Part II: Convexity and constructive infima

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Let $A$ be a subset of $\mathbb{R}$ and $x \in \mathbb{R}$. $x$ is the infimum of $A$ if

$$\forall a \in A \ (x \leq a) \quad \text{and} \quad \forall z \ (x < z \Rightarrow \exists a \in A \ (a < z)).$$

In this case, we write $x = \inf A$.

Let $\alpha$ be a binary sequence. Set

$$A = \{ \alpha_n \mid n \in \mathbb{N} \}.$$

Let $x$ be the infimum of $A$. Then either $x < 1$, in this case there exists an $n$ with $\alpha_n = 0$. Or $0 < x$, in this case $\alpha_n = 1$ for all $n$. 

scalar product $\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot y_i$

norm $\|x\| = \sqrt{\langle x, x \rangle}$

metric $d(x, y) = \|y - x\|$

$C \subseteq \mathbb{R}^n$ is convex if

$$\lambda \cdot x + (1 - \lambda) \cdot y \in C$$

for all $x, y \in C$ and $\lambda \in [0, 1]$. 
Fix $\varepsilon > 0$ and sets $D \subseteq C \subseteq \mathbb{R}^n$. $D$ is an $\varepsilon$-approximation of $C$ if for every $c \in C$ there exists $d \in D$ with $d(c, d) < \varepsilon$.

$C$ is *totally bounded* if for every $\varepsilon > 0$ there exist elements $x_1, \ldots, x_m$ of $C$ such that $\{x_1, \ldots, x_m\}$ is an $\varepsilon$-approximation of $C$.

$C$ is *closed* if

$$c_n \in C \land c_n \to x \implies x \in C.$$ 

$C$ is *compact* if it is totally bounded and closed.
Lemma 1
If $X \subseteq \mathbb{R}^n$ is totally bounded and $f : X \to \mathbb{R}$ is uniformly continuous, then
$$\{ f(x) \mid x \in X \}$$
is totally bounded.

Lemma 2
If $A \subseteq \mathbb{R}$ is totally bounded, then $\inf A$ exists.

Lemma 3
If $X \subseteq \mathbb{R}^n$ is totally bounded and $f : X \to \mathbb{R}$ is uniformly continuous, then
$$\inf f = \inf \{ f(x) \mid x \in X \}$$
exists.
Lemma 4

Fix a compact convex set $C \subseteq \mathbb{R}^2$ and suppose that there are $y, z \in C$ with $y_1 < 0 < z_1$. Then the set

$$\mathcal{M} = \{ x \in C \mid x_1 = 0 \}$$

is convex and compact.
Proof.

Set
\[ \mathcal{L} = \{ x \in C \mid x_1 \leq 0 \} \] and \[ \mathcal{R} = \{ x \in C \mid x_1 \geq 0 \} \].

For \[ \kappa : \mathbb{R} \to \mathbb{R}, \; s \mapsto \max(-s, 0) \]

\[ f : \mathbb{R}^n \to \mathbb{R}^n, \; x \mapsto \frac{z_1}{z_1 + \kappa(x_1)} x + \frac{\kappa(x_1)}{z_1 + \kappa(x_1)} z \]

\begin{itemize}
  \item [\textcircled{1}] \( f \) is uniformly continuous
  \item [\textcircled{2}] \( f \) maps \( C \) onto \( \mathcal{R} \)
  \item [\textcircled{3}] \( f \) maps \( \mathcal{L} \) onto \( \mathcal{M} \)
\end{itemize}
Proposition 1

If $C \subseteq \mathbb{R}^n$ is compact and convex and

$$f : C \rightarrow \mathbb{R}^+$$

is quasi-convex and uniformly continuous, then $\inf f > 0$.

Quasi-convex means:

$$\forall x, y \in C \forall \lambda \in [0, 1] (f(\lambda \cdot x + (1 - \lambda) \cdot y) \leq \max (f(x), f(y)))$$
Lemma 5
Let $C$ be an inhabited convex subset of $\mathbb{R}^n$ such that

$$\delta = \inf \{ \|y\| \mid y \in C \}$$

exists. Then there exists a unique $a \in \overline{C}$ such that $\|a\| = \delta$. Furthermore, we have

$$\langle a, c - a \rangle \geq 0$$

and therefore

$$\langle a, c \rangle \geq \delta^2$$

for all $c \in C$. 
Proof.
Fix a sequence \((c_l)\) in \(C\) such that \(\|c_l\| \to \delta\). Then \((c_l)\) is Cauchy. Let \(a\) be the limit of the sequence \((c_l)\).

For every \(\lambda \in (0, 1)\) and \(c \in Y\) we have

\[
\|a\|^2 \leq \|(1 - \lambda) \cdot a + \lambda \cdot c\|^2 = \|a + \lambda \cdot (c - a)\|^2 = \|a\|^2 + \lambda^2 \cdot \|c - a\|^2 + 2 \cdot \lambda \cdot \langle a, c - a \rangle
\]

and therefore

\[
0 \leq \lambda \|c - a\|^2 + 2 \langle a, c - a \rangle.
\]

This implies

\[
\langle a, c - a \rangle \geq 0.
\]
Proposition 2

Let $C, Y \subseteq \mathbb{R}^n$ such that

- $C$ is convex and compact
- $Y$ is convex, closed, and located
- $d(c, y) > 0$ for all $c \in C$ and $y \in Y$.

Then there exist $p \in \mathbb{R}^n$ and reals $\alpha, \beta$ such that

$$\langle p, c \rangle < \alpha < \beta < \langle p, y \rangle$$

for all $c \in C$ and $y \in Y$.

*located* means that

$$d(x, Y) = \inf \{ d(x, y) | y \in Y \}$$

exist for all $x \in \mathbb{R}^n$. 
Proof.
The function $f : C \to \mathbb{R}, c \mapsto d(c, Y)$ is positive-valued, uniformly continuous and convex. The set

$$Z = \{ y - c \mid y \in Y, c \in C \}$$

is convex and

$$\delta = d(0, Z) = \inf f > 0.$$  

By Lemma 5, there exists $p \in \mathbb{R}^n$ with

$$\langle p, y \rangle \geq \delta^2 + \langle p, c \rangle$$

for all $c \in C$ and $y \in Y$. Setting

$$\eta = \sup \{ \langle p, c \rangle \mid c \in C \}, \quad \alpha = \frac{\delta^2}{3} + \eta, \quad \beta = \frac{\delta^2}{2} + \eta,$$

we obtain

$$\langle p, c \rangle < \alpha < \beta < \langle p, y \rangle$$

for all $c \in C$ and $y \in Y$. \qed