# Constructive (functional) analysis

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# Contents (lectures 1 - 3)

- Intuitionistic logic
- Real numbers
- Metric spaces
- Normed and Banach spaces

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Hilbert spaces

# Contents (lecture 1)

- The BHK interpretation
- Natural deduction
  - Minimal logic
  - Intuitionistic logic
  - Classical logic
- Omniscience principles
- Number systems
- Real numbers
- Ordering relation
- Apartness and equality
- Arithmetical operations

# A history of constructivism

### History

- Arithmetization of mathematics (Kronecker, 1887)
- Three kinds of intuition (Poincaré, 1905)
- French semi-intuitionism (Borel, 1914)
- Intuitionism (Brouwer, 1914)
- Predicativity (Weyl, 1918)
- Finitism (Skolem, 1923; Hilbert-Bernays, 1934)
- ► Constructive recursive mathematics (Markov, 1954)
- Constructive mathematics (Bishop, 1967)
- Logic
  - Intuitionistic logic (Heyting, 1934; Kolmogorov, 1932)

We use the standard language of (many-sorted) first-order predicate logic based on

▶ primitive logical operators  $\land, \lor, \rightarrow, \bot, \forall, \exists$ .

We introduce the abbreviations

$$\blacktriangleright \neg A \equiv A \rightarrow \bot;$$

$$\bullet \ A \leftrightarrow B \equiv (A \rightarrow B) \land (B \rightarrow A).$$

## The BHK interpretation

The Brouwer-Heyting-Kolmogorov (BHK) interpretation of the logical operators is the following.

- A proof of A ∧ B is given by presenting a proof of A and a proof of B.
- A proof of A ∨ B is given by presenting either a proof of A or a proof of B.
- A proof of A → B is a construction which transform any proof of A into a proof of B.
- Absurdity  $\perp$  has no proof.
- A proof of ∀xA(x) is a construction which transforms any t into a proof of A(t).
- A proof of ∃xA(x) is given by presenting a t and a proof of A(t).

# Natural Deduction System

We shall use  $\mathcal{D}$ , possibly with a subscript, for arbitrary deduction. We write  $\Gamma \\ \mathcal{D} \\ \mathcal{A}$ 

to indicate that  ${\mathcal D}$  is deduction with conclusion A and assumptions  $\Gamma.$ 

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# Deduction (Basis)

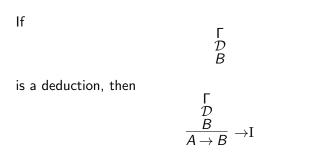
For each formula A,

#### Α

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is a deduction with conclusion A and assumptions  $\{A\}$ .

Deduction (Induction step,  $\rightarrow$ I)

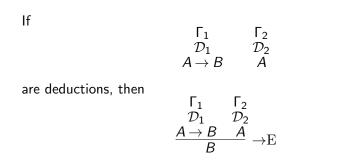


is a deduction with conclusion  $A \rightarrow B$  and assumptions  $\Gamma \setminus \{A\}$ . We write

$$\frac{\begin{bmatrix} A \end{bmatrix}}{B} \\ \frac{B}{A \to B} \to I$$

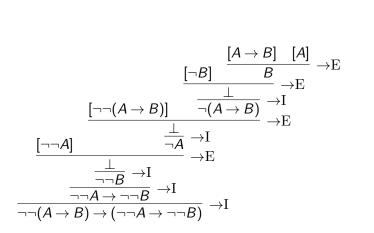
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Deduction (Induction step,  $\rightarrow E$ )



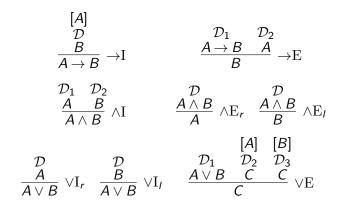
is a deduction with conclusion *B* and assumptions  $\Gamma_1 \cup \Gamma_2$ .

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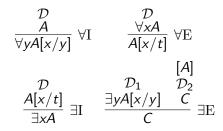
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## Minimal logic



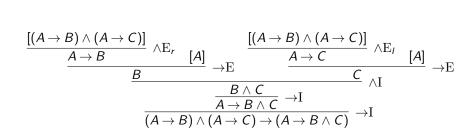
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## Minimal logic



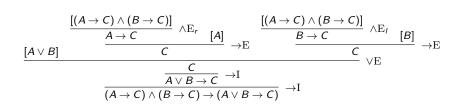
- ▶ In  $\forall$ E and  $\exists$ I, *t* must be free for *x* in *A*.
- In ∀I, D must not contain assumptions containing x free, and y ≡ x or y ∉ FV(A).

In ∃E, D<sub>2</sub> must not contain assumptions containing x free except A, x ∉ FV(C), and y ≡ x or y ∉ FV(A).



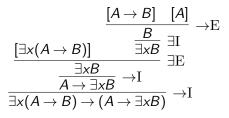
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$$\frac{\begin{bmatrix} A \to \forall xB \end{bmatrix} \quad \begin{bmatrix} A \end{bmatrix}}{\begin{bmatrix} \forall xB \\ B \\ \forall E \end{bmatrix}} \to E$$
$$\frac{\frac{B}{A \to B} \to I}{\begin{bmatrix} A \to B \\ \forall E \\ \hline A \to B \end{bmatrix}} \forall I$$
$$\frac{\forall x(A \to B)}{\forall A \to B} \to I$$

where  $x \notin FV(A)$ .



where  $x \notin FV(A)$ .

# Intuitionistic logic

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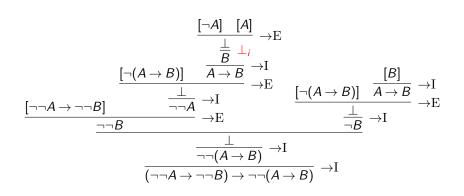
Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule (ex falso quodlibet).

is a deduction, then

is a deduction with conclusion A and assumptions  $\Gamma$ .

$$\mathcal{D}$$

 $\begin{array}{c} \Gamma \\ \mathcal{D} \\ \underline{\perp} \\ \underline{\Lambda} \ \perp_i \end{array}$ 



$$\frac{\begin{bmatrix} [\neg A] & [A] \\ \hline \pm & \bot_i \end{bmatrix}}{\begin{bmatrix} B \\ \neg A \to B \end{bmatrix}} \to E$$
$$\frac{B}{\neg A \to B} \to I$$
$$\frac{B}{\neg A \to B} \to I$$

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# **Classical** logic

Classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule (reductio ad absurdum).

 $\Gamma \mathcal{D}$ 

 $\begin{array}{c} \Gamma \\ \mathcal{D} \\ \frac{\perp}{A} \perp_c \end{array}$ 

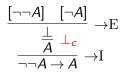
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is a deduction, then

is a deduction with conclusion A and assumption  $\Gamma \setminus \{\neg A\}$ .

# Example (classical logic)

The double negation elimination (DNE):



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# Example (classical logic)

The principle of excluded middle (PEM):

$$\frac{\begin{bmatrix} \neg (A \lor \neg A) \end{bmatrix} \quad \frac{\begin{bmatrix} A \end{bmatrix}}{A \lor \neg A} \lor I_r}{\begin{bmatrix} \neg A \\ \neg A \end{bmatrix}} \to E}$$
$$\frac{\begin{bmatrix} \neg (A \lor \neg A) \end{bmatrix} \quad \frac{\downarrow}{A \lor \neg A} \lor I_l}{\downarrow} \to E}{\frac{\downarrow}{A \lor \neg A} \downarrow c} \to E}$$

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Example (classical logic)

De Morgan's law (DML):

 $\frac{\begin{bmatrix} \neg (A \land B) \end{bmatrix}}{\frac{\bot}{A \land B}} \stackrel{[A]}{\rightarrow E} \stackrel{[B]}{\rightarrow E} \stackrel{\land I}{\rightarrow E} \\ \frac{\frac{\bot}{\neg A} \rightarrow I}{\frac{\neg A \lor \neg B}{\neg A \lor \neg B}} \stackrel{\lor I_r}{\rightarrow F}$  $[\neg(\neg A \lor \neg B)]$  $\frac{\frac{\bot}{\neg B} \to \mathrm{I}}{\neg A \lor \neg B} \lor \mathrm{I}_{I}$  $[\neg(\neg A \lor \neg B)]$  $\frac{\frac{\bot}{\neg A \lor \neg B} \bot_{c}}{\neg (A \land B) \to \neg A \lor \neg B} \to I$ 

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### $\mathsf{RAA} \mathsf{\,vs} \to I$

 $\perp_c$ : deriving *A* by deducing absurdity ( $\perp$ ) from  $\neg A$ .

 $\begin{bmatrix} \neg A \\ \mathcal{D} \\ \frac{\bot}{A} \bot_c \end{bmatrix}$ 

 $\rightarrow$ I: deriving  $\neg A$  by deducing absurdity ( $\perp$ ) from A.

$$\begin{array}{c} [A] \\ \mathcal{D} \\ \frac{\perp}{\neg \mathcal{A}} \rightarrow \mathbf{I} \end{array}$$

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# Notations

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## Omniscience principles

• The limited principle of omniscience (LPO,  $\Sigma_1^0$ -PEM):

$$\forall \alpha [\alpha \# \mathbf{0} \lor \neg \alpha \# \mathbf{0}]$$

• The weak limited principle of omniscience (WLPO,  $\Pi_1^0$ -PEM):

$$\forall \alpha [\neg \neg \alpha \ \# \ \mathbf{0} \lor \neg \alpha \ \# \ \mathbf{0}]$$

• The lesser limited principle of omniscience (LLPO,  $\Sigma_1^0$ -DML):

$$\forall \alpha \beta [\neg (\alpha \# \mathbf{0} \land \beta \# \mathbf{0}) \rightarrow \neg \alpha \# \mathbf{0} \lor \neg \beta \# \mathbf{0}]$$

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# Markov's principle

• Markov's principle (MP,  $\Sigma_1^0$ -DNE):

$$\forall \alpha [\neg \neg \alpha \ \# \ \mathbf{0} \rightarrow \alpha \ \# \ \mathbf{0}]$$

• Markov's principle for disjunction ( $MP^{\vee}$ ,  $\Pi_1^0$ -DML):

$$\forall \alpha \beta [\neg (\neg \alpha \# \mathbf{0} \land \neg \beta \# \mathbf{0}) \rightarrow \neg \neg \alpha \# \mathbf{0} \lor \neg \neg \beta \# \mathbf{0}]$$

Weak Markov's principle (WMP):

$$\forall \alpha [\forall \beta (\neg \neg \beta \# \mathbf{0} \lor \neg \neg \beta \# \alpha) \to \alpha \# \mathbf{0}]$$

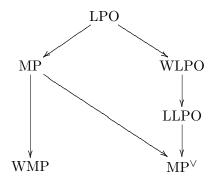
## Remark

We may assume without loss of generality that  $\alpha$  (and  $\beta)$  are ranging over

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- binary sequences,
- nondecreasing sequences,
- sequences with at most one nonzero term, or
- sequences with  $\alpha(0) = 0$ .

# Relationship among principles



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- $\blacktriangleright \text{ LPO} \Leftrightarrow \text{WLPO} + \text{MP}$
- ▶  $MP \Leftrightarrow WMP + MP^{\vee}$

## Remark

► MP (and hence WMP and MP<sup>∨</sup>) holds in constructive recursive mathematics.

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• WMP holds in intuitionism.

## CZF and choice axioms

The materials in the lectures could be formalized in

the constructive Zermelo-Fraenkel set theory (CZF)

without the powerset axiom and the full separation axiom, together with the following choice axioms.

► The axiom of countable choice (AC<sub>0</sub>):

$$\forall n \exists y \in YA(n, y) \rightarrow \exists f \in Y^{\mathbf{N}} \forall nA(n, f(n))$$

▶ The axiom of dependent choice (DC):

$$\forall x \in X \exists y \in XA(x, y) \rightarrow \\ \forall x \in X \exists f \in X^{\mathbf{N}}[f(0) = x \land \forall nA(f(n), f(n+1))]$$

### Number systems

• The set **Z** of integers is the set  $\mathbf{N} \times \mathbf{N}$  with the equality

$$(n,m) =_{\mathsf{Z}} (n',m') \Leftrightarrow n+m'=n'+m.$$

The arithmetical relations and operations are defined on Z in a straightforwad way; natural numbers are embedded into Z by the mapping  $n \mapsto (n, 0)$ .

• The set **Q** of rationals is the set  $\mathbf{Z} \times \mathbf{N}$  with the equality

$$(a,m) =_{\mathbf{Q}} (b,n) \Leftrightarrow a \cdot (n+1) =_{\mathbf{Z}} b \cdot (m+1).$$

The arithmetical relations and operations are defined on  $\mathbf{Q}$  in a straightforwad way; integers are embedded into  $\mathbf{Q}$  by the mapping  $a \mapsto (a, 0)$ .

#### Definition

A real number is a sequence  $(p_n)_n$  of rationals such that

$$\forall mn \left( |p_m - p_n| < 2^{-m} + 2^{-n} \right).$$

We shall write  $\mathbf{R}$  for the set of real numbers as usual.

#### Remark

Rationals are embedded into **R** by the mapping  $p \mapsto p^* = \lambda n.p$ .

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## Ordering relation

#### Definition

Let < be the ordering relation between real numbers  $x = (p_n)_n$ and  $y = (q_n)_n$  defined by

$$x < y \Leftrightarrow \exists n \left( 2^{-n+2} < q_n - p_n 
ight).$$

#### Proposition

Let  $x, y, z \in \mathbf{R}$ . Then

$$\neg (x < y \land y < x),$$

 $x < y \to x < z \lor z < y.$ 

## Ordering relation

#### Proof.

Let  $x = (p_n)_n$ ,  $y = (q_n)_n$  and  $z = (r_n)_n$ , and suppose that x < y. Then there exists *n* such that  $2^{-n+2} < q_n - p_n$ . Setting N = n+3, either  $(p_n + q_n)/2 < r_N$  or  $r_N \le (p_n + q_n)/2$ . In the former case, we have

$$2^{-N+2} < 2^{-n+1} - (2^{-(n+3)} + 2^{-n}) < \frac{q_n - p_n}{2} - (p_N - p_n)$$
  
=  $\frac{p_n + q_n}{2} - p_N < r_N - p_N,$ 

and hence x < z. In the latter case, we have

$$2^{-N+2} < -(2^{-(n+3)}+2^{-n})+2^{-n+1} < (q_N-q_n)+\frac{q_n-p_n}{2}$$
  
=  $q_N - \frac{p_n+q_n}{2} \le q_N - r_N,$ 

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and hence z < y.

#### Definition

We define the apartness #, the equality =, and the ordering relation  $\leq$  between real numbers x and y by

• 
$$x \# y \Leftrightarrow (x < y \lor y < x),$$

• 
$$x = y \Leftrightarrow \neg (x \# y),$$

• 
$$x \leq y \Leftrightarrow \neg (y < x)$$
.

#### Lemma

Let  $x, y, z \in \mathbf{R}$ . Then

- $\blacktriangleright x \# y \leftrightarrow y \# x,$
- $\blacktriangleright x \# y \to x \# z \lor z \# y.$

Proposition

Let  $x, y, z \in \mathbf{R}$ . Then

► x = x,

$$\blacktriangleright x = y \rightarrow y = x,$$

 $x = y \land y = z \to x = z.$ 

#### Proposition

Let  $x, x', y, y' \in \mathbf{R}$ . Then  $x = x' \land y = y' \land x < y \rightarrow x' < y',$   $\neg \neg (x < y \lor x = y \lor y < x),$  $x < y \land y < z \rightarrow x < z.$ 

#### Corollary

Let  $x, x', y, y', z \in \mathbf{R}$ . Then  $x = x' \land y = y' \land x \# y \to x' \# y',$  $x = x' \land y = y' \land x < y \rightarrow x' < y'.$  $x \leq y \leftrightarrow \neg \neg (x < y \lor x = y),$  $\neg \neg (x < y \lor y < x),$  $\land x < y \land y < x \rightarrow x = y$ ,  $x < y \land y < z \rightarrow x < z,$  $x < y \land y < z \rightarrow x < z,$  $x < y \land y < z \to x < z.$ 

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# Proposition $\forall xy \in \mathbf{R}(x \# y \lor x = y) \Leftrightarrow LPO,$

Proof.

( $\Leftarrow$ ): Let  $x = (p_n)_n$  and  $y = (q_n)_n$ , and define a binary sequence  $\alpha$  by

$$\alpha(n) = 1 \Leftrightarrow 2^{-n+2} < |q_n - p_n|.$$

Then  $\alpha \# \mathbf{0} \leftrightarrow x \# y$ , and hence  $x \# y \lor x = y$ , by LPO. ( $\Rightarrow$ ): Let  $\alpha$  be a binary sequence  $\alpha$  with at most one nonzero term, and define a sequence  $(p_n)_n$  of rationals by

$$p_n = \sum_{k=0}^n \alpha(k) \cdot 2^{-k}.$$

Then  $x = (p_n)_n \in \mathbf{R}$ , and  $x \# 0 \leftrightarrow \alpha \# \mathbf{0}$ . Therefore  $\alpha \# \mathbf{0} \lor \neg \alpha \# \mathbf{0}$ , by  $x \# 0 \lor x = 0$ .

### Proposition

$$\forall xy \in \mathbf{R}(\neg x = y \lor x = y) \Leftrightarrow \text{WLPO},$$

$$\flat \forall xy \in \mathbf{R} (x \leq y \lor y \leq x) \Leftrightarrow \text{LLPO},$$

$$\forall xy \in \mathbf{R}(\neg x = y \to x \ \# \ y) \Leftrightarrow \mathrm{MP},$$

$$\flat \forall xyz \in \mathbf{R}(\neg x = y \rightarrow \neg x = z \lor \neg z = y) \Leftrightarrow \mathrm{MP}^{\lor},$$

$$\forall xy \in \mathbf{R} (\forall z \in \mathbf{R} (\neg x = z \lor \neg z = y) \to x \# y) \Leftrightarrow \text{WMP}.$$

## Arithmetical operations

The arithmetical operations are defined on  ${\bf R}$  in a straightforwad way.

For  $x = (p_n), y = (q_n) \in \mathbf{R}$ , define  $x + y = (p_{n+1} + q_{n+1});$   $-x = (-p_n);$   $|x| = (|p_n|);$   $\max\{x, y\} = (\max\{p_n, q_n\});$  $\vdots$ 

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