}(\sigma')}(\boldsymbol{x}, \boldsymbol{y}) + \tilde{d}_{\tilde{\alpha}(\sigma')}(\boldsymbol{y}, \iota(y_\tau)) \\
&\leq 2^{-|\alpha(\sigma')^{+1}|} + 2^{-|\alpha(\sigma')^{+1}|} = 2^{-|\alpha(\sigma')|},
\end{aligned}$$

we have $y_\tau \in U_{\alpha(\sigma')}(\boldsymbol{x})$. Consider $\boldsymbol{y} = (y_\rho), \boldsymbol{z} = (z_\rho) \in \tilde{U}_{\tilde{\alpha}(\sigma')}((\boldsymbol{x}_\rho))$ and $\sigma' \in J^*$. Given an n, since $(f(y_\rho))$ and $(f(z_\rho))$ converge to $\tilde{f}(\boldsymbol{y})$ and $\tilde{f}(\boldsymbol{z})$, respectively, there exist $\check{\tau}, \check{\upsilon} \in I^*$ such that

$$d'_{\sigma'^{+n}}(f(y_\tau), \tilde{f}(\boldsymbol{y})) \leq 2^{-|\sigma'^{+n}|} \leq 2^{-n}, \quad d'_{\sigma'^{+n}}(f(z_\upsilon), \tilde{f}(\boldsymbol{z})) \leq 2^{-|\sigma'^{+n}|} \leq 2^{-n}$$

for each $\tau, \upsilon \in I^*$ with $\check{\tau} \preceq_I \tau$ and $\check{\upsilon} \preceq_I \upsilon$. Let $\tau = \check{\tau} * \tilde{\alpha}(\sigma')$ and $\upsilon = \check{\upsilon} * \tilde{\alpha}(\sigma')$. Then, since $y_\tau, z_\upsilon \in U_{\alpha(\sigma')}(\boldsymbol{x})$, we have $d'_{\sigma'}(f(y_\tau), f(z_\upsilon)) \leq 2^{-|\sigma'|}$, and hence

$$\begin{aligned}
d'_{\sigma'}(\tilde{f}(\boldsymbol{y}), \tilde{f}(\boldsymbol{z})) &\leq d'_{\sigma'}(\tilde{f}(\boldsymbol{y}), f(y_\tau) + d'_{\sigma'}(f(y_\tau), f(z_\upsilon)) + d'_{\sigma'}(f(z_\upsilon), \tilde{f}(\boldsymbol{z})) \\
&\leq d'_{\sigma'^{+n}}(\tilde{f}(\boldsymbol{y}), f(y_\tau) + d'_{\sigma'}(f(y_\tau), f(z_\upsilon)) + d'_{\sigma'^{+n}}(f(z_\upsilon), \tilde{f}(\boldsymbol{z})) \\
&\leq 2^{-n} + 2^{-|\sigma'|} + 2^{-n}.
\end{aligned}$$

Therefore, letting $n \to \infty$, we have $d'_{\sigma'}(\tilde{f}(\boldsymbol{y}), \tilde{f}(\boldsymbol{z})) \leq 2^{-|\sigma'|}$. Thus f is locally uniformly continuous at $(\boldsymbol{x}_\rho)$ with a modulus $\tilde{\alpha}$.

As to the uniqueness, let $g : \tilde{X} \to Y$ be any other locally uniformly continuous extension of f. Consider $\boldsymbol{x} \in \tilde{X}$, and let $\alpha : J^* \to I^*$ and $\beta : J^* \to I^*$ be moduli of uniform continuity for $\tilde{f}$ and g, respectively, at $\iota_{\tilde{X}}(\boldsymbol{x})$. Given $j \in J$ and n, let $\upsilon = \alpha(j^n) * \beta(j^n)$. Then, since

$$\tilde{\tilde{d}}_{\alpha(j^n)}(\iota_{\tilde{X}}(\boldsymbol{x}), \iota_{\tilde{X}}(\iota_X(x_\upsilon))) = \tilde{d}_{\alpha(j^n)}(\boldsymbol{x}, \iota_X(x_\upsilon)) \leq 2^{-|\upsilon|} \leq 2^{-|\alpha(j^n)|},$$

by Lemma 11, we have

$$d'_j(\tilde{f}(\boldsymbol{x}), f(x_\upsilon)) = d'_j(\tilde{f}(\boldsymbol{x}), \tilde{f}(\iota_X(x_\upsilon))) = d'_{j^n}(\tilde{f}(\boldsymbol{x}), \tilde{f}(\iota_X(x_\upsilon))) \leq 2^{-n}.$$

Similarly, we have $d'_j(g(\boldsymbol{x}), f(x_\upsilon)) = d'_j(g(\boldsymbol{x}), g(\iota_X(x_\upsilon))) \leq 2^{-n}$. Therefore

$$d'_j(\tilde{f}(\boldsymbol{x}), g(\boldsymbol{x})) \leq d'_j(\tilde{f}(\boldsymbol{x}), f(x_\upsilon)) + d'_j(f(x_\upsilon), g(\boldsymbol{x})) \leq 2^{-n} + 2^{-n}.$$

Thus, letting $n \to \infty$, we have $d'_j(\tilde{f}(\boldsymbol{x}), g(\boldsymbol{x})) = 0$ for each $j \in J$. □

Corollary 22. *Let (X, D) be a uniform space. Then $(\tilde{X}, \tilde{D})$ and $(\tilde{\tilde{X}}, \tilde{\tilde{D}})$ are uniformly equivalent.*

Proof. By Theorem 21, there exists a uniformly continuous mapping $\widetilde{\mathrm{id}}_{\tilde{X}} : (\tilde{\tilde{X}}, \tilde{\tilde{D}}) \to (\tilde{X}, \tilde{D})$ which makes the following diagram commute.

$$\begin{tikzcd}
(\tilde{\tilde{X}}, \tilde{\tilde{D}}) \arrow[r, "\widetilde{\mathrm{id}}_{\tilde{X}}"] & (\tilde{X}, \tilde{D}) \arrow[r, "\iota_{\tilde{X}}"] & (\tilde{\tilde{X}}, \tilde{\tilde{D}}) \\
 & (\tilde{X}, \tilde{D}) \arrow[ul, "\iota_{\tilde{X}}"] \arrow[u, "\mathrm{id}_{\tilde{X}}"] \arrow[ur, "\iota_{\tilde{X}}"'] &
\end{tikzcd}$$

Since $\iota_{\tilde{X}} : (\tilde{X}, \tilde{D}) \to (\tilde{\tilde{X}}, \tilde{\tilde{D}})$ is uniformly continuous, by Lemma 10, $\iota_{\tilde{X}} \circ \widetilde{\mathrm{id}}_{\tilde{X}} : (\tilde{\tilde{X}}, \tilde{\tilde{D}}) \to (\tilde{\tilde{X}}, \tilde{\tilde{D}})$ is uniformly continuous. Therefore we have $\iota_{\tilde{X}} \circ \widetilde{\mathrm{id}}_{\tilde{X}} = \mathrm{id}_{\tilde{\tilde{X}}}$. □

4 Product uniform spaces

Definition 23. Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of uniform spaces such that $D_k = \{d^k_i \mid i \in I_k\}$ for each $k \in K$. Then the *product uniform space* $\prod_{k \in K}(X_k, D_k)$ is a uniform space (X, D) such that $X = \prod_{k \in K} X_k$ and $D = \{d_{(k,i)} \mid (k, i) \in I\}$, where $I = \sum_{k \in K} I_k$ and

$$d_{(k,i)}(\xi, \zeta) = d^k_i(\xi(k), \zeta(k))$$

for each $(k, i) \in I$ and $\xi, \zeta \in \prod_{k \in K} X_k$.

Remark 24. For a family $\{(X_k, D_k) \mid k \in K\}$ of uniform spaces, where $D_k = \{d^k_i \mid i \in I_k\}$ for each $k \in K$, we tacitly assume that $\prod_{k \in K} I_k$ is inhabited.

Definition 25. Let $\{S_k \mid k \in K\}$ be a family of sets indexed by a set K, and let $S = \sum_{k \in K} S_k$. Then for each $k \in K$ and $\sigma \in S^*_k$, define $\{k\} \times \sigma \in S^*$ with $|\{k\} \times \sigma| = |\sigma|$ by

$$(\{k\} \times \sigma)(l) = (k, \sigma(l))$$

for each $l < |\sigma|$. Note that for each $k \in K$, if $\sigma \preceq_{S_k} \tau$, then $\{k\} \times \sigma \preceq_S \{k\} \times \tau$.

Proposition 26. *Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of complete uniform spaces. Then the product uniform space $\prod_{k\in K}(X_k, D_k)$ is complete.*

Proof. Let $D_k = \{d_i^k \mid i \in I_k\}$ for each $k \in K$, and let $I = \sum_{k\in K} I_k$. Suppose that (ξ_λ) is a Cauchy net on $(\Lambda, \preccurlyeq)$ in $\prod_{k\in K} X_k$ with a modulus $\alpha : I^* \to \Lambda$. Given a $k \in K$, let (x_λ^k) be a net on $(\Lambda, \preccurlyeq)$ in X_k defined by

$$x_\lambda^k = \xi_\lambda(k).$$

Then, for each $\sigma \in I_k^*$ and $\mu, \nu \in \Lambda$ with $\alpha(\{k\} \times \sigma) \preccurlyeq \mu, \nu$, we have

$$d_\sigma^k(x_\tau^k, x_\upsilon^k) = d_\sigma^k(\xi_\mu(k), \xi_\nu(k)) = d_{\{k\}\times\sigma}(\xi_\mu, \xi_\nu) \le 2^{-|\{k\}\times\sigma|} = 2^{-|\sigma|}.$$

Therefore (x_λ^k) is a Cauchy net with a modulus $\sigma \mapsto \alpha(\{k\}\times\sigma)$, and so converges to the unique limit $z_k \in X_k$ with a modulus $\beta_k : I_k^* \to \Lambda$.

Define $\zeta \in X$ by $\zeta(k) = z_k$ for each $k \in K$, and consider $\sigma \in I^*$ and $\lambda \in \Lambda$ with $\alpha(\sigma) \preccurlyeq \lambda$. Then for each $(k, i) \in \sigma$ and n, choosing $\mu \in \Lambda$ so that $\alpha(\sigma) \preccurlyeq \mu$ and $\beta_k(i^n) \preccurlyeq \mu$, we have

$$\begin{aligned} d_{(k,i)}(\xi_\lambda, \zeta) &= d_{(k,i)}(\xi_\lambda, \xi_\mu) + d_{(k,i)}(\xi_\mu, \zeta) \le d_\sigma(\xi_\lambda, \xi_\mu) + d_i^k(\xi_\mu(k), \zeta(k)) \\ &\le 2^{-|\sigma|} + d_{i^n}^k(x_\mu(k), z_k) \le 2^{-|\sigma|} + 2^{-n}, \end{aligned}$$

and hence, letting $n \to \infty$, we have $d_{(k,i)}(\xi_\lambda, \zeta) \le 2^{-|\sigma|}$. Therefore $d_\sigma(\xi_\lambda, \zeta) \le 2^{-|\sigma|}$ for each $\sigma \in I^*$ and $\lambda \in \Lambda$ with $\alpha(\sigma) \preccurlyeq \lambda$, and so (ξ_λ) converges to ζ with the modulus α. □

Lemma 27. *Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of uniform spaces such that $D_k = \{d_i^k \mid i \in I_k\}$ for each $k \in K$, and let $I = \sum_{k\in K} I_k$. Define the* inclusion map $\vec{\iota}$ *of $\prod_{k\in K} X_k$ into $\prod_{k\in K} \tilde{X}_k$ by*

$$\vec{\iota}(\xi) = (\iota_{X_k}(x^k))_{k\in K}$$

for each $\xi = (x^k)_{k\in K} \in \prod_{k\in K} X_k$. Then

$$d_\sigma(\xi, \zeta) = \tilde{d}_\sigma(\vec{\iota}(\xi), \vec{\iota}(\zeta))$$

for each $\sigma \in I$ and $\xi, \zeta \in \prod_{k\in K} X_k$, and $\vec{\iota} : \prod_{k\in K}(X_k, D_k) \to \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k)$ is uniformly continuous.

Proof. Straightforward, by Lemma 10. □

Definition 28. A set K is *discrete* if $k = k' \vee \neg k = k'$ for each $k, k' \in K$. Let $\{S_k \mid k \in K\}$ be a family of sets indexed by a discrete set K such that $(s_k)_{k\in K} \in \prod_{k\in K} S_k$, and let $S = \sum_{k\in K} S_k$. Then for each $k \in K$ and $\sigma \in S^*$, define $\sigma \restriction k \in S_k^*$ with $|\sigma \restriction k| = |\sigma|$ by

$$(\sigma \restriction k)(l) = \begin{cases} s & \text{if } \sigma(l) = (k', s) \text{ and } k = k', \\ s_k & \text{otherwise} \end{cases}$$

for each $l < |\sigma|$. Note that for each $k \in K$, since K is discrete, $\sigma \restriction k$ is well defined and if $\sigma \preceq_S \tau$, then $\sigma \restriction k \preceq_{S_k} \tau \restriction k$.

Lemma 29. *Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of uniform spaces indexed by a discrete set K, and let $(X, D) = \prod_{k\in K}(X_k, D_k)$ be the product uniform space. Then there exists a uniformly continuous injection $\kappa : \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k) \to (\tilde{X}, \tilde{D})$ such that $\kappa \circ \vec{\iota} = \iota_X$.*

Proof. Let $D_k = \{d^k_i \mid i \in I_k\}$ for each $k \in K$, and let $I = \sum_{k\in K} I_k$. Note that $\prod_{k\in K} I_k$ is inhabited. Consider $\vec{\boldsymbol{x}} \in \prod_{k\in K} \tilde{X}_k$ with $\vec{\boldsymbol{x}} = (\boldsymbol{x}^k)_{k\in K} = ((x^k_\rho)_{\rho\in I^*_k})_{k\in K}$, and let $[\vec{\boldsymbol{x}}]_\rho = (x^k_{\rho\restriction k})_{k\in K}$ for each $\rho \in I^*$, where $\rho \restriction k$ is defined for each element ρ of I^*. Then for each $\sigma, \tau, \upsilon \in I^*$ with $\sigma \preceq_I \tau, \upsilon$, since $\boldsymbol{x}^k$ is regular and $\sigma \restriction k \preceq_{I_k} \tau \restriction k, \upsilon \restriction k$, we have

$$d_{(k,i)}([\vec{\boldsymbol{x}}]_\tau, [\vec{\boldsymbol{x}}]_\upsilon) = d^k_i(x^k_{\tau\restriction k}, x^k_{\upsilon\restriction k}) \le d^k_{\sigma\restriction k}(x^k_{\tau\restriction k}, x^k_{\upsilon\restriction k}) \le 2^{-|\sigma\restriction k|} = 2^{-|\sigma|}$$

for each $(k, i) \in \sigma$, and hence $d_\sigma([\vec{\boldsymbol{x}}]_\tau, [\vec{\boldsymbol{x}}]_\upsilon) \le 2^{-|\sigma|}$. Therefore $([\vec{\boldsymbol{x}}]_\rho)_{\rho\in I^*}$ is regular in X. For each $\vec{\boldsymbol{x}} \in \prod_{k\in K} \tilde{X}_k$, define $\kappa(\vec{\boldsymbol{x}}) \in \tilde{X}$ by

$$\kappa(\vec{\boldsymbol{x}}) = ([\vec{\boldsymbol{x}}]_\rho)_{\rho\in I^*}.$$

Then for each $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in \prod_{k\in K} \tilde{X}_k$ with $\vec{\boldsymbol{x}} = (\boldsymbol{x}^k)_{k\in K} = ((x^k_{\rho'})_{\rho'\in I^*_k})_{k\in K}$ and $\vec{\boldsymbol{y}} = (\boldsymbol{y}^k)_{k\in K} = ((y^k_{\rho'})_{\rho'\in I^*_k})_{k\in K}$, we have

$$\begin{aligned}\tilde{d}_{(k,i)}(\kappa(\vec{\boldsymbol{x}}), \kappa(\vec{\boldsymbol{y}})) &= \lim_{n\to\infty} d_{(k,i)}([\vec{\boldsymbol{x}}]_{(k,i)^n}, [\vec{\boldsymbol{y}}]_{(k,i)^n}) = \lim_{n\to\infty} d^k_i(x^k_{(k,i)^n\restriction k}, y^k_{(k,i)^n\restriction k}) \\ &= \lim_{n\to\infty} d^k_i(x^k_{i^n}, y^k_{i^n}) = \tilde{d}^k_i(\boldsymbol{x}^k, \boldsymbol{y}^k) = \tilde{d}_{(k,i)}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}})\end{aligned}$$

for each $(k, i) \in I$. Therefore $\kappa : \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k) \to (\tilde{X}, \tilde{D})$ is a uniformly continuous injection with a modulus id_{I^*}. Since

$$\kappa(\vec{\iota}(\xi)) = ([\vec{\iota}(\xi)]_\rho)_{\rho\in I^*} = ([(\iota_{X_k}(x^k))_{k\in K}]_\rho)_{\rho\in I^*} = ((x^k)_{k\in K})_{\rho\in I^*} = \iota_X(\xi)$$

for each $\xi \in X$ with $\xi = (x^k)_{k\in K}$, we have $\kappa \circ \vec{\iota} = \iota_X$. □

Theorem 30. *Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of uniform spaces indexed by a discrete set K, and let $(X, D) = \prod_{k\in K}(X_k, D_k)$ be the product uniform space. Then $(\tilde{X}, \tilde{D})$ and $\prod_{k\in K}(\tilde{X}_k, \tilde{D}_k)$ are uniformly equivalent.*

Proof. Since $\vec{\iota} : (X, D) \to \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k)$ is uniformly continuous, by Lemma 27, there exists a uniformly continuous mapping $\tilde{\vec{\iota}} : (\tilde{X}, \tilde{D}) \to \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k)$ which makes the following diagram commute, by Theorem 21.

$$\begin{tikzcd}
(\tilde{X}, \tilde{D}) \arrow[r, "\tilde{\vec{\iota}}"] & \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k) \arrow[r, "\kappa"] & (\tilde{X}, \tilde{D}) \\
& (X, D) \arrow[ul, "\iota_X"] \arrow[u, "\vec{\iota}"] \arrow[ur, "\iota_X"'] &
\end{tikzcd}$$

Since $\kappa \circ \tilde{\vec{\iota}} : (\tilde{X}, \tilde{D}) \to (\tilde{X}, \tilde{D})$ is uniformly continuous, we have $\kappa \circ \tilde{\vec{\iota}} = \mathrm{id}_{\tilde{X}}$, and hence κ is surjective. Therefore, since κ is bijective, we have $\tilde{\vec{\iota}} = \kappa^{-1}$. □

Remark 31. In particular, Theorem 30 holds for finite or countable products of uniform spaces.

Corollary 32. *Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of uniform spaces indexed by a discrete set K, and let (Y, D') be a complete uniform space. Then for each uniformly (respectively, locally uniformly) continuous mapping $f : \prod_{k\in K}(X_k, D_k) \to (Y, D')$, there exists a unique uniformly (respectively, locally uniformly) continuous mapping $\tilde{f} : \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k) \to (Y, D')$ which makes the following diagram commute.*

$$\begin{tikzcd}
\prod_{k\in K}(\tilde{X}_k, \tilde{D}_k) \arrow[r, "\tilde{f}"] & (Y, D') \\
\prod_{k\in K}(X_k, D_k) \arrow[u, "\vec{\iota}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proof. Let $(X, D) = \prod_{k\in K}(X_k, D_k)$. Then, by Theorem 21, there exists a uniformly continuous mapping $g : (\tilde{X}, \tilde{D}) \to (Y, D')$ which makes the following diagram commute.

$$\begin{tikzcd}
\prod_{k\in K}(\tilde{X}_k, \tilde{D}_k) \arrow[r, "\kappa"] & (\tilde{X}, \tilde{D}) \arrow[r, "g"] & (Y, D') \\
& (X, D) \arrow[ul, "\vec{\iota}"] \arrow[u, "\iota_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Define $\tilde{f} : \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k) \to (Y, D')$ by $\tilde{f} = g \circ \kappa$, and assume that $h : \prod_{k\in K}(\tilde{X}_k, \tilde{D}_k) \to (Y, D')$ is a uniformly continuous mapping with $h \circ \vec{\iota} = f$. Then $h \circ \tilde{\vec{\iota}} : (\tilde{X}, \tilde{D}) \to (Y, D')$ is a uniformly continuous mapping with $h \circ \tilde{\vec{\iota}} \circ \iota_X = h \circ \vec{\iota} = f$, and hence $h \circ \tilde{\vec{\iota}} = g$. Therefore $\tilde{f} = g \circ \kappa = h \circ \tilde{\vec{\iota}} \circ \kappa = h$. $\square$

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