On Predicativity

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Aims

Discuss a **theme**, *Predicativity*, that emerged within last century's debates on the foundations of mathematics and then shaped substantial portions of contemporary mathematical logic.

Predicativity

Predicativity imposes constraints to the notion of **set**.

Metaphor: ***predicative = built up from within.***

A definition is **impredicative** if it *quantifies* on a totality that includes the object to be defined, it is **predicative** otherwise.

Predicativity – a slogan

Predicativity points towards a **constructive and computational** way of understanding mathematics where the **natural number system** has central stage.

Why predicativity?

Predicativity gave rise to **type theory** (Russell 1908), and is gaining renewed attention within constructive mathematics and the area of assisted theorem proving.

Predicativity has been at the centre of intensive study within **proof-theory**, therefore further witnessing the fruitfulness of the interaction between mathematics and philosophy.

Predicativity is a so far unexploited tool for the **philosophy of mathematics**.

Plan

- Origins of predicativity
- Plurality of notions
- On a predicative concept of set

19th Century: the second birth of mathematics

19th Century: **Radical changes to the methodology of mathematics.** Emergence of infinitary mathematics.

- Growing preference for **conceptual reasoning** and **abstract** characterizations of **mathematical concepts**, and corresponding **de–emphasis on calculation**.
- Increasing confidence in dealing with the **infinite**.
- Emergence of the **axiomatic method**.
- Fruitful cross–fertilization between the various branches of mathematics, in particular emerging forms of **"algebrisation" of mathematics**.

Mathematical style

- Two **different mathematical *styles* emerge**:
  - one (old) more **computationally explicit**, grounded on the symbols used for calculation and representation;
  - one (new) more **general and abstract**, dealing with general mathematical structures. Actual infinity.

Paradoxes: together with the deep *methodological* changes to the mathematics, the discovery of the *paradoxes* (e.g. Russell's paradox) prompted reflection. Especially so in the case of predicativity.

Die Grundlagenkrise: Justification?

Foundational programmes in the philosophy of mathematics were put forward between the end of the 19th and the beginning of the 20th centuries. They centred around the question of the *justification* of the new mathematics: Logicism, Hilbert's programme, Intuitionism and ***Predicativism***.

These programmes shaped the philosophy of mathematics but also mathematical logic: Predicate Calculus (Frege), Type Theory (Russell) and Proof Theory (Hilbert's School).

Foundational Programmes

Foundational Programmes: 2 main attitudes:

- Accept and attempt to justify the new kind of mathematics (Logicism, Hilbert's Programme);
- Restrict the new mathematics (Predicativism, Intuitionism). Here a pessimism regarding the possibility of correctly justifying the whole new mathematics.

Note: the second kind of attitude has opened up a wealth of technical problems and new ideas.
Only by adopting a different (weaker) perspective we can see more.

Poincaré and Russell's analysis of the paradoxes

The term **predicativity** originates within the discussions between Russell and Poincaré and their analysis of the paradoxes.

Russell and Poincaré agreed that impredicativity was to be held responsible for the onset of the paradoxes, and explored a number of ways of clarifying the notion of (im)predicativity.

Poincaré and Russell's analysis of the paradoxes: circularity

The most well-known diagnosis by Poincaré and Russell:

- **Vicious Circularity/self-reference**: The paradoxes originate because of a vicious circularity: we define an object by reference to a totality that includes the very object we wish to define.

  A definition is **impredicative** if it *quantifies* on a totality that includes the object to be defined, it is **predicative** otherwise.

Example: The logicist definition of natural number

$$N(n) := \forall F[F(0) \wedge \forall x(F(x) \rightarrow F(Suc(x))) \rightarrow F(n)]$$

Check N(5):

$\forall F[F(0) \wedge \forall x(F(x) \rightarrow F(Suc(x))) \rightarrow F(5)]$?

$N(0) \wedge \forall x(N(x) \rightarrow N(Suc(x))) \rightarrow N(5)$?

Back 1 Back 2

We seem to require to already know what $N(n)$ means prior to its definition.

More examples

- The Liar: the sentence "I'm lying".
  "It is not true of all propositions *p* that if I affirm *p* then *p* is true."

- The sentence: ***Napoleon had all the qualities of a great general.***

  Compare with: *Napoleon was Corsican*.

  Here it seems that the statement/definition quantifies on a collection of entities and in so doing generates a new element of that collection.

  Russell: collections as "all propositions" or "all qualities" are **illegitimate**, and quantification over them is **meaningless**.

More examples

- $R = \{x \mid x \notin x\}$.
- Least Upper Bound principle (LUB):
  Every bounded, non–empty subset M of the real numbers has a least upper bound.

Bertrand Russell

Russell formulates the **Vicious Circle Principle (VCP)** (1908) and introduces type theory.

In one formulation the VCP reads:

"... whatever in any way concerns *all* or *any* or *some* of a class must not be itself one of the members of a class"

To block the paradoxes Russell makes *two moves simultaneously* (retrospectively).

First move: *Simple* Type theory

(1) Types: Each propositional function has a **range of significance**: i.e. a collection of all arguments for which it is *meaningful*.

We can think of a propositional function as a formula with a free variable $\varphi(x)$.

Simple Type theory

In modern terminology:

- Individuals are of type 0;
- Sets of individuals are of type 1;
- Sets of sets of individuals are of type 2, etc..
- The $\in$ relation is to hold only between members of one type and those of the next type.

This amounts to **simple type theory** and seems sufficient to block Russell's paradox (Chwistek 1923, Ramsey 1926).
Check Russell's Paradox: $R = \{x \mid x \notin x\}$.

Second move: *Ramified* type theory

(2) Ramification

- Recall the example: *Napoleon had all the qualities of a great general.*
  Here the property: “to have all the qualities of a great general” refers to a totality to which it belongs, thus it is impredicative.
  Note the difference with: *Napoleon was Corsican*.
- “Ramification”: Idea: introduce also a notion of **order** for propositional functions: **a propositional function can only quantify on propositional functions of lower order than its own**.

Stratification of propositional functions

Essentially we distinguish between:

- the **first order properties**, that do not refer to the totality of properties (e.g. being Corsican),
- the **second-order properties**, that refer only to the totality of first-order properties,
- the **third-order properties**, that can refer to the second-order properties, but not refer to the fourth-order properties etc...

Reducibility

Ramified type theory was motivated by the desire to eliminate all forms of impredicativity; not only the set-theoretic paradoxes (e.g. Russell's), but also "semantic" paradoxes.

Check the Liar Paradox: "It is not true of all propositions p of order n, that if I affirm p then p is true." But this is a proposition of order $n+1$, and no contradiction arises.

Ramification, however, makes the mathematics unworkable: Russell obtains natural numbers and real numbers of different orders.

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Axiom of reducibility.

Hermann Weyl

Weyl: "Das Kontinuum" (1918, p. 1):

> *The house of analysis [...] is to a large degree built on sand. I believe that I can replace this shifting foundation with pillars of enduring strength. They will not, however, support everything which today is generally considered to be securely grounded. I give up the rest, since I see no other possibility.*

Weyl's Mathematical Process

Due to the difficulty of working with ramifications, Weyl looked at what could be recovered *on the sole arithmetical comprehension*.

- Start from **the natural numbers** and some simple properties and relations over them.
- Use the standard logical operations to obtain more complex sets.
- Crucial requirement: **quantification** is only allowed **on the natural numbers** (to avoid vicious circularity).
- This corresponds to applying arithmetical comprehension.
- Weyl's predicative analysis can be carried out within $\mathbf{ACA}_0$, which is a conservative extension of **PA**.

Weyl: Predicativism given the natural numbers

Herman Weyl: "Das Kontinuum" (1918):

- *Mathematical practice:* recognition that more could be framed in predicative terms than previously thought. Weyl showed that large portions of 19th Century analysis can be developed on the basis of a system analogous to $\mathbf{ACA}_0$.
- *Philosophical stance:* Weyl *rejected as unjustified* what can not be predicatively reduced. In addition, he saw the natural number structure with mathematical induction **as an ultimate foundation of mathematical thought, which can not be further reduced.** Restrictions motivated by predicative concerns were imposed at the next level of idealization: the continuum (powerset of $\mathbb{N}$).

The natural numbers: Poincaré and Weyl

Poincaré: Primacy of the natural number structure with the principle of induction. This is given in intuition and does not require a justification/foundation. (Kantian inspiration).

Predicativity *given the natural numbers* (Weyl, Feferman): accepting the natural numbers, $\mathbb{N}$, as a **completed totality**, but not the totality of all subsets of $\mathbb{N}$.

Feferman (2005):

> *That there is a fundamental difference between our understanding of the concept of natural numbers and our understanding of the set concept, even for sets of natural numbers, is undeniable.*

Ramsey and Gödel – the end of predicativism?

Ramsey (1926): "The tallest man in this room": there is circularity, but it is harmless.

Gödel (1944): The problem with impredicative definitions only arises if we see the definitions as constructing, not as singling out the mathematical objects.

> *If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members which can be [...] uniquely characterized only by reference to this totality.*

The re–emergence of predicativity (given the natural numbers)

Predicativity re–emerges as a chapter in mathematical logic (1950's).

Two main questions:

- **How far** does predicativity goes **mathematically**? Which portion of contemporary mathematics is predicative?
- What is the **limit** of predicativity? How **strong** are theories codifying predicative mathematics?

Beyond Weyl: predicativity and mathematical practice

Which parts of *ordinary* mathematics can be reduced to the predicative?

Reverse mathematics (see Simpson 1999) and Feferman's work (see e.g. Feferman 1988) indicate that large portions of "ordinary mathematics" are in fact predicative.

In practice it turns out that large portions of ordinary mathematics can be carried out in systems with the strength of a fragment of PA, known as PRA–usually taken to represent finitary reasoning (Tait 1981).

See e.g. (Simpson 2002, Schwichtenberg and Wainer 2012) for independence results.

Predicativity given the natural numbers and scientifically applicable mathematics

Feferman has proposed the following **working hypothesis**: *all of scientifically applicable analysis can be developed in a system as weak as Peano Arithmetic.*

This rises significant **philosophical** questions on issues of ***indispensability*** of mathematics to science.
Which mathematical concepts are needed for scientific applications?
What does a precise reply to this question tell us about the nature of the mathematical objects that are postulated by the scientifically necessary part of mathematics?

Kreisel, Feferman and Schütte: predicatively provable well–orderings and the systems of ramified analysis.

Predicatively provable ordinal.

Kreisel's idea: Transfinite progression of systems $\mathbf{RA}_\alpha$, in which an ordinal α is to be accepted as the index for a system if a well-ordering of that type has been proved in a previous system.

Feferman and Schütte (independently): The limit is **the ordinal** Γ_0: the least non–zero ordinal closed under the Veblen function.

A number of alternative ways of characterizing predicativity given the natural numbers have been proposed by Kreisel and Feferman and their analysis has converged to the ordinal Γ_0.

Predicatively reducible systems

*Which **un-ramified** systems can be justified on predicative grounds?*

A formal system T is **predicatively reducible** iff every arithmetical sentence provable in T is also provable in $\mathbf{RA}_{<\Gamma_0}$.

Example: Martin-Löf's type theory with universes $U_1, U_2, \ldots$ is predicatively reducible (proof of Hancock's conjecture).

Predicativity and proof theory: Kreisel and Feferman

The study of predicativity becomes now part of a wider program of *conceptual clarification*, addressing the question: **"what rests on what"** in mathematics? Which operations and proof principles ought to be accepted if one has accepted certain given concepts?

E.g. what is implicit in our acceptance of the structure of the natural numbers with full induction?

Note the descriptive character of this enterprise: there is now no claim that only predicative reasoning is justified. In addition, the tools utilized in the analysis are typically impredicative.

Plurality of notions: Constructive Predicativity

Other forms of predicativity appear within the literature: Constructive and Strict predicativity.

Constructive Predicativity: Predicative themes re-emerge in the writings of Lorenzen, Myhill and Wang, and in particular in connection with constructive forms of mathematics in Martin-Löf type theory.

The appearance in Martin-Löf type theory is particularly significant: ***Girard's paradox***.

Constructive Predicativity in Martin-Löf type theory

In type theory predicativity has two manifestations:

- **Inductive definitions**;
- **Curry-Howard isomorphism**.

The role of the Curry-Howard isomorphism for predicativity in Martin-Löf type theory was clarified by Girard's paradox. One reading of the paradox is as highlighting the incompatibility of impredicativity with the identification of propositions and sets that is at the heart of Martin Löf type theory.

One way of getting impredicativity is to weaken the latter: to each proposition corresponds a type.

Constructive Predicativity: Inductive definitions

Inductive definitions are considered predicative, as they are expressed by **finite rules** and describe the construction of a set "**from within and from the bottom up**". One only uses previously constructed fragments of the set under construction to define new larger fragments, and so on.

The proof-theoretic analysis of theories of inductive definitions shows that this notion of predicativity is quite generous compared with predicativity given the natural numbers.

Plurality of notions: Strict Predicativity

Strict Predicativity: In a classical context, Edward Nelson and Charles Parsons have independently argued for the **impredicativity** already of the **natural number structure with full induction**.

They have therefore introduced severe restrictions to the induction principle. This allows for the justification only of weak fragments of Peano Arithmetic (bounded arithmetic).

Strict Predicativity: Nelson

Nelson writes:

> *The induction principle assumes that the natural number system is given. A number is conceived to be an object satisfying every inductive formula; for a particular inductive formula, therefore, the bound variables are conceived to range over objects satisfying every inductive formula, including the one in question.*

(Nelson, Predicative arithmetic, p. 1)

Strict predicativity: Nelson

Nelson's principal worry seems the following:

Consider the induction principle of **PA**:

$$[\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Suc(x)))] \to \forall x \varphi(x).$$

$\varphi(x)$ is here unrestricted: it may contain quantifiers on the natural numbers.
For those quantifiers to make sense, the natural number set needs to be definite, or completed.
But we need to use instances of induction with unrestricted quantifiers to clarify what belongs to the domain of natural numbers. Therefore there is a vicious circularity.

Strict predicativity: Nelson

Why *vicious* circularity? Because for Nelson:

> *... numbers are symbolic constructions; a construction does not exist until it is made; when something new is made, it is something new and not a selection from a pre-existing collection. There is no map to the world because the world is coming into being.*

(Nelson, Predicative arithmetic, p. 2)

Example: is $5^{5^{5^{5^{5}}}}$ a natural number? It is not of the form 0 or $Suc(0)$, or $Suc(Suc(0))$, ... We need to use induction to show that this is a natural number.

Plurality of notions: Strict Predicativity

(Parsons 1992,Parsons2008): not only the logicist definition of natural number is impredicative, but any informal explanation of the notion of natural number is impredicative.

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The reason, for Parsons, is that **the induction principle** is a fundamental component of the **definition** of natural number. But then considerations analogous to those by Nelson apply.

Plurality of notions of predicativity

We have seen that there are **different ways of understanding (im)predicativity**.

- Strict predicativity (Nelson, Parsons).
- Predicativity given the natural numbers (Weyl, Feferman).
- Constructive (generalized) predicativity (Lorenzen, Myhill, Martin–Löf, ...).
- Gödel's Constructible hierarchy: predicativity given the ordinals.

Predicativity as a **relative** concept

This plurality of notions of predicativity suggests that predicativity is a **relative** rather than an absolute concept (Feferman 2005).

That is, predicative restrictions are **applied** on top of various (conceptual) **bases** which are taken for granted: e.g. the natural number structure, or a finitary fragment of it.

Here we see the potential impact of the discussions and analysis of predicativity for the philosophy of mathematics: distinguish **different kinds** of mathematics and clarify their **implicit assumptions** and their **limit**.

On a predicative Concept of Set

We have seen Nelson's constructivistic view of mathematical entities. This is very different from a static understanding of mathematical entities that is often taken for granted.

I have claimed that by assuming a "weaker" perspective we can better analyze mathematical notions and uncover implicit assumptions.

I now wish to address a constructivistic notion of set that appears both in Poincaré and Weyl (and Parsons) and has perhaps similarities with the notion of set in constructive type theory.

On a predicative Concept of Set

I shall further suggest that this represents a revival of older ways of doing mathematics: more explicit, computational and less abstract.

Claim: the present analysis suggests that we have different portions of mathematics that rely on different concepts, **a number of distinct notions of set**, and a number of mathematical styles.

Poincaré and Russell's analysis of the paradoxes: circularity

We have seen a characterization of predicativity in terms of vicious circularity:

Vicious Circularity/self-reference: The paradoxes originate because of a vicious circularity: we define an object by quantification on a totality that includes the very object we wish to define.

Russell introduced his ramified type theory as a way of eliminating any form of vicious circularity.

Weyl (1918): there is a vicious circularity at the very heart of mathematics. We need to rectify its foundations.

Henri Poincaré

Poincaré (1909,1912) proposes a second characterization of predicativity: **Invariance under extension**:

A collection is predicative if any extension of it does not disorder the collection itself.

Examples (invariance)

Example of predicative collection: the class E of all natural numbers less than 10.

Example of impredicative collection: the collection D of all definable real numbers.
Richard's paradox: Given D, define by diagonalisation a "new" real number, r, different from all definable real numbers. Then $r \in D$ iff $r \notin D$.
Usual analysis: circularity.
New analysis: D is unstable, unfinished, but we treat it as if it were "finished".

What is Poincaré's notion of set?

The logical notion of set

Traditional/Logical notion of set: **set as extension of a concept**, i.e. the collection of all the objects that satisfy a given concept.

One way of reading this: concepts are *epistemologically* prior to sets: first you need to have an understanding of the concept and then you can grasp what the set is.

Traditionally sets are uniformly formed – according to a law (given by the concept).

The set theoretic paradoxes put strain on the logical notion of set.

Predicativity as reaction to the concept of arbitrary set

Two reactions to the paradoxes:

- Paradoxes were seen as a warning that more care was needed in formalizing the notion of set (Hilbert school). Zermelo's axiomatization of set theory. Set is emancipated from concepts.
- Paradoxes were manifestations of deep problems caused by the new methodology introduced in mathematics. In particular they demonstrated the unreliability of the new notion of "arbitrary" set. Poincaré and Weyl pointed towards a notion of set that is deeply rooted in (i) intuition and (ii) a definition (property). In this lecture I shall focus on the second aspect only.

Second half of the 19th Century: a new notion of set emerges

With the new mathematics (e.g. analysis, set theory) a new notion of set emerges:

Arbitrary set: independent from the availability *even in principle* of an explicit law or a *rule* of formation.

Paradigmatic example: Powerset of the Natural Numbers.

Arbitrary sets

Bernays (1934): Quasi-Combinatorialism: in analogy to the finitary case, we take e.g. *all* subsets of the natural numbers, as if each were produced by an individual and arbitrary choice. No requirement of law-like generation of the set.

Predicativity as a rejection of the notion of arbitrary set

The representation of an infinite set as a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled, and then surveyed as a whole by consciousness, is nonsensical; "inexhaustibility" is essential to the infinite. (Weyl 1918, p. 23)

Arbitrary sets: Powerset of an infinite set

Myhill 1975 writes:

> *Power set seems expecially nonconstructive and impredicative compared with the other axioms [of set theory]: it does not involve, as the others do, putting together or taking apart sets that one has already constructed but rather selecting out of the totality of all sets, all those that stand in the relation of inclusion with a given set. One could make the same, admittedly vague, objection to the existence of the set $A \to B$ of mappings of A into B but I do not think the situation is parallel – a mapping or function is a rule, a finite object which can actually be be given; in general this is not the case for infinite sets.*

Finite and infinite sets

Both Poincaré and Weyl emphasize that the problematic case for impredicative definitions is that of infinite sets.

> *Finite sets can be described in two ways: either in individual terms, by exhibiting each of their elements, or in general terms, on the basis of a rule, i.e., by indicating properties which apply to the elements of the set and to no other objects. In the case of infinite sets, the first way is impossible (and this is the very essence of the infinite). (Weyl 1918, p. 20)*

Contrary to quasi-combinatorialism there is here a *deep disanalogy between the finitary and infinitary cases*.

Predicative notion of set

Predicative concept of set: **set as extension of a definition** (Poincaré) **or a property** (Weyl).

We have a refinement of the traditional notion of set that preserves the tie of the set with something prior to it: a definition (or property) which expresses a rule or a law.

The hard work is to clarify which definitions (properties) are safe. Here predicativity comes into play.

A dynamic concept of set

We saw Poincaré's characterisation of predicativity in terms of "invariance":
A collection is predicative if any extension of it does not disorder the collection itself.

The very idea of a set "disturbed" by the addition of "new" elements is difficult to grasp from a standard/static understanding of sets, but makes sense if a set is the extension of a definition.

A dynamic concept of set

An analysis of (Poincaré 1912) suggests the following points:

- All mathematical objects need to be introduced by a **finitary definition**.
- **Constructivistic perspective**: sets come into being through our definitions, step by step.
- **Predicativity**: This view of sets causes difficulties with impredicative definitions: they seem to require the assumption of those sets that we are about to define. Here invariance is crucial.

Weyl's Mathematical Process

We can now review Weyl's mathematical process as a description of how to obtain sets as "extensions of properties".

- Start from the natural numbers structure and some primitive properties and relations between the natural numbers.
- Use the standard logical operations to obtain more complex properties.
- Crucial requirement: **quantification** is only allowed **on the primitive category of objects** (to avoid vicious circularity).
- **Sets** are the **extension of the resulting complex properties**.
- Iteration.
- Mirroring of properties and sets.
- Open-ended nature of sets and intensionality.

Poincaré on genus

Poincaré assimilates the definition of a set to traditional classifications by *"genus proximum et differentiam specificam"*: when defining a set, one specifies a ***"genus"*** *and further individuating characteristics of the elements of the set*.

He notes that a mathematical platonist (Cantorian) can be satisfied by a specification of a genus, G. He will take G with all the elements that satisfy it as "given". A definition then selects out of the 'set theoretic universe' those sets that satisfy the genus.

But the genus does not suffice as a definition from a "constructivistic" perspective, as the elements of the set need to be constructed or produced by the definition and can not therefore be presupposed.

Martin-Löf's distinction between sets and categories (Martin-Löf 1984)

It is tempting to see a similarity between Poincaré's genre and Martin-Löf's notion of category.

Martin-Löf (1984) distinguishes between sets and categories. Sets are defined by giving formation, introduction and elimination rules (with corresponding equality rules.)

Martin-Löf’s distinction between sets and categories (Martin-Löf 1984)

Martin-Löf writes (Martin-Löf 1984, p. 21):

> *A category is defined by explaining what an object of the category is and when two such objects are equal. A category need not be a set, since we can grasp what it means to be an object of a given category even without exhaustive rules for forming its objects.*
>
> *Each set determines a category, namely the category of elements of the set, but not conversely: for instance, the category of sets and the category of propositions are not sets, since we cannot describe how all their elements are formed.*

A predicative notion of set

Poincaré and Weyl propose a predicative notion of set that is dependent on a (finite) rule that tells us how to construct the elements of the set (and when two elements are equal).

This is radically different from the notion of set that ZF aims as codifying.

For the predicative notion of set invariance is crucial: we need to ensure that the dependence on the rule is not too strong, and we can build safely from definitions.

Thank you!