Constructive (functional) analysis

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Omniscience principles

- The limited principle of omniscience (LPO, Σ^0_1-PEM):

$$\forall\alpha[\alpha \# \mathbf{0} \vee \neg\alpha \# \mathbf{0}]$$

- The weak limited principle of omniscience (WLPO, Π^0_1-PEM):

$$\forall\alpha[\neg\neg\alpha \# \mathbf{0} \vee \neg\alpha \# \mathbf{0}]$$

- The lesser limited principle of omniscience (LLPO, Σ^0_1-DML):

$$\forall\alpha\beta[\neg(\alpha \# \mathbf{0} \wedge \beta \# \mathbf{0}) \to \neg\alpha \# \mathbf{0} \vee \neg\beta \# \mathbf{0}]$$

Markov's principle

- Markov's principle (MP, Σ^0_1-DNE):

$$\forall\alpha[\neg\neg\alpha \# \mathbf{0} \to \alpha \# \mathbf{0}]$$

- Markov's principle for disjunction ($\mathrm{MP}^{\vee}$, Π^0_1-DML):

$$\forall\alpha\beta[\neg(\neg\alpha \# \mathbf{0} \wedge \neg\beta \# \mathbf{0}) \to \neg\neg\alpha \# \mathbf{0} \vee \neg\neg\beta \# \mathbf{0}]$$

- Weak Markov's principle (WMP):

$$\forall\alpha[\forall\beta(\neg\neg\beta \# \mathbf{0} \vee \neg\neg\beta \# \alpha) \to \alpha \# \mathbf{0}]$$

Relationship among principles

$$
\begin{tikzcd}
 & \mathrm{LPO} \arrow[dl] \arrow[dr] & \\
\mathrm{MP} \arrow[ddd] \arrow[dddrr] & & \mathrm{WLPO} \arrow[d] \\
 & & \mathrm{LLPO} \arrow[dd] \\
 & & \\
\mathrm{WMP} & & \mathrm{MP}^{\vee}
\end{tikzcd}
$$

- $\mathrm{LPO} \Leftrightarrow \mathrm{WLPO} + \mathrm{MP}$
- $\mathrm{MP} \Leftrightarrow \mathrm{WMP} + \mathrm{MP}^{\vee}$

Apartness and equality

Proposition

- $\forall xy \in \mathbf{R}(x \# y \vee x = y) \Leftrightarrow \mathrm{LPO}$,
- $\forall xy \in \mathbf{R}(\neg x = y \vee x = y) \Leftrightarrow \mathrm{WLPO}$,
- $\forall xy \in \mathbf{R}(x \leq y \vee y \leq x) \Leftrightarrow \mathrm{LLPO}$,
- $\forall xy \in \mathbf{R}(\neg x = y \to x \# y) \Leftrightarrow \mathrm{MP}$,
- $\forall xyz \in \mathbf{R}(\neg x = y \to \neg x = z \vee \neg z = y) \Leftrightarrow \mathrm{MP}^{\vee}$,
- $\forall xy \in \mathbf{R}(\forall z \in \mathbf{R}(\neg x = z \vee \neg z = y) \to x \# y) \Leftrightarrow \mathrm{WMP}$.

Cauchy completeness

Definition

A sequence (x_n) of real numbers converges to $x \in \mathbf{R}$ if

$$\forall k \exists N_k \forall n \geq N_k [|x_n - x| < 2^{-k}].$$

Definition

A sequence (x_n) of real numbers is a Cauchy sequence if

$$\forall k \exists N_k \forall mn \geq N_k [|x_m - x_n| < 2^{-k}].$$

Theorem

A sequence of real numbers converges if and only if it is a Cauchy sequence.

Classical order completeness

Theorem

If S is an inhabited subset of $\mathbf{R}$ with an upper bound, then $\sup S$ exists.

Proposition

If every inhabited subset S of $\mathbf{R}$ with an upper bound has a supremum, then WLPO holds.

Proof.

Let α be a binary sequence. Then $S = \{\alpha(n) \mid n \in \mathbf{N}\}$ is an inhabited subset of $\mathbf{R}$ with an upper bound 2. If $\sup S$ exists, then either $0 < \sup S$ or $\sup S < 1$; in the former case, we have $\neg\neg\alpha \# \mathbf{0}$; in the latter case, we have $\neg\alpha \# \mathbf{0}$. $\square$

Constructive order completeness

Theorem

Let S be an inhabited subset of $\mathbf{R}$ with an upper bound. If either $\exists s \in S(a < s)$ or $\forall s \in S(s < b)$ for each $a, b \in \mathbf{R}$ with $a < b$, then $\sup S$ exists.

Proof.

Let $s_0 \in S$ and u_0 be an upper bound of S with $s_0 < u_0$. Define sequences (s_n) and (u_n) of real numbers by

$$\begin{array}{ll} s_{n+1} = (2s_n + u_n)/3, u_{n+1} = u_n & \text{if } \exists s \in S[(2s_n + u_n)/3 < s]; \\ s_{n+1} = s_n, u_{n+1} = (s_n + 2u_n)/3 & \text{if } \forall s \in S[s < (s_n + 2u_n)/3]. \end{array}$$

Note that $s_n < u_n$, $\exists s \in S(s_n \leq s)$ and $\forall s \in S(s \leq u_n)$ for each n. Then (s_n) and (u_n) converge to the same limit which is a supremum of S. □

Constructive order completeness

Definition

A set S of real numbers is totally bounded if for each k there exist $s_0, \ldots, s_{n-1} \in S$ such that

$$\forall y \in S \exists m < n[|s_m - y| < 2^{-k}].$$

Constructive order completeness

Proposition

An inhabited totally bounded set S of real numbers has a supremum.

Proof.

Let $a, b \in \mathbf{R}$ with $a < b$, and let k be such that $2^{-k} < (b-a)/2$. Then there exists $s_0, \ldots, s_{n-1} \in S$ such that

$$\forall y \in S \exists m < n[|s_m - y| < 2^{-k}].$$

Either $a < \max\{s_m \mid m < n\}$ or $\max\{s_m \mid m < n\} < (a+b)/2$. In the former case, there exists $s \in S$ such that $a < s$. In the latter case, for each $s \in S$ there exists m such that $|s - s_m| < 2^{-k}$, and hence

$$s < s_m + |s - s_m| < (a+b)/2 + (b-a)/2 = b.$$

□

Classical intermediate value theorem

Definition

A function f from $[0,1]$ into $\mathbf{R}$ is uniformly continuous if

$$\forall k \exists M_k \forall xy \in [0,1][|x-y| < 2^{-M_k} \to |f(x) - f(y)| < 2^{-k}].$$

Theorem

If f is a uniformly continuous function from $[0,1]$ into $\mathbf{R}$ with $f(0) \leq 0 \leq f(1)$, then there exists $x \in [0,1]$ such that $f(x) = 0$.

Classical intermediate value theorem

Proposition

The classical intermediate value theorem implies LLPO*.*

Proof.

Let $a \in \mathbf{R}$, and define a function f from $[0, 1]$ into $\mathbf{R}$ by

$$f(x) = \min\{3(1 + a)x - 1, 0\} + \max\{0, 3(1 - a)x + (3a - 2)\}.$$

Then f is uniformly continuous, and $f(0) = -1$ and $f(1) = 1$. If there exists $x \in [0, 1]$ such that $f(x) = 0$, then either $1/3 < x$ or $x < 2/3$; in the former case, we have $a \leq 0$; in the latter case, we have $0 \leq a$. $\square$

Constructive intermediate value theorem

Theorem

If f is a uniformly continuous function from $[0,1]$ into $\mathbf{R}$ with $f(0) \leq 0 \leq f(1)$, then for each k there exists $x \in [0,1]$ such that $|f(x)| < 2^{-k}$.

Constructive intermediate value theorem

Proof.
For given a k, let $l_0 = 0$ and $r_0 = 1$, and define sequences (l_n) and (r_n) by

$$\begin{array}{ll} l_{n+1} = (l_n + r_n)/2, r_{n+1} = r_n & \text{if } f((l_n + r_n)/2) < 0, \\ l_{n+1} = l_n, r_{n+1} = (l_n + r_n)/2 & \text{if } 0 < f((l_n + r_n)/2), \\ l_{n+1} = (l_n + r_n)/2, r_{n+1} = (l_n + r_n)/2 & \text{if } |f((l_n + r_n)/2)| < 2^{-(k+1)}. \end{array}$$

Note that $f(l_n) < 2^{-(k+1)}$ and $-2^{-(k+1)} < f(r_n)$ for each n. Then (l_n) and (r_n) converge to the same limit $x \in [0, 1]$. Either $2^{-(k+1)} < |f(x)|$ or $|f(x)| < 2^{-k}$. In the former case, if $2^{-(k+1)} < f(x)$, then $2^{-(k+1)} < f(l_n) < 2^{-(k+1)}$ for some n, a contradiction; if $f(x) < -2^{-(k+1)}$, then $-2^{-(k+1)} < f(r_n) < -2^{-(k+1)}$ for some n, a contradiction. Therefore the latter must be the case. $\square$

Metric spaces

Definition

A metric space is a set X equipped with a metric $d : X \times X \to \mathbf{R}$ such that

- $d(x, y) = 0 \leftrightarrow x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, y) \leq d(x, z) + d(z, y)$,

for each $x, y, z \in X$.

For $x, y \in X$, we write $x \# y$ for $0 < d(x, y)$.

Open and closed subsets

Definition
A subset S of a metric space X is

- open if $\forall x \in S \exists k \forall y \in X[d(x,y) < 2^{-k} \to y \in S]$;
- closed if $\forall x \in X[\forall k \exists y \in S(d(x,y) < 2^{-k}) \to x \in S]$.

Definition
Let S be a subset of a metric space X. Then

- the interior S° of S is defined by

$$S^\circ = \{x \in S \mid \exists k \forall y \in X[d(x,y) < 2^{-k} \to y \in S]\};$$

- the closure $\overline{S}$ of S is defined by

$$\overline{S} = \{x \in X \mid \forall k \exists y \in S[d(x,y) < 2^{-k}]\}.$$

Separable metric spaces

Definition

- A subset S of a metric space X is dense in X if $\overline{S} = X$.
- A metric space is separable if there exists a countable dense subset. (A set S is countable if there exists a mapping from $\mathbf{N}$ onto S.)

Convergent sequences

Definition

A sequence (x_n) of X converges to $x \in X$ if

$$\forall k \exists N_k \forall n \geq N_k [d(x_n, x) < 2^{-k}].$$

Remark

A subset S of a metric space X is closed if and only if $x \in S$ whenever there exists a sequence (x_n) of S converging to x.

Closed finite sets

Proposition

Let $x, y \in \mathbf{R}$. *Then* $\forall z \in \mathbf{R}(\neg x = z \vee \neg z = y)$ *if and only if* $\neg x = y$ *and* $\{x, y\}$ *is closed.*

Proof.

For "only if part", suppose that $\forall z \in \mathbf{R}(\neg x = z \vee \neg z = y)$. Then trivially $\neg x = y$. Let (z_n) is a sequence of $\{x, y\}$ converging to $z \in \mathbf{R}$. Then either $\neg x = z$ or $\neg z = y$. In the former case, if $z \# y$, then there exists N such that $\forall n \geq N(z_n \# y)$, and hence $x = z$, a contradiction. Therefore $z = y$. In the latter case, similarly we have $z = y$. $\square$

Closed finite sets

Proof.

For "if part", suppose that $\neg x = y$ and $\{x, y\}$ is closed. Define binary sequences α and β with at most one nonzero term such that

$$\alpha(n) = 0 \to |x - y| < 2^{-(n+1)}, \quad \alpha(n) = 1 \to x \# y,$$
$$\beta(n) = 0 \to \alpha(n) = 0 \vee x \# z, \quad \beta(n) = 1 \to \alpha(n) = 1 \wedge z \# y$$

for each n, and define a sequence (u_n) of $\{x, y\}$ by

$$u_n = \begin{cases} x & \text{if } \forall k \le n(\alpha(k) = 0), \\ x & \text{if } \exists k \le n(\alpha(k) = 1 \wedge \beta(k) = 0), \\ y & \text{if } \exists k \le n(\alpha(k) = 1 \wedge \beta(k) = 1). \end{cases}$$

Then (u_n) is a Cauchy sequence, and hence converges to $u \in \{x, y\}$. Therefore either $u = x$ or $u = y$. In the former case, assume that $x = z$. If $x \# y$, then $u = y$, and hence $x = y$, a contradiction. Therefore $x = y$, a contradiction, and so $\neg x = z$. In the latter case, similarly we have $\neg z = y$. $\square$

Closed finite sets

Proposition (Mandelkern 1988)

- $\forall xy \in \mathbf{R}[\neg x = y \to \{x, y\} \text{ is closed}] \Leftrightarrow \mathrm{MP}^{\vee}$,
- $\forall xy \in \mathbf{R}[\{x, y\} \text{ is closed} \to (\neg x = y \to x \# y)] \Leftrightarrow \mathrm{WMP}$.

Strong extensionality and sequential continuity

Definition

A mapping f between metric spaces X and Y is

- strongly extensional if

$$\forall xy \in X[f(x) \# f(y) \to x \# y];$$

- sequentially continuous if

$$f(x_n) \to f(x)$$

for each sequence (x_n) converging to x.

Strong extensionality and sequential continuity

Proposition

If every mapping between metric spaces is strongly extensional, then MP *holds.*

Proof.

Let x and y be real numbers such that $\neg x = y$, and define a mapping from $\{x, y\}$ into $\{0, 1\}$ by

$$f(x) = 0, \quad f(y) = 1.$$

Then, since $\neg x = y$, f is well defined. If f is strongly extensional, then we have $x \# y$. $\square$

Strong extensionality and sequential continuity

Proposition

Assume MP. *Then every mapping between metric spaces is strongly extensional.*

Proof.

If $d(x, y) = 0$ for $x, y \in X$ with $f(x) \# f(y)$, then we have a contradiction, and $\neg d(x, y) = 0$; whence $0 < d(x, y)$, by MP. $\square$

Strong extensionality and sequential continuity

Proposition

If every mapping from a complete metric space into a metric sapce is strongly extensional, then WMP *holds.*

Proposition

Assume WMP*. Then every mapping from a complete metric space into a metric sapce is strongly extensional.*

Strong extensionality and sequential continuity

Proposition

Every sequentially continuous mapping f between metric spaces X and Y is strongly extensional.

Proof.

Let $x, y \in X$ with $f(x) \# f(y)$, and define a sequnece (x_n) of X by

$$x_n = \begin{cases} y & \text{if } d(x,y) < 2^{-n}, \\ x & \text{if } 2^{n+1} < d(x,y). \end{cases}$$

Then (x_n) converges to x. Since f is sequentially continuous, there exists N such that $d(f(x), f(x_N)) < d(f(x), f(y))$.
If $d(x,y) < 2^{N+1}$, then $x_N = y$, a contradiction; whence $2^{N+1} \leq d(x,y)$. $\square$

Continuity and uniform continuity

Definition

A mapping f between metric spaces X and Y is

- (pointwise) continuous if

$$\forall x \in X \forall k \exists M_k \forall y \in X[d(x,y) < 2^{-M_k} \to d(f(x), f(y)) < 2^{-k}];$$

- uniformly continuous if

$$\forall k \exists M_k \forall xy \in X[d(x,y) < 2^{-M_k} \to d(f(x), f(y)) < 2^{-k}].$$

Continuity and uniform continuity

Definition

A subset S of $\mathbf{N}$ is pseudobounded if $\lim_{n\to\infty} s_n/n = 0$ for each sequence (s_n) of S.

$\mathrm{BD\text{-}N}$: Every countable pseudobounded subset of $\mathbf{N}$ is bounded.

Remark

$\mathrm{BD\text{-}N}$ does not hold in (Bishop's) constructive mathematics, but holds in classical mathematics, constructive recuresive mathematics and intuitionism.

Continuity and uniform continuity

Proposition (I 1992)

If evey sequentially continuous mapping from a separable metric space into a metric space is continuous, then BD-N *holds.*

Proof.

Let $S = \{s_n \mid n \in \mathbf{N}\}$ be a countable pseudobounded subset of $\mathbf{N}$, and let $X = \{0\} \cup \{2^{-s_n} \mid n \in \mathbf{N}\}$. Then X is a separable metric space. Defin a mapping f from X into $\{0, 1\}$ by

$$f(0) = 0, \quad f(2^{-s_n}) = 1.$$

Then, since S is pseudobounded, we can show that f is sequentially continuous. If f is continuous, then there exists M_1 such that $\forall y \in X[|y| < 2^{-M_1} \to |f(y)| < 2^{-1}]$, and therefore $s_n \leq M_1$ for each n. $\square$

Continuity and uniform continuity

Proposition (I 1992)

Assume BD-N. *Then evey sequentially continuous mapping from a separable metric space into a metric space is continuous.*

Complete metric spaces

Definition

A sequence (x_n) of a metric space is a Cauchy sequence if

$$\forall k \exists N_k \forall mn \geq N_k [d(x_m, x_n) < 2^{-k}].$$

A metric space is complete if every Cauchy sequence converes.

Classical Baire theorem

Theorem (the Baire theorem)

If (G_n) is a sequence of closed subsets of a complete metric space X such that $X = \cup_{n=0}^{\infty} G_n$, then there exists n such that G_n has an inhabited interior.

Proposition (I-Schuster 2008)

The classical Baire theorem implies $\mathrm{BD\text{-}N}$.

Open Problem

Does $\mathrm{BD\text{-}N}$ *imply the classical Baire theorem?*

Constructive Baire theorem

Theorem

If (U_n) is a sequence of open dense subsets of a complete metric space X, then $\cap_{n=0}^{\infty} U_n$ is dense in X.

Corollary

If (x_n) is a sequence of real numbers, then there exists $a \in \mathbf{R}$ such that $x_n \# a$ for each n.

Proof.

Let $U_n = \{x \in \mathbf{R} \mid x \# x_n\}$ for each n. Then U_n is open and dense. Therefore there exists $a \in \cap_{n=0}^{\infty} U_n$. $\square$

Totally bounded metric spaces

Definition

A metric space X is totally bounded if for each k there exist $x_0, \ldots, x_{n-1} \in X$ such that

$$\forall y \in X \exists m < n[d(x_m, y) < 2^{-k}].$$

Note that every totally bounded metric space is separable.

Proposition

If f is a uniformly continuous mapping from a totally bounded metric space X into a metric space, then

$$f(X) = \{f(x) \mid x \in X\}$$

is totally bounded.

Totally bounded metric spaces

Proposition

If every subset S of a totally bounded metric space is totally bounded, then LPO holds.

Proof.

Let α be a binary sequence. Then $S = \{\alpha(n) \mid n \in \mathbf{N}\}$ is a subset of a totally bounded metric space $\{0, 1\}$. If S is totally bounded, then there exist $s_0, \ldots, s_{n-1} \in S$ such that

$$\forall y \in S \exists m < n[|s_m - y| < 2^{-1}],$$

and therefore either $0 < \max\{s_m \mid m < n\}$ or $\max\{s_m \mid m < n\} < 2^{-1}$; in the former case, we have $\alpha \# \mathbf{0}$; in the later case, we have $\neg\alpha \# \mathbf{0}$. $\square$

Located subsets

Definition

A subset S of a metric space X is located if

$$d(x, S) = \inf\{d(x, y) \mid y \in S\}$$

exists for each $x \in X$.

Located subsets

Proposition

Every located subset S of a totally bounded metric space X is totally bounded.

Proof.

For given a k, there exist $x_0, \ldots, x_{n-1} \in X$ such that

$$\forall y \in X \exists m < n[d(x_m, y) < 2^{-(k+2)}].$$

Let $s_0, \ldots, s_{n-1} \in S$ be such that $d(x_m, s_m) < d(x_m, S) + 2^{-(k+1)}$ for each $m < n$. Then for each $y \in S$ there exists $m < n$ such that $d(x_m, y) < 2^{-(k+2)}$, and therefore

$$\begin{aligned} d(s_m, y) &\leq d(s_m, x_m) + d(x_m, y) < d(x_m, S) + 2^{-(k+1)} + d(x_m, y) \\ &\leq d(x_m, y) + 2^{-(k+1)} + d(x_m, y) < 2^{-k}. \end{aligned}$$

$\square$

Located subsets

Proposition

A totally bounded subset S of a metric space X is located.

Proof.

Let $x \in X$. Then the mapping $y \mapsto d(x, y)$ is a uniformly continuous mapping from S into $\mathbf{R}$, and hence $\{d(x, y) \mid y \in S\}$ is totally bounded. Therefore $\inf\{d(x, y) \mid y \in S\}$ exists. $\square$

Compact metric spaces

Definition

A metric space is compact if it is totally bounded and complete.

Proposition

If for every sequence (U_n) of open subsets of a compact metric space X with $X = \cup_{n=0}^{\infty} U_n$ there exists N such that $X = \cup_{n=0}^{N} U_n$, then the fan theorem (FAN) holds.

Proposition

If every sequence (G_n) of closed subsets of a compact metric space X with finite intersection property has an inhabited intersection, then LLPO holds.

Compact metric spaces

Proposition

If every continuous mapping from a compact metric space into a metric space is uniformly continuous, then FAN *holds.*

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