Part II: Convexity and constructive infima

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Let A be a subset of $\mathbb{R}$ and $x \in \mathbb{R}$. x is the *infimum* of A if

$$\forall a \in A \,(x \leq a) \quad \text{and} \quad \forall z \,(x < z \;\Rightarrow\; \exists a \in A \,(a < z))\,.$$

In this case, we write $x = \inf A$.

Let α be a binary sequence. Set

$$A = \{\alpha_n \mid n \in \mathbb{N}\}\,.$$

Let x be the infimum of A. Then either $x < 1$, in this case there exists an n with $\alpha_n = 0$. Or $0 < x$, in this case $\alpha_n = 1$ for all n.

Josef Berger and Gregor Svindland, *Convexity and constructive infima*, Archive for Mathematical Logic (2016)

http://link.springer.com/article/10.1007/s00153-016-0502-y

- scalar product $\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot y_i$
- norm $\|x\| = \sqrt{\langle x, x \rangle}$
- metric $d(x, y) = \|y - x\|$

$C \subseteq \mathbb{R}^n$ is *convex* if

$$\lambda \cdot x + (1 - \lambda) \cdot y \in C$$

for all $x, y \in C$ and $\lambda \in [0, 1]$.

- Fix $\varepsilon > 0$ and sets $D \subseteq C \subseteq \mathbb{R}^n$. D is an ε-*approximation* of C if for every $c \in C$ there exists $d \in D$ with $d(c, d) < \varepsilon$.
- C is *totally bounded* if for every $\varepsilon > 0$ there exist elements $x_1, \ldots, x_m$ of C such that $\{x_1, \ldots, x_m\}$ is an ε-approximation of C.
- C is *closed* if

$$c_n \in C \ \wedge \ c_n \to x \ \Rightarrow \ x \in C.$$

- C is *compact* if it is totally bounded and closed.

Lemma 1

If $X \subseteq \mathbb{R}^n$ is totally bounded and $f : X \to \mathbb{R}$ is uniformly continuous, then

$$\{f(x) \mid x \in X\}$$

is totally bounded.

Lemma 2

If $A \subseteq \mathbb{R}$ is totally bounded, then $\inf A$ *exists.*

Lemma 3

If $X \subseteq \mathbb{R}^n$ is totally bounded and $f : X \to \mathbb{R}$ is uniformly continuous, then

$$\inf f = \inf \{f(x) \mid x \in X\}$$

exists.

Lemma 4

Fix a compact convex set $C \subseteq \mathbb{R}^2$ and suppose that there are $y, z \in C$ with $y_1 < 0 < z_1$. Then the set

$$\mathcal{M} = \{x \in C \mid x_1 = 0\}$$

is convex and compact.

Proof.

Set

$$\mathcal{L} = \{x \in C \mid x_1 \leq 0\} \text{ and } \mathcal{R} = \{x \in C \mid x_1 \geq 0\}.$$

For

$$\kappa : \mathbb{R} \to \mathbb{R}, \; s \mapsto \max(-s, 0)$$

$$f : \mathbb{R}^n \to \mathbb{R}^n, \; x \mapsto \frac{z_1}{z_1 + \kappa(x_1)} x + \frac{\kappa(x_1)}{z_1 + \kappa(x_1)} z$$

- f is uniformly continuous
- f maps C onto $\mathcal{R}$
- f maps $\mathcal{L}$ onto $\mathcal{M}$

□

Proposition 1

If $C \subseteq \mathbb{R}^n$ is compact and convex and

$$f : C \to \mathbb{R}^+$$

is quasi-convex and uniformly continuous, then $\inf f > 0$.

quasi-convex means:

$$\forall x, y \in C \, \forall \lambda \in [0,1] \, (f(\lambda \cdot x + (1-\lambda) \cdot y) \leq \max(f(x), f(y)))$$

Lemma 5

Let C be an inhabited convex subset of $\mathbb{R}^n$ such that

$$\delta = \inf \{ \|y\| \mid y \in C \}$$

exists. Then there exists a unique $a \in \overline{C}$ such that $\|a\| = \delta$. Furthermore, we have

$$\langle a, c - a \rangle \geq 0$$

and therefore

$$\langle a, c \rangle \geq \delta^2$$

for all $c \in C$.

Proof.

Fix a sequence (c_l) in C such that $\|c_l\| \to \delta$. Then (c_l) is Cauchy. Let a be the limit of the sequence (c_l).

For every $\lambda \in (0,1)$ and $c \in Y$ we have

$$\|a\|^2 \leq \|(1-\lambda)\cdot a + \lambda \cdot c\|^2 = \|a + \lambda \cdot (c-a)\|^2 =$$

$$\|a\|^2 + \lambda^2 \cdot \|c-a\|^2 + 2 \cdot \lambda \cdot \langle a, c-a \rangle$$

and therefore

$$0 \leq \lambda \|c-a\|^2 + 2\langle a, c-a \rangle.$$

This implies

$$\langle a, c-a \rangle \geq 0.$$

□

Proposition 2

Let $C, Y \subseteq \mathbb{R}^n$ *such that*

- C *is convex and compact*
- Y *is convex, closed, and located*
- $d(c, y) > 0$ *for all* $c \in C$ *and* $y \in Y$*.*

Then there exist $p \in \mathbb{R}^n$ *and reals* α, β *such that*

$$\langle p, c \rangle < \alpha < \beta < \langle p, y \rangle$$

for all $c \in C$ *and* $y \in Y$*.*

located means that

$$d(x, Y) = \inf \{ d(x, y) \mid y \in Y \}$$

exist for all $x \in \mathbb{R}^n$

Proof.

The function $f : C \to \mathbb{R}$, $c \mapsto d(c, Y)$ is positive-valued, uniformly continuous and convex. The set

$$Z = \{y - c \mid y \in Y, c \in C\}$$

is convex and

$$\delta = d(0, Z) = \inf f > 0.$$

By Lemma 5, there exists $p \in \mathbb{R}^n$ with

$$\langle p, y \rangle \geq \delta^2 + \langle p, c \rangle$$

for all $c \in C$ and $y \in Y$. Setting

$$\eta = \sup \{\langle p, c \rangle \mid c \in C\}, \ \alpha = \frac{\delta^2}{3} + \eta, \ \beta = \frac{\delta^2}{2} + \eta,$$

we obtain

$$\langle p, c \rangle < \alpha < \beta < \langle p, y \rangle$$

for all for all $c \in C$ and $y \in Y$. □