Provable (and Unprovable) Computability

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  For PA, ATR_0, Π^1_1-CA_0.

§1. The Computable Hierarchy

Definition

$\{e\}_s^f$ = the value after s steps in the computation of program e on oracle f.

Definition

The computable jump operator is $f \mapsto f'$ where $f'(n) = \{n_0\}_{n_1}^f$.

Definition

(i) The "sub-elementary" functions are those definable by compositions of $+$, $\dot{-}$ and $\Sigma_{i<k}$.
(Polynomially bounded; same as TM computable in linear space.)
(ii) For the "elementary" functions add $\Pi_{i<k}$.

Lemma

For "honest" functions f, f' is (sub)-elementarily inter-reducible with $n \mapsto f^n(n)$.

Fast Growing Hierarchy F and Slow Growing G

Definition

For "tree ordinals" α, with specified fundamental sequences assigned at limits λ, F_α is obtained by iterating the jump.

$$F_0(n) = n + 1\,;\; F_{\alpha+1}(n) = F_\alpha^n(n)\,;\; F_\lambda(n) = F_{\lambda_n}(n).$$

NB: this is highly sensitive to the choice of fundamental sequences.

Theorem

For arithmetical theories T with "proof-theoretic ordinal" $\|T\|$, the functions provably computable in T are exactly those elementary in the F_α for $\alpha \prec \|T\|$. (Schwichtenberg-W. around 1970 for PA.)

Definition (Slow Growing Hierarchy)

$$G_0(n) = 0\,;\; G_{\alpha+1}(n) = G_\alpha(n) + 1\,;\; G_\lambda(n) = G_{\lambda_n}(n).$$

Goodstein Sequences and the Hardy Hierarchy H

- Take any number a, for example $a = 16$.
- Write a in "complete base-2", thus $a = 2^{2^2}$.
- Subtract 1, so the base-2 representation is $a - 1 = 2^{2+1} + 2^2 + 2^1 + 1$.
- Increase the base by 1, to produce the next stage $a_1 = 3^{3+1} + 3^3 + 3^1 + 1 = 112$.
- Continue subtracting 1 and increasing the base: $a, a_1, a_2, a_3, \ldots$. Example: 16, 112, 1284, 18753, 326594, ...

Theorem (1. Goodstein 1944, 2. Kirby & Paris 1982)

1. Every Goodstein sequence eventually terminates in 0.

2. But this is not provable in Peano Arithmetic (PA).

Proof – The Hardy Functions

Throughout any Cantor Normal Form $\alpha \prec \varepsilon_0$, replace ω by n.
Then we obtain a "complete base-n" representation.
Subtract 1 and put ω back: one gets a smaller ordinal $P_n(\alpha)$.
Hence part 1 of the theorem, by well-foundedness.

E.G. With $\alpha = \omega^{\omega^\omega}$ and $n = 2$ we get $a = 2^{2^2} = 16$.
Then $a - 1 = 2^{2+1} + 2^2 + 2^1 + 1$ and $P_n(\alpha) = \omega^{\omega+1} + \omega^\omega + \omega^1 + 1$.

Definition (Hardy Hierarchy)

$$H_0(n) = n\,;\ H_{\alpha+1}(n) = H_\alpha(n+1)\,;\ H_\lambda(n) = H_{\lambda_n}(n)\,.$$

Theorem (Cichon (1983))

$H_\alpha(n) \ = \ n +$ the length of a Goodstein sequence on a, n.
A proof that all G-sequences terminate says H_{ε_0} is recursive.
But $H_{\varepsilon_0} \simeq F_{\varepsilon_0}$ is not provably recursive in PA. Hence part 2.

Some Relationships: $F_\alpha := H_{\omega^\alpha}$, $B_\alpha := H_{2^\alpha}$

- $H_{\alpha+\beta} = H_\alpha \circ H_\beta$.
- So $H_{\omega^{\alpha+1}}(n) = H_{\omega^\alpha \cdot n}(n) = H^n_{\omega^\alpha}(n) = F^n_\alpha(n) = F_{\alpha+1}(n)$.
- Similarly if $B_\alpha = H_{2^\alpha}$ then

$$B_\alpha(n) = \begin{cases} n+1 & \text{if } \alpha = 0 \\ B_\beta(B_\beta(n)) & \text{if } \alpha = \beta + 1 \\ B_{\alpha_n}(n) & \text{if } \alpha \text{ is a limit} \end{cases}$$

Theorem
$\{B_\alpha : \alpha \prec \|T\|\}$ also classifies provable recursion in arithmetical theories T, i.e. provides bounds for witnesses of provable Σ^0_1 formulas. Roughly, $F_\alpha \simeq B_{\omega\cdot\alpha}$.

The Basic Witness-Bounding Principle

Suppose $A(n)$ is a Σ_1 formula: $A(n) \equiv \exists a D(n,a)$.
Suppose $A(k) \to A(n)$ is derivable by Cuts with "height" α:

$$\frac{\vdash^{\beta} A(k) \to A(m) \qquad \vdash^{\beta} A(m) \to A(n)}{\vdash^{\alpha} A(k) \to A(n)} \ (\beta \prec \alpha)$$

Then $\models \exists a \leq b. D(k,a)$ implies $\models \exists a \leq B_\alpha(b). D(n,a)$.

Proof.

Sketch: by induction on α. Since $\beta \prec \alpha$, the premises give

$$\models \exists a \leq b. D(k,a) \ \text{implies} \ \models \exists a \leq B_\beta(b). D(m,a)$$

$$\models \exists a \leq b'. D(m,a) \ \text{implies} \ \models \exists a \leq B_\beta(b'). D(n,a)$$

Put $b' = B_\beta(b)$ to obtain $B_\beta(B_\beta(b)) = B_{\beta+1}(b) \leq B_\alpha(b)$.
NB. This requires $\beta + 1 \preceq_b \alpha$ where $\gamma \prec_b \gamma + 1$ and $\lambda_b \prec_b \lambda$. $\square$

The Majorization Lemma

Lemma

If $\beta \preceq_b \alpha$ *then* $B_\beta(b) \leq B_\alpha(b)$.

Proof.

By transfinite induction on α:.

- If $\alpha = 0$ then trivial.
- If α is a limit and $\beta \prec_b \alpha$ then $\beta \preceq_b \alpha_b$.
  By the induction hypothesis,
  $$B_\beta(b) \leq B_{\alpha_b}(b) = B_\alpha(b)\,.$$
- If $\alpha = \gamma + 1$ and $\beta \prec_b \alpha$ then $\beta \preceq_b \gamma$.
  By the induction hypothesis,
  $$B_\beta(b) \leq B_\gamma(b) \leq B_\gamma B_\gamma(b) = B_\alpha(b)\,.$$

□

§2. Provable Recursion in "Input-Output" Arithmetics

Definition (of EA(I;O))

- EA(I;O) has the language of arithmetic, with (quantified, "output") variables $a, b, c, \ldots$.
- In addition there are numerical constants ("inputs") $x, y, \ldots$.
- There are defining equations for (prim.) recursive functions.
- *Basic terms* are those built from the constants and variables by successive application of successor and predecessor.
- Only *basic terms* are allowed as "witnesses" in the logical rules for $\forall$ and $\exists$. E.g. $A(t) \to \exists a A(a)$ only for basic t.
- However the equality axioms give $t = a \wedge A(t) \to A(a)$, hence $\exists a(t = a) \wedge A(t) \to \exists a A(a)$ and $\exists a(t = a) \wedge \forall a A(a) \to A(t)$.
- "Predicative" induction axioms, for closed basic terms $t(x)$:

$$A(0) \wedge \forall c(A(c) \to A(c+1)) \to A(t(x)) .$$

Working in EA(I;O)

Definition

Write $t\downarrow$ for $\exists a(t=a)$.

Note: if t is not basic one cannot pass directly from $t=t$ to $t\downarrow$. But $a+1$ is basic, and $t=a\rightarrow t+1=a+1$ so $t\downarrow\rightarrow t+1\downarrow$.

Example

- From $b+c\downarrow\rightarrow b+(c+1)\downarrow$ one gets $b+x\downarrow$ by Σ_1-induction "up to" x. Then $\forall b(b+x\downarrow)$.
- Then $b+x\cdot c\downarrow\rightarrow b+x\cdot(c+1)\downarrow$. Therefore, by another Σ_1-induction, $b+x\cdot x\downarrow$.
- Hence $\forall b(b+x^2\downarrow)$, $\forall b(b+x^3\downarrow)$ etc.
- Similarly, $I\Sigma_1(I;O)\vdash\forall b(b+p(\vec{x})\downarrow)$ for any polynomial p.
- Exponential requires a Π_2 induction on $\forall b(b+2^c\downarrow)$:

Proving $\forall b(b + 2^x \downarrow)$ with Π_2 induction - an argument going back to Gentzen.

Assume

$$\forall b(b + 2^c \downarrow) .$$

Then, for arbitrary b, we have, by the assumption:

$$b + 2^c \downarrow \quad \text{and again} \quad (b + 2^c) + 2^c \downarrow$$

Therefore

$$\forall b(b + 2^c \downarrow) \to \forall b(b + 2^{c+1} \downarrow)$$

and $\forall b(b + 2^0 \downarrow)$ because $b + 1$ is basic.

Therefore $I\Pi_2(I;O) \vdash \forall b(b + 2^x \downarrow)$.

Similarly $I\Pi_2(I;O) \vdash \forall b(b + 2^{p(\vec{x})} \downarrow)$.

Then $I\Pi_3(I;O) \vdash \forall b(b + 2^{2^x} \downarrow)$ etc.

Bounding Σ_1-Inductions

Theorem

Witnesses for Σ_1 theorems $A(n) \equiv \exists a D(n, a)$, proved by Σ_1-inductions up to $x := n$, are bounded by B_h where $h = \log n$.

Proof.

Sketch: first, any induction up to $x := n$ can be unravelled, inside EA(I;O), to a binary tree of Cuts of height $h = \log n$:
For any c, $\vdash A(c) \to A(c + 2^h)$ with cut-height h.

$$\frac{A(c) \to A(c + 2^h) \qquad A(c + 2^h) \to A(c + 2^h + 2^h)}{A(c) \to A(c + 2^{h+1})}$$

Therefore $\vdash^h A(0) \to A(n)$ with cut-height $h = \log n$.
The Witness-Bounding Principle then gives $\exists a \leq B_h(b).D(n, a)$ where b is the witness for $A(0)$. □

Provably Computable Functions in EA(I;O)

Definition

A provably computable/recursive function of EA(I;O) is one which is Σ_1^0-definable and provably total on inputs, i.e. $\vdash f(\vec{x})\downarrow$.

Theorem (Leivant 1995, Ostrin-W. 2005)

The provable functions of $I\Sigma_1(I;O)$ are sub-elementary. Equiv: TM-computable in linear space, or Grzegorczyk's $\mathcal{E}^2$.

The provable functions of $I\Pi_2(I;O)$ are those computable in exp-time $2^{p(n)}$.
Etcetera, up the Ritchie-Schwichtenberg hierarchy for $\mathcal{E}^3$.

(See Leivant's Ramified Inductions (1995) where such characterizations were first obtained. Also Nelson's Predicative Arithmetic (1986). Spoors (Ph.D. 2010) develops hierarchies of ramified extensions of EA(I;O) classifying primitive recursion.)

Proof

- Fix $x := n$ in $\vdash f(x)\downarrow$ say with d nested inductions.
- Partial cut-elimination yields a "free-cut-free" proof, so after unravelling, only cuts on the induction formulas remain.
- The height of the proof-tree will be (of the order of) $h = \log n \cdot d$.
- For $I\Sigma_1(I;O)$ the Bounding Principle applies immediately to give complexity bounds

$$B_{\log n \cdot d}(b) = b + 2^{\log n \cdot d} = b + n^d \text{ for some constant } b.$$

- For $I\Pi_2$ one must first reduce all cuts to Σ_1 form by Gentzen cut-reduction, which further increases the height by an exponential, so in that case the complexity bounds will be

$$B_{2^{\log n \cdot d}}(b) = B_{n^d}(b) = b + 2^{n^d}.$$

PA – by adding an Inductive Definition

Definition

$ID_1(I;O)$ is obtained from $EA(I;O)$ by adding, for each uniterated positive inductive form $F(X, a)$, a new predicate P, and Closure and Least-Fixed-Point axioms:

$$\forall a(F(P, a) \to P(a))$$

$$\forall a(F(A, a) \to A(a)) \to \forall a(P(a) \to A(a))$$

for each formula A.

Example

Associate the predicate N with the inductive form:

$$F(X, a) \;:\equiv\; a = 0 \lor \exists b(X(b) \land a = b + 1).$$

Embedding Peano Arithmetic

Theorem

If $PA \vdash A$ then $ID_1(I;O) \vdash A^N$.

- Since the LFP axiom gives:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \ \rightarrow \ \forall a(N(a) \rightarrow A(a)).$$

- Hence Peano Arithmetic is interpreted in $\mathrm{ID}_1(\mathrm{I;O})$ by relativizing quantifiers to N.
- Note that $N(0) \wedge \forall a(N(a) \rightarrow N(a+1))$ by the Closure Axiom, so by "predicative" induction, $\mathrm{ID}_1(\mathrm{I;O}) \vdash N(x)$.
- Hence if f is provably recursive in PA then, by the embedding,

$$ID_1(I;O) \vdash \forall a(N(a) \rightarrow \exists b(N(b) \wedge f(a) = b))$$

and therefore, $ID_1(I;O) \vdash f(x) \downarrow \wedge N(f(x))$.

Unravelling LFP-Axiom by Buchholz' Ω-Rule

- We are still working in the I/O context, so can fix $\vec{x} := \vec{n}$ and unravel inductions into iterated Cuts as before.
- However the resulting ID_1(I;O)-derivations will be further complicated by the presence of Least-Fixed-Point axioms.
- These must be "unravelled" as well, by the Ω-Rule.

The infinitary system $\mathrm{ID}_1(\mathrm{I;O})^\infty$ has Tait-style sequents $n : N \vdash^\alpha \Gamma$ and rules (where $\beta \prec_n \alpha$) :

$$(\exists)\ \frac{k \leq B_\beta(n) \qquad n : N \vdash^\beta \Gamma, A(k)}{n : N \vdash^\alpha \Gamma, \exists a A(a)} \qquad (\forall)\ \frac{n : N \vdash^\beta \Gamma, A(i) \ \text{ for all } \ i}{n : N \vdash^\alpha \Gamma, \forall a A(a)}$$

$$(\Omega)\ \frac{\vdash^{\lambda_0} N(m), \Gamma_0 \qquad \vdash^h_0 N(m), \Delta \ \Rightarrow \vdash^{\lambda_h} \Gamma_1, \Delta}{\vdash^\lambda \Gamma_0, \Gamma_1}$$

where Δ denotes an arbitrary set of "positive-in-N" formulas.

Ω Proves LFP-Axiom

The basic idea.

- For the left-hand premise of the Ω-rule choose $\vdash^0 N(m), \neg N(m)$.
- For the right-hand premise, first assume $\vdash^h_0 N(m), \Delta$.
- Each step of this (direct) cut-free proof can be mimicked to prove $A(m)$ if we assume that A is "inductive".
- Thus $\vdash^{k+h} \neg\forall a(F(A,a) \to A(a)), A(m), \Delta$ where $k = |A|$.
- The standard fundamental sequence for ω gives $\omega_h = h$.
- Ω-rule gives $\vdash^{k+\omega} \neg\forall a(F(A,a) \to A(a)), \neg N(m), A(m)$ and this holds for every number m.
- Therefore by $\vee$ and the infinitary $\forall$-rule obtain LFP-Ax:

$$\vdash^{k+\omega+3} \neg\forall a(F(A,a) \to A(a)) \vee \forall a(\neg N(a) \vee A(a))\,.$$

$\square$

Cut Elimination in $\mathrm{ID}_1(\mathrm{I};\mathrm{O})^\infty$

As usual, Gentzen-style cut-reduction raises height exponentially. It cannot be done directly in PA because of the induction axioms.

Lemma (Cut-reduction)

(i) If $\vdash^{\gamma} \Gamma, \forall a \neg A(a)$ *and* $\vdash^{\alpha} \Gamma, \exists a A(a)$ *both with cut-rank* r*, and* $|A| = r$ *then* $\vdash^{\gamma+\alpha} \Gamma$ *with cut-rank* r*.*
(ii) Hence if $\vdash^{\alpha}_{r+1} \Gamma$ *then* $\vdash^{2^\alpha}_{r} \Gamma$*.*

Proof.

(i) By induction on α. If the second premise comes from $\vdash^{\beta} \Gamma, \exists a A(a), A(t)$ then $\vdash^{\gamma+\beta} \Gamma, A(t)$ by the induction hypothesis. Inverting the first premise gives $\vdash^{\gamma} \Gamma, \neg A(t)$. Then $\vdash^{\gamma+\alpha} \Gamma$ by a cut on $A(t)$, still with rank r.
(ii) By another induction on α: at a cut on $C = \exists a A(a)$ of size $r+1$ apply the induction hypothesis to both premises. Then apply (i) with $\gamma = \beta = 2^{\alpha'}$ where $\alpha' < \alpha$. Clearly $\gamma + \beta \leq 2^\alpha$. □

Collapsing in $\mathrm{ID}_1(\mathrm{I;O})^\infty$

Lemma (Collapsing)

Suppose, for fixed input $x := n > 1$, we have a cut-free derivation $\vdash_0^\alpha \Gamma$ with Γ positive in N.
Then there is a derivation of finite height $\vdash_0^k \Gamma$ where $k \leq B_{\alpha+1}(n)$.

Proof.

- For Ω-rule, assume it holds for the premises, choosing $\Delta = \Gamma_0$:

$$\vdash_0^{\alpha_0} N(m), \Gamma_0 \quad \text{and} \quad \vdash_0^h N(m), \Gamma_0 \Rightarrow \vdash_0^{\alpha_h} \Gamma.$$

- Then for the left premise, $\vdash_0^h N(m), \Gamma_0$ where $h \leq B_{\alpha_0+1}(n)$.
- And for the right premise, $\vdash_0^k \Gamma$ where $k \leq B_{\alpha_h+1}(n)$.
- Hence $k \leq B_{\alpha_h+1}(n) \leq B_\alpha(h+1) \leq B_\alpha B_\alpha(n) = B_{\alpha+1}(n)$.

$B_{\alpha_h+1}(n) \leq B_\alpha(\max(n, h+1))$ is a standard property at limits. □

"Another" Proof of an Old Theorem

Theorem

Every Σ^0_1 theorem of PA has witnesses bounded by B_α for some $\alpha \prec \varepsilon_0$. Therefore the provably recursive functions of PA are those computable in B_α-bounded resource for some $\alpha \prec \varepsilon_0$.

Proof.

- Embed as $\mathrm{ID}_1(\mathrm{I;O}) \vdash \exists a(N(a) \wedge A(n,a))$ with $x := n$ input.
- Translate this into $\mathrm{ID}_1(\mathrm{I;O})^\infty$ with proof-height $\omega + k$, cut-rank r.
- Eliminate cuts to obtain proof-height $\alpha = 2_r(\omega + k) \prec \varepsilon_0$.
- Collapse to obtain $\vdash^h_0 \exists a(N(a) \wedge A(n,a))$ with $h = B_{\alpha+1}(n)$.
- Use original Bounding Principle to bound witness a below $B_h(n) \leq B_h(h) = B_\omega(h) = B_\omega B_{\alpha+1}(n) \leq B_{\alpha+2}(n)$.

□

Generalizing to $\mathrm{ID}_{<\omega}$

- Williams' thesis (Leeds 2004) generalizes the foregoing to theories of finitely iterated inductive definitions $\mathrm{ID}_i(\mathrm{I;O})$. E.g. $\mathrm{ID}_2(\mathrm{I;O})$ defines Kleene's $\mathcal{O}$:

$$a \in \mathcal{O} \;\leftrightarrow\; a = 0 \vee \forall n \in N\left(\{a\}(n) \in \mathcal{O}\right).$$

- Higher-level Ω-rules are then needed, and they require ordinals in successively higher number-classes $\Omega_1, \Omega_2, \ldots, \Omega_i$.
- Collapsing (and Bounding) from one level $i+1$ down to the one below is then computed in terms of higher-level extensions of the B_α hierarchy: $\varphi_\alpha^{(i)}(\beta)$ for $\alpha \in \Omega_{i+1}, \beta \in \Omega_i$.
- The ordinal bound of $\mathrm{ID}_2(\mathrm{I;O})$ is then the Bachmann-Howard:

$$\tau_3 \;=\; \varphi^{(1)}_{\varepsilon_{\omega_1+1}}(\omega) \;=\; \varphi^{(1)}_{\varphi^{(2)}_{\varphi^{(3)}_\omega(\omega_2)}(\omega_1)}(\omega)$$

Bounding Functions for $ID_{<\omega}$ and Π^1_1-CA_0

Define $\varphi^{(k)} : \Omega_{k+1} \times \Omega_k \to \Omega_k$ by:

$$\varphi^{(k)}_\alpha(\beta) = \begin{cases} \beta + 1 & \text{if } \alpha = 0 \\ \varphi^{(k)}_\gamma \circ \varphi^{(k)}_\gamma(\beta) & \text{if } \alpha = \gamma + 1 \\ \varphi^{(k)}_{\alpha_\beta}(\beta) & \text{if } \alpha = \sup \alpha_\xi \ (\xi \in \Omega_k) \\ \sup \varphi^{(k)}_{\alpha_\xi}(\beta) & \text{if } \alpha = \sup \alpha_\xi \ (\xi \in \Omega_{<k}) \end{cases}$$

Define $\tau = \sup \tau_i$ where $\tau_0 = \omega$ and

$$\tau_1 = \varphi^{(1)}_\omega(\omega) \ , \ \tau_2 = \varphi^{(1)}_{\varphi^{(2)}_\omega(\omega_1)}(\omega) \ , \ \tau_3 = \varphi^{(1)}_{\varphi^{(2)}_{\varphi^{(3)}_\omega(\omega_2)}(\omega_1)}(\omega) \ , \ \ldots$$

Theorem

The proof-theoretic ordinal of ID_i is τ_{i+2}. The provably computable functions of Π^1_1-CA_0 are those computably-bounded by $\{B_\alpha\}_{\alpha \prec \tau}$.

§3. Independence Results
(i) Kruskal's Theorem with Labels

Theorem (Friedman's Miniaturized Version)

For each constant c there is a number $K(c)$ so large that in every sequence $\{T_j\}_{j<K(c)}$ of finite trees with labels from a given finite set, and such that $|T_j| \leq c \cdot 2^j$, there are $j_1 < j_2$ where $T_{j_1} \hookrightarrow T_{j_2}$. The embedding must preserve infs, labels, and satisfy a certain "gap condition".

Lemma

The (natural) computation sequence for $B_{\tau_i}(n)$ satisfies the size-bound above, and is a "bad" sequence, i.e. no embeddings.

Corollary

For a simple c_n we must have $B_\tau(n) = B_{\tau_n}(n) < K(c_n)$ for all n. Therefore K is not provably recursive in $ID_{<\omega}$, nor in Π^1_1-CA_0.

The Computation Sequence for τ_n

By reducing/rewriting τ_n according to the defining equations of the φ-functions, we pass through all the ordinals $\prec_n \tau_n$. Each term is a binary tree with labels $\leq n$, and each one-step-reduction at most doubles the size of the tree. E.g. with $n = 2$ the sequence begins:

$$\tau_2 \to \varphi^{(1)}_{\varphi^{(2)}_2(\omega_1)}(\omega) \to \varphi^{(1)}_{\varphi^{(2)}_1 \varphi^{(2)}_1(\omega_1)}(\omega) \to \varphi^{(1)}_{\varphi^{(2)}_0 \varphi^{(2)}_0 \varphi^{(2)}_1(\omega_1)}(\omega) \to$$

$$\varphi^{(1)}_{\varphi^{(2)}_0 \varphi^{(2)}_1(\omega_1)}(\varphi^{(1)}_{\varphi^{(2)}_0 \varphi^{(2)}_1(\omega_1)}(\omega)) \to \varphi^{(1)}_{\varphi^{(2)}_1(\omega_1)}(\varphi^{(1)}_{\varphi^{(2)}_1(\omega_1)}(\varphi^{(1)}_{\varphi^{(2)}_0 \varphi^{(2)}_1(\omega_1)}(\omega)))$$

$$\to \varphi^{(1)}_{\omega_1} \varphi^{(1)}_{\omega_1} \varphi^{(1)}_{\varphi^{(2)}_0(\omega_1)} \varphi^{(1)}_{\varphi^{(2)}_1(\omega_1)} \varphi^{(1)}_{\varphi^{(2)}_0 \varphi^{(2)}_1(\omega_1)}(\omega) \to \varphi^{(1)}_{\varphi^{(1)}_{\omega_1}(-)}(\varphi^{(1)}_{\omega_1}(-)) \ldots$$

The length of the entire sequence (down to zero) is therefore $\geq G_n(\tau_n) = B_{\tau_{n-1}}(n)$. Furthermore, the sequence is bad - no term is gap-embeddable in any follower.

Recall the Slow Growing Hierarchy G_α

Definition

For each countable "tree ordinal" α, define the finite set $\alpha[n]$ of its "n-predecessors" as follows:

$$0\,[n] = \phi \qquad \alpha + 1\,[n] = \alpha\,[n] \cup \{\alpha\} \qquad \lambda\,[n] = \lambda_n\,[n]\;.$$

Call α "standard" if $\alpha = \bigcup\{\alpha\,[0] \subset \alpha\,[1] \subset \alpha\,[2] \subset \alpha\,[3] \subset \ldots\}$.

Then the "slow growing hierarchy" is $\{G_\alpha\}$ where $G_\alpha(n) = |\alpha\,[n]|$. With n fixed we often write $G_n(\alpha)$ instead of $G_\alpha(n)$. Thus

$$G_n(0) = 0\,;\;\; G_n(\alpha+1) = G_n(\alpha) + 1\,;\;\; G_n(\lambda) = G_n(\lambda_n).$$

Theorem

Let $\varphi = \varphi^{(1)}$. Then for "well behaved" $\alpha \in \Omega_2, \beta \in \Omega_1$,

$$G_n(\varphi_\alpha(\beta)) = B_{G_n(\alpha)}(G_n(\beta))\;.$$

Proof by induction on α

- If $\alpha = 0$,
  $G_n(\varphi_0(\beta)) = G_n(\beta + 1) = G_n(\beta) + 1 = B_0(G_n(\beta))$.
- For $\alpha \mapsto \alpha + 1$,
  $G_n(\varphi_{\alpha+1}(\beta)) = G_n(\varphi_\alpha\varphi_\alpha(\beta)) = B_{G_n(\alpha)}B_{G_n(\alpha)}(G_n(\beta)) = B_{G_n(\alpha)+1}(G_n(\beta)) = B_{G_n(\alpha+1)}(G_n(\beta))$.
- If $\alpha = \sup_i \alpha_i$,
  $G_n(\varphi_\alpha(\beta)) = G_n(\sup_i \varphi_{\alpha_i}(\beta)) = G_n(\varphi_{\alpha_n}(\beta)) = B_{G_n(\alpha_n)}(G_n(\beta)) = B_{G_n(\alpha)}(G_n(\beta))$.
- If $\alpha = \sup_\xi \alpha_\xi$,
  $G_n(\varphi_\alpha(\beta)) = G_n(\varphi_{\alpha_\beta}(\beta)) = B_{G_n(\alpha_\beta)}(G_n(\beta)) =^* B_{G_n(\alpha)_{G_n(\beta)}}(G_n(\beta)) = B_{G_n(\alpha)}(G_n(\beta))$.

Example

With $\tau_2 = \varphi^{(1)}_{\varphi^{(2)}_\omega(\omega_1)}(\omega)$, $G_n(\tau_2) = B_{\varphi^{(1)}_n(\omega)}(n) = B_{\tau_1}(n)$.

(ii) Goodstein-style Independence Results
Tree Ordinals $\alpha \prec \Gamma_0$ (joint with Arai & Weiermann)

A fundamental sequence $\{\lambda_x\}$ is assigned to each $\lambda = \varphi_\alpha(\beta)$ in the Veblen hierarchy of normal functions:

Definition

- If $\lambda = \varphi_0(\beta + 1) = \omega^{\beta+1}$ then $\lambda_x = \omega^\beta \cdot x$
- If $\lambda = \varphi_\alpha(0)$ then $\lambda_x = \psi_\alpha^{(x)}(1)$
- If $\lambda = \varphi_\alpha(\beta + 1)$ then $\lambda_x = \psi_\alpha^{(x)}(\varphi_\alpha(\beta) + 1)$
- If $\lambda = \varphi_\alpha(\beta)$ and $\mathrm{Lim}(\beta)$ then $\lambda_x = \varphi_\alpha(\beta_x)$

where

$$\psi_\alpha = \begin{cases} \varphi_{\alpha-1} & \text{if } \mathrm{Succ}(\alpha) \\ \psi_{\alpha_x} & \text{if } \mathrm{Lim}(\alpha). \end{cases}$$

G_x Collapses Veblen onto Ackermann

Theorem

$$G_x(\varphi_\alpha(\beta)) \;=\; A(x; G_x(\alpha), G_x(\beta))$$

where $A(x; a, b))$ is a parametrized-at-x version of Ackermann:

$$\begin{array}{lcl} A(x;0,b) & = & x^b \\ A(x;a+1,0) & = & A(x;a)^{(x)}(1) \\ A(x;a+1,b+1) & = & A(x,a)^{(x)}(A(x;a+1,b)+1)\,. \end{array}$$

This is easily checked by induction on α, for example:

$$G_x(\varphi_{\alpha+1}(0)) \;=\; G_x(\sup_x \varphi_\alpha^{(x)}(1)) \;=\; G_x(\varphi_\alpha^{(x)}(1))$$

$$=\; A(x; G_x(\alpha))^{(x)}(1) \;=\; A(x; G_x(\alpha+1), 0)\,.$$

And if α is a limit:

$$G_x(\varphi_\alpha(0)) \;=\; G_x(\varphi_{\alpha_x}(0)) \;=\; A(x; G_x(\alpha_x), 0) \;=\; A(x; G_x(\alpha), 0)\,.$$

An Independence result for ATR_0

The x-**representation of** n is formed as follows:

- Choose n and a fixed base $x \geq 2$.
  Write $A_a(b)$ for $A(x; a, b)$
- Find greatest a and then greatest b such that $A_a(b) \leq n$
- If not equal, find greatest b' such that $A_{a-1}(b') \leq n$
- Continue until $= n$ or $A_0(b'') < n < A_0(b'' + 1)$
  Then $n = x^{b''} \cdot y_1 + x^{b'''} \cdot y_2 + \ldots$ with y's $< x$.
- Now, hereditarily find x-representations of the a's and b's
- This x-representation of n is now $G_x(\alpha)$ where α is obtained by replacing $A(x; -, -)$ by $\varphi(-, -)$ throughout.

- The **Goodstein process** is:
  Subtract 1 and update the base to $x + 1$. Then repeat.

Termination of Goodstein implies $\forall\alpha \preceq \Gamma_0 .\, H_\alpha \downarrow$

Note: $G_x(\, P_x(\alpha)\,) = G_x(\alpha) \dot{-} 1$ where (Cichon)

$$P_x(0) = 0,\; P_x(\alpha+1) = \alpha,\; P_x(\lambda) = P_x(\lambda_x)\,.$$

- Start with the x-representation of n
- Then by Collapsing, $n = G_x(\alpha)$ where α is a φ-term $\prec \Gamma_0$
- Goodstein: $n := n - 1 = G_x(P_x(\alpha))$; $x := x + 1$; $n := n_1$
- Then $n_1 = G_{x+1}(\alpha_1)$ where $\alpha_1 := P_x(\alpha)$
- Repeat: $n_2 = G_{x+2}(\alpha_2)$ where $\alpha_2 := P_{x+1}(\alpha_1)$ etcetera
- Termination at stage y when $P_{y-1} \cdots P_{x+2} P_{x+1} P_x(\alpha) = 0$
- But the least such y is $H_\alpha(x)$

EG. $n = A(x; A(x; \cdots A(x; 1, 0) \cdots , 0), 0)$ gives $y = H_{\Gamma_0}(x)$.

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