Efficient algorithms from proofs in constructive analysis - Mark Bickford

Meike Beatrice Lorenz
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1 Representation of Real Numbers

Definition 1. (Bishop)
A real number is a sequence $(q_n)_n, q_n \in \mathbb{Q}$ with $\forall n, m \in \mathbb{N}^+ : |q_n - q_m| \leq \frac{1}{n} + \frac{1}{m}$

Bishop's definition of real numbers has some drawbacks when it comes to computing real numbers up to a certain accuracy, i.e. we would need to compute way more decimal digits than we actually care about (for details see the Bickford paper, p.832). To circumvent this issue, Nuprl uses *regular sequences* for the reals, where the n-th rational number is normalized to have the denominator 2n. As a consequence, regular sequences are integer sequences and avoid some of the pitfalls of rational arithmetic.

Definition 2. (*Regular Sequence*)
A sequence $(x_n)_n$ of integers is regular if $\forall n, m \in \mathbb{N}^+ : |mx_n - nx_m| \leq 2(n + m)$

Note that in contrast to Cauchy sequences with modulo functions, here the modulus is simply the identity. The real numbers are then defined as all the regular sequences.

Definition 3. $\mathbb{R} = \{x : \mathbb{N}^+ \to \mathbb{Z}, x \text{ is regular}\}$

2 Equivalence of Real Numbers

Similar to the case when we define real numbers as Cauchy sequences and modulus, we can define equivalence $\equiv$ between real numbers and show that $\equiv$ is an equivalence relation on $\mathbb{R}$.

Definition 4. (Equivalence)
The real numbers x and y are equivalent ($x \equiv y$) iff $\forall n \in \mathbb{N}^+ : |x_n - y_n| \leq 4$.

Definition 5. (bdiff) The relation $\text{bdiff}(x, y) \equiv \exists B \in \mathbb{N} \forall n \in \mathbb{N}^+ : |x_n - y_n| \leq B$ says that the pointwise difference between x and y is bounded.

Lemma 1. For all $x, y \in \mathbb{R}, (x \equiv y) \Leftrightarrow \text{bdiff}(x, y)$

Proof. "$\Rightarrow$" Simply take $B = 4$.
"$\Leftarrow$" Suppose that $\forall n \in \mathbb{N}^+ : |x_n - y_n| \leq B$, and suppose that there exists an $n \in \mathbb{N}^+$ with $5 \leq |x_n - y_n|$. Then $\forall m \in \mathbb{N}^+$:

$$5m \leq m|x_n - y_n| \leq |mx_n - nx_m| + |nx_m - ny_m| + |ny_m - my_n| \leq 4(n + m) + nB \tag{1}$$

We get the first inequality by adding and subtracting x_m and y_m and using triangle inequality. We get the second inequality by our definition of regular sequences and the assumption. Since $5m \leq 4(n+m) + nB$ we obtain $m \leq (4+B)n$ for all $m \in \mathbb{N}^+$. We obtain a contradiction by choosing $m = 1 + (4+B)n$ and hence there cannot exist $n \in \mathbb{N}^+$ with $5 \leq |x_n - y_n|$. Thus we obtain the desired $\forall n \in \mathbb{N}^+ : |x_n - y_n| \leq 4$. $\square$

We finally show that $\equiv$ is an equivalence relation on $\mathbb{R}$.

Corollary 1. $\equiv$ is an equivalence relation on $\mathbb{R}$.

Proof. $\mathrm{bdiff}(x, y)$ is an equivalence relation on the regular sequences. Reflexivity and symmetry are easy to see and for transitivity we note that if $x, y, z \in \mathbb{R}$ with $(x \equiv y)$ and $(y \equiv z)$ we have that $\exists B \in \mathbb{N} \forall n \in \mathbb{N}^+ : |x_n - y_n| \leq B$ and $\exists C \in \mathbb{N} \forall n \in \mathbb{N}^+ : |y_n - z_n| \leq C$ by Lemma 1. Therefore, $\forall n \in \mathbb{N}^+ |x_n - z_n| \leq |x_n - y_n| + |y_n - z_n| \leq B + C$. Thus, by Lemma 1: $(x \equiv z)$. $\square$

3 Bibliography

M. Bickford, "Efficient algorithms from proofs in constructive analysis", Handbook of Constructive Mathematics (ed. D.Bridges, H.Ishihara, M.Rathjen and H.Schwichtenberg), Cambridge University Press, 2021, p. 829-858.