A constructive integration theory: a topological approach

Hajime Ishihara

July 2, 2020

Abstract

The aim of this paper is to develop a constructive integration theory from a topological point of view. We constructively prove several convergence theorems in integration theory including Fatou's lemma, and the monotone and dominated convergence theorems of Lebesgue in a totally topological framework developed in the paper.

1 Introduction

One of the motivations Lebesgue developed his integration theory was to make integration and limit commute:

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n,$$

which does not hold for the Riemann integral. The Lebesgue integral is based on the Lebesgue measure which is a generalization of the notions of a length, an area and a volume. Nowadays, integration theory is based on measure theory, and both theories are crucial especially in analysis, functional analysis and theory of probability.

Since a measure is defined on a σ-algebra which is closed under the complementation, the lack of the principle of excluded middle in constructive mathematics brings us a difficulty to define constructively an appropriate domain of a measure. Bishop overcame the difficulty by introducing the notion of a complemented set, and developed a constructive measure and integration theory; see [4, Chapters 3, 6, 7 and 8]. One can find a full and much improved account, in [5, Chapter 6], of the constructive integration theory based on Bishop-Cheng [6]; see also [7] for constructive aspects of measure theory.

However, the original motivation of Lebesgue is concerned with the topological notion of a limit. Since the notion of a convergence with appropriate properties induces a closure operation and hence classically defines a topology, we may reconsider it totally from a topological, instead of a measure theoretical, point of view. As far as we are concerned with convergence theorems such as the monotone and dominated convergence theorems of Lebesgue, we may be able to constructively deal with them topologically without invoking the notion of a measure and hence the notion of a complemented set.

The aim of this paper is to develop a constructive integration theory from a topological point of view. Although Spitters [10] took a similar approach using the notion of a uniform space, he adopted a rather unsatisfactory notion of a complete uniform space: a uniform space is complete if it is uniformly (metrically) equivalent to a complete *metric* space; see [10, Definition 5.2]. Berger et al. [3] gave a predicative completion of a uniform space given by entourages in **CZF**, the constructive and predicative Zermelo-Fraenkel set theory founded by Aczel [1], and, recently, the author [8] gave yet another predicative completion of a uniform space given by pseudometrics in a subsystem of **CZF**; see also [9] for a localic completion of a uniform space. Therefore we are now ready to deal with complete uniform spaces in a full generality.

Furthermore, we develop our integration theory on an abstract integration space which consists of a vector lattice and a positive linear functional on it. In Spitters [10], the simple functions on a Boolean ring forms a vector lattice, and a measure on the Boolean ring gives a positive linear functional on it. Therefore our integration theory may be seen as a generalization of Spitters' approach; see also [2] for a constructive treatment of a vector lattice.

The paper is organized as follows. In Section 2, we review a completion of a uniform space given in [8], as preliminaries. A universal property of the completion (Theorem 23) with uniformly continuous and locally uniformly continuous mappings plays an important role in the following sections. In Section 3, we give a constructive definition of a vector lattices, and show its basic properties. In Section 4, we introduce an abstract integration space as a pair (L, E) of a vector lattice L and a positive linear functional E on L, and define a seminorm on L by E. We show that the completion $\mathfrak{L}$ of L with the metric induced by the seminorm is a vector lattice, and define an integral for each element of $\mathfrak{L}$ which has the usual properties of integral. In Section 5, we define a family D_L of pseudometrics, indexed by L, on L using E. We show that the completion $\mathfrak{M}$ of a uniform space (L, D_L) forms a vector lattice, and construct a uniformly continuous embedding of $\mathfrak{L}$ into $\mathfrak{M}$ (Theorem 50). We also construct a locally uniformly continuous mapping of $\mathfrak{L} \times \mathfrak{M}$ into $\mathfrak{L}$ (Proposition 51) which is crucial to prove the convergence theorems in the next section. In Section 6, we define measurable functions and integrable functions on an abstract integration space as elements of $\mathfrak{M}$ and $\mathfrak{L}$, respectively. We prove several convergence theorems including Fatou's lemma (Corollary 58), and the monotone and dominated convergence theorems of Lebesgue (Theorem 60 and Theorem 61) in a totally topological framework we have developed in the previous sections. We conclude the paper with remarks on the classical and constructive definitions of a vector lattices and on a possible metrization of $\mathfrak{M}$, in Section 7.

2 A completion of a uniform space

In this section, we review a completion of a uniform space given in [8], as preliminaries. A universal property of the completion (Theorem 23) with uniformly continuous and locally uniformly continuous mappings plays an important role in the following sections.

Definition 1. A *pseudometric* d on a set X is a mapping $d : X \times X \to \mathbf{R}$ such that

1. $d(x, x) = 0$,
2. $d(x, y) = d(y, x)$,

3. $d(x, y) \leq d(x, z) + d(z, y)$

for each $x, y, z \in X$.

Definition 2. A (metrically) *uniform space* is a pair (X, D) of a set X and a family $D = \{d_i \mid i \in I\}$ of pseudometrics indexed by an inhabited set I such that

$$\forall i \in I(d_i(x, y) = 0) \Rightarrow x = y$$

for each $x, y \in X$. If I is a singleton, then (X, D) is called a *metric space*.

Definition 3. For a set S, we write S^* for the set of finite sequences of S with the following notations:

1. $|\sigma|$ denotes the *length* of $\sigma \in S^*$;
2. ϵ denotes the *empty sequence* with $|\epsilon| = 0$;
3. $\sigma(l)$ denotes the *l-th element* of $\sigma \in S^*$, where $l < |\sigma|$;
4. $s \in \sigma$ denotes that $s = \sigma(l)$ for some $l < |\sigma|$;
5. $\sigma * \tau$ denotes the *concatenation* of $\sigma \in S^*$ and $\tau \in S^*$;
6. s^n denotes the the *constant sequence* $\langle s, \ldots, s\rangle$ of the length n.

We define a binary relation $\preceq_S$ on S^* by

$$\sigma \preceq_S \tau \Leftrightarrow |\sigma| \leq |\tau| \wedge \forall s \in S(s \in \sigma \to s \in \tau)$$

for each $\sigma, \tau \in S^*$. If S is inhabited by $s_0 \in S$, then for each n, we write σ^{+n} for the sequence $\sigma * s_0^n$; note that $\sigma^{+n} \preceq_S \tau^{+n}$ whenever $\sigma \preceq_S \tau$.

Lemma 4. *Let S be a set. Then $(S^*, \preceq_S)$ is a directed preordered set.*

Remark 5. If S is a singleton $\{s\}$, then $(S^*, \preceq_S)$ is order isomorphic to $(\mathbf{N}, \leq)$ by the mapping $\sigma \mapsto |\sigma|$ and its inverse $n \mapsto s^n$.

Definition 6. Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$, and for each $\sigma \in I^*$, let d_σ be a pseudometric on X given by

$$d_\sigma(x, y) = \max\{d_i(x, y) \mid i \in \sigma\}$$

for each $x, y \in X$; if $\sigma = \epsilon$, then let $d_\sigma(x, y) = 0$. Let $(\Lambda, \preccurlyeq)$ be a directed preordered set. Then a map $\lambda \mapsto x_\lambda$ of Λ into X is called a *net* (or *Moore-Smith sequence*) on $(\Lambda, \preccurlyeq)$ in X, and is denoted by $(x_\lambda)_{\lambda \in \Lambda}$, or simply (x_λ). A net (x_λ) *converges* to an element x of X with a *modulus* $\beta : I^* \to \Lambda$ if

$$\beta(\sigma) \preccurlyeq \lambda \Rightarrow d_\sigma(x_\lambda, x) \leq 2^{-|\sigma|}$$

for each $\sigma \in I^*$ and $\lambda \in \Lambda$. We then write $x_\lambda \to x$, and x is called a *limit* of (x_λ). A net (x_λ) is a *Cauchy net* with a *modulus* $\alpha : I^* \to \Lambda$ if

$$\alpha(\sigma) \preccurlyeq \mu, \nu \Rightarrow d_\sigma(x_\mu, x_\nu) \leq 2^{-|\sigma|}$$

for each $\sigma \in I^*$ and $\mu, \nu \in \Lambda$. A uniform space (X, D) is *complete* if every Cauchy net converges.

Definition 7. Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$. A *regular* net in X is a Cauchy net on $(I^*, \preceq_I)$ with the modulus id_{I^*}. We write $\tilde{X}$ for the set of all regular nets in X.

Lemma 8. *Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$, and let $i \in I$. Then the limit*

$$\tilde{d}_i(\boldsymbol{x}, \boldsymbol{y}) = \lim_{n\to\infty} d_i(x_{i^n}, y_{i^n})$$

exists for each $\boldsymbol{x} = (x_\rho), \boldsymbol{y} = (y_\rho) \in \tilde{X}$, and $\tilde{d}_i$ is a pseudometric on $\tilde{X}$.

Lemma 9. *Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$. Define the* inclusion map *ι_X of X into $\tilde{X}$ by*

$$(\iota_X(x))(\sigma) = x$$

for each $x \in X$ and $\sigma \in I^*$. Then

$$d_\sigma(x, y) = \tilde{d}_\sigma(\iota_X(x), \iota_X(y))$$

for each $\sigma \in I^*$ and $x, y \in X$.

Lemma 10. *Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$, and let $\boldsymbol{x} = (x_\rho) \in \tilde{X}$. Then*

$$\tilde{d}_\sigma(\boldsymbol{x}, \iota_X(x_\tau)) \leq 2^{-|\tau|}$$

for each $\sigma, \tau \in I^*$ with $\sigma \preceq_I \tau$.

Definition 11. The *completion* of a uniform space (X, D) with $D = \{d_i \mid i \in I\}$ is the uniform space $(\tilde{X}, \tilde{D})$ with $\tilde{D} = \{\tilde{d}_i \mid i \in I\}$ and with the equality $=_{\tilde{X}}$ given by

$$\boldsymbol{x} =_{\tilde{X}} \boldsymbol{y} \Leftrightarrow \forall i \in I(\tilde{d}_i(\boldsymbol{x}, \boldsymbol{y}) = 0)$$

for each $\boldsymbol{x}, \boldsymbol{y} \in \tilde{X}$.

Theorem 12. *The completion $(\tilde{X}, \tilde{D})$ of a uniform space (X, D) is complete.*

Definition 13. Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of uniform spaces such that $D_k = \{d_i^k \mid i \in I_k\}$ for each $k \in K$. Then the *product uniform space* $\prod_{k\in K}(X_k, D_k)$ is a uniform space (X, D) such that $X = \prod_{k\in K} X_k$ and $D = \{d_{(k,i)} \mid (k, i) \in I\}$, where $I = \sum_{k\in K} I_k$ and

$$d_{(k,i)}(\xi, \zeta) = d_i^k(\xi(k), \zeta(k))$$

for each $(k, i) \in I$ and $\xi, \zeta \in \prod_{k\in K} X_k$.

Definition 14. Let $\{S_k \mid k \in K\}$ be a family of sets indexed by a set K, and let $S = \sum_{k\in K} S_k$. Then for each $k \in K$ and $\sigma \in S_k^*$, define $\{k\} \times \sigma \in S^*$ with $|\{k\} \times \sigma| = |\sigma|$ by

$$(\{k\} \times \sigma)(l) = (k, \sigma(l))$$

for each $l < |\sigma|$. Note that for each $k \in K$, if $\sigma \preceq_{S_k} \tau$, then $\{k\} \times \sigma \preceq_S \{k\} \times \tau$.

Proposition 15. *Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of complete uniform spaces. Then the product uniform space $\prod_{k\in K}(X_k, D_k)$ is complete.*

Lemma 16. *Let* $\{(X_k, D_k) \mid k \in K\}$ *be an inhabited family of uniform spaces such that* $D_k = \{d_i^k \mid i \in I_k\}$ *for each* $k \in K$*, and let* $I = \sum_{k \in K} I_k$*. Define the* inclusion map $\vec{\iota}$ *of* $\prod_{k \in K} X_k$ *into* $\prod_{k \in K} \tilde{X}_k$ *by*

$$\vec{\iota}(\xi) = (\iota_{X_k}(x^k))_{k \in K}$$

for each $\xi = (x^k)_{k \in K} \in \prod_{k \in K} X_k$*. Then*

$$d_\sigma(\xi, \zeta) = \tilde{d}_\sigma(\vec{\iota}(\xi), \vec{\iota}(\zeta))$$

for each $\sigma \in I$ *and* $\xi, \zeta \in \prod_{k \in K} X_k$*.*

Definition 17. Let (X, D) and (Y, D') be uniform spaces with $D = \{d_i \mid i \in I\}$ and $D' = \{d'_j \mid j \in J\}$. Then a mapping $f : X \to Y$ is *uniformly continuous* with a monotone *modulus* $\alpha : J^* \to I^*$ (that is, $\sigma' \preceq_J \tau'$ implies $\alpha(\sigma') \preceq_I \alpha(\tau')$ for each $\sigma', \tau' \in J^*$) if

$$d_{\alpha(\sigma')}(x, y) \leq 2^{-|\alpha(\sigma')|} \Rightarrow d'_{\sigma'}(f(x), f(y)) \leq 2^{-|\sigma'|}$$

for each $\sigma' \in J^*$ and $x, y \in X$. A uniformly continuous mapping $f : X \to Y$ is a *uniform isomorphism* if it has a uniformly continuous inverse, and (X, D) and (Y, D') are *uniformly equivalent* if there exists a uniform isomorphism between X and Y.

A mapping $f : X \to Y$ is *locally uniformly continuous* if for each $\boldsymbol{x} \in \tilde{X}$ there exists a monotone *modulus* $\alpha : J^* \to I^*$ such that

$$y, z \in U_{\alpha(\sigma')}(\boldsymbol{x}) \Rightarrow d'_{\sigma'}(f(y), f(z)) \leq 2^{-|\sigma'|}$$

for each $\sigma' \in J^*$ and $y, z \in X$, where

$$U_\sigma(\boldsymbol{x}) = \{z \in X \mid \tilde{d}_\sigma(\boldsymbol{x}, \iota_X(z)) \leq 2^{-|\sigma|}\}$$

for each $\sigma \in I^*$.

Remark 18. For each $\rho' \in J^*$, the set $\{\sigma' \in J^* \mid \sigma' \preceq_J \rho'\}$ is finitely enumerable; for $\{\sigma' \in J^* \mid \sigma' \preceq_J \rho'\} = \{\rho' \circ \pi \mid \pi \in |\rho'|^n, n \leq |\rho'|\}$, where $|\rho'|^n$ is the finite set of functions from $\{0, \ldots, n-1\}$ into $\{0, \ldots, |\rho'| - 1\}$. Therefore we may replace any modulus $\alpha : J^* \to I^*$ of uniform continuity by a *monotone* modulus $\alpha' : J^* \to I^*$ given by

$$\alpha'(\rho') = \text{the concatenation of } \{\alpha(\sigma') \mid \sigma' \preceq_J \rho'\}$$

for each $\rho' \in J^*$.

Lemma 19. *Let* (X, D) *and* (Y, D') *be uniform spaces. Then every uniformly continuous mapping* $f : X \to Y$ *is locally uniformly continuous.*

Lemma 20. *Let* (X, D) *and* (Y, D') *be uniform spaces with* $D = \{d_i \mid i \in I\}$ *and* $D' = \{d'_j \mid j \in J\}$*. Then a locally uniformly continuous mapping* $f : (X, D) \to (Y, D')$ *is pointwise continuous, in the sense that for each* $x \in X$ *there exists* $\alpha : J^* \to I^*$ *such that*

$$d_{\alpha(\sigma')}(x, y) \leq 2^{-|\alpha(\sigma')|} \Rightarrow d'_{\sigma'}(f(x), f(y)) \leq 2^{-|\sigma'|}$$

for each $y \in X$ *and* $\sigma' \in J^*$*. Especially,*

$$x_\lambda \to x \Rightarrow f(x_\lambda) \to f(x)$$

for each net (x_λ) *in* X*.*

Lemma 21. *Let (X, D), (Y, D') and (Z, D'') be uniform spaces. Then the composition $g \circ f$ of uniformly (respectively, locally uniformly) continuous mappings $f : (X, D) \to (Y, D')$ and $g : (Y, D') \to (Z, D'')$ is uniformly (respectively, locally uniformly) continuous.*

Definition 22. A set K is *discrete* if $k = k' \vee \neg k = k'$ for each $k, k' \in K$.

Theorem 23. *Let $\{(X_k, D_k) \mid k \in K\}$ be an inhabited family of uniform spaces indexed by a discrete set K, and let (Y, D') be a complete uniform space. Then for each uniformly (respectively, locally uniformly) continuous mapping $f : \prod_{k \in K}(X_k, D_k) \to (Y, D')$, there exists a unique uniformly (respectively, locally uniformly) continuous mapping $\tilde{f} : \prod_{k \in K}(\tilde{X}_k, \tilde{D}_k) \to (Y, D')$ which makes the following diagram commute.*

$$\begin{tikzcd}
\prod_{k \in K}(\tilde{X}_k, \tilde{D}_k) \arrow[r, "\tilde{f}"] & (Y, D') \\
\prod_{k \in K}(X_k, D_k) \arrow[u, "\vec{\iota}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

3 Vector lattices

In this section, we give a constructive definition of a vector lattices, and show its basic properties.

Definition 24. A (join) *semilattice* is a pair $(L, \vee)$ of a set L and a binary operation $\vee$ on L such that

$$x \vee (y \vee z) = (x \vee y) \vee z, \qquad x \vee y = y \vee x, \qquad x \vee x = x$$

for each $x, y, z \in L$. A trivial example of semilattice is the reals $\mathbf{R}$ with a binary operation $\vee$ given by $a \vee b = \max\{a, b\}$ for each $a, b \in \mathbf{R}$.

Remark 25. Let $(L, \vee)$ be a semilattice. Then it is well known that L becomes a partially ordered set with a partial order $\leq$ given by

$$x \leq y \Leftrightarrow x \vee y = y$$

for each $x, y \in L$, and we have

$$x, y \leq x \vee y; \qquad x \leq z, y \leq z \Rightarrow x \vee y \leq z$$

for each $x, y, z \in L$. Hence for each $x, y \in L$, $x \vee y$ is the least upper bound of $\{x, y\}$.

Definition 26. A *vector lattice* (or *Riesz space*) is a linear space L with a binary operation $\vee$ on L such that $(L, \vee)$ is a semilattice, and

1. $(x + z) \vee (y + z) = x \vee y + z$,
2. if $0 \leq a$, then $a(x \vee y) = (ax) \vee (ay)$,
3. if $0 \leq x$, then $(a \vee b)x = (ax) \vee (bx)$

for each $x, y, z \in L$ and $a, b \in \mathbf{R}$.

Example 27. Recall that a function $f : \mathbf{R} \to \mathbf{R}$ has a compact support if there exists M such that $M \leq |x|$ implies $f(x) = 0$ for each $x \in \mathbf{R}$, and let $C_0(\mathbf{R})$ be the set of uniformly continuous functions from $\mathbf{R}$ into $\mathbf{R}$ having compact supports. Then $C_0(\mathbf{R})$ is a vector lattice with the operations

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = af(x), \quad (f \vee g)(x) = \max\{f(x), g(x)\}$$

for each $f, g \in C_0(\mathbf{R})$ and $a, x \in \mathbf{R}$.

Lemma 28. *Let L be a vector lattice. Then*

1. *if $x \leq y$, then $x + z \leq y + z$,*
2. *if $x \leq y$ and $0 \leq a$, then $ax \leq ay$,*
3. *if $0 \leq x$ and $a \leq b$, then $ax \leq bx$*

for each $x, y, z \in L$ and $a, b \in \mathbf{R}$.

Proof. If $x \leq y$, then $(x+z) \vee (y+z) = x \vee y + z = y + z$, and hence $x + z \leq y + z$. If $x \leq y$ and $0 \leq a$, then $ax \vee ay = a(x \vee y) = ay$, and hence $ax \leq ay$. If $0 \leq x$ and $a \leq b$, then $ax \vee bx = (a \vee b)x = bx$, and hence $ax \leq bx$. □

Lemma 29. *Let L be a vector lattice, and let $\wedge$ be a binary operation on L given by*

$$x \wedge y = -(-x \vee -y)$$

for each $x, y \in L$. Then

1. *$x + y = x \vee y + x \wedge y$,*
2. *$(x + z) \wedge (y + z) = (x \wedge y) + z$,*
3. *if $0 \leq a$, then $a(x \wedge y) = ax \wedge ay$,*
4. *if $0 \leq x$, then $(a \wedge b)x = ax \wedge bx$,*

for each $x, y, z \in L$ and $a, b \in \mathbf{R}$.

Proof. Since $x \vee y - (x+y) = -y \vee -x = -(x \wedge y)$, we have $x+y = x \vee y + x \wedge y$. We have $(x+z) \wedge (y+z) = -((-x-z) \vee (-y-z)) = -((-x \vee -y) - z) = x \wedge y + z$. If $0 \leq a$, then $a(x \wedge y) = a(-(-x \vee -y)) = -(a(-x \vee -y)) = -(-ax \vee -ay) = ax \wedge ay$. If $0 \leq x$, then $(a \wedge b)x = -(-a \vee -b)x = -((-a \vee -b)x) = -(-ax \vee -bx) = ax \wedge bx$. □

Proposition 30. *Let L be a vector lattice. Then $(L, \vee, \wedge)$ is a distributive lattice.*

Proof. Note that for each $x, y \in L$, if $x \leq y$, then $-y = x - (x+y) \leq y - (x+y) = -x$, by Lemma 28 (1). Then it is straightforward to see that $x \wedge y$ is the greatest lower bound of $\{x, y\}$. As to the distributivity, since $(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z$, it suffices to show that $(x \vee y) \wedge z \leq (x \wedge z) \vee (y \wedge z)$; see [12, 13.4]. Let $u = (x \wedge z) \vee (y \wedge z)$. Then, since $x \wedge z \leq u$ and $y \wedge z \leq u$, we have $x + z =$

$x \vee z + x \wedge z \leq x \vee z + u$ and $y + z = y \vee z + y \wedge z \leq y \vee z + u$, by Lemma 29 (1) and Lemma 28 (1), and hence

$$x \vee y + z = (x + z) \vee (y + z) \leq (x \vee z + u) \vee (y \vee z + u) = (x \vee y) \vee z + u.$$

Therefore $(x \vee y) \wedge z = (x \vee y) + z - (x \vee y) \vee z \leq u$, and so $(x \vee y) \wedge z \leq (x \wedge z) \vee (y \wedge z)$. □

Definition 31. Let L be a vector lattice, and let $(\text{-})^+ : L \to L$, $(\text{-})^- : L \to L$ and $|\text{-}| : L \to L$ be unary operations given by

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0, \qquad |x| = x \vee (-x),$$

respectively, for each $x \in L$.

Remark 32. Since $0 \leq a \Leftrightarrow a = a^+$ and $0 \leq x \Leftrightarrow x = x^+$ for each $a \in \mathbf{R}$ and $x \in L$, we may replace the implications (2) and (3) of Definition 26 by the equations

$$a^+(x \vee y) = (a^+x) \vee (a^+y), \qquad (a \vee b)x^+ = (ax^+) \vee (bx^+),$$

respectively.

Lemma 33. *Let L be a vector lattice. Then*

1. $x = x^+ - x^-$, $x^+ \wedge x^- = 0$ *and* $|x| = x^+ + x^- = x^+ \vee x^- \geq 0$;
2. $|x + y| \leq |x| + |y|$ *and* $||x| - |y|| \leq |x - y|$;
3. $|ax| = |a||x|$;
4. $|x \vee z - y \vee z| \leq |x - y|$ *and* $|x \wedge z - y \wedge z| \leq |x - y|$;
5. *if* $0 \leq x, y$, *then* $(x + y) \wedge |z| \leq x \wedge |z| + y \wedge |z|$;
6. *if* $0 \leq x, y$, *then* $|x \wedge z - y \wedge z| \leq |x - y| \wedge |z|$

for each $x, y, z \in L$ *and* $a \in \mathbf{R}$.

Proof. (1): Since $x^- + x = (-x \vee 0) + x = 0 \vee x = x^+$, we have $x = x^+ - x^-$. We have $x^+ \wedge x^- = (x + x^-) \wedge x^- = (x \wedge 0) + x^- = -(-x \vee 0) + x^- = -x^- + x^- = 0$, by Lemma 29 (2). We have $|x| = (-x) \vee x = x + (-2x \vee 0) = x + 2(-x \vee 0) = x + 2x^- = x^+ + x^- = x^+ \vee x^- + x^+ \wedge x^- = x^+ \vee x^- \geq x^+ \geq 0$, by Lemma 29 (1).

(2): Since $0 \leq 2y^+ = |y| + y$, by (1), we have $x \leq |x| \leq |x| + |y| + y$, by Lemma 28 (1), and hence $x - y \leq |x| + |y|$. Similarly we have $y - x \leq |x| + |y|$. Therefore $|x - y| \leq |x| + |y|$. Since $|x| \leq |x - y| + |y|$, we have $|x| - |y| \leq |x - y|$. Similarly, we have $|y| - |x| \leq |x - y|$. Therefore $||x| - |y|| \leq |x - y|$.

(3): Note that

$$\begin{aligned}|a||x| &= |a|(x^+ + x^-) = |a|x^+ + |a|x^- = (a^+ + a^-)x^+ + (a^+ + a^-)x^- \\ &= a^+x^+ + a^-x^+ + a^+x^- + a^-x^-,\end{aligned}$$

by (1), on the one hand, and

$$\begin{aligned}|a||x| &= |a|(x^+ \vee x^-) = |a|x^+ \vee |a|x^- = (a^+ \vee a^-)x^+ \vee (a^+ \vee a^-)x^- \\ &= a^+x^+ \vee a^-x^+ \vee a^+x^- \vee a^-x^-,\end{aligned}$$

by (1) and Definition 26 (2) and (3), on the other. Then, since $0 \leq x^+, x^-$ and $0 \leq a^+, a^-$, we have $0 \leq a^+x^+, a^-x^-, a^-x^+, a^+x^-$, by Lemma 28 (2), and hence $0 \leq a^+x^+ \wedge a^-x^- \leq |a|x^+ \wedge |a|x^- = |a|(x^+ \wedge x^-) = 0$ and $0 \leq a^-x^+ \wedge a^+x^- \leq |a|x^+ \wedge |a|x^- = |a|(x^+ \wedge x^-) = 0$, by (1) and Lemma 28 (3). Therefore, since

$$a^+x^+ + a^-x^- = a^+x^+ \vee a^-x^- \quad \text{and} \quad a^-x^+ + a^+x^- = a^-x^+ \vee a^+x^-,$$

by Lemma 29 (1), we have

$$\begin{aligned}
(ax)^+ &= ax \vee 0 = (a^+ - a^-)(x^+ - x^-) \vee 0 \\
&= (a^+x^+ - a^-x^+ - a^+x^- + a^-x^-) \vee 0 \\
&= (a^+x^+ + a^-x^-) \vee (a^-x^+ + a^+x^-) - (a^-x^+ + a^+x^-) \\
&= (a^+x^+ \vee a^-x^- \vee a^-x^+ \vee a^+x^-) - (a^-x^+ + a^+x^-) \\
&= (a^+x^+ + a^-x^- + a^-x^+ + a^+x^-) - (a^-x^+ + a^+x^-) \\
&= a^+x^+ + a^-x^-.
\end{aligned}$$

Similarly, we have $(ax)^- = a^-x^+ + a^+x^-$. Thus

$$|ax| = (ax)^+ + (ax)^- = a^+x^+ + a^-x^- + a^-x^+ + a^+x^- = |a||x|.$$

(4): Since $x - y \leq |x - y|$, we have $x = x - y + y \leq |x - y| + y \leq |x - y| + y \vee z$, by Lemma 28 (1). Since $0 \leq |x - y|$, we have $z \leq y \vee z \leq |x - y| + y \vee z$, by Lemma 28 (1). Therefore $x \vee z \leq |x - y| + y \vee z$, and so $x \vee z - y \vee z \leq |x - y|$. Similarly we have $y \vee z - x \vee z \leq |x - y|$. Thus $|x \vee z - y \vee z| \leq |x - y|$. Since $x \wedge z \leq x \leq |x - y| + y$ and $x \wedge z \leq z \leq |x - y| + z$, we have $x \wedge z \leq (|x - y| + y) \wedge (|x - y| + z) = |x - y| + y \wedge z$, by Lemma 29 (2), and hence $x \wedge z - y \wedge z \leq |x - y|$. Similarly we have $y \wedge z - x \wedge z \leq |x - y|$. Therefore $|x \wedge z - y \wedge z| \leq |x - y|$.

(5): Suppose that $0 \leq x, y$. Then, since $(x+y) \wedge |z| \leq x+y$ and $(x+y) \wedge |z| \leq |z| \leq |z|+y$, by Lemma 28 (1), we have $(x+y) \wedge |z| \leq (x+y) \wedge (|z|+y) = x \wedge |z| + y$, by Lemma 29 (2). Since $0 \leq x \wedge |z|$, we have $(x+y) \wedge |z| \leq |z| \leq x \wedge |z| + |z|$, by Lemma 28 (1). Therefore $(x+y) \wedge |z| \leq (x \wedge |z| + y) \wedge (x \wedge |z| + |z|) = x \wedge |z| + y \wedge |z|$, by Lemma 29 (2).

(6): Suppose that $0 \leq x, y$. Then, since $x \wedge z \leq z \leq |z| + z$ and $x \wedge z \leq z \leq |z| \leq |z| + y$, by Lemma 28 (1), we have $x \wedge z \leq (|z| + z) \wedge (|z| + y) = |z| + y \wedge z$, by Lemma 29 (2), and hence $x \wedge z - y \wedge z \leq |z|$. Similarly, we have $y \wedge z - x \wedge z \leq |z|$. Therefore $|x \wedge z - y \wedge z| \leq |z|$, and so $|x \wedge z - y \wedge z| \leq |x - y| \wedge |z|$ with (4). $\square$

4 The first completion

In this section, we introduce an abstract integration space as a pair (L, E) of a vector lattice L and a positive linear functional E on L, and define a seminorm on L by E. We show that the completion $\mathfrak{L}$ of L with the metric induced by the seminorm is a vector lattice, and define an integral for an element of $\mathfrak{L}$ which has the usual properties of integral.

Definition 34. An *abstract integration space* is a pair (L, E) of a vector lattice L and a positive linear functional E on L, that is, $E : L \to \mathbf{R}$ and

1. $E(x + y) = E(x) + E(y)$,

2. $E(ax) = aE(x)$,

3. $0 \leq x \Rightarrow 0 \leq E(x)$

for each $x, y \in L$ and $a \in \mathbf{R}$.

Example 35. For each $f \in C_0(\mathbf{R})$, let $E(f)$ be the Riemann integral

$$E(f) = \int_{-\infty}^{\infty} f(x)dx;$$

see [4, 2.6], [5, 2.6] and [11, 6.2]. Then $(C_0(\mathbf{R}), E)$ is an abstract integration space.

In what follows, we fix an abstract integration space (L, E).

Lemma 36. *Let $\|\text{-}\| : L \to \mathbf{R}$ be a mapping defined by $\|x\| = E(|x|)$ for each $x \in L$. Then $\|\text{-}\|$ is a seminorm on L, that is,*

1. $\|ax\| = |a|\|x\|$,

2. $\|x + y\| \leq \|x\| + \|y\|$

for each $x, y \in L$ and $a \in \mathbf{R}$, and induces a pseudometric $d_s : L \times L \to \mathbf{R}$ given by

$$d_s(x, y) = \|x - y\|$$

for each $x, y \in L$.

Proof. We have $\|ax\| = E(|ax|) = E(|a||x|) = |a|E(|x|) = |a|\|x\|$, by Lemma 33 (3), and $\|x + y\| = E(|x + y|) \leq E(|x| + |y|) = E(|x|) + E(|y|) = \|x\| + \|y|$, by Lemma 33 (2). It is straightforward to see that d_s is a pseudometric. □

Lemma 37. *For each $x, x', y, y' \in L$,*

1. $d_s(x + y, x' + y') \leq d_s(x, x') + d_s(y, y')$,

2. $d_s(x \vee y, x' \vee y') \leq d_s(x, x') + d_s(y, y')$,

3. $d_s(x \wedge y, x' \wedge y') \leq d_s(x, x') + d_s(y, y')$.

Proof. We have

$$\begin{aligned} d_s(x + y, x' + y') &= \|(x + y) - (x' + y')\| = \|(x - x') + (y - y')\| \\ &\leq \|x - x'\| + \|y - y'\| = d_s(x, x') + d_s(y, y'), \end{aligned}$$

and

$$\begin{aligned} d_s(x \vee y, x' \vee y') &= \|x \vee y - x' \vee y'\| = E(|x \vee y - x' \vee y'|) \\ &= E(|(x \vee y - x' \vee y) + (x' \vee y - x' \vee y')|) \\ &\leq E(|x \vee y - x' \vee y| + |x' \vee y - x' \vee y'|) \\ &= E(|x \vee y - x' \vee y|) + E(|x' \vee y - x' \vee y'|) \\ &\leq E(|x - x'|) + E(|y - y'|) = \|x - x'\| + \|y - y'\| \\ &= d_s(x, x') + d_s(y, y'), \end{aligned}$$

by Lemma 33 (2) and (4). Similarly, we have

$$d_s(x \wedge y, x' \wedge y') \leq d_s(x, x') + d_s(y, y').$$

□

Definition 38. Let $(\mathfrak{L}, \tilde{d}_s)$ be the completion of the metric space (L, d_s) with the equality $=_s$ given by $x =_s y \Leftrightarrow d_s(x, y) = 0$ for each $x, y \in L$. We write $\iota_{\mathfrak{L}}$ for the inclusion map $\iota_L : L \to \mathfrak{L}$.

Lemma 39. *The mappings $\iota_{\mathfrak{L}} \circ + : (L, d_s) \times (L, d_s) \to \mathfrak{L}$, $\iota_{\mathfrak{L}} \circ \vee : (L, d_s) \times (L, d_s) \to \mathfrak{L}$ and $\iota_{\mathfrak{L}} \circ \wedge : (L, d_s) \times (L, d_s) \to \mathfrak{L}$ are uniformly continuous, and the mapping $\iota_{\mathfrak{L}} \circ (\cdot\cdot\cdot) : \mathbf{R} \times (L, d_s) \to \mathfrak{L}$, where $(\cdot\cdot\cdot) : (a, x) \mapsto ax$ is the scalar multiplication, is locally uniformly continuous.*

Proof. We assume that the family $\{d_s\}$ of metrics for L is indexed by a singleton $\{s\}$. Note that $\{s\}^*$ is order isomorphic to $\mathbf{N}$.

Define $\alpha : \mathbf{N} \to (\{s\} + \{s\})^*$ by

$$\alpha(n) = (0, s)^{n+1} * (1, s)^1$$

for each $n \in \mathbf{N}$, where t^n is the constant sequence given in Definition 3. Consider n and $(x, y), (x', y') \in L \times L$ with

$$d_{\alpha(n)}((x, y), (x', y')) \leq 2^{-|\alpha(n)|}.$$

Then, since $(0, s), (1, s) \in \alpha(n)$, we have

$$d_s(x, x') = d_{(0,s)}((x, y), (x', y')) \leq d_{\alpha(n)}((x, y), (x', y')) \leq 2^{-|\alpha(n)|} = 2^{-(n+2)}$$

and $d_s(y, y') = d_{(1,s)}((x, y), (x', y')) \leq d_{\alpha(n)}((x, y), (x', y')) \leq 2^{-(n+2)}$, and hence

$$\begin{aligned}\tilde{d}_s(\iota_{\mathfrak{L}}(x + y), \iota_{\mathfrak{L}}(x' + y')) &= d_s(x + y, x' + y') \leq d_s(x, x') + d_s(y, y') \\ &\leq 2^{-(n+2)} + 2^{-(n+2)} < 2^{-n}\end{aligned}$$

and $\tilde{d}_s(\iota_{\mathfrak{L}}(x \vee y), \iota_{\mathfrak{L}}(x' \vee y')) = d_s(x \vee y, x' \vee y') \leq d_s(x, x') + d_s(y, y') < 2^{-n}$, by Lemma 37 (1) and (2). Similarly, we have $\tilde{d}_s(\iota_{\mathfrak{L}}(x \wedge y), \iota_{\mathfrak{L}}(x' \wedge y')) < 2^{-n}$, by Lemma 37 (3). Therefore $\iota_{\mathfrak{L}} \circ + : (L, d_s) \times (L, d_s) \to \mathfrak{L}$, $\iota_{\mathfrak{L}} \circ \vee : (L, d_s) \times (L, d_s) \to \mathfrak{L}$ and $\iota_{\mathfrak{L}} \circ \wedge : (L, d_s) \times (L, d_s) \to \mathfrak{L}$ are uniformly continuous with the modulus α.

We assume that the family $\{d_r\}$ of metrics for $\mathbf{R}$, given by $d_r(a, b) = |a - b|$ for each $a, b \in \mathbf{R}$, is indexed by a singleton $\{r\}$. Let $\boldsymbol{\xi}$ be a regular net in $(\mathbf{R}, d_r) \times (L, d_s)$ with $\boldsymbol{\xi} = ((c_\rho, z_\rho))_{\rho \in (\{r\}+\{s\})^*}$. Let $\rho = (0, r)^1 * (1, s)^1$, and choose N so that

$$\max\{|c_\rho|, \|z_\rho\|\} \leq 2^N - 1.$$

Define $\beta : \mathbf{N} \to (\{r\} + \{s\})^*$ by

$$\beta(n) = (0, r)^{N+n+1} * (1, s)^1$$

for each n, and consider n and $(a, x), (b, y) \in U_{\beta(n)}(\boldsymbol{\xi})$. Then

$$\begin{aligned}d_r(a, b) &= d_{(0,r)}((a, x), (b, y)) = d_{\beta(n)}((a, x), (b, y)) \\ &\leq \tilde{d}_{\beta(n)}(\iota_{\mathbf{R}\times L}((a, x)), \iota_{\mathbf{R}\times L}((b, y))) \\ &\leq \tilde{d}_{\beta(n)}(\iota_{\mathbf{R}\times L}((a, x)), \boldsymbol{\xi}) + \tilde{d}_{\beta(n)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}((b, y))) \\ &\leq 2^{-|\beta(n)|} + 2^{-|\beta(n)|} = 2^{-(N+n+2)} + 2^{-(N+n+2)} = 2^{-(N+n+1)}\end{aligned}$$

and

$$\begin{aligned}
d_s(x,y) &= d_{(1,s)}((a,x),(b,y)) = d_{\beta(n)}((a,x),(b,y)) \\
&\leq \tilde{d}_{\beta(n)}(\iota_{\mathbf{R}\times L}((a,x)), \iota_{\mathbf{R}\times L}((b,y))) \\
&\leq \tilde{d}_{\beta(n)}(\iota_{\mathbf{R}\times L}((a,x)), \boldsymbol{\xi}) + \tilde{d}_{\beta(n)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}((b,y))) \leq 2^{-(N+n+1)},
\end{aligned}$$

by Lemma 9. Since

$$\begin{aligned}
|c_\rho - a| &= d_{(0,r)}((c_\rho, z_\rho),(a,x)) = \tilde{d}_{(0,r)}(\iota_{\mathbf{R}\times L}(c_\rho, z_\rho), \iota_{\mathbf{R}\times L}(a,x)) \\
&\leq \tilde{d}_{(0,r)}(\iota_{\mathbf{R}\times L}(c_\rho, z_\rho), \boldsymbol{\xi}) + \tilde{d}_{(0,r)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}(a,x)) \\
&\leq \tilde{d}_{\rho}(\iota_{\mathbf{R}\times L}(c_\rho, z_\rho), \boldsymbol{\xi}) + \tilde{d}_{\beta(n)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}(a,x)) \\
&\leq 2^{-|\rho|} + 2^{-|\beta(n)|} \leq 2^{-2} + 2^{-2} < 1,
\end{aligned}$$

by Lemma 10, we have $|a| \leq |c_\rho| + |c_\rho - a| \leq |c_\rho| + 1 \leq 2^N$, and, since

$$\begin{aligned}
\|z_\rho - y\| &= d_{(1,s)}((c_\rho, z_\rho),(b,y)) = \tilde{d}_{(1,s)}(\iota_{\mathbf{R}\times L}(c_\rho, z_\rho), \iota_{\mathbf{R}\times L}(b,y)) \\
&\leq \tilde{d}_{(1,s)}(\iota_{\mathbf{R}\times L}(c_\rho, z_\rho), \boldsymbol{\xi}) + \tilde{d}_{(1,s)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}(b,y)) \\
&\leq \tilde{d}_{\rho}(\iota_{\mathbf{R}\times L}(c_\rho, z_\rho), \boldsymbol{\xi}) + \tilde{d}_{\beta(n)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}(b,y)) < 1,
\end{aligned}$$

we have $\|y\| \leq \|z_\rho\| + \|z_\rho - y\| \leq \|z_\rho\| + 1 \leq 2^N$. Hence

$$\begin{aligned}
\tilde{d}_s(\iota_{\mathfrak{L}}(ax), \iota_{\mathfrak{L}}(by)) &= d_s(ax, by) = E(|ax - by|) \leq E(|ax - ay| + |ay - by|) \\
&= E(|a||x-y| + |a-b||y|) = |a| d_s(x,y) + \|y\| d_r(a,b) \\
&\leq 2^N \cdot 2^{-(N+n+1)} + 2^N \cdot 2^{-(N+n+1)} = 2^{-n}.
\end{aligned}$$

Therefore $\iota_{\mathfrak{L}} \circ (\text{-} \cdot \text{-}) : \mathbf{R} \times (L, d_s) \to \mathfrak{L}$, is locally uniformly continuous at $\boldsymbol{\xi}$ with a modulus β. $\square$

Proposition 40. *$\mathfrak{L}$ is a vector lattice.*

Proof. Since the mapping $\iota_{\mathfrak{L}} \circ + : (L, d_s) \times (L, d_s) \to \mathfrak{L}$ is uniformly continuous, by Lemma 39, there exists a unique uniformly continuous extension $+_{\mathfrak{L}} : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ of $\iota_{\mathfrak{L}} \circ +$, by Theorem 23. Similarly, there exist locally uniformly continuous extension $(\text{-} \cdot_{\mathfrak{L}} \text{-}) : \mathbf{R} \times \mathfrak{L} \to \mathfrak{L}$ of $\iota_{\mathfrak{L}} \circ (\text{-} \cdot \text{-})$ and uniformly continuous extension $\vee_{\mathfrak{L}} : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ of $\iota_{\mathfrak{L}} \circ \vee$.

To see that $+_{\mathfrak{L}}$ and $(\text{-} \cdot_{\mathfrak{L}} \text{-})$ are linear operations, $\vee_{\mathfrak{L}}$ is a semilattice operation, and $\mathfrak{L}$ is a vector lattice, we only show that

$$(f +_{\mathfrak{L}} h) \vee_{\mathfrak{L}} (g +_{\mathfrak{L}} h) =_{\mathfrak{L}} f \vee_{\mathfrak{L}} g +_{\mathfrak{L}} h$$

for each $f, g, h \in \mathfrak{L}$. Other equations are similar. Let $\varphi : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ and $\psi : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ be mappings defined by $\varphi(f,g,h) = (f +_{\mathfrak{L}} h) \vee_{\mathfrak{L}} (g +_{\mathfrak{L}} h)$ and $\psi(f,g,h) = f \vee_{\mathfrak{L}} g +_{\mathfrak{L}} h$ for each $f, g, h \in \mathfrak{L}$, respectively. Note that φ and ψ are uniformly continuous. Then, since

$$\begin{aligned}
\varphi(\iota_{\mathfrak{L}}(x), \iota_{\mathfrak{L}}(y), \iota_{\mathfrak{L}}(z)) &= (\iota_{\mathfrak{L}}(x) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(z)) \vee_{\mathfrak{L}} (\iota_{\mathfrak{L}}(y) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(z)) \\
&= \iota_{\mathfrak{L}}(x+z) \vee_{\mathfrak{L}} \iota_{\mathfrak{L}}(y+z) = \iota_{\mathfrak{L}}((x+z) \vee (y+z)) \\
&= \iota_{\mathfrak{L}}(x \vee y + z) = \iota_{\mathfrak{L}}(x \vee y) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(z) \\
&= \iota_{\mathfrak{L}}(x) \vee_{\mathfrak{L}} \iota_{\mathfrak{L}}(y) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(z) = \psi(\iota_{\mathfrak{L}}(x), \iota_{\mathfrak{L}}(y), \iota_{\mathfrak{L}}(z))
\end{aligned}$$

for each $x, y, z \in L$, we have $\varphi \circ \vec{\iota} = \psi \circ \vec{\iota}$, and hence $\varphi = \psi$, by Theorem 23. Therefore $(f +_{\mathfrak{L}} h) \vee_{\mathfrak{L}} (g +_{\mathfrak{L}} h) =_{\mathfrak{L}} \varphi(f, g, h) =_{\mathfrak{L}} \psi(f, g, h) =_{\mathfrak{L}} f \vee_{\mathfrak{L}} g +_{\mathfrak{L}} h$ for each $f, g, h \in \mathfrak{L}$. $\square$

Lemma 41. *The maps $E : (L, d_s) \to \mathbf{R}$ and $\| \text{-} \| : (L, d_s) \to \mathbf{R}$ are uniformly continuous.*

Proof. Let $x, y \in L$. Then, since $x \leq |x - y| + y$, we have

$$E(x) \leq E(|x - y|) + E(y) = \|x - y\| + E(y) = d_s(x, y) + E(y),$$

and hence $E(x) - E(y) \leq d_s(x, y)$. Similarly, we have $E(y) - E(x) \leq d_s(x, y)$. Therefore $|E(y) - E(x)| \leq d_s(x, y)$. Since $\|x\| \leq \|x - y\| + \|y\| = d_s(x, y) + \|y\|$, we have $\|x\| - \|y\| \leq d_s(x, y)$, and similarly, we have $\|y\| - \|x\| \leq d_s(x, y)$. Therefore $|\|x\| - \|y\|| \leq d_s(x, y)$. $\square$

Definition 42. Let $\int : \mathfrak{L} \to \mathbf{R}$ and $\| \text{-} \|_{\mathfrak{L}} : \mathfrak{L} \to \mathbf{R}$ be the uniformly continuous extensions of E and $\| \text{-} \|$, respectively. For each $f \in \mathfrak{L}$, $\int f$ is called the *integral* of f, and $\|f\|_{\mathfrak{L}}$ is called the *norm* of f.

Lemma 43. *For each $f, g \in \mathfrak{L}$ and $a \in \mathbf{R}$,*

1. $\int (f +_{\mathfrak{L}} g) = \int f + \int g$ *and* $\int (a \cdot_{\mathfrak{L}} f) = a \int f$;
2. *if* $0 \leq_{\mathfrak{L}} f$, *then* $0 \leq \int f$;
3. $\|f\|_{\mathfrak{L}} = \int |f|_{\mathfrak{L}}$ *and* $\tilde{d}_s(f, g) = \|f - g\|_{\mathfrak{L}}$.

Proof. (1): Since

$$\begin{aligned}\int (\iota_{\mathfrak{L}}(x) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(y)) &= \int \iota_{\mathfrak{L}}(x + y) = E(x + y) = E(x) + E(y) \\ &= \int \iota_{\mathfrak{L}}(x) + \int \iota_{\mathfrak{L}}(y)\end{aligned}$$

and $\int (a \cdot_{\mathfrak{L}} \iota_{\mathfrak{L}}(x)) = \int \iota_{\mathfrak{L}}(ax) = E(ax) = aE(x) = a \int \iota_{\mathfrak{L}}(x)$ for each $x, y \in L$ and $a \in \mathbf{R}$, we have $\int (f +_{\mathfrak{L}} g) = \int f + \int g$ and $\int (a \cdot_{\mathfrak{L}} f) = a \int f$ for each $f, g \in \mathfrak{L}$ and $a \in \mathbf{R}$, by Theorem 23.

(2): For each $x \in L$, since $0 \leq x^+$, we have

$$0 \leq E(x^+) = \int (\iota_{\mathfrak{L}}(x^+)) = \int (\iota_{\mathfrak{L}}(x))^+,$$

and hence $\max\{0, \int (\iota_{\mathfrak{L}}(x))^+\} = \int (\iota_{\mathfrak{L}}(x))^+$. Therefore $\max\{0, \int f^+\} = \int f^+$, by Theorem 23, and so $0 \leq \int f^+$ for each $f \in \mathfrak{L}$. If $0 \leq_{\mathfrak{L}} f$, then $f =_{\mathfrak{L}} f^+$, and hence $0 \leq \int f$.

(3): Since $\|\iota_{\mathfrak{L}}(x)\|_{\mathfrak{L}} = \|x\| = E(|x|) = \int \iota_{\mathfrak{L}}(|x|) = \int |\iota_{\mathfrak{L}}(x)|_{\mathfrak{L}}$ and

$$\begin{aligned}\tilde{d}_s(\iota_{\mathfrak{L}}(x), \iota_{\mathfrak{L}}(y)) = d_s(x, y) = \|x - y\| = E(|x - y|) &= \int \iota_{\mathfrak{L}}(|x - y|) \\ = \int |\iota_{\mathfrak{L}}(x) - \iota_{\mathfrak{L}}(y)|_{\mathfrak{L}} &= \|\iota_{\mathfrak{L}}(x) - \iota_{\mathfrak{L}}(y)\|_{\mathfrak{L}}\end{aligned}$$

for each $x, y \in \mathfrak{L}$, we have $\|f\|_{\mathfrak{L}} = \int |f|_{\mathfrak{L}}$ and $\tilde{d}_s(f, g) = \|f - g\|_{\mathfrak{L}}$, by Theorem 23. $\square$

Remark 44. In particular, $\mathfrak{L}$ is a Banach lattice, in the sense that $(\mathfrak{L}, \| \text{-} \|_{\mathfrak{L}})$ is a Banach space and
$$|f| \leq |g| \Rightarrow \|f\|_{\mathfrak{L}} \leq \|g\|_{\mathfrak{L}}$$
for each $f, g \in \mathfrak{L}$.

5 The second completion

In this section, we define a family D_L of pseudometrics, indexed by L, on L using E. We show that the completion $\mathfrak{M}$ of a uniform space (L, D_L) forms a vector lattice, and construct a uniformly continuous embedding of $\mathfrak{L}$ into $\mathfrak{M}$ (Theorem 50). We also construct a locally uniformly continuous mapping of $\mathfrak{L} \times \mathfrak{M}$ into $\mathfrak{L}$ (Proposition 51) which is crucial to prove the convergence theorems in the next section.

Lemma 45. *Let $u \in L$, and let $d_u : L \times L \to \mathbf{R}$ be a mapping defined by*
$$d_u(x, y) = E(|x - y| \wedge |u|)$$
for each $x, y \in L$ Then d_u is a pseudometric on L.

Proof. It is obvious that $d_u(x, x) = 0$ and $d_u(x, y) = d_u(y, x)$ for each $u, x, y \in L$. For the triangle inequality, we have
$$\begin{aligned} d_u(x, y) &= E(|x - y| \wedge |u|) \leq E((|x - z| + |z - y|) \wedge |u|) \\ &\leq E(|x - z| \wedge |u| + |z - y| \wedge |u|) = E(|x - z| \wedge |u|) + E(|z - y| \wedge |u|) \\ &= d_u(x, z) + d_u(z, y) \end{aligned}$$
for each $u, x, y, z \in L$, by Lemma 33 (2) and (5). $\square$

Lemma 46. *For each $u, x, x', y, y' \in L$,*

1. $d_u(x + y, x' + y') \leq d_u(x, x') + d_u(y, y')$,
2. $d_u(x \vee y, x' \vee y') \leq d_u(x, x') + d_u(y, y')$,
3. $d_u(x \wedge y, x' \wedge y') \leq d_u(x, x') + d_u(y, y')$.

Proof. For each $u, x, x', y, y' \in L$, we have
$$\begin{aligned} d_u(x + y, x' + y') &= E(|(x + y) - (x' + y')| \wedge |u|) \\ &\leq E((|x - x'| + |y - y'|) \wedge |u|) \\ &\leq E(|x - x'| \wedge |u| + |y - y'| \wedge |u|) \\ &= E(|x - x'| \wedge |u|) + E(|y - y'| \wedge |u|) \\ &= d_u(x, x') + d_u(y, y') \end{aligned}$$
and
$$\begin{aligned} d_u(x \vee y, x' \vee y') &= E(|x \vee y - x' \vee y'| \wedge |u|) \\ &\leq E((|x \vee y - x' \vee y| + |x' \vee y - x' \vee y'|) \wedge |u|) \\ &\leq E(|x \vee y - x' \vee y| \wedge |u| + |x' \vee y - x' \vee y'| \wedge |u|) \\ &\leq E(|x - x'| \wedge |u| + |y - y'| \wedge |u|) \\ &= E(|x - x'| \wedge |u|) + E(|y - y'| \wedge |u|) \\ &= d_u(x, x') + d_u(y, y'), \end{aligned}$$

by Lemma 33 (2), (4) and (5). Similarly, we have

$$d_u(x \wedge y, x' \wedge y') \leq d_u(x, x') + d_u(y, y').$$

$\square$

Definition 47. Let $(\mathfrak{M}, \tilde{D}_L)$ be the completion of the uniform space (L, D_L) with $D_L = \{d_u \mid u \in L\}$ and with the equality $=_{D_L}$ on L given by

$$x =_{D_L} y \Leftrightarrow \forall u \in L(d_u(x, y) = 0)$$

for each $x, y \in L$. We write $\iota_{\mathfrak{M}}$ for the inclusion map $\iota_L : L \to \mathfrak{M}$.

Lemma 48. *The mappings $\iota_{\mathfrak{M}} \circ + : (L, D_L) \times (L, D_L) \to \mathfrak{M}$, $\iota_{\mathfrak{M}} \circ \vee : (L, D_L) \times (L, D_L) \to \mathfrak{M}$ and $\iota_{\mathfrak{M}} \circ \wedge : (L, D_L) \times (L, D_L) \to \mathfrak{M}$ are uniformly continuous, and the mapping $\iota_{\mathfrak{M}} \circ (\cdot\cdot\cdot) : \mathbf{R} \times (L, D_L) \to \mathfrak{M}$ is locally uniformly continuous.*

Proof. Define $\alpha : L^* \to (L + L)^*$ by

$$\alpha(\sigma) = (\{0\} \times \sigma)^{+1} * (\{1\} \times \sigma)$$

for each $\sigma \in L^*$, where $\times$ and $^{+1}$ are the operations given in Definition 14 and Definition 3, respectively. Consider $\sigma \in L^*$ and $(x, y), (x', y') \in L \times L$ with

$$d_{\alpha(\sigma)}((x, y), (x', y')) \leq 2^{-|\alpha(\sigma)|}.$$

Then, since

$$\begin{aligned} d_\sigma(x, x') = d_{\{0\}\times\sigma}((x, y), (x', y')) &\leq d_{\alpha(\sigma)}((x, y), (x', y')) \\ &\leq 2^{-|\alpha(\sigma)|} = 2^{-(2|\sigma|+1)} \end{aligned}$$

and $d_\sigma(y, y') = d_{\{1\}\times\sigma}((x, y), (x', y')) \leq d_{\alpha(\sigma)}((x, y), (x', y')) \leq 2^{-(2|\sigma|+1)}$, we have

$$\begin{aligned} \tilde{d}_u(\iota_{\mathfrak{M}}(x + y), \iota_{\mathfrak{M}}(x' + y')) &= d_u(x + y, x' + y') \leq d_u(x, x') + d_u(y, y') \\ &\leq d_\sigma(x, x') + d_\sigma(y, y') \leq 2^{-(2|\sigma|+1)} + 2^{-(2|\sigma|+1)} \\ &= 2^{-2|\sigma|} \leq 2^{-|\sigma|} \end{aligned}$$

for each $u \in \sigma$, by Lemma 46 (1), and hence $\tilde{d}_\sigma(\iota_{\mathfrak{M}}(x + y), \iota_{\mathfrak{M}}(x' + y')) \leq 2^{-|\sigma|}$. Similarly, we have

$$\tilde{d}_\sigma(\iota_{\mathfrak{M}}(x \vee y), \iota_{\mathfrak{M}}(x' \vee y')) \leq 2^{-|\sigma|} \quad \text{and} \quad \tilde{d}_\sigma(\iota_{\mathfrak{M}}(x \wedge y), \iota_{\mathfrak{M}}(x' \wedge y')) \leq 2^{-|\sigma|},$$

by Lemma 46 (2) and (3). Therefore $\iota_{\mathfrak{M}} \circ + : (L, D_L) \times (L, D_L) \to \mathfrak{M}$, $\iota_{\mathfrak{M}} \circ \vee : (L, D_L) \times (L, D_L) \to \mathfrak{M}$ and $\iota_{\mathfrak{M}} \circ \wedge : (L, D_L) \times (L, D_L) \to \mathfrak{M}$ are uniformly continuous with the modulus α.

Let $\boldsymbol{\xi}$ be a regular net in $(\mathbf{R}, d_r) \times (L, D_L)$ with $\boldsymbol{\xi} = ((c_\rho, z_\rho))_{\rho \in (\{r\}+L)^*}$. For each $\sigma \in L^*$, let $\rho_\sigma = (0, r)^2 * (\{1\} \times \sigma)$, and let $\sigma \mapsto N_\sigma$ be a mapping of L^* into $\mathbf{N}$ such that

$$\max\{|c_{\rho_\sigma}|, \|z_{\rho_\sigma}\|\} \leq 2^{N_\sigma - 1} - 1.$$

Define $\beta : L^* \to (\{r\} + L)^*$ by

$$\beta(\sigma) = (0, r)^{N_\sigma + 2} * (\{1\} \times (2^{N_\sigma + |\sigma| + 1} \cdot \sigma))$$

for each $\sigma \in L^*$, where $2^{N_\sigma+|\sigma|+1} \cdot \sigma \in L^*$ is given by $|2^{N_\sigma+|\sigma|+1} \cdot \sigma| = |\sigma|$ and

$$(2^{N_\sigma+|\sigma|+1} \cdot \sigma)(l) = 2^{N_\sigma+|\sigma|+1}\sigma(l)$$

for each $l < |\sigma|$. Note that, since

$$\begin{aligned} d_u(x,y) &= E(|x-y| \wedge |u|) \leq E(|x-y| \wedge 2^{N_\sigma+|\sigma|+1}|u|) \\ &= E(|x-y| \wedge |2^{N_\sigma+|\sigma|+1}u|) = d_{2^{N_\sigma+|\sigma|+1}u}(x,y) \end{aligned}$$

for each $u \in \sigma$ and $x, y \in L$, we have

$$d_{(1,u)}((a,x),(b,y)) \leq d_{(1,2^{N_\sigma+|\sigma|+1})}((a,x),(b,y)) \leq d_{\beta(\sigma)}((a,x),(b,y))$$

for each $u \in \sigma$ and $(a,x),(b,y) \in \mathbf{R} \times L$, and hence $\tilde{d}_{(1,u)}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \leq \tilde{d}_{\beta(\sigma)}(\boldsymbol{\zeta}, \boldsymbol{\eta})$ for each $u \in \sigma$ and regular net $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ in $(\mathbf{R}, d_r) \times (L, D_L)$. Consider $\sigma \in L^*$ and $(a,x),(b,y) \in U_{\beta(\sigma)}(\boldsymbol{\xi})$. Then

$$\begin{aligned} |a-b| &= d_{(0,r)}((a,x),(b,y)) \leq d_{\beta(\sigma)}((a,x),(b,y)) \\ &= \tilde{d}_{\beta(\sigma)}(\iota_{\mathbf{R}\times L}((a,x)), \iota_{\mathbf{R}\times L}((b,y))) \\ &\leq \tilde{d}_{\beta(\sigma)}(\iota_{\mathbf{R}\times L}((a,x)), \boldsymbol{\xi}) + \tilde{d}_{\beta(\sigma)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}((b,y))) \\ &\leq 2^{-|\beta(\sigma)|} + 2^{-|\beta(\sigma)|} = 2^{-(N_\sigma+|\sigma|+1)} \end{aligned}$$

and

$$\begin{aligned} d_{2^{N_\sigma+|\sigma|+1}u}(x,y) &= d_{(1,2^{N_\sigma+|\sigma|+1}u)}((a,x),(b,y)) \leq d_{\beta(\sigma)}((a,x),(b,y)) \\ &= \tilde{d}_{\beta(\sigma)}(\iota_{\mathbf{R}\times L}((a,x)), \iota_{\mathbf{R}\times L}((b,y))) \\ &\leq \tilde{d}_{\beta(\sigma)}(\iota_{\mathbf{R}\times L}((a,x)), \boldsymbol{\xi}) + \tilde{d}_{\beta(\sigma)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}((b,y))) \\ &\leq 2^{-(N_\sigma+|\sigma|+1)} \end{aligned}$$

for each $u \in \sigma$, by Lemma 9. Since

$$\begin{aligned} |c_{\rho_\sigma} - a| &= d_{(0,r)}((c_{\rho_\sigma}, z_{\rho_\sigma}), (a,x)) = \tilde{d}_{(0,r)}(\iota_{\mathbf{R}\times L}(c_{\rho_\sigma}, z_{\rho_\sigma}), \iota_{\mathbf{R}\times L}(a,x)) \\ &\leq \tilde{d}_{(0,r)}(\iota_{\mathbf{R}\times L}(c_{\rho_\sigma}, z_{\rho_\sigma}), \boldsymbol{\xi}) + \tilde{d}_{(0,r)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}(a,x)) \\ &\leq \tilde{d}_{\rho_\sigma}(\iota_{\mathbf{R}\times L}(c_{\rho_\sigma}, z_{\rho_\sigma}), \boldsymbol{\xi}) + \tilde{d}_{\beta(\sigma)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}(a,x)) \\ &\leq 2^{-|\rho_\sigma|} + 2^{-|\beta(\sigma)|} < 1, \end{aligned}$$

by Lemma 10, we have $|a| \leq |c_{\rho_\sigma}| + |c_{\rho_\sigma} - a| \leq |c_{\rho_\sigma}| + 1 \leq 2^{N_\sigma - 1} < 2^{N_\sigma}$, and for each $u \in \sigma$, since

$$\begin{aligned} d_u(z_{\rho_\sigma}, y) &= d_{(1,u)}((c_{\rho_\sigma}, z_{\rho_\sigma}), (b,y)) \leq= \tilde{d}_{(1,u)}(\iota_{\mathbf{R}\times L}((c_{\rho_\sigma}, z_{\rho_\sigma})), \iota_{\mathbf{R}\times L}((b,y))) \\ &\leq \tilde{d}_{(1,u)}(\iota_{\mathbf{R}\times L}((c_{\rho_\sigma}, z_{\rho_\sigma})), \boldsymbol{\xi}) + \tilde{d}_{(1,u)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}((b,y))) \\ &\leq \tilde{d}_{\rho_\sigma}(\iota_{\mathbf{R}\times L}((c_{\rho_\sigma}, z_{\rho_\sigma})), \boldsymbol{\xi}) + \tilde{d}_{\beta(\sigma)}(\boldsymbol{\xi}, \iota_{\mathbf{R}\times L}((b,y)))) \\ &\leq 2^{-|\rho_\sigma|} + 2^{-|\beta(\sigma)|} = 2^{-(|\sigma|+2)} + 2^{-(N_\sigma+|\sigma|+2)}, \end{aligned}$$

by Lemma 10, we have

$$
\begin{aligned}
E(|y| \wedge |2^{N_\sigma+|\sigma|+1}u|) &\le E((|z_{\rho_\sigma}| + |z_{\rho_\sigma} - y|) \wedge |2^{N_\sigma+|\sigma|+1}u|) \\
&\le E(|z_{\rho_\sigma}| \wedge |2^{N_\sigma+|\sigma|+1}u| + |z_{\rho_\sigma} - y| \wedge |2^{N_\sigma+|\sigma|+1}u|) \\
&= E(|z_{\rho_\sigma}| \wedge |2^{N_\sigma+|\sigma|+1}u|) + E(|z_{\rho_\sigma} - y| \wedge |2^{N_\sigma+|\sigma|+1}u|) \\
&\le E(|z_{\rho_\sigma}|) + E((2^{N_\sigma+|\sigma|+1}|z_{\rho_\sigma} - y|) \wedge (2^{N_\sigma+|\sigma|+1}|u|)) \\
&\le E(|z_{\rho_\sigma}|) + 2^{N_\sigma+|\sigma|+1}E(|z_{\rho_\sigma} - y| \wedge |u|) \\
&= \|z_{\rho_\sigma}\| + 2^{N_\sigma+|\sigma|+1}d_u(z_{\rho_\sigma}, y) \\
&\le 2^{N_\sigma-1} - 1 + 2^{N_\sigma+|\sigma|+1} \cdot (2^{-(|\sigma|+2)} + 2^{-(N_\sigma+|\sigma|+2)}) \\
&< 2^{N_\sigma}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\tilde{d}_u(\iota_{\mathfrak{M}}(ax), \iota_{\mathfrak{M}}(by)) &= d_u(ax, by) = E(|ax - by| \wedge |u|) \\
&= E((|ax - ay| + |ay - by|) \wedge |u|) \\
&= E((|a||x - y|) \wedge |u|) + E((|a - b||y|) \wedge |u|) \\
&\le E((2^{N_\sigma}|x - y|) \wedge |u|) + E((2^{-(N_\sigma+|\sigma|+1)}|y|) \wedge |u|) \\
&\le E((2^{N_\sigma}|x - y|) \wedge (2^{2N_\sigma+|\sigma|+1}|u|)) \\
&\qquad + E((2^{-(N_\sigma+|\sigma|+1)}|y|) \wedge (2^{-(N_\sigma+|\sigma|+1)}|2^{N_\sigma+|\sigma|+1}u|)) \\
&= 2^{N_\sigma} d_{2^{N_\sigma+|\sigma|+1}u}(x, y) + 2^{-(N_\sigma+|\sigma|+1)}E(|y| \wedge |2^{N_\sigma+|\sigma|+1}u|) \\
&\le 2^{-(|\sigma|+1)} + 2^{-(|\sigma|+1)} = 2^{-|\sigma|}
\end{aligned}
$$

for each $u \in \sigma$, and hence $\tilde{d}_\sigma(\iota_{\mathfrak{L}}(ax), \iota_{\mathfrak{L}}(by)) \le 2^{-|\sigma|}$. Therefore $\iota_{\mathfrak{M}} \circ (- \cdot -) : \mathbf{R} \times (L, D_L) \to \mathfrak{M}$, is uniformly continuous with the modulus β. $\Box$

Proposition 49. *$\mathfrak{M}$ is a vector lattice.*

Proof. Similar to the proof of Proposition 40 using Lemma 48. $\Box$

Theorem 50. *There exists a uniformly continuous embedding $\kappa : \mathfrak{L} \to \mathfrak{M}$ such that $\kappa \circ \iota_{\mathfrak{L}} = \iota_{\mathfrak{M}}$.*

Proof. For each $x, y \in L$, since

$$\tilde{d}_u(\iota_{\mathfrak{M}}(x), \iota_{\mathfrak{M}}(y)) = d_u(x, y) = E(|x - y| \wedge |u|) \le E(|x - y|) = d_s(x, y)$$

for each $u \in L$, we have

$$d_s(x, y) \le 2^{-|\sigma|} \Rightarrow \tilde{d}_\sigma(\iota_{\mathfrak{M}}(x), \iota_{\mathfrak{M}}(y)) \le 2^{-|\sigma|}$$

for each $\sigma \in L^*$. Therefore $\iota_{\mathfrak{M}}$ is a uniformly continuous mapping of (L, d_s) into $\mathfrak{M}$, and so there exists a uniformly continuous extension $\kappa : \mathfrak{L} \to \mathfrak{M}$ such that $\kappa \circ \iota_{\mathfrak{L}} = \iota_{\mathfrak{M}}$, by Theorem 23.

To show that κ is injective, let $f = (x_n), g = (y_n) \in \mathfrak{L}$, and suppose that $\kappa(f) =_{\mathfrak{M}} \kappa(g)$. Note that $\iota_{\mathfrak{L}}(x_n) \to f$ and $\iota_{\mathfrak{L}}(y_n) \to g$ in $\mathfrak{L}$, by Lemma 10, and hence $\kappa(\iota_{\mathfrak{L}}(x_n)) \to \kappa(f)$ and $\kappa(\iota_{\mathfrak{L}}(y_n)) \to \kappa(g)$ in $\mathfrak{M}$, as $n \to \infty$. Then for each m and n, setting $z = x_m - y_m$, since

$$\begin{aligned} |x_n - y_n| = |x_n - y_n| \wedge |x_n - y_n| &= |z + (x_n - x_m) + (y_m - y_n)| \wedge |x_n - y_n| \\ &\leq (|z| + |x_n - x_m| + |y_m - y_n|) \wedge |x_n - y_n| \\ &\leq |z| \wedge |x_n - y_n| + |x_n - x_m| \wedge |x_n - y_n| + |y_m - y_n| \wedge |x_n - y_n| \\ &\leq |z| \wedge |x_n - y_n| + |x_n - x_m| + |y_m - y_n|, \end{aligned}$$

we have

$$\begin{aligned} d_s(x_n, y_n) &\leq E(|z| \wedge |x_n - y_n|) + d_s(x_n, x_m) + d_s(y_m, y_n) \\ &= d_z(x_n, y_n) + d_s(x_n, x_m) + d_s(y_m, y_n) \\ &= \tilde{d}_z(\iota_{\mathfrak{M}}(x_n), \iota_{\mathfrak{M}}(y_n)) + \tilde{d}_s(\iota_{\mathfrak{L}}(x_n), \iota_{\mathfrak{L}}(x_m)) + \tilde{d}_s(\iota_{\mathfrak{L}}(y_m), \iota_{\mathfrak{L}}(y_n)) \\ &\leq \tilde{d}_z(\kappa(\iota_{\mathfrak{L}}(x_n)), \kappa(f)) + \tilde{d}_z(\kappa(f), \kappa(g)) + \tilde{d}_z(\kappa(g), \kappa(\iota_{\mathfrak{L}}(y_n))) \\ &\quad + \tilde{d}_s(\iota_{\mathfrak{L}}(x_n), \iota_{\mathfrak{L}}(x_m)) + \tilde{d}_s(\iota_{\mathfrak{L}}(y_m), \iota_{\mathfrak{L}}(y_n)) \\ &= \tilde{d}_z(\kappa(\iota_{\mathfrak{L}}(x_n)), \kappa(f)) + \tilde{d}_z(\kappa(g), \kappa(\iota_{\mathfrak{L}}(y_n))) \\ &\quad\quad + \tilde{d}_s(\iota_{\mathfrak{L}}(x_n), \iota_{\mathfrak{L}}(x_m)) + \tilde{d}_s(\iota_{\mathfrak{L}}(y_m), \iota_{\mathfrak{L}}(y_n)), \end{aligned}$$

and hence, letting $n \to \infty$, we have $\tilde{d}_s(f, g) \leq \tilde{d}_s(f, \iota_{\mathfrak{L}}(x_m)) + \tilde{d}_s(\iota_{\mathfrak{L}}(y_m), g)$. Therefore, letting $m \to \infty$, we have $\tilde{d}_s(f, g) = 0$.

Since

$$\begin{aligned} \kappa(\iota_{\mathfrak{L}}(x) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(y)) &=_{\mathfrak{M}} \kappa(\iota_{\mathfrak{L}}(x + y)) =_{\mathfrak{M}} \iota_{\mathfrak{M}}(x + y) =_{\mathfrak{M}} \iota_{\mathfrak{M}}(x) +_{\mathfrak{M}} \iota_{\mathfrak{M}}(y) \\ &=_{\mathfrak{M}} \kappa(\iota_{\mathfrak{L}}(x)) +_{\mathfrak{M}} \kappa(\iota_{\mathfrak{L}}(y)) \end{aligned}$$

for each $x, y \in L$, we have $\kappa(f +_{\mathfrak{L}} g) =_{\mathfrak{M}} \kappa(f) +_{\mathfrak{M}} \kappa(g)$ for each $f, g \in \mathfrak{L}$, by Theorem 23. Similarly, $\kappa(a \cdot_{\mathfrak{L}} f) =_{\mathfrak{M}} a \cdot_{\mathfrak{M}} \kappa(f)$ and $\kappa(f \vee_{\mathfrak{L}} g) =_{\mathfrak{M}} \kappa(f) \vee_{\mathfrak{M}} \kappa(g)$ for each $f, g \in \mathfrak{L}$ and $a \in \mathbf{R}$. □

Proposition 51. *There exists a locally uniformly continuous mapping $\theta : \mathfrak{L} \times \mathfrak{M} \to \mathfrak{L}$ such that*

$$\theta(f, \kappa(g)) =_{\mathfrak{L}} f \wedge_{\mathfrak{L}} |g|_{\mathfrak{L}} \quad \text{and} \quad \kappa(\theta(f, h)) =_{\mathfrak{M}} \kappa(f) \wedge_{\mathfrak{M}} |h|_{\mathfrak{M}}$$

for each $f, g \in \mathfrak{L}$ and $h \in \mathfrak{M}$.

Proof. Let $\theta_0 : L \times L \to \mathfrak{L}$ be a mapping defined by

$$\theta_0(x, u) = \iota_{\mathfrak{L}}(x \wedge |u|)$$

for each $x, u \in L$. Then we show that θ_0 is a locally uniformly continuous mapping of $(L, d_s) \times (L, D_L)$ into $\mathfrak{L}$. To this end, let $\boldsymbol{\xi}$ be a regular net in $(L, d_s) \times (L, D_L)$ with $\boldsymbol{\xi} = ((z_\rho, w_\rho))_{\rho \in (\{s\}+L)^*}$, and define $\alpha : \mathbf{N} \to (\{s\} + L)^*$ by

$$\alpha(n) = \sigma_n * (1, z_{\sigma_n})^1,$$

where $\sigma_n = (0, s)^{n+2}$, for each n. Consider n and $(x, u), (y, v) \in U_{\alpha(n)}(\boldsymbol{\xi})$. Then we have

$$\begin{aligned} d_s(x, z_{\sigma_n}) = d_{(0,s)}((x, u), (z_{\sigma_n}, w_{\sigma_n})) &= \tilde{d}_{(0,s)}(\iota_{L\times L}((x, u)), \iota_{L\times L}((z_{\sigma_n}, w_{\sigma_n}))) \\ &\leq \tilde{d}_{(0,s)}(\iota_{L\times L}((x, u)), \boldsymbol{\xi}) + \tilde{d}_{(0,s)}(\boldsymbol{\xi}, \iota_{L\times L}((z_{\sigma_n}, w_{\sigma_n}))) \\ &\leq \tilde{d}_{\alpha(n)}(\iota_{L\times L}((x, u)), \boldsymbol{\xi}) + \tilde{d}_{\sigma_n}(\boldsymbol{\xi}, \iota_{L\times L}((z_{\sigma_n}, w_{\sigma_n}))) \\ &\leq 2^{-|\alpha(n)|} + 2^{-|\sigma_n|} = 2^{-(n+3)} + 2^{-(n+2)}, \end{aligned}$$

and, similarly, we have $d_s(y, z_{\sigma_n}) \leq 2^{-(n+3)} + 2^{-(n+2)}$. Since

$$\begin{aligned}
d_s(z_{\sigma_n} \wedge |u|, z_{\sigma_n} \wedge |v|) &= E(||u| \wedge z_{\sigma_n} - |v| \wedge z_{\sigma_n}|) \leq E(||u| - |v|| \wedge |z_{\sigma_n}|) \\
&\leq E(|u - v| \wedge |z_{\sigma_n}||) = d_{z_{\sigma_n}}(u, v) \\
&= d_{(1,z_{\sigma_n})}((x, u), (y, v)) \leq d_{\alpha(n)}((x, u), (y, v)) \\
&= \tilde{d}_{\alpha(n)}(\iota_{L\times L}((x, u)), \iota_{L\times L}((y, v))) \\
&\leq \tilde{d}_{\alpha(n)}(\iota_{L\times L}((x, u)), \boldsymbol{\xi}) + \tilde{d}_{\alpha(n)}(\boldsymbol{\xi}, \iota_{L\times L}((y, v))) \\
&\leq 2^{-|\alpha(n)|} + 2^{-|\alpha(n)|} = 2^{-(n+3)} + 2^{-(n+3)} = 2^{-(n+2)}
\end{aligned}$$

by Lemma 33 (2) and (6), we have

$$\begin{aligned}
\tilde{d}_s(\theta_0(x, u), \theta_0(y, v)) &= \tilde{d}_s(\iota_{\mathfrak{L}}(x \wedge |u|), \iota_{\mathfrak{L}}(y \wedge |v|)) = d_s(x \wedge |u|, y \wedge |v|) \\
&\leq d_s(x \wedge |u|, z_{\sigma_n} \wedge |u|) + d_s(z_{\sigma_n} \wedge |u|, z_{\sigma_n} \wedge |v|) \\
&\qquad + d_s(z_{\sigma_n} \wedge |v|, y \wedge |v|) \\
&\leq d_s(x, z_{\sigma_n}) + 2^{-(n+2)} + d_s(z_{\sigma_n}, y) \\
&\leq 2^{-(n+3)} + 2^{-(n+2)} + 2^{-(n+2)} + 2^{-(n+3)} + 2^{-(n+2)} \\
&= 2^{-n},
\end{aligned}$$

by Lemma 37 (3). Therefore θ_0 is locally uniformly continuous, and so there exists a locally uniformly continuous extension $\theta : \mathfrak{L} \times \mathfrak{M} \to \mathfrak{L}$ such that $\theta \circ (\iota_{\mathfrak{L}} \times \iota_{\mathfrak{M}}) = \theta_0$, by Theorem 23. Since

$$\begin{aligned}
\theta(\iota_{\mathfrak{L}}(x), \kappa(\iota_{\mathfrak{L}}(u))) &=_{\mathfrak{L}} \theta(\iota_{\mathfrak{L}}(x), \iota_{\mathfrak{M}}(u)) =_{\mathfrak{L}} \theta_0(x, u) \\
&=_{\mathfrak{L}} \iota_{\mathfrak{L}}(x \wedge |u|) =_{\mathfrak{L}} \iota_{\mathfrak{L}}(x) \wedge_{\mathfrak{L}} |\iota_{\mathfrak{L}}(u)|_{\mathfrak{L}}
\end{aligned}$$

and

$$\begin{aligned}
\kappa(\theta(\iota_{\mathfrak{L}}(x), \iota_{\mathfrak{M}}(u))) &=_{\mathfrak{M}} \kappa(\theta_0(x, u)) =_{\mathfrak{M}} \kappa(\iota_{\mathfrak{L}}(x \wedge |u|)) \\
&=_{\mathfrak{M}} \iota_{\mathfrak{M}}(x) \wedge_{\mathfrak{M}} |\iota_{\mathfrak{M}}(u)|_{\mathfrak{M}} =_{\mathfrak{M}} \kappa(\iota_{\mathfrak{L}}(x)) \wedge_{\mathfrak{M}} |\iota_{\mathfrak{M}}(u)|_{\mathfrak{M}}
\end{aligned}$$

for each $x, u \in L$, we have $\theta(f, \kappa(g)) =_{\mathfrak{L}} f \wedge_{\mathfrak{L}} |g|_{\mathfrak{L}}$ and $\kappa(\theta(f, h)) =_{\mathfrak{M}} \kappa(f) \wedge_{\mathfrak{M}} |h|_{\mathfrak{M}}$ for each $f, g \in \mathfrak{L}$ and $h \in \mathfrak{M}$, by Theorem 23. □

6 Measurable and integrable functions

In this section, we define measurable functions and integrable functions on an abstract integration space as elements of $\mathfrak{M}$ and $\mathfrak{L}$, respectively. We prove several convergence theorems including Fatou's lemma (Corollary 58), and the monotone and dominated convergence theorems of Lebesgue (Theorem 60 and Theorem 61) in a totally topological framework we have developed in the previous sections.

Definition 52. An element f of $\mathfrak{M}$ is called a *measurable function* over an abstract integration space (L, E). A net (f_λ) of measurable functions *converges in measure* to a measurable function f if (f_λ) converges to f in $\mathfrak{M}$.

A measurable function f is *integrable* if there exists $g \in \mathfrak{L}$ such that $f =_{\mathfrak{M}} \kappa(g)$; note that, since κ is injective, such a $g \in \mathfrak{L}$ is unique. We identify an integrable function f and $g \in \mathfrak{L}$ with $f =_{\mathfrak{M}} \kappa(g)$, and the *integral* $\int f$ of f is given by $\int f = \int g$. With this identification, we will omit the subscripts $_{\mathfrak{L}}$ and $_{\mathfrak{M}}$ for the relations and the operations. A net (f_λ) of integrable functions *converges in norm* to an integrable function f if (f_λ), as a net in $\mathfrak{L}$, converges to f in $\mathfrak{L}$.

Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net of measurable functions on $(\Lambda, \preccurlyeq)$. Then (f_λ) is *increasing* if

$$\lambda \preccurlyeq \mu \Rightarrow f_\lambda \leq f_\mu$$

for each $\lambda, \mu \in \Lambda$. For a predicate $P(f)$ on the measurable functions, we say that $P(f_\lambda)$ *holds eventually* if there exists $\lambda \in \Lambda$ such that $P(f_\mu)$ holds for each $\mu \in \Lambda$ with $\lambda \preccurlyeq \mu$.

Lemma 53. *Let f be a measurable function with $0 \leq f$, and let g be an integrable function. Then $f \wedge g$ is an integrable function.*

Proof. We have $f \wedge g = |f| \wedge g = \theta(g, f)$, and $\theta(g, f)$ is integrable, by Proposition 51. □

Theorem 54. *Let f be a measurable function. If there exists an integrable function g such that $|f| \leq g$, then f is integrable.*

Proof. Suppose that $|f| \leq g$ for some integrable function g. Then, since $f^+ \leq |f| \leq g$ and $f^- \leq |f| \leq g$, $f^+ = f^+ \wedge g$ and $f^- = f^- \wedge g$ are integrable, by Lemma 53, and hence $f = f^+ - f^-$ is integrable. □

Lemma 55. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net of measurable functions converging in measure to a measurable function f, and let g be a measurable function. If $f_\lambda \leq g$ holds eventually, then $f \leq g$, and if $g \leq f_\lambda$ holds eventually, then $g \leq f$.*

Proof. Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net of measurable functions on $(\Lambda, \preccurlyeq)$ converging in measure to a measurable function f, and consider $u \in L$ and n. Then, since $\vee : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ is uniformly continuous, there exists $\lambda \in \Lambda$ such that

$$\tilde{d}_u(f_\mu \vee g, f \vee g) \leq 2^{-n}$$

for each $\mu \in \Lambda$ with $\lambda \preccurlyeq \mu$. Assume that $f_\lambda \leq g$ holds eventually. Then there exists $\lambda' \in \Lambda$ such that $g = f_\mu \vee g$ for each $\mu \in \Lambda$ with $\lambda' \preccurlyeq \mu$, and hence, choosing $\mu \in \Lambda$ so that $\lambda \preccurlyeq \mu$ and $\lambda' \preccurlyeq \mu$, we have

$$\tilde{d}_u(g, f \vee g) = \tilde{d}_u(f_\mu \vee g, f \vee g) \leq 2^{-n}.$$

Therefore, letting $n \to \infty$, we have $\tilde{d}_u(g, f \vee g) = 0$ for each u, and so $g = f \vee g$.

Similarly, if $g \leq f_\lambda$ holds eventually, then $g \leq f$. □

Proposition 56. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be an increasing net of measurable functions converging in measure to a measurable function f. Then $f_\lambda \leq f$ for each $\lambda \in \Lambda$.*

Proof. Consider $\lambda \in \Lambda$, and a net $(f_\lambda \wedge f_\mu)_{\mu\in\Lambda}$. Then, since $\wedge : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ is uniformly continuous, $(f_\lambda \wedge f_\mu)_{\mu\in\Lambda}$ converges in measure to $f_\lambda \wedge f$, and, since $f_\lambda \leq f_\lambda \wedge f_\mu$ holds eventually (in μ), we have $f_\lambda \leq f_\lambda \wedge f \leq f_\lambda$, by Lemma 55. Therefore $f_\lambda \leq f$. □

Lemma 57. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net of measurable functions converging in measure to an integrable function f such that $0 \leq f_\lambda$ for each $\lambda \in \Lambda$. Then $(f_\lambda \wedge f)_{\lambda\in\Lambda}$ converges in norm to f.*

Proof. Note that $f_\lambda \wedge f$ is integrable for each $\lambda \in \Lambda$, by Lemma 53, and $0 \leq f$, by Lemma 55. Then, since $\theta : \mathfrak{L} \times \mathfrak{M} \to \mathfrak{L}$ is locally uniformly continuous, we have

$$f_\lambda \wedge f = |f_\lambda| \wedge f = \theta(f, f_\lambda)) \to \theta(f, f) = |f| \wedge f = f$$

in $\mathfrak{L}$. $\square$

Corollary 58 (Fatou's Lemma). *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net of integrable functions converging in measure to an integrable function f such that $0 \leq f_\lambda$ and $\int f_\lambda \leq A$ for each $\lambda \in \Lambda$. Then $\int f \leq A$.*

Proof. Note that $f_\lambda \wedge f \to f$ in $\mathfrak{L}$, by Lemma 57. Then, since $\int : \mathfrak{L} \to \mathbf{R}$ is uniformly continuous, we have $\int f_\lambda \wedge f \to \int f$, and, since $\int f_\lambda \wedge f \leq \int f_\lambda \leq A$ for each $\lambda \in \Lambda$, we have $\int f \leq A$. $\square$

Lemma 59. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be an increasing net of integrable functions. If $(\int f_\lambda)$ converges, then (f_λ) converges in norm.*

Proof. Suppose that $(\int f_\lambda)_{\lambda\in\Lambda}$ converges to $a \in \mathbf{R}$ with a modulus $\alpha : \mathbf{N} \to \Lambda$. For each n and $\mu, \nu \in \Lambda$ with $\alpha(n+2) \preccurlyeq \mu, \nu$, we have

$$\begin{aligned}
\tilde{d}_s(f_\mu, f_\nu) &\leq \tilde{d}_s(f_\mu, f_{\alpha(n+1)}) + \tilde{d}_s(f_{\alpha(n+1)}, f_\nu) \\
&= \int |f_\mu - f_{\alpha(n+1)}| + \int |f_{\alpha(n+1)} - f_\nu| \\
&= \int (f_\mu - f_{\alpha(n+1)}) + \int (f_\nu - f_{\alpha(n+1)}) \\
&= \left(\int f_\mu - \int f_{\alpha(n+1)}\right) + \left(\int f_\nu - \int f_{\alpha(n+1)}\right) \\
&= \left(\int f_\mu - a\right) + \left(a - \int f_{\alpha(n+1)}\right) \\
&\qquad + \left(\int f_\nu - a\right) + \left(a - \int f_{\alpha(n+1)}\right) \\
&\leq \left|\int f_\mu - a\right| + \left|a - \int f_{\alpha(n+1)}\right| + \left|\int f_\nu - a\right| + \left|a - \int f_{\alpha(n+1)}\right| \\
&\leq 2^{-(n+2)} + 2^{-(n+2)} + 2^{-(n+2)} + 2^{-(n+2)} = 2^{-n}.
\end{aligned}$$

Therefore (f_λ) is a Cauchy net in $\mathfrak{L}$ with a modulus $n \mapsto \alpha(n+2)$, and so converges to a limit $f \in \mathfrak{L}$. $\square$

Theorem 60 (Lebesgue's Monotone Convergence Theorem). *Let $(f_\lambda)_{\lambda\in\Lambda}$ be an increasing net of measurable functions such that $0 \leq f_\lambda$ for each $\lambda \in \Lambda$. Then (f_λ) converges in measure to an integrable function f if and only if each f_λ is integrable and $(\int f_\lambda)_{\lambda\in\Lambda}$ converges; in which case*

$$\int f_\lambda \to \int f.$$

Proof. Suppose that (f_λ) converges in measure to an integrable function f. Then, since $f_\lambda \leq f$ for each $\lambda \in \Lambda$, by Proposition 56, each $f_\lambda = f_\lambda \wedge f$ is integrable by Lemma 53. Since $f_\lambda = f_\lambda \wedge f \to f$ in $\mathfrak{L}$, by Lemma 57, and $\int : \mathfrak{L} \to \mathbf{R}$ is uniformly continuous, we have $\int f_\lambda \to \int f$.

Conversely, suppose that each f_λ is integrable and $(\int f_\lambda)_{\lambda\in\Lambda}$ converges. Then (f_λ) converges in norm (and hence in measure) to an integrable function f, by Lemma 59, and, since $\int : \mathfrak{L} \to \mathbf{R}$ is uniformly continuous, we have $\int f_\lambda \to \int f$. $\square$

Theorem 61 (Lebesgue's Dominated Convergence Theorem)**.** *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net of measurable functions converging in measure to a measurable function f, and let g be an integrable function such that $|f_\lambda| \leq g$ for each $\lambda \in \Lambda$. Then each f_λ and f are integrable, and (f_λ) converges in norm to f.*

Proof. Since $(\text{-})^+ : \mathfrak{M} \to \mathfrak{M}$ is uniformly continuous, $(f_\lambda^+)_{\lambda\in\Lambda}$ converges in measure to f^+, and, since $f_\lambda^+ \leq |f_\lambda| \leq g$ for each $\lambda \in \Lambda$, we have $f^+ \leq g$, by Lemma 55. Therefore each f_λ^+ and f^+ are integrable, by Lemma 53. Since $\theta : \mathfrak{L} \times \mathfrak{M} \to \mathfrak{L}$ is locally uniformly continuous, we have

$$f_\lambda^+ = f_\lambda^+ \wedge g = |f_\lambda^+| \wedge g = \theta(g, f_\lambda^+) \to \theta(g, f^+) = |f^+| \wedge g = f^+ \wedge g = f^+$$

in $\mathfrak{L}$, and hence (f_λ^+) converges in norm to f^+. Similarly, each f_λ^- and f^- are integrable, and (f_λ^-) converges in norm to f^-. Therefore each f_λ $(= f_\lambda^+ - f_\lambda^-)$ and f $(= f^+ - f^-)$ are integrable, and (f_λ) converges in norm to f. $\square$

7 Concluding remarks

We conclude the paper with remarks on the classical and constructive definitions of a vector lattices and on a possible metrization of $\mathfrak{M}$.

Remark 62. An *ordered linear space* is a linear space L equipped with a partial order $\leq$ satisfying (1) and (2) of Lemma 28. Note that Lemma 28 (3) follows from (1) and (2) of Lemma 28. Classically, a vector lattice is an ordered linear space such that for each $x, y \in L$, the least upper bound $x \vee y$ of $\{x, y\}$ exists.

In a classical vector lattice L, it is straightforward to see that

1. $(x + z) \vee (y + z) = x \vee y + z$,
2. if $0 \leq a$, then $a(x \vee y) \geq (ax) \vee (ay)$,
3. if $0 \leq x$, then $(a \vee b) \geq (ax) \vee (bx)$

for each $x, y, z \in L$ and $a, b \in \mathbf{R}$. However, a classical vector lattice L is a vector lattice in the sense of Definition 26 if the equality $=$ on L is *stable*, that is,

$$\neg\neg x = y \to x = y$$

for each $x, y \in L$.

In fact, assume that $0 \leq a$ and $\neg(a(x \vee y) = ax \vee ay)$. If $0 < a$, then, since $ax, ay \leq (ax) \vee (ay)$, we have $x, y \leq a^{-1}((ax) \vee (ay))$, and hence $x \vee y \leq a^{-1}((ax) \vee (ay))$; whence $a(x \vee y) \leq (ax) \vee (ay)$, a contradiction. Therefore $a \leq 0$, and, since $a = 0$, we have $a(x \vee y) = ax \vee ay$, a contradiction. Thus $\neg\neg(a(x \vee y) = ax \vee ay)$, and so $a(x \vee y) = ax \vee ay$, by the stability. Assume that

$0 \leq x$ and $\neg((a \vee b)x = (ax) \vee (bx))$. If $a \leq b$, then $(a \vee b)x = bx \leq (ax) \vee (bx)$, a contradiction, and if $b \leq a$, then $(a \vee b)x = ax \leq (ax) \vee (bx)$, a contradiction. Therefore $\neg(a \leq b) \wedge \neg(b \leq a)$, that is, $\neg\neg(b < a \wedge a < b)$, a contradiction. Thus $\neg\neg((a \vee b)x = (ax) \vee (bx))$, and so $(a \vee b)x = (ax) \vee (bx)$, by the stability.

Note that the conditions (2) and (3) of Definition 26 are crucial in the proof of Lemma 33 (3).

Remark 63. A subset L_0 of L is *dense* in an abstract integration space (L, E) if there exists a mapping $\delta : L \times N \to L_0$ such that

$$E(|x - \delta(x, n)|) \leq 2^{-n}$$

for each $x \in L$ and n. Let L_0 be a dense subset of L, and let $(\mathfrak{M}_0, \tilde{D}_{L_0})$ be the completion of the uniform space (L, D_{L_0}) with $D_{L_0} = \{d_u \mid u \in L_0\}$. Then, since $L_0 \subseteq L$, the identity mapping $\mathrm{id}_L : (L, D_L) \to (L, D_{L_0})$ is uniformly continuous with a modulus $\sigma \mapsto \sigma$ as a mapping of L_0^* into L^*. For each $\sigma \in L^*$, define $\alpha : L^* \to L_0^*$ by

$$\alpha(\sigma) = \langle \delta(0, 0), \delta(\sigma(0), |\sigma| + 1), \ldots, \delta(\sigma(|\sigma| - 1), |\sigma| + 1) \rangle.$$

Then for each $\sigma \in L^*$ and $x, y \in L$, if $d_{\alpha(\sigma)}(x, y) \leq 2^{-|\alpha(\sigma)|}$, then, since

$$\begin{aligned}
d_u(x, y) &= E(|x - y| \wedge |u|) = E(|x - y| \wedge |\delta(u, |\sigma| + 1) + (u - \delta(u, |\sigma| + 1))|) \\
&\leq E(|x - y| \wedge (|\delta(u, |\sigma| + 1)| + |u - \delta(u, |\sigma| + 1)|)) \\
&\leq E(|x - y| \wedge |\delta(u, |\sigma| + 1)| + |x - y| \wedge |u - \delta(u, |\sigma| + 1)|) \\
&\leq E(|x - y| \wedge |\delta(u, |\sigma| + 1)| + |u - \delta(u, |\sigma| + 1)|) \\
&= E(|x - y| \wedge |\delta(u, |\sigma| + 1)|) + E(|u - \delta(u, |\sigma| + 1)|) \\
&\leq d_{\delta(u, |\sigma|+1)}(x, y) + 2^{-(|\sigma|+1)} \leq d_{\alpha(\sigma)}(x, y) + 2^{-(|\sigma|+1)} \\
&\leq 2^{-(|\sigma|+1)} + 2^{-(|\sigma|+1)} = 2^{-|\sigma|}
\end{aligned}$$

for each $u \in \sigma$, we have $d_\sigma(x, y) \leq 2^{-|\sigma|}$. Therefore $\mathrm{id}_L : (L, D_{L_0}) \to (L, D)$ is uniformly continuous with the modulus α. Let $\tilde{f} : \mathfrak{M} \to \mathfrak{M}_0$ and $\tilde{g} : \mathfrak{M}_0 \to \mathfrak{M}$ be the uniformly continuous extensions of $\iota_{\mathfrak{M}_0} \circ \mathrm{id}_L$ and $\iota_{\mathfrak{M}} \circ \mathrm{id}_L$, respectively. Then, since $\tilde{f} \circ \iota_{\mathfrak{M}} = \iota_{\mathfrak{M}_0} \circ \mathrm{id}_L = \iota_{\mathfrak{M}_0}$ and $\tilde{g} \circ \iota_{\mathfrak{M}_0} = \iota_{\mathfrak{M}} \circ \mathrm{id}_L = \iota_{\mathfrak{M}}$, we have $\tilde{f} \circ \tilde{g} \circ \iota_{\mathfrak{M}_0} = \iota_{\mathfrak{M}_0}$ and $\tilde{g} \circ \tilde{f} \circ \iota_{\mathfrak{M}} = \iota_{\mathfrak{M}}$, and hence $\tilde{f} \circ \tilde{g} = \mathrm{id}_{\mathfrak{M}_0}$ and $\tilde{g} \circ \tilde{f} = \mathrm{id}_{\mathfrak{M}}$. Thus $\mathfrak{M}$ and $\mathfrak{M}_0$ are uniformly equivalent.

Note that if (L, E) has a countable dense subset L_0, then $\mathfrak{M}_0$ is metrizable, and hence $\mathfrak{M}$ is metrizable; see [8, Remark 17].

Acknowledgment

The author hearty thanks Helmut Schwichtenberg for encouraging him to make an early rough draft in 2004 into the paper, and Samuele Maschio for stimulus discussions. He also thanks the Japan Society for the Promotion of Science (JSPS), Core-to-Core Program (A. Advanced Research Networks) for supporting the research.

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Hajime Ishihara
School of Information Science
Japan Advanced Institute of Science and Technology
Nomi, Ishikawa 923-1292, Japan
E-mail: ishihara@jaist.ac.jp