PROOF THEORY: From arithmetic to set theory

Michael Rathjen
Leverhulme Fellow

Proof, Truth, Computation

Summer School on the Interactions between Modern Foundations of Mathematics and Contemporary Philosophy

Fraueninsel den 23. Juli 2014
Plan of the Talks

• First Lecture
  1. From Hilbert to Gentzen.
  2. Gentzen’s Hauptsatz and applications
  3. The general form of ordinal analysis
  4. A brief history of early ordinal representation systems

• Second Lecture:
  1. Proof theory of (sub)systems of second order arithmetic.
  2. Applications of Ordinal Analysis
  3. Proof theory of systems of set theory.
Theory of Proofs

- Aristotle
- Frege
Beweistheorie (Proof Theory)

- Hilbert’s second problem (1900): Consistency of Analysis
- Hilbert’s Programme (1922, 1925)
The Origins of Proof theory?

• Dedekind 1888, 1890. Canonical requirement for a structural definition: Prove the existence of a system of things falling under the notion to ensure it does not contain internal contradictions.

• Hilbert 1904 (Heidelberg talk): Syntactic consistency proof for a weak system of arithmetic.

• Hilbert 1917 (Axiomatisches Denken): we must turn the concept of a specifically mathematical proof itself into an object of investigation.

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• Hilbert's finitist consistency program only emerged in the winter term 1921/22.
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- II. Prove the consistency of \( T \) by finitistic means.

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In Hilbert’s Proof Theory, proofs become mathematical objects sui generis.
Gödel’s 1938 lecture at Zilsel’s

How then shall we extend? (Extension is necessary.)

Three ways are known up to now:

1. Higher types of functions (functions of functions of numbers, etc.).
2. The modal-logical route (introduction of an absurdity applied to universal sentences and a notion of “consequence”).
3. Transfinite induction, that is, inference by induction is added for certain concretely defined ordinal numbers of the second number class.
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Extended Hilbert Programs

(a) Arithmetical Predicativism.
(b) Theories of higher type functionals.
(c) Takeuti's "Hilbert-Gentzen finitist standpoint".
(d) Feferman's explicit mathematics.
(e) Martin-Löf's intuitionistic type theory.
(f) Constructive set theory (Myhill, Friedman, Beeson, Aczel).

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Consistency proof for a second-order version of Primitive Recursive Arithmetic.

Uses a finitistic version of transfinite induction up to the ordinal $\omega^\omega$. 
Gerhard Gentzen showed that transfinite induction up to the ordinal

\[ \varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} = \text{least } \alpha. \omega^\alpha = \alpha \]

suffices to prove the consistency of Peano Arithmetic, PA.
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Gentzen’s applied transfinite induction up to \( \epsilon_0 \) solely to primitive recursive predicates and besides that his proof used only finitistically justified means.
Gentzen’s Result in Detail

\[ F + PR-TI(\varepsilon_0) \vdash \text{Con}(PA), \]

where \( F \) signifies a theory that is acceptable in finitism (e.g. \( F = \text{PRA} = \text{Primitive Recursive Arithmetic} \)) and \( PR-TI(\varepsilon_0) \) stands for transfinite induction up to \( \varepsilon_0 \) for primitive recursive predicates.
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• Gentzen also showed that his result is best possible: \( \text{PA} \) proves transfinite induction up to \( \alpha \) for arithmetic predicates for any \( \alpha < \varepsilon_0 \).
The non-finitist part of \( \text{PA} \) is encapsulated in \( \text{PR-TI}(\varepsilon_0) \) and therefore “measured” by \( \varepsilon_0 \), thereby tempting one to adopt the following definition of \textit{proof-theoretic ordinal} of a theory \( T \):

\[
\left| T \right|_{\text{Con}} = \text{least } \alpha. \quad \text{PRA} + \text{PR-TI}(\alpha) \vdash \text{Con}(T).
\]
We are interested in representing specific ordinals $\alpha$ as relations on $\mathbb{N}$.

Natural ordinal representation systems are frequently derived from structures of the form

$$\mathcal{A} = \langle \alpha, f_1, \ldots, f_n, <_{\alpha} \rangle$$

where $\alpha$ is an ordinal, $<_{\alpha}$ is the ordering of ordinals restricted to elements of $\alpha$ and the $f_i$ are functions

$$f_i : \underbrace{\alpha \times \cdots \times \alpha}_{k_i \text{ times}} \rightarrow \alpha$$

for some natural number $k_i$. 
A = ⟨A, g_1, \ldots, g_n, \prec⟩

is a **computable** (or **recursive**) representation of

\( A = \langle \alpha, f_1, \ldots, f_n, \prec_\alpha \rangle \) if the following conditions hold:

1. \( A \subseteq \mathbb{N} \) and \( A \) is a computable set.
Ordinal Representation Systems

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1. \( A \subseteq \mathbb{N} \) and \( A \) is a computable set.
2. \( \prec \) is a computable total ordering on \( A \) and the functions \( g_i \)
   are computable.
\[ \dot{\mathcal{A}} = \langle \mathcal{A}, g_1, \ldots, g_n, \prec \rangle \]

is a **computable** (or **recursive**) representation of \( \mathcal{A} = \langle \alpha, f_1, \ldots, f_n, <_\alpha \rangle \) if the following conditions hold:

1. \( A \subseteq \mathbb{N} \) and \( A \) is a computable set.
2. \( \prec \) is a computable total ordering on \( A \) and the functions \( g_i \) are computable.
3. \( \mathcal{A} \cong \dot{\mathcal{A}} \), i.e. the two structures are isomorphic.
Theorem  (Cantor, 1897) For every ordinal $\beta > 0$ there exist unique ordinals $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ such that

$$\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_n}. \quad (1)$$

The representation of $\beta$ in (1) is called the Cantor normal form.

We shall write $\beta =_{\text{CNF}} \omega^{\beta_1} + \cdots \omega^{\beta_n}$ to convey that $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_k$. 

FROM ARITHMETIC TO SET THEORY
A Representation for $\varepsilon_0$

- $\varepsilon_0$ denotes the least ordinal $\alpha > 0$ such that

$$\beta < \alpha \implies \omega^\beta < \alpha.$$
A Representation for $\varepsilon_0$

- $\varepsilon_0$ denotes the least ordinal $\alpha > 0$ such that $\beta < \alpha \Rightarrow \omega^\beta < \alpha$.

- $\varepsilon_0$ is the least ordinal $\alpha$ such that $\omega^\alpha = \alpha$. 
A Representation for $\varepsilon_0$

- $\varepsilon_0$ denotes the least ordinal $\alpha > 0$ such that
  \[ \beta < \alpha \implies \omega^{\beta} < \alpha. \]

- $\varepsilon_0$ is the least ordinal $\alpha$ such that $\omega^{\alpha} = \alpha$.

- $\beta < \varepsilon_0$ has a Cantor normal form with exponents $\beta_i < \beta$ and these exponents have Cantor normal forms with yet again smaller exponents. As this process must terminate, ordinals $< \varepsilon_0$ can be coded by natural numbers.
Coding $\varepsilon_0$ in $\mathbb{N}$

Define a function

$$[ . ] : \varepsilon_0 \longrightarrow \mathbb{N}$$

by

$$[\delta] = \begin{cases} 0 & \text{if } \delta = 0 \\ \langle [\delta_1], \ldots, [\delta_n] \rangle & \text{if } \delta =_{\text{CNF}} \omega^{\delta_1} + \cdots + \omega^{\delta_n} \end{cases}$$

where $\langle k_1, \ldots, k_n \rangle := 2^{k_1+1} \cdot \ldots \cdot p_n^{k_n+1}$ with $p_i$ being the $i$th prime number (or any other coding of tuples). Further define

$$A_0 := \text{ran}([ . ]),$$

$$[\delta] < [\beta] :\iff \delta < \beta,$$

$$[\delta] \hat{+} [\beta] := [\delta + \beta],$$

$$[\delta] \hat{\cdot} [\beta] := [\delta \cdot \beta],$$

$$\hat{\omega}[^{\delta}] := [\omega^{\delta}].$$
Then

\[ \langle \varepsilon_0, +, \cdot, \delta \mapsto \omega^\delta, \prec \rangle \cong \langle A_0, +, \cdot, x \mapsto \hat{\omega}^x, \prec \rangle. \]

\( A_0, +, \cdot, x \mapsto \hat{\omega}^x, \prec \) are recursive, in point of fact, they are all elementary recursive.
Transfinite Induction

- Let \( \langle A, \prec, \ldots \rangle \) be a primitive recursive ordinal representation system.

\[ \text{TI}_{qf}(A, \prec) \text{ is the schema} \]

\[ \forall \alpha \in A [\forall \beta \prec \alpha P(\beta) \rightarrow P(\alpha)] \rightarrow \forall \alpha \in A P(\alpha) \]

with \( P \) quantifier-free.
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- For \( \alpha \in A \) let \( \prec_\alpha \) be \( \prec \) restricted to \( A_\alpha := \{ \beta \in A \mid \beta \prec \alpha \} \).
Proof-theoretic reductions

Let $T_1, T_2$ be a pair of theories with languages $L_1$ and $L_2$, respectively, and let $\Phi$ be a (primitive recursive) collection of formulae common to both languages. Furthermore, $\Phi$ should contain the closed equations of the language of PRA.

$T_1$ is proof-theoretically $\Phi$-reducible to $T_2$, written $T_1 \leq_{\Phi} T_2$, if there exists a primitive recursive function $f$ such that $\text{PRA} \vdash \forall \phi \in \Phi \forall x [\text{Proof}_{T_1}(x, \phi) \rightarrow \text{Proof}_{T_2}(f(x), \phi)]$.

$T_1$ and $T_2$ are said to be proof-theoretically $\Phi$-equivalent, written $T_1 \equiv_{\Phi} T_2$, if $T_1 \leq_{\Phi} T_2$ and $T_2 \leq_{\Phi} T_1$. 
Let $T_1$, $T_2$ be a pair of theories with languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and let $\Phi$ be a (primitive recursive) collection of formulae common to both languages. Furthermore, $\Phi$ should contain the closed equations of the language of PRA.
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FROM ARITHMETIC TO SET THEORY
Proof-theoretic ordinals

• In practice, if \( T_1 \equiv T_2 \) is shown through an ordinal analysis this always entails that the two theories prove at least the same \( \Pi_0^2 \) sentences.

• Given a natural ordinal representation system \( \langle A, \prec, \ldots \rangle \) of order type \( \tau \) let \( \text{PRA} + \text{TI}^{\text{qf}}(<\tau) \) be \( \text{PRA} \) augmented by quantifier-free induction over all initial (externally indexed) segments of \( \prec \), i.e., \( \text{TI}^{\text{qf}}(A^\alpha, \prec^\alpha) \) for \( \alpha \in A \).

• We say that a theory \( T \) has proof-theoretic ordinal \( \tau \), written \( |T| = \tau \), if \( T \) can be proof-theoretically reduced to \( \text{PRA} + \text{TI}^{\text{qf}}(<\tau) \), i.e., \( T \equiv \Pi_0^2 \text{PRA} + \text{TI}^{\text{qf}}(<\tau) \).
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• Given a natural ordinal representation system $\langle A, \prec, \ldots \rangle$ of order type $\tau$ let $\text{PRA} + \text{TI}_{qf}(\prec \tau)$ be $\text{PRA}$ augmented by quantifier-free induction over all initial (externally indexed) segments of $\prec$, i.e.,

$$\text{TI}_{qf}(A_\alpha, \prec_\alpha)$$

for $\alpha \in A$. 

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  \[
  T \equiv_{\Pi_2^0} \text{PRA} + \text{TI}_{qf}(\prec \tau).
  \]
The two main strands of research are:

- **Cut Elimination** (and Proof Collapsing Techniques)
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- Development of ever stronger **Ordinal Representation Systems**
A sequent is an expression \( \Gamma \Rightarrow \Delta \) where \( \Gamma \) and \( \Delta \) are finite sequences of formulae \( A_1, \ldots, A_n \) and \( B_1, \ldots, B_m \), respectively.
The Sequent Calculus

**SEQUENTS**

- A **sequent** is an expression $\Gamma \Rightarrow \Delta$ where $\Gamma$ and $\Delta$ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively.

- $\Gamma \Rightarrow \Delta$ is read, informally, as $\Gamma$ yields $\Delta$ or, rather, the conjunction of the $A_i$ yields the disjunction of the $B_j$. 
The Sequent Calculus

LOGICAL INFERENCES I

Negation

\[
\Gamma \Rightarrow \Delta, A \\
\neg A, \Gamma \Rightarrow \Delta \\
\neg L
\]

\[
B, \Gamma \Rightarrow \Delta \\
\Gamma \Rightarrow \Delta, \neg B \\
\neg R
\]

Implication

\[
\Gamma \Rightarrow \Delta, A \\
B, \land \Rightarrow \Theta \\
A \rightarrow B, \Gamma, \land \Rightarrow \Delta, \Theta \\
\rightarrow L
\]

\[
A, \Gamma \Rightarrow \Delta, B \\
\Gamma \Rightarrow \Delta, A \rightarrow B \\
\rightarrow R
\]
**Conjunction**

\[
\begin{align*}
A, \Gamma & \Rightarrow \Delta \\
\hline
A \land B, \Gamma & \Rightarrow \Delta \\
\end{align*}
\]

\(\land L_1\)

\[
\begin{align*}
B, \Gamma & \Rightarrow \Delta \\
\hline
A \land B, \Gamma & \Rightarrow \Delta \\
\end{align*}
\]

\(\land L_2\)

\[
\begin{align*}
\Gamma & \Rightarrow \Delta, A \\
\hline
\Gamma & \Rightarrow \Delta, B \\
\end{align*}
\]

\(\land R\)

\[
\begin{align*}
\Gamma & \Rightarrow \Delta, A \land B \\
\hline
\end{align*}
\]

**Disjunction**

\[
\begin{align*}
A, \Gamma & \Rightarrow \Delta \\
\hline
B, \Gamma & \Rightarrow \Delta \\
\end{align*}
\]

\(\lor L\)

\[
\begin{align*}
\Gamma & \Rightarrow \Delta, A \lor B \\
\hline
\end{align*}
\]

\(\lor R_1\)

\[
\begin{align*}
\Gamma & \Rightarrow \Delta, B \\
\hline
\end{align*}
\]

\(\lor R_2\)
Quantifiers

\[
\frac{F(t), \Gamma \Rightarrow \Delta}{\forall x \ F(x), \Gamma \Rightarrow \Delta} \quad \forall L
\]

\[
\frac{\Gamma \Rightarrow \Delta, \ F(a)}{\Gamma \Rightarrow \Delta, \forall x \ F(x)} \quad \forall R
\]

\[
\frac{F(a), \Gamma \Rightarrow \Delta}{\exists x \ F(x), \Gamma \Rightarrow \Delta} \quad \exists L
\]

\[
\frac{\Gamma \Rightarrow \Delta, \ F(t)}{\Gamma \Rightarrow \Delta, \exists x \ F(x)} \quad \exists R
\]

In \(\forall L\) and \(\exists R\), \(t\) is an arbitrary term. The variable \(a\) in \(\forall R\) and \(\exists L\) is an eigenvariable of the respective inference, i.e. \(a\) is not to occur in the lower sequent.
Identity Axiom

\[ A \Rightarrow A \]

where \( A \) is any formula.

One could limit this axiom to the case of atomic formulae \( A \).
The Sequent Calculus

**CUTS**

CUT

\[ \Gamma \Rightarrow \Delta, A \quad A, \Lambda \Rightarrow \Theta \]

\[ \Gamma, \Lambda \Rightarrow \Delta, \Theta \]  \hspace{1cm} \text{Cut}  

A is called the *cut formula* of the inference.

**Example**

\[ B \Rightarrow A \quad A \Rightarrow C \]

\[ B \Rightarrow C \]  \hspace{1cm} \text{Cut}
The Sequent Calculus

STRUCTURAL RULES

Structural Rules

\[
\frac{\Gamma, A, B, \Lambda \Rightarrow \Delta}{\Gamma, B, A, \Lambda \Rightarrow \Delta} \quad \mathcal{X}_l
\]

\[
\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \quad \mathcal{W}_l
\]

\[
\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \quad \mathcal{C}_l
\]

Exchange, Weakening, Contraction

\[
\frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda} \quad \mathcal{X}_r
\]

\[
\frac{\Gamma \Rightarrow \Delta, B, A, \Lambda}{\Gamma \Rightarrow \Delta, A, \Lambda} \quad \mathcal{W}_r
\]

\[
\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \quad \mathcal{C}_r
\]
The intuitionistic sequent calculus is obtained by requiring that all sequents be intuitionistic. A sequent $\Gamma \Rightarrow \Delta$ is said to be intuitionistic if $\Delta$ consists of at most one formula. Specifically, in the intuitionistic sequent calculus there are no inferences corresponding to contraction right or exchange right.
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Classical Example

Our first example is a deduction of the law of excluded middle.

\[ A \Rightarrow A \]

\[ \neg R \Rightarrow A \]
\[ \neg A \lor R \Rightarrow A \]
\[ A \lor \neg A \]

\[ r \Rightarrow A \lor \neg A \]
\[ A \lor \neg A \]
\[ C \Rightarrow A \lor \neg A \]

Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.
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\[
\begin{align*}
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\Rightarrow & A, \neg A \\
\neg R & \\
\Rightarrow & A, A \lor \neg A \\
\lor R & \\
\Rightarrow & A \lor \neg A, A \\
\lor_r & \\
\Rightarrow & A \lor \neg A, A \lor \neg A \\
\lor R & \\
\Rightarrow & A \lor \neg A \\
\lor C_r & 
\end{align*}
\]
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\Rightarrow A \lor \neg A \\
\Rightarrow A \lor \neg A
\end{align*}
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Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.
Gentzen’s Hauptsatz (1934)

Cut Elimination

If a sequent

\[ \Gamma \Rightarrow \Delta \]

is provable, then it is provable \textit{without cuts}. 
The Hauptsatz has an important corollary:
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The **Subformula Property**

*If a sequent $\Gamma \Rightarrow \Delta$ is provable, then it has a deduction all of whose formulae are subformulae of the formulae in $\Gamma$ and $\Delta$.***
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*If a sequent $\Gamma \Rightarrow \Delta$ is provable, then it has a deduction all of whose formulae are subformulae of the formulae in $\Gamma$ and $\Delta$.*

**Corollary**  
A contradiction, i.e. the empty sequent, is not deducible.
Applications of the Haupsatz

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  $\vdash \exists x R(x)$ implies $\vdash R(t_1) \lor \ldots \lor R(t_n)$ for some $t_i$ (R quantifier-free).

• Extended Herbrand's Theorem in LK:
  $\vdash \Gamma \Rightarrow \exists x R(x)$ implies $\vdash \Gamma \Rightarrow R(t_1) \lor \ldots \lor R(t_n)$ for some $t_i$ (R quantifier-free, $\Gamma$ purely universal).

• In LI (intuitionistic predicate logic):
  $\vdash \Gamma \Rightarrow \exists x R(x)$ implies $\vdash R(t)$ for some term $t$ where $\Gamma$ is $\lor$ and $\exists$ free.

• Hilbert-Ackermann Consistency

• If $T$ is a geometric theory and $T$ classically proves a geometric implication $A$, then $T$ intuitionistically proves $A$. 

FROM ARITHMETIC TO SET THEORY
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Theories and Cut Elimination

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- Axioms are detrimental to this procedure. It breaks down because the symmetry of the sequent calculus is lost. In general, we cannot remove cuts from deductions in a theory $T$ when the cut formula is an axiom of $T$.
- However, sometimes the axioms of a theory are of bounded syntactic complexity. Then the procedure applies partially in that one can remove all cuts that exceed the complexity of the axioms of $T$. 
Partial Cut Elimination

- Gives rise to **partial cut elimination**.
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- This is a very important tool in proof theory. For example, it works very well if the axioms of a theory can be presented as atomic intuitionistic sequents (also called Horn clauses), yielding the completeness of Robinsons resolution method.
Partial cut elimination also pays off in the case of fragments of PA and set theory with restricted induction schemes, be it induction on natural numbers or sets. This method can be used to extract bounds from proofs of $\Pi^0_2$ statements in such fragments.
Gentzen’s way out

• Gentzen defined an assignment of ordinals to derivations of PA such for every derivation D of PA in his sequent calculus, \( \text{ord}(D) < \varepsilon_0 \).
• He then defined a reduction procedure \( R \) such that whenever \( D \) is a derivation of the empty sequent in PA then \( R(D) \) is another derivation of the empty sequent in PA but with a smaller ordinal assigned to it, i.e., \( \text{ord}(R(D)) < \text{ord}(D) \).
• Moreover, both \( \text{ord} \) and \( R \) are primitive recursive functions and only finitist means are used in showing (3).
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- Moreover, both \( \text{ord} \) and \( R \) are primitive recursive functions and only finitist means are used in showing (3).
If \( \text{PRWO}(\varepsilon_0) \) is the statement that there are no infinitely descending primitive recursive sequences of ordinals below \( \varepsilon_0 \), then the following are immediate consequences of Gentzen's work.

**Theorem:** (Gentzen 1936, 1938)

1. The theory of primitive recursive arithmetic, \( \text{PRA} \), proves that \( \text{PRWO}(\varepsilon_0) \) implies the 1-consistency of \( \text{PA} \).
2. Assuming that \( \text{PA} \) is consistent, \( \text{PA} \) does not prove \( \text{PRWO}(\varepsilon_0) \).

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Second Order Arithmetic, $\mathbb{Z}_2$
• $\mathbb{Z}_2$ is an extension of $\text{PA}$ with quantifiers $\forall X$ and $\exists Y$ imagined to range over subsets of $\mathbb{N}$ and full comprehension ($\text{CA}$):

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- Impredicativity

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Y = \{ n \mid \forall X \exists Z G(X, Z, n) \}
\]
Comments

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Birth of Second Order Proof Theory: The Fundamental Conjecture on \textbf{GLC}

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Quantifiers

\[
\frac{F(\{v \mid A(v)\}), \Gamma \Rightarrow \Delta}{\forall X F(X), \Gamma \Rightarrow \Delta} \quad \forall_2 \text{L}
\]

\[
\frac{\Gamma \Rightarrow \Delta, F(U)}{\Gamma \Rightarrow \Delta, \forall X F(X)} \quad \forall_2 \text{R}
\]

\[
\frac{F(U), \Gamma \Rightarrow \Delta}{\exists X F(X), \Gamma \Rightarrow \Delta} \quad \exists_2 \text{L}
\]

\[
\frac{\Gamma \Rightarrow \Delta, F(\{v \mid A(v)\})}{\Gamma \Rightarrow \Delta, \exists X F(X)} \quad \exists_2 \text{R}
\]

In \(\forall_2 \text{L}\) and \(\exists_2 \text{R}\), \(A(a)\) is an arbitrary formula. The variable \(U\) in \(\forall_2 \text{R}\) and \(\exists_2 \text{L}\) is an eigenvariable of the respective inference, i.e. \(U\) is not to occur in the lower sequent.
Non-constructive proofs of FC


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Mariko Yasugi and I tried to resolve the fundamental conjecture for the system with the \( \Delta_{1} \) comprehension axiom within our extended version of the finite standpoint. Ultimately, our success was limited to the system with provably \( \Delta_{1} \) comprehension axiom. This was my last successful result in this area.

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A brief history of ordinal representation systems

1904-1950

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O. Veblen, 1908

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- He applied two new operations to continuous increasing functions on ordinals:
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- Let \( \text{ON} \) be the class of ordinals. A (class) function \( f : \text{ON} \rightarrow \text{ON} \) is said to be increasing if \( \alpha < \beta \) implies \( f(\alpha) < f(\beta) \) and continuous (in the order topology on \( \text{ON} \)) if

\[
\lim_{\xi<\lambda} f(\alpha_{\xi}) = \lim_{\xi<\lambda} f(\alpha_{\xi})
\]

holds for every limit ordinal \( \lambda \) and increasing sequence \( (\alpha_{\xi})_{\xi<\lambda} \).
Derivations

- \( f \) is called **normal** if it is increasing and continuous.

\[
\begin{align*}
\beta &\mapsto \omega + \beta \\
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\end{align*}
\]

The function \( \beta \mapsto \omega + \beta \) is normal while \( \beta \mapsto \beta + \omega \) is not continuous at \( \omega \) since

\[
\lim_{\xi \to \omega} (\xi + \omega) = \omega \\
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\]

- The derivative \( f' \) of a function \( f: \mathbb{N} \to \mathbb{N} \) is the function which enumerates in increasing order the solutions of the equation \( f(\alpha) = \alpha \), also called the **fixed points** of \( f \).

- If \( f \) is a normal function, \( \{ \alpha : f(\alpha) = \alpha \} \) is a proper class and \( f' \) will be a normal function, too.
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The Feferman-Schütte Ordinal $\Gamma_0$

- From the normal function $f$ we get a two-place function,

$$\varphi_f(\alpha, \beta) := f_\alpha(\beta).$$

Veblen then discusses the hierarchy when

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  \]

- The least ordinal \( \gamma > 0 \) closed under \( \varphi_\ell \), i.e. the least ordinal \( \gamma > 0 \) satisfying
  \[
  (\forall \alpha, \beta < \gamma) \varphi_\ell(\alpha, \beta) < \gamma
  \]

  is the famous ordinal \( \Gamma_0 \) which Feferman and Schütte determined to be the least ordinal ‘unreachable’ by predicative means.
Veblen extended this idea first to arbitrary **finite numbers of arguments**, but then also to **transfinite numbers of arguments**, with the proviso that in, for example

\[ \Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta), \]

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The Big Veblen Number

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• Veblen singled out the ordinal $E(0)$, where $E(0)$ is the least ordinal $\delta > 0$ which cannot be named in terms of functions

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with $\eta < \delta$, and each $\alpha_\gamma < \delta$. 
Bachmann’s novel idea: Use uncountable ordinals to keep track of the functions defined by diagonalization.
Bachmann’s novel idea: Use **uncountable ordinals** to keep track of the functions defined by **diagonalization**.

Define a set of ordinals $\mathcal{B}$ closed under successor such that with each limit $\lambda \in \mathcal{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathcal{B}$ of length $\tau_\lambda \leq \mathcal{B}$ and $\lim_{\xi < \tau_\lambda} \lambda[\xi] = \lambda$. 

$\Omega$ be the first uncountable ordinal. A hierarchy of functions $(\phi^B_\alpha(\beta))_{\alpha \in \mathcal{B}}$ is then obtained as follows:

$$\phi^B_0(\beta) = 1 + \beta \phi^B_{\alpha+1}(\beta) = (\phi^B_\alpha)'$$

$\phi^B_\lambda$ enumerates $\bigcap_{\xi < \tau_\lambda} \lambda[\xi]$ if $\lambda$ is limit, $\tau_\lambda < \Omega$; $\phi^B_\lambda$ enumerates $\{\beta < \Omega : \phi^B_\lambda[\beta](0) = \beta\}$ if $\lambda$ is limit, $\tau_\lambda = \Omega$. 

**From arithmetic to set theory**
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$$\varphi_0^\mathcal{B}(\beta) = 1 + \beta \quad \varphi_{\alpha+1}^\mathcal{B} = \left(\varphi_\alpha^\mathcal{B}\right)'$$

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$\varphi_\lambda^\mathcal{B}$ enumerates $\{\beta < \Omega : \varphi_\lambda^\mathcal{B}(0) = \beta\} \quad \lambda \text{ limit, } \tau_\lambda = \Omega$. 
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1960-1974

After Bachmann, the story of ordinal representation systems becomes very complicated.

- **Isles, Bridge, Gerber, Pfeiffer, Schütte** extended Bachmann’s approach. Drawback: Horrendous computations.

- **Aczel** and **Weyhrauch** combined Bachmann’s approach with uses of higher type functionals.

- **Feferman**’s new proposal: Bachmann-type hierarchy without fundamental sequences.

- **Bridge** and **Buchholz** showed computability of systems obtained by Feferman’s approach.
How much of $\mathbb{Z}_2$ is needed?

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- **Hermann Weyl** 1918 “Das Kontinuum" Predicative Analysis.
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- **Hilbert, Bernays** 1938: $\mathbb{Z}_2$ sufficient for “Ordinary Mathematics"
- Minimal foundational frameworks for Ordinary Mathematics:
  Feferman, Lorenzen, Takeuti ....
- **Reverse Mathematics**, early 1970s-now
  H. Friedman, S. Simpson, ....

Given a specific theorem $\tau$ of ordinary mathematics, which set existence axioms are needed in order to prove $\tau$?
Subsystems of $\mathbb{Z}_2$

- Basic arithmetical axioms in all subtheories of $\mathbb{Z}_2$ are: defining axioms for $0, 1, +, \times, E, <$ (as for PA) and the induction axiom

$$\forall X [0 \in X \land \forall n(n \in X \rightarrow n + 1 \in X) \rightarrow \forall n (n \in X)].$$
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- For each axiom scheme \( \text{Ax} \), \( (\text{Ax})_0 \) denotes the theory consisting of the basic arithmetical axioms plus the scheme \( \text{Ax} \).
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- For each axiom scheme $Ax$, $(Ax)_0$ denotes the theory consisting of the basic arithmetical axioms plus the scheme $Ax$.
- $(Ax)$ stands for the theory $(Ax)_0$ augmented by the scheme of induction for all $\mathcal{L}_2$-formulae.
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- $(\mathbf{Ax})$ stands for the theory $(\mathbf{Ax})_0$ augmented by the scheme of induction for all $\mathcal{L}_2$-formulae.
- Let $\mathcal{F}$ be a collection of formulae of $\mathbb{Z}_2$.
  Another important axiom scheme for formulae $F$ in $\mathcal{C}$ is

  $\mathcal{C} - \mathbf{AC} \quad \forall n \exists YF(n, Y) \rightarrow \exists Y \forall nF(x, Y_n),$

  where $Y_n := \{ m : 2^n 3^m \in Y \}$. 

FROM ARITHMETIC TO SET THEORY
For many mathematical theorems $\tau$, there is a weakest natural subsystem $S(\tau)$ of $\mathbb{Z}_2$ such that $S(\tau)$ proves $\tau$. Moreover, it has turned out that $S(\tau)$ often belongs to a small list of specific subsystems of $\mathbb{Z}_2$. Reverse Mathematics has singled out five subsystems of $\mathbb{Z}_2$:

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Mathematical Equivalences: Examples

- **RCA**₀  “Every countable field has an algebraic closure"; “Every countable ordered field has a real closure"
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  “Every countable commutative ring with a unit has a maximal ideal"

- **ATR\(_0\)**  
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- **(Π\(_1\)_1−CA\(_0\))**  
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\[ |\text{ATR}_0| = \Gamma_0 \]

\[ |\text{ACA}_0| = \varepsilon_0 \]

\[ |\text{RCA}_0| = \omega^\omega = |\text{WKL}_0| \]
\[ |(\Sigma^1_2 - AC) + BI| = \psi_\Omega_1 \varepsilon_{i+1} \]

\[ |(\Delta^1_2 - CA)| = \psi_\Omega_1 \Omega \varepsilon_0 \]

\[ |(\Pi^1_1 - CA)_0| = \psi_\Omega_1 \Omega \omega \]

\[ |ATR_0| = \Gamma_0 \]
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\[ |\text{KPM}| = \psi_{\Omega_1} \varepsilon_{M+1} \]
$|KP + \Pi_3\text{-Reflection}| = \psi_{\Omega_1} \varepsilon_{K+1}$
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A Brief History of Ordinal Analysis

- Gentzen 1936
  theory $\text{PA}$
  ordinal $\varepsilon_0$

- Feferman, Schütte 1963
  Predicative Second Order Arithmetic
  ordinal $\Gamma_0$

- Takeuti 1967
  $(\Pi_1^1 - \text{CA})_0$, $(\Pi_1^1 - \text{CA})_0 + \text{BI}$
  ordinals $\psi_\Omega_1 \omega$, $\psi_\Omega_1 \varepsilon_\omega + 1$
  cardinal analogue: $\omega$-many regular cardinals

- Takeuti, Yasugi 1983
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  $(\Delta^1_2\text{-CA})$
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  cardinal analogue: $\varepsilon_0$-many regular cardinals
Buchholz, Pohlers, Sieg 1977
Theories of Iterated Inductive Definitions
ordinals $\psi_{\Omega_1} \varepsilon_{\Omega_\nu+1}$
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  Method of Local Predicativity
Buchholz, Pohlers, Sieg 1977
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Girard 1979
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Jäger 1979
Constructible Hierarchy in Proof Theory
A Brief History of Ordinal Analysis cont’d

- Jäger, Pohlers 1982
  \((\Sigma^1_2 - \text{AC}) + \text{BI, KPi}\)
  ordinal \(\psi_{\Omega_1 \varepsilon_{l+1}}\)
  cardinal analogue: \(l\) inaccessible cardinal
Jäger, Pohlers 1982
$(\Sigma^1_2 \text{-} \text{AC}) + \text{BI, KPi}$
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1989
KPM
ordinal $\psi_{\Omega_1 \varepsilon_{M+1}}$
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Jäger, Pohlers 1982
$(\Sigma^1_2 \text{-AC}) + \text{BI, KPi}$
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cardinal analogue: $\lambda$ inaccessible cardinal

1989
KPM
ordinal $\psi_{\Omega_1 \varepsilon_{\mu+1}}$
cardinal analogue: $\mu$ Mahlo cardinal

Buchholz 1990
Operator Controlled Derivations
1992

$\Pi_3$-reflection

ordinal $\psi_{\Omega_1} \varepsilon_{K+1}$

cardinal analogue: $K$ weakly compact cardinal
A Brief History of Ordinal Analysis cont’d

• 1992
  \( \Pi_3 \)-reflection
  ordinal \( \psi_{\Omega_1 \varepsilon_{K+1}} \)
  cardinal analogue: \( K \) weakly compact cardinal

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• 1995
  \( \Pi^1_2 \)-Comprehension
  cardinal analogue: \( \omega \)-many reducible cardinals
A Brief History of Ordinal Analysis cont’d

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  \( \Pi_2^1 \)-Comprehension
  cardinal analogue: \( \omega \)-many reducible cardinals

- Arai
  Ordinal Analysis of Theories up to \( \Pi_2^1 \)-Comprehension
  using Reductions on Finite Proof Figures and Ordinal Diagrams.
Rewards of Ordinal Analyses

• I. Hilbert’s Programme Extended: Constructive Consistency Proofs
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• I. Hilbert’s Programme Extended: Constructive Consistency Proofs
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• III. Classification of Provable Functions and Functionals of Theories
• IV. Combinatorial Independence Results
Examples

Theorem: (Jäger)

*Feferman’s intuitionistic $T_0$ is of the same strength as $(\Sigma^1_2\text{-AC}) + BI$.***
Examples

Theorem: (Jäger)

Feferman’s intuitionistic $T_0$ is of the same strength as $(\Sigma^1_2$-AC) + BI.

Theorem: (R; Setzer)

A consistency proof for $(\Sigma^1_2$-AC) + BI can be carried out in Martin-Löf Type Theory.


Combinatorial Independence Results

- A *finite tree* is a finite partially ordered set

\[ \mathbb{B} = (B, \leq) \]

such that:

(i) \( B \) has a smallest element (called the *root* of \( \mathbb{B} \));
(ii) for each \( s \in B \) the set \( \{ t \in B : t \leq s \} \) is a totally ordered subset of \( B \).
A finite tree is a finite partially ordered set $\mathbb{B} = (B, \leq)$ such that:

(i) $B$ has a smallest element (called the root of $\mathbb{B}$);

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For finite trees $\mathbb{B}_1$ and $\mathbb{B}_2$, an embedding of $\mathbb{B}_1$ into $\mathbb{B}_2$ is a one-to-one mapping $f : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ such that

$$f(a \land b) = f(a) \land f(b)$$

for all $a, b \in \mathbb{B}_1$, where $a \land b$ denotes the infimum of $a$ and $b$. 
• **Kruskal’s Theorem.** For every infinite sequence of trees $(\mathcal{B}_k : k < \omega)$, there exist $i$ and $j$ such that $i < j < \omega$ and $\mathcal{B}_i$ is embeddable into $\mathcal{B}_j$. (In particular, there is no infinite set of pairwise nonembeddable trees.)
• **Kruskal’s Theorem.** For every infinite sequence of trees 
\((B_k : k < \omega)\), there exist \(i\) and \(j\) such that \(i < j < \omega\) and \(B_i\) is embeddable into \(B_j\). 
(In particular, there is no infinite set of pairwise nonembeddable trees.)

• **Theorem** (H. Friedman, D. Schmidt) Kruskal’s Theorem is not provable in \(\text{ATR}_0\).
• **Kruskal’s Theorem.** For every infinite sequence of trees \( (\mathcal{B}_k : k < \omega) \), there exist \( i \) and \( j \) such that \( i < j < \omega \) and \( \mathcal{B}_i \) is embeddable into \( \mathcal{B}_j \). (In particular, there is no infinite set of pairwise nonembeddable trees.)

• **Theorem** (H. Friedman, D. Schmidt) Kruskal’s Theorem is not provable in \( \text{ATR}_0 \).

• The proof utilizes that Kruskal’s Theorem implies that \( \Gamma_0 \) is well-founded.
The Extended Kruskal Theorem

- For $n < \omega$, let $\mathcal{B}_n$ be the set of all finite trees with labels from $n$, i.e. $(\mathcal{B}, \ell) \in \mathcal{B}_n$ if $\mathcal{B}$ is a finite tree and
  $\ell : B \to \{0, \ldots, n - 1\}$.

  The set $\mathcal{B}_n$ is quasiordered by putting $(\mathcal{B}_1, \ell_1) \leq (\mathcal{B}_2, \ell_2)$ if there exists an embedding
  $f : \mathcal{B}_1 \to \mathcal{B}_2$ such that:

  - $\ell_1(b) = \ell_2(f(b))$ for each $b \in \mathcal{B}_1$;
  - if $b$ is an immediate successor of $a \in \mathcal{B}_1$, then for each $c \in \mathcal{B}_2$ in the interval $f(a) < c < f(b)$,
    $\ell_2(c) \geq \ell_2(f(b))$.
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$$\ell_2(c) \geq \ell_2(f(b)).$$

This condition is called a gap condition.
The Extended Kruskal Theorem

**Theorem.** (Friedman) For each $n < \omega$, $\mathcal{B}_n$ is a well quasi ordering (abbreviated $\text{WQO}(\mathcal{B}_n)$), i.e. there is no infinite set of pairwise nonembeddable trees.
The Extended Kruskal Theorem

Theorem. (Friedman) For each $n < \omega$, $B_n$ is a well quasi ordering (abbreviated $WQO(B_n)$), i.e. there is no infinite set of pairwise nonembeddable trees.

Theorem $\forall n < \omega \ WQO(B_n)$ is not provable in $\Pi^1_1 \text{-- CA}_0$. 
**The Extended Kruskal Theorem**

**Theorem.** (Friedman) For each $n < \omega$, $B_n$ is a **well quasi ordering** (abbreviated $WQO(B_n)$), i.e. there is no infinite set of pairwise nonembeddable trees.

**Theorem** $\forall n < \omega \ WQO(B_n)$ is not provable in $\Pi^1_1 - CA_0$.

- The proof employs an ordinal representation system for the proof-theoretic ordinal of $\Pi^1_1 - CA_0$. The ordinal is $\psi_{\Omega_1}(\Omega_\omega)$. 

*From arithmetic to set theory*
The Graph Minor Theorem

• $G$, $H$ graphs. If $H$ is obtained from $G$ by first deleting some vertices and edges, and then contracting some further edges, $H$ is a minor of $G$. 
The Graph Minor Theorem

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GMT Theorem. (Robertson and Seymour 1986-1997) If $G_0, G_1, G_2, \ldots$ is an infinite sequence of finite graphs, then there exist $i < j$ so that $G_i$ is isomorphic to a minor of $G_j$.
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- The proof of GMT uses the EKT.
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- The proof of GMT uses the EKT.

- Corollary. (Vázsonyi’s conjecture) If all the $G_k$ are trivalent, then there exist $i < j$ so that $G_i$ is embeddable into $G_j$. 
The Graph Minor Theorem

- \( G, H \) graphs. If \( H \) is obtained from \( G \) by first deleting some vertices and edges, and then contracting some further edges, \( H \) is a **minor** of \( G \).

**GMT Theorem.** *(Robertson and Seymour 1986-1997)* If \( G_0, G_1, G_2, \ldots \) is an infinite sequence of finite graphs, then there exist \( i < j \) so that \( G_i \) is isomorphic to a minor of \( G_j \).

- The proof of GMT uses the EKT.

- **Corollary.** *(Vázsonyi’s conjecture)* If all the \( G_k \) are trivalent, then there exist \( i < j \) so that \( G_i \) is embeddable into \( G_j \).

- **Corollary.** *(Wagner’s conjecture)* For any 2-manifold \( M \) there are only finitely many graphs which are not embeddable in \( M \) and are minimal with this property.
• **Theorem.** (Friedman, Robertson, Seymour)
The Graph Minor Theorem

- **Theorem.** (Friedman, Robertson, Seymour)
  - GMT implies EKT within, say, RCA₀.
The Graph Minor Theorem

- **Theorem.** (Friedman, Robertson, Seymour)
  
  - GMT implies EKT within, say, $\text{RCA}_0$.
  
  - GMT is not provable in $\Pi^1_1 - \text{CA}_0$. 
Future Work

- Simplify the Ordinal Representation Systems for $\Pi^1_2$-Comprehension
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- Simplify the Ordinal Representation Systems for $\Pi_2^1$-Comprehension
- Find new Combinatorial Principles related to Ordinal Representation System for $\Pi_2^1$-Comprehension.
- Develop an abstract theory of ordinal representation that takes reflection configurations into account.
- Carry out ordinal analysis for $\Pi_{1^n}^1$-Comprehension for all $n$.
- Is $\Pi_2^1$-Comprehension the generic case?
- Conjecture: $\Pi_3^1$-Comprehension is the generic case.
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• Is $\Pi^1_2$-Comprehension the generic case?
Future Work

- Simplify the Ordinal Representation Systems for \( \Pi_2^1 \)-Comprehension

- Find new Combinatorial Principles related to Ordinal Representation System for \( \Pi_2^1 \)-Comprehension.

- Develop an abstract theory of ordinal representation that takes reflection configurations into account.

- Carry out ordinal analysis for \( \Pi_n^1 \)-Comprehension for all \( n \), i.e. \( \mathbb{Z}_2 \).

- Is \( \Pi_2^1 \)-Comprehension the generic case?

- Conjecture: \( \Pi_3^1 \)-Comprehension is the generic case.
Das Ende

Danke!