

CHAPTER 4

Gödel's Theorems

4.1. Undefinability of the notion of truth

4.1.1. Elementary functions on Gödel numbers. Let $\mathcal{L}$ be a countable first-order language. Assume that we have injectively assigned to every n -ary relation symbol R a *symbol number* $\text{sn}(R)$ of the form $\langle 1, n, i \rangle$ and to every n -ary function symbol f a symbol number $\text{sn}(f)$ of the form $\langle 2, n, j \rangle$. Call $\mathcal{L}$ *elementarily presented* if the set $\text{Symb}_{\mathcal{L}}$ of all these symbol numbers is elementary. In what follows we shall always assume that the languages $\mathcal{L}$ considered are elementarily presented. In particular this applies to every language with finitely many relation and function symbols.

Let $\text{sn}(\text{Var}) := \langle 0 \rangle$. For every $\mathcal{L}$ -term r we define recursively its Gödel number $\ulcorner r \urcorner$ by

$$\begin{aligned}\ulcorner x_i \urcorner &:= \langle \text{sn}(\text{Var}), i \rangle, \\ \ulcorner f r_1 \dots r_n \urcorner &:= \langle \text{sn}(f), \ulcorner r_1 \urcorner, \dots, \ulcorner r_n \urcorner \rangle.\end{aligned}$$

Assign numbers to the logical symbols by $\text{sn}(\rightarrow) := \langle 3, 0 \rangle$ and $\text{sn}(\forall) := \langle 3, 1 \rangle$. For simplicity we leave out the logical connectives $\wedge$, $\vee$ and $\exists$ here; they could be treated similarly. We define for every $\mathcal{L}$ -formula A its Gödel number $\ulcorner A \urcorner$ by

$$\begin{aligned}\ulcorner R r_1 \dots r_n \urcorner &:= \langle \text{sn}(R), \ulcorner r_1 \urcorner, \dots, \ulcorner r_n \urcorner \rangle, \\ \ulcorner A \rightarrow B \urcorner &:= \langle \text{sn}(\rightarrow), \ulcorner A \urcorner, \ulcorner B \urcorner \rangle, \\ \ulcorner \forall_{x_i} A \urcorner &:= \langle \text{sn}(\forall), i, \ulcorner A \urcorner \rangle.\end{aligned}$$

Assume that 0 is a constant and S is a unary function symbol in $\mathcal{L}$. For every $a \in \mathbb{N}$ the *numeral* $\underline{a} \in \text{Ter}_{\mathcal{L}}$ is defined by $\underline{0} := 0$ and $\underline{n+1} := S\underline{n}$. We can define an elementary function s such that for every formula $C = C(z)$ with $z := x_0$,

$$s(\ulcorner C \urcorner, k) = \ulcorner C(\underline{k}) \urcorner;$$

the proof is an exercise.

4.1.2. Undefinability of the notion of truth. Let $\mathcal{M}$ be an $\mathcal{L}$ -structure. A relation $R \subseteq |\mathcal{M}|^n$ is called *definable* in $\mathcal{M}$ if there is an $\mathcal{L}$ -formula $A(x_1, \dots, x_n)$ such that

$$R = \{ (a_1, \dots, a_n) \in |\mathcal{M}|^n \mid \mathcal{M} \models A(x_1, \dots, x_n)[x_1 := a_1, \dots, x_n := a_n] \}.$$

We assume in this section that $|\mathcal{M}| = \mathbb{N}$, 0 is a constant in $\mathcal{L}$ and S is a unary function symbol in $\mathcal{L}$ with $0^{\mathcal{M}} = 0$ and $S^{\mathcal{M}}(a) = a + 1$. Recall that for every $a \in \mathbb{N}$ the *numeral* $\underline{a} \in \text{Ter}_{\mathcal{L}}$ is defined by $\underline{0} := 0$ and $\underline{n+1} := S\underline{n}$. Observe that in this case the definability of $R \subseteq \mathbb{N}^n$ by $A(x_1, \dots, x_n)$ is equivalent to

$$R = \{ (a_1, \dots, a_n) \in \mathbb{N}^n \mid \mathcal{M} \models A(\underline{a_1}, \dots, \underline{a_n}) \}.$$

Furthermore let $\mathcal{L}$ be an elementarily presented language. We assume in this section that every elementary relation is definable in $\mathcal{M}$. A set S of formulas is called *definable* in $\mathcal{M}$ if $\ulcorner S \urcorner := \{ \ulcorner A \urcorner \mid A \in S \}$ is.

We shall show that already from these assumptions it follows that the notion of truth for $\mathcal{M}$, more precisely the set $\text{Th}(\mathcal{M})$ of all closed formulas valid in $\mathcal{M}$, is undefinable in $\mathcal{M}$. From this it will follow that the notion of truth is in fact undecidable, for otherwise the set $\text{Th}(\mathcal{M})$ would be recursive, hence recursively enumerable, and hence definable, because we have assumed already that all elementary relations are definable in $\mathcal{M}$ and so their projections are definable also. For the proof we shall need the following fixed point lemma, which will be generalized in 4.2.2.

LEMMA (Semantical fixed point lemma). *If every elementary relation is definable in $\mathcal{M}$, then for every $\mathcal{L}$ -formula $B(z)$ we can find a closed $\mathcal{L}$ -formula A such that*

$$\mathcal{M} \models A \quad \text{if and only if} \quad \mathcal{M} \models B(\ulcorner A \urcorner).$$

PROOF. Let s be the elementary function satisfying for every formula $C = C(z)$ with $z := x_0$,

$$s(\ulcorner C \urcorner, k) = \ulcorner C(\underline{k}) \urcorner$$

mentioned above. Then in particular

$$s(\ulcorner C \urcorner, \ulcorner C \urcorner) = \ulcorner C(\ulcorner C \urcorner) \urcorner.$$

By assumption the graph G_s of s is definable in $\mathcal{M}$, by $A_s(x_1, x_2, x_3)$ say. Let

$$C(z) := \forall x (A_s(z, z, x) \rightarrow B(x)), \quad A := C(\ulcorner C \urcorner),$$

and therefore

$$A = \forall x (A_s(\ulcorner C \urcorner, \ulcorner C \urcorner, x) \rightarrow B(x)).$$

Hence $\mathcal{M} \models A$ if and only if $\forall_{a \in \mathbb{N}} (a = \ulcorner C(\ulcorner C \urcorner) \urcorner \rightarrow \mathcal{M} \models B(a))$, which is the same as $\mathcal{M} \models B(\ulcorner A \urcorner)$. $\square$

THEOREM (Tarski's undefinability theorem). *Assume that every elementary relation is definable in $\mathcal{M}$. Then $\text{Th}(\mathcal{M})$ is undefinable in $\mathcal{M}$, hence in particular not recursively enumerable.*

PROOF. Assume that $\ulcorner \text{Th}(\mathcal{M}) \urcorner$ is definable by $B_W(z)$. Then for all closed formulas A

$$\mathcal{M} \models A \quad \text{if and only if} \quad \mathcal{M} \models B_W(\ulcorner A \urcorner).$$

Now consider the formula $\neg B_W(z)$ and choose by the fixed point lemma a closed $\mathcal{L}$ -formula A such that

$$\mathcal{M} \models A \quad \text{if and only if} \quad \mathcal{M} \models \neg B_W(\ulcorner A \urcorner).$$

This contradicts the equivalence above.

We already have noticed that all recursively enumerable relations are definable in $\mathcal{M}$. Hence it follows that $\ulcorner \text{Th}(\mathcal{M}) \urcorner$ cannot be recursively enumerable. $\square$

4.2. The notion of truth in formal theories

We now want to generalize the arguments of the previous section. There we have made essential use of the notion of truth in a structure $\mathcal{M}$, i.e., of the relation $\mathcal{M} \models A$. The set of all closed formulas A such that $\mathcal{M} \models A$ has been called the theory of $\mathcal{M}$, denoted $\text{Th}(\mathcal{M})$.

Now instead of $\text{Th}(\mathcal{M})$ we shall start more generally from an arbitrary theory T . We consider the question as to whether in T there is a *notion of truth* (in the form of a *truth formula* $B(z)$), such that $B(z)$ “means” that z is “true”. A consequence is that we have to explain all the notions used without referring to semantical concepts at all.

- (i) z ranges over closed formulas (or sentences) A , or more precisely over their Gödel numbers $\ulcorner A \urcorner$.
- (ii) A “true” is to be replaced by $T \vdash A$.
- (iii) C “equivalent” to D is to be replaced by $T \vdash C \leftrightarrow D$.

Hence the question now is whether there is a truth formula $B(z)$ such that $T \vdash A \leftrightarrow B(\ulcorner A \urcorner)$ for all sentences A . The result will be that this is impossible, under rather weak assumptions on the theory T . Technically, the issue will be to replace the notion of definability by the notion of “representability” within a formal theory. We begin with a discussion of this notion.

In this section we assume that $\mathcal{L}$ is an elementarily presented language with 0, S and = in $\mathcal{L}$, and T an $\mathcal{L}$ -theory containing the equality axioms $\text{Eq}_{\mathcal{L}}$.

4.2.1. Representable relations and functions.

DEFINITION. A relation $R \subseteq \mathbb{N}^n$ is *representable* in T if there is a formula $A(x_1, \dots, x_n)$ such that

$$\begin{aligned} T \vdash A(\underline{a_1}, \dots, \underline{a_n}) & \quad \text{if } (a_1, \dots, a_n) \in R, \\ T \vdash \neg A(\underline{a_1}, \dots, \underline{a_n}) & \quad \text{if } (a_1, \dots, a_n) \notin R. \end{aligned}$$

A function $f: \mathbb{N}^n \rightarrow \mathbb{N}$ is called *representable* in T if there is a formula $A(x_1, \dots, x_n, y)$ representing the graph $G_f \subseteq \mathbb{N}^{n+1}$ of f , i.e., such that

$$(4.1) \quad T \vdash A(\underline{a_1}, \dots, \underline{a_n}, \underline{f(a_1, \dots, a_n)}),$$

$$(4.2) \quad T \vdash \neg A(\underline{a_1}, \dots, \underline{a_n}, \underline{c}) \quad \text{if } c \neq f(a_1, \dots, a_n)$$

and such that in addition

$$(4.3) \quad T \vdash A(\underline{a_1}, \dots, \underline{a_n}, y) \wedge A(\underline{a_1}, \dots, \underline{a_n}, z) \rightarrow y = z \text{ for all } a_1, \dots, a_n \in \mathbb{N}.$$

Note that in case $T \vdash \underline{b} \neq \underline{c}$ for $b < c$ condition (4.2) follows from (4.1) and (4.3).

LEMMA. *If the characteristic function c_R of a relation $R \subseteq \mathbb{N}^n$ is representable in T , then so is the relation R itself.*

PROOF. For simplicity assume $n = 1$. Let $A(x, y)$ be a formula representing c_R . We show that $A(x, \underline{1})$ represents the relation R . Assume $a \in R$. Then $c_R(a) = 1$, hence $(a, 1) \in G_{c_R}$, hence $T \vdash A(\underline{a}, \underline{1})$. Conversely, assume $a \notin R$. Then $c_R(a) = 0$, hence $(a, 1) \notin G_{c_R}$, hence $T \vdash \neg A(\underline{a}, \underline{1})$. $\square$

4.2.2. Undefinability of the notion of truth in formal theories.

LEMMA (Fixed point lemma). *Assume that all elementary functions are representable in T . Then for every formula $B(z)$ we can find a closed formula A such that*

$$T \vdash A \leftrightarrow B(\ulcorner A \urcorner).$$

PROOF. The proof is similar to the proof of the semantical fixed point lemma. Let s be the elementary function introduced there and $A_s(x_1, x_2, x_3)$ a formula representing s in T . Let

$$C(z) := \forall x (A_s(z, z, x) \rightarrow B(x)), \quad A := C(\ulcorner C \urcorner),$$

and therefore

$$A = \forall_x (A_s(\ulcorner C \urcorner, \ulcorner C \urcorner, x) \rightarrow B(x)).$$

Because of $s(\ulcorner C \urcorner, \ulcorner C \urcorner) = \ulcorner C(\ulcorner C \urcorner) \urcorner = \ulcorner A \urcorner$ we can prove in T

$$A_s(\ulcorner C \urcorner, \ulcorner C \urcorner, x) \leftrightarrow x = \ulcorner A \urcorner,$$

hence by definition of A also

$$A \leftrightarrow \forall_x (x = \ulcorner A \urcorner \rightarrow B(x))$$

and therefore

$$A \leftrightarrow B(\ulcorner A \urcorner). \quad \square$$

Note that for $T = \text{Th}(\mathcal{M})$ we obtain the semantical fixed point lemma above as a special case.

THEOREM. *Let T be a consistent theory such that all elementary functions are representable in T . Then there cannot exist a formula $B(z)$ defining the notion of truth, i.e., such that for all closed formulas A*

$$T \vdash A \leftrightarrow B(\ulcorner A \urcorner).$$

PROOF. Assume we would have such a $B(z)$. Consider the formula $\neg B(z)$ and choose by the fixed point lemma a closed formula A such that

$$T \vdash A \leftrightarrow \neg B(\ulcorner A \urcorner).$$

For this A we obtain $T \vdash A \leftrightarrow \neg A$, contradicting the consistency of T . $\square$

With $T := \text{Th}(\mathcal{M})$ Tarski's undefinability theorem is a special case.

4.2.3. Undecidability and incompleteness. Consider a consistent formal theory T with the property that all recursive functions are representable in T . This is a very weak assumption, as we shall show in the next section: it is always satisfied if the theory allows to develop a certain minimum of arithmetic. We shall show that such a theory necessarily is undecidable. Then we prove Gödel's (first) incompleteness theorem saying that every axiomatized such theory must be incomplete.

In this section let $\mathcal{L}$ be an elementarily presented language with $0, S, =$ in $\mathcal{L}$ and T a theory containing the equality axioms $\text{Eq}_{\mathcal{L}}$. Call a relation *recursive* if its (total) characteristic function is recursive. A set S of formulas is called *recursive* (*recursively enumerable*), if $\ulcorner S \urcorner := \{\ulcorner A \urcorner \mid A \in S\}$ is recursive (recursively enumerable).

THEOREM (Undecidability). *Assume that T is a consistent theory such that all recursive functions are representable in T . Then T is not recursive.*

PROOF. Assume that T is recursive. By assumption there exists a formula $B(z)$ representing $\ulcorner T \urcorner$ in T . Choose by the fixed point lemma a closed formula A such that

$$T \vdash A \leftrightarrow \neg B(\ulcorner A \urcorner).$$

We shall prove $(*) T \not\vdash A$ and $(**) T \vdash A$; this is the desired contradiction.

Ad $(*)$. Assume $T \vdash A$. Then $A \in T$, hence $\ulcorner A \urcorner \in \ulcorner T \urcorner$, hence $T \vdash B(\ulcorner A \urcorner)$ (because $B(z)$ represents in T the set $\ulcorner T \urcorner$). By the choice of A it follows that $T \vdash \neg A$, which contradicts the consistency of T .

Ad $(**)$. By $(*)$ we know $T \not\vdash A$. Therefore $A \notin T$, hence $\ulcorner A \urcorner \notin \ulcorner T \urcorner$ and therefore $T \vdash \neg B(\ulcorner A \urcorner)$. By the choice of A it follows that $T \vdash A$. $\square$

We now aim at Gödel's (first) incompleteness theorem. A theory T is *consistent* if $\perp \notin T$; otherwise T is *inconsistent*. Recall from 2.2.3 that a theory T is *complete* if for every closed formula $A \in \mathcal{L}$ we have $A \in T$ or $\neg A \in T$.

THEOREM (First incompleteness theorem). *Assume that T is a recursively enumerable consistent theory with the property that all recursive functions are representable in T . Then T is incomplete.*

PROOF. Let T be such a theory. Clearly $F := \{\ulcorner A \urcorner \mid A \in \overline{\mathcal{L}}\}$ is elementary. Since T is complete, we have

$$a \notin \ulcorner T \urcorner \leftrightarrow a \notin F \vee \dot{a} \in \ulcorner T \urcorner$$

with $\dot{a} := \langle \text{sn}(\rightarrow), a, \text{sn}(\perp) \rangle$. Hence the complement of $\ulcorner T \urcorner$ is recursively enumerable as well, which means that $\ulcorner T \urcorner$ is recursive. Now the claim follows from the undecidability theorem above. $\square$

4.3. Undecidable theories

We show in this section that already very simple theories have the property that all recursive functions are representable in them; an example is a finitely axiomatized arithmetical theory Q due to Robinson (1950). A consequence will be the (even “essential”) undecidability of Q .

4.3.1. Weak arithmetical theories. Call T a *theory with defined natural numbers* if there is a formula $N(x)$ – written Nx – such that $T \vdash N0$ and $T \vdash \forall_{x \in N} N(Sx)$ where $\forall_{x \in N} A$ is short for $\forall_x (Nx \rightarrow A)$. Representing a function in such a theory of course means that the free variables in (4.3) are relativized to N :

$$T \vdash \forall_{y, z \in N} (A(\underline{a}_1, \dots, \underline{a}_n, y) \wedge A(\underline{a}_1, \dots, \underline{a}_n, z) \rightarrow y=z) \text{ for all } a_1, \dots, a_n \in \mathbb{N}.$$

THEOREM. Let $\mathcal{L}$ be an elementarily presented language with $0, S, =$ in $\mathcal{L}$ and T a consistent theory with defined natural numbers containing the equality axioms $\text{Eq}_{\mathcal{L}}$ and the ex-falso-quodlibet axiom $\forall_{x,y \in N} (\perp \rightarrow x = y)$. Assume that

- (4.4) $T \vdash S\underline{a} \neq 0$ for all $a \in \mathbb{N}$,
- (4.5) $T \vdash S\underline{a} = S\underline{b} \rightarrow \underline{a} = \underline{b}$ for all $a, b \in \mathbb{N}$,
- (4.6) the functions $+$ and $\cdot$ are representable in T

and that there is a formula $L(x, y)$ – written $x < y$ – such that

- (4.7) $T \vdash \forall_{x \in N} (x \not< 0),$
- (4.8) $T \vdash \forall_{x \in N} (x < S\underline{b} \rightarrow x < \underline{b} \vee x = \underline{b})$ for all $b \in \mathbb{N}$,
- (4.9) $T \vdash \forall_{x \in N} (x < \underline{b} \vee x = \underline{b} \vee \underline{b} < x)$ for all $b \in \mathbb{N}$.

Then every recursive function is representable in T .

PROOF. First note that the formulas $x = y$ and $x < y$ actually do represent in T the equality and the less-than relations, respectively. From (4.4) and (4.5) we can see immediately that $T \vdash \underline{a} \neq \underline{b}$ when $a \neq b$. Assume $a \not< b$. We show $T \vdash \underline{a} \not< \underline{b}$ by induction on b . $T \vdash \underline{a} \not< 0$ follows from (4.7). In the step we have $a \not< b + 1$, hence $a \not< b$ and $a \neq b$, hence by induction hypothesis and the representability (above) of the equality relation, $T \vdash \underline{a} \not< \underline{b}$ and $T \vdash \underline{a} \neq \underline{b}$, hence by (4.8) $T \vdash \underline{a} \not< S\underline{b}$. Now assume $a < b$. Then $T \vdash \underline{a} \neq \underline{b}$ and $T \vdash \underline{b} \not< \underline{a}$, hence by (4.9) $T \vdash \underline{a} < \underline{b}$.

We now show by induction on the definition of μ -recursive functions that every recursive function is representable in T . Recall (from 4.2.1) that the second condition (4.2) in the definition of representability of a function automatically follows from the other two (and hence need not be checked further). This is because $T \vdash \underline{a} \neq \underline{b}$ for $a \neq b$.

The *initial functions* constant 0, successor and projection (onto the i -th coordinate) are trivially represented by the formulas $0 = y$, $Sx = y$ and $x_i = y$ respectively. Addition and multiplication are represented in T by assumption. Recall that the one remaining initial function of μ -recursiveness is $\div$, but this is definable from the characteristic function of $<$ by $a \div b = \mu_i(b + i \geq a) = \mu_i(c_{<}(b + i, a) = 0)$. We now show that the characteristic function of $<$ is representable in T . (It will then follow that $\div$ is representable, once we have shown that the representable functions are closed under μ .) We show that

$$A(x_1, x_2, y) := (x_1 < x_2 \wedge y = 1) \vee (x_1 \not< x_2 \wedge y = 0)$$

represents $c_<$. First notice that $\forall_{y,z \in N} (A(\underline{a_1}, \underline{a_2}, y) \wedge A(\underline{a_1}, \underline{a_2}, z) \rightarrow y = z)$ already follows logically from the equality axiom and the ex-falso-quodlibet axiom for equality (by cases on the alternatives of A). Assume $a_1 < a_2$. Then $T \vdash \underline{a_1} < \underline{a_2}$, hence $T \vdash A(\underline{a_1}, \underline{a_2}, 1)$. Now assume $a_1 \not< a_2$. Then $T \vdash \underline{a_1} \not< \underline{a_2}$, hence $T \vdash A(\underline{a_1}, \underline{a_2}, 0)$.

For the *composition* case, suppose f is defined from $h, g_1, \dots, g_m$ by

$$f(\vec{a}) = h(g_1(\vec{a}), \dots, g_m(\vec{a})).$$

By induction hypothesis we already have representing formulas $A_{g_i}(\vec{x}, y_i)$ and $A_h(\vec{y}, z)$. As representing formula for f we take

$$A_f := \exists_{\vec{y} \in N} (A_{g_1}(\vec{x}, y_1) \wedge \dots \wedge A_{g_m}(\vec{x}, y_m) \wedge A_h(\vec{y}, z)).$$

Assume $f(\vec{a}) = c$. Then there are $b_1, \dots, b_m$ such that $T \vdash A_{g_i}(\vec{a}, \underline{b_i})$ for each i , and $T \vdash A_h(\vec{b}, \underline{c})$ so by logic $T \vdash A_f(\vec{a}, \underline{c})$. It remains to show uniqueness $T \vdash \forall_{z_1, z_2 \in N} (A_f(\vec{a}, z_1) \wedge A_f(\vec{a}, z_2) \rightarrow z_1 = z_2)$. But this follows by logic from the induction hypothesis for g_i , which gives

$$T \vdash \forall_{y_{1i}, y_{2i} \in N} (A_{g_i}(\vec{a}, y_{1i}) \wedge A_{g_i}(\vec{a}, y_{2i}) \rightarrow y_{1i} = y_{2i} = \underline{g_i(\vec{a})})$$

and the induction hypothesis for h , which gives

$$T \vdash \forall_{z_1, z_2 \in N} (A_h(\vec{b}, z_1) \wedge A_h(\vec{b}, z_2) \rightarrow z_1 = z_2) \quad \text{with } b_i = g_i(\vec{a}).$$

For the μ case, suppose f is defined from g (taken here to be binary for notational convenience) by $f(a) = \mu_i(g(i, a) = 0)$, assuming $\forall_a \exists_i (g(i, a) = 0)$. By induction hypothesis we have a formula $A_g(y, x, z)$ representing g . In this case we represent f by the formula

$$A_f(x, y) := Ny \wedge A_g(y, x, 0) \wedge \forall_{v \in N} (v < y \rightarrow \exists_{u \in N; u \neq 0} A_g(v, x, u)).$$

We first show the representability condition (4.1), that is $T \vdash A_f(\underline{a}, \underline{b})$ when $f(a) = b$. Because of the form of A_f this follows from the assumed representability of g together with $T \vdash \forall_{v \in N} (v < \underline{b} \rightarrow v = \underline{0} \vee \dots \vee v = \underline{b-1})$.

We now tackle the uniqueness condition (4.3). Given a , let $b := f(a)$ (thus $g(b, a) = 0$ and b is the least such). It suffices to show

$$T \vdash \forall_{y \in N} (A_f(\underline{a}, y) \rightarrow y = \underline{b}).$$

We prove $T \vdash \forall_{y \in N} (y < \underline{b} \rightarrow \neg A_f(\underline{a}, y))$ and $T \vdash \forall_{y \in N} (\underline{b} < y \rightarrow \neg A_f(\underline{a}, y))$, and then appeal to the trichotomy law and the ex-falso-quodlibet axiom for equality.

We first show $T \vdash \forall_{y \in N} (y < \underline{b} \rightarrow \neg A_f(\underline{a}, y))$. Now since, for any $i < b$, $T \vdash \neg A_g(\underline{i}, \underline{a}, 0)$ by the assumed representability of g , we obtain immediately $T \vdash \neg A_f(\underline{a}, \underline{i})$. Hence because of $T \vdash \forall_{y \in N} (y < \underline{b} \rightarrow y = \underline{0} \vee \dots \vee y = \underline{b-1})$ the claim follows.

Secondly, $T \vdash \forall_{y \in N}(\underline{b} < y \rightarrow \neg A_f(\underline{a}, y))$ follows almost immediately from $T \vdash \forall_{y \in N}(\underline{b} < y \rightarrow A_f(\underline{a}, y) \rightarrow \exists_{u \in N; u \neq 0} A_g(\underline{b}, \underline{a}, u))$ and the uniqueness for g , $T \vdash \forall_{u \in N}(A_g(\underline{b}, \underline{a}, u) \rightarrow u = 0)$. $\square$

4.3.2. Robinson's theory Q . We conclude this section by considering a special and particularly simple arithmetical theory due originally to Robinson (1950). Let $\mathcal{L}_1$ be the language given by $0, S, +, \cdot$ and $=$, and let Q be the theory determined by the axioms $\text{Eq}_{\mathcal{L}_1}$, ex-falso-quodlibet for equality $\perp \rightarrow x = y$ and

$$(4.10) \quad Sx \neq 0,$$

$$(4.11) \quad Sx = Sy \rightarrow x = y,$$

$$(4.12) \quad x + 0 = x,$$

$$(4.13) \quad x + Sy = S(x + y),$$

$$(4.14) \quad x \cdot 0 = 0,$$

$$(4.15) \quad x \cdot Sy = x \cdot y + x,$$

$$(4.16) \quad \exists_z(x + Sz = y) \vee x = y \vee \exists_z(y + Sz = x).$$

THEOREM (Robinson's Q). *Every consistent theory $T \supseteq Q$ fulfills the assumptions of the previous theorem w.r.t. the definition $L(x, y) := \exists_z(x + Sz = y)$ of the $<$ -relation. Hence every recursive function is representable in T .*

PROOF. We show that T satisfies the conditions of the previous theorem. For (4.4) and (4.5) this is clear. For (4.6) we can take $x + y = z$ and $x \cdot y = z$ as representing formulas. For (4.7) we have to show $\neg \exists_z(x + Sz = 0)$; this follows from (4.13) and (4.10). For the proof of (4.8) we need the auxiliary proposition

$$(4.17) \quad x = 0 \vee \exists_y(x = 0 + Sy),$$

which will be attended to below. Assume $x + Sz = S\underline{b}$, hence also $S(x + z) = S\underline{b}$ and therefore $x + z = \underline{b}$. We must show $\exists_{y'}(x + Sy' = \underline{b}) \vee x = \underline{b}$. But this follows from (4.17) for z . In case $z = 0$ we obtain $x = \underline{b}$, and in case $\exists_y(z = 0 + Sy)$ we have $\exists_{y'}(x + Sy' = \underline{b})$, since $0 + Sy = S(0 + y)$. Thus (4.8) is proved. (4.9) follows immediately from (4.16). For the proof of (4.17) we use (4.16) with $y = 0$. It clearly suffices to exclude the first case $\exists_z(x + Sz = 0)$. But this means $S(x + z) = 0$, contradicting (4.10). $\square$

COROLLARY (Essential undecidability of Q). *Every consistent theory $T \supseteq Q$ in an elementarily presented language is non-recursive.*

PROOF. This follows from the theorem above and the undecidability theorem in 4.2.3. $\square$

COROLLARY (Undecidability of logic). *The set of formulas derivable in the classical fragment of minimal logic is non-recursive.*

PROOF. Otherwise Q would be recursive, because a formula A is derivable in Q if and only if the implication $B \rightarrow A$ is derivable, where B is the conjunction of the finitely many axioms and equality axioms of Q . $\square$

REMARK. Note that it suffices that the underlying language contains one binary relation symbol (for $=$), one constant symbol (for 0), one unary function symbol (for S) and two binary functions symbols (for $+$ and $\cdot$). The study of decidable fragments of first-order logic is one of the oldest research areas of mathematical logic. For more information see Börger et al. (1997).