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1 Topology, Metric, Norm

1.1 Motivation, Definitions, Convergence

One major goal of this class is to study convergence in more general contexts than the sets $\mathbb{R}$ or $\mathbb{C}$. We have already encountered the convergence of $\mathbb{K}$-valued functions in [Phi16, Sec. 8], where we considered the distinct concepts of pointwise and uniform convergence. The notion of topology provides an abstract concept suitable for all the abovementioned convergences as well as for convergence in countless other situations of interest to us (convergence in $\mathbb{K}^n$ being just one important example).

In Analysis I, we said that a sequence $(z_k)_{k \in \mathbb{N}}$ in $\mathbb{K}$ converged to $z \in \mathbb{K}$ if, and only if, every neighborhood of $z$ contains almost all $z_k$ (cf. [Phi16, Rem. 7.8]). As it turns out, this is the concept that one can still use in the most abstract situation – one merely needs a suitable abstract notion of neighborhood. Recall from [Phi16, Def. 7.7(a)] that a neighborhood of a point $z \in \mathbb{K}$ is a set $U \subseteq \mathbb{K}$, containing an open $\epsilon$-ball with center $z$. In the situation of an abstract topological space $X$, to be defined in Def. 1.1 below, one specifies all the open subsets of $X$, calling $U \subseteq X$ a neighborhood of $x \in X$ if, and only if, there is an open set $O \subseteq X$ such that $x \in O \subseteq U$.

**Definition 1.1.** Let $X$ be a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ a set of subsets of $X$. Then $\mathcal{T}$ is called a topology on $X$ if, and only if, the following three conditions are satisfied:

(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

(ii) $\mathcal{T}$ is closed under finite intersections, i.e. the intersection of finitely many sets in $\mathcal{T}$ is again in $\mathcal{T}$: If $n \in \mathbb{N}$ and $O_i \in \mathcal{T}$ for each $i = 1, \ldots, n$, then

$$\bigcap_{i=1}^{n} O_i \in \mathcal{T}.$$

(iii) $\mathcal{T}$ is closed under arbitrary unions, i.e. the union of arbitrarily many (i.e. of finitely of infinitely many) sets in $\mathcal{T}$ is again in $\mathcal{T}$: If $I$ is an arbitrary index set and $O_i \in \mathcal{T}$ for each $i \in I$, then

$$\bigcup_{i \in I} O_i \in \mathcal{T}.$$

If $\mathcal{T}$ constitutes a topology on $X$, then the pair $(X, \mathcal{T})$ is called a topological space. Moreover, a set $O \subseteq X$ is called $\mathcal{T}$-open or open with respect to $\mathcal{T}$ if, and only if, $O \in \mathcal{T}$. One simply calls $O$ open in case the topology is understood. Given $x \in X$, a set $U \subseteq X$ is called a neighborhood of $x$ if, and only if, there is an open set $O \in \mathcal{T}$ such that $x \in O \subseteq U$ (note that $U$ does not have to be in $\mathcal{T}$). The set of all neighborhoods of $x$ is denoted by $\mathcal{U}(x)$; it is also called the neighborhood system or the neighborhood filter of $x$. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are both topologies on $X$ such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then we call $\mathcal{T}_1$ smaller or coarser than $\mathcal{T}_2$, and we call $\mathcal{T}_2$ bigger or finer than $\mathcal{T}_1$. 
Lemma 1.2. Let \((X, \mathcal{T})\) be a topological space. Then \(O \subseteq X\) is open if, and only if, for each \(x \in O\), there exists an open set \(O_x \in \mathcal{T}\) such that \(x \in O_x \subseteq O\).

Proof. If \(O \in \mathcal{T}\) and \(x \in O\), then one can just choose \(O_x := O\). Conversely, if, for each \(x \in O\), there exists an open set \(O_x \in \mathcal{T}\) such that \(x \in O_x \subseteq O\), then \(O = \bigcup_{x \in O} O_x\), proving \(O \in \mathcal{T}\) by Def. 1.1(iii).

Example 1.3. (a) For each set \(X\), \(\mathcal{T} := \mathcal{P}(X)\) constitutes a topology on \(X\), called the discrete topology on \(X\).

(b) For each set \(X\), \(\mathcal{T} := \{\emptyset, X\}\) constitutes a topology on \(X\), called the indiscrete or trivial topology on \(X\).

(c) For each set \(X\),

\[\mathcal{T} := \{O \subseteq X : O = \emptyset \text{ or } O^c \text{ is finite}\}\]

(where \(O^c := X \setminus O\) denotes the complement) constitutes a topology on \(X\), called the cofinite topology on \(X\) (clearly, \(\emptyset, X \in \mathcal{T}\); if \(O_1^c\) and \(O_2^c\) are both finite, then \((O_1 \cap O_2)^c = O_1^c \cup O_2^c\) is finite; if \(O_i^c, i \in I\), are all finite, then \((\bigcup_{i \in I} O_i)^c = \bigcap_{i \in I} O_i^c\) is finite).

(d) We call \(O \subseteq K\) open if, and only if,

\[\forall z \in O \exists \epsilon \in \mathbb{R}^+ \quad B_\epsilon(z) \subseteq O,\]

where, as in [Phil16, Def. 7.7(a)], \(B_\epsilon(z) := \{w \in K : |w - z| < \epsilon\}\). Then \(\mathcal{T} := \{O \subseteq K : O \text{ open}\}\) constitutes a topology on \(K\) (cf. Th. 1.16 below).

Proposition 1.4. Arbitrary intersections of topologies yield again topologies: Let \(X\) be a set, let \(I\) be a nonempty index set, and let \((\mathcal{T}_i)_{i \in I}\) be a family of topologies on \(X\). Then

\[\mathcal{T} := \bigcap_{i \in I} \mathcal{T}_i\]

is a again a topology on \(X\).

Proof. Since \(\emptyset\) and \(X\) are in each \(\mathcal{T}_i\), they are also in \(\mathcal{T}\). If \(O_1, O_2 \in \mathcal{T}\) and \(i \in I\), then \(O_1, O_2 \in \mathcal{T}_i\). Thus, \(O_1 \cap O_2 \in \mathcal{T}_i\) as well, proving \(O_1 \cap O_2 \in \mathcal{T}\). Similarly, if \(J\) is an index set and \(O_j \in \mathcal{T}\) for each \(j \in J\), then \(O_j \in \mathcal{T}_i\) for each \(i \in I\) and each \(j \in J\). Thus, \(O := \bigcup_{j \in J} O_j \in \mathcal{T}_i\) for each \(i \in I\), showing \(O \in \mathcal{T}\).

As in \(K\), topologies often arise from so-called metrics or norms, which we define next:

Definition 1.5. Let \(X\) be a set. A function \(d : X \times X \rightarrow \mathbb{R}_0^+\) is called a metric on \(X\) if, and only if, the following three conditions are satisfied:
(i) $d$ is positive definite, i.e., for each $(x, y) \in X \times X$, $d(x, y) = 0$ if, and only if, $x = y$.

(ii) $d$ is symmetric, i.e., for each $(x, y) \in X \times X$, $d(y, x) = d(x, y)$.

(iii) $d$ satisfies the triangle inequality, i.e., for each $(x, y, z) \in X^3$, $d(x, z) \leq d(x, y) + d(y, z)$.

If $d$ constitutes a metric on $X$, then the pair $(X, d)$ is called a metric space. One then often refers to the elements of $X$ as points and to the number $d(x, y)$ as the $d$-distance between the points $x$ and $y$. If the metric $d$ on $X$ is understood, one also refers to $X$ itself as a metric space.

**Remark 1.6.** The requirement that a metric be nonnegative is included in Def. 1.5 merely for emphasis. Nonnegativity actually follows from the remaining properties of a metric: For each $x, y \in X$, one computes

$$0 \overset{\text{Def. 1.5(i)}}{=} d(x, x) \overset{\text{Def. 1.5(ii)}}{\leq} d(x, y) + d(y, x) \overset{\text{Def. 1.5(ii)}}{=} 2d(x, y),$$

showing $d(x, y) \geq 0$.

**Definition 1.7.** Let $X$ be a vector space over the field $\mathbb{K}$. Then a function $\| \cdot \| : X \rightarrow \mathbb{R}_0^+$ is called a norm on $X$ if, and only if, the following three conditions are satisfied:

(i) $\| \cdot \|$ is positive definite, i.e.

$$\left( \|x\| = 0 \iff x = 0 \right) \quad \text{for each } x \in X.$$

(ii) $\| \cdot \|$ is homogeneous of degree 1, i.e.

$$\|\lambda x\| = |\lambda|\|x\| \quad \text{for each } \lambda \in \mathbb{K}, x \in X.$$

(iii) $\| \cdot \|$ satisfies the triangle inequality, i.e.

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{for each } x, y \in X.$$

If $\| \cdot \|$ constitutes a norm on $X$, then the pair $(X, \| \cdot \|)$ is called a normed vector space or just normed space. If the norm $\| \cdot \|$ on $X$ is understood, then one also refers to $X$ itself as a normed space.

**Lemma 1.8.** If $(X, \| \cdot \|)$ is a normed space, then the function

$$d : X \times X \rightarrow \mathbb{R}_0^+, \quad d(x, y) := \|x - y\|,$$

constitutes a metric on $X$. One also calls $d$ the metric induced by the norm $\| \cdot \|$. Thus, the induced metric $d$ makes $X$ into a metric space.
1 TOPOLOGY, METRIC, NORM

Proof. Consider \( x, y \in X \). If \( x = y \), then \( d(x, y) = \| x - y \| = \| 0 \| = 0 \). Conversely, if \( 0 = d(x, y) = \| x - y \| \), then \( x - y = 0 \), i.e. \( x = y \). Symmetry is verified by the computation

\[
d(y, x) = \| y - x \| = | -1 | \| x - y \| = d(x, y).
\]

Finally, for the triangle inequality, one lets \( x, y, z \in X \) and estimates

\[
d(x, y) = \| x - y \| = \| x - z + z - y \| \leq \| x - z \| + \| z - y \| = d(x, z) + d(z, y),
\]

which establishes the case. \( \blacksquare \)

Remark 1.9. Throughout this class, a multitude of notions will be introduced for metric spaces \((X, d)\), including open sets, balls, closed sets, etc. Subsequently, we will then also use these notions in normed spaces \((X, \| \cdot \|)\), always implicitly assuming that they are meant with respect to the metric space \((X, d)\), where \( d \) is the metric induced by the norm \( \| \cdot \| \), i.e. where \( d \) is given by \((1.2)\).

Definition 1.10. Let \((X, d)\) be a metric space. Given \( x \in X \) and \( r \in \mathbb{R}^+ \), define

\[
B_r(x) := \{ y \in X : d(x, y) < r \}, \tag{1.3a}
\]

\[
\overline{B}_r(x) := \{ y \in X : d(x, y) \leq r \}, \tag{1.3b}
\]

\[
S_r(x) := \{ y \in X : d(x, y) = r \}. \tag{1.3c}
\]

The set \( B_r(x) \) is called the open ball with center \( x \) and radius \( r \), also known as the \( r \)-ball with center \( x \). The set \( \overline{B}_r(x) \) is called the closed ball with center \( x \) and radius \( r \). The set \( S_r(x) \) is called the sphere with center \( x \) and radius \( r \). A set \( U \subseteq X \) is called a neighborhood of \( x \) if, and only if, there is \( \epsilon \in \mathbb{R}^+ \) such that \( B_{\epsilon}(x) \subseteq U \). We call \( O \subseteq X \) open if, and only if,

\[
\forall \quad x \in O \quad \exists \epsilon \in \mathbb{R}^+ \quad B_{\epsilon}(x) \subseteq O.
\]

Definition 1.11. For \( n \in \mathbb{N} \), \( p \in [1, \infty] \), the function

\[
\| \cdot \|_p : \mathbb{K}^n \rightarrow \mathbb{R}^+_0, \quad \| z \|_p := \left( \sum_{j=1}^n |z_j|^p \right)^{1/p}, \tag{1.4}
\]

is called the \( p \)-norm on \( \mathbb{K}^n \) (here, and in the following, we write vectors \( z \in \mathbb{K}^n \) in the form \( z = (z_1, \ldots, z_n) \), where \( z_1, \ldots, z_n \in \mathbb{K} \) are the coordinates of \( z \)). For \( p = 2 \) and \( \mathbb{K} = \mathbb{R} \), one also speaks of the Euclidean norm.

We want to show that the \( p \)-norms are, indeed, norms in the sense of Def. 1.7. Before we can do that in Cor. 1.14 below, we need to establish some important inequalities:

Theorem 1.12 (Hölder inequality). If \( n \in \mathbb{N} \) and \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[
\left| \sum_{j=1}^n a_j \overline{b}_j \right| \leq \| a \|_p \| b \|_q \quad \text{for each } a, b \in \mathbb{K}^n. \tag{1.5}
\]
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Proof. If \( a = 0 \) or \( b = 0 \), then there is nothing to prove. So let \( a \neq 0 \) and \( b \neq 0 \). For each \( j \in \{1, \ldots, n\} \), apply inequality between the weighted arithmetic mean and the weighted geometric mean [Phi16, (9.43)], i.e.

\[
x_1^{\lambda_1} \cdots x_n^{\lambda_n} \leq \lambda_1 x_1 + \cdots + \lambda_n x_n,
\]

with \( \lambda_1 = 1/p, \lambda_2 = 1/q, x_1 = |a_j|^p/\|a\|_p^p \) and \( x_2 = |b_j|^q/\|b\|_q^q \), to get

\[
\|a\|_p \|b\|_q \leq \frac{1}{p} |a_j|^p + \frac{1}{q} |b_j|^q.
\]

(1.6a)

Summing (1.6a) over \( j \in \{1, \ldots, n\} \) yields 1 on the right-hand side, and, thus,

\[
|a \cdot b| = \left| \sum_{j=1}^n a_j b_j \right| \leq \sum_{j=1}^n |a_j| |b_j| \leq \|a\|_p \|b\|_q,
\]

proving (1.5).

Theorem 1.13 (Minkowski inequality). For each \( p \geq 1, z, w \in \mathbb{K}^n, n \in \mathbb{N} \), one has

\[
\|z + w\|_p \leq \|z\|_p + \|w\|_p.
\]

(1.7)

Proof. For \( p = 1 \), (1.7) follows directly from the triangle inequality for the absolute value in \( \mathbb{K} \). It remains to consider the case \( p > 1 \). In that case, define \( q := p/(p-1) \), i.e. \( 1/p + 1/q = 1 \). Also define \( a \in \mathbb{R}^n \) by letting \( a_j := |z_j + w_j|^{p-1} \in \mathbb{R}^+ \) for each \( j \in \{1, \ldots, n\} \), and notice

\[
|z_j + w_j|^p = |z_j + w_j| a_j \leq |z_j| a_j + |w_j| a_j.
\]

(1.8a)

Summing (1.8a) over \( j \in \{1, \ldots, n\} \) and applying the Hölder inequality (1.5), one obtains

\[
\|z + w\|_p^p \leq (|z_1|, \ldots, |z_n|) \cdot (|w_1|, \ldots, |w_n|) \cdot a \leq \|z\|_p \|a\|_q + \|w\|_p \|a\|_q.
\]

(1.8b)

As \( q(p-1) = p \), it is \( a_j^q = |z_j + w_j|^p \), and, thus

\[
\|a\|_q = \left( \sum_{j=1}^n |z_j + w_j|^p \right)^{\frac{1}{p} \cdot \frac{q}{p}} = \|z + w\|_p^{p-1},
\]

(1.8c)

where \( p/q = p-1 \) was used in the last step. Finally, combining (1.8b) with (1.8c) yields (1.7).

Corollary 1.14. For each \( n \in \mathbb{N}, p \in [1, \infty], \) the \( p \)-norm on \( \mathbb{K}^n \) constitutes, indeed, a norm on \( \mathbb{K}^n \).

Proof. If \( z = 0 \), then \( \|z\|_p = 0 \) follows directly from (1.4). If \( z \neq 0 \), then there is \( j \in \{1, \ldots, n\} \) such that \( |z_j| > 0 \). Then (1.4) provides \( \|z\|_p \geq |z_j| > 0 \). If \( \lambda \in \mathbb{K} \) and \( z \in \mathbb{K}^n \), then \( \|\lambda z\|_p = (\sum_{j=1}^n |\lambda z_j|^p)^{1/p} = (|\lambda|^p)^{1/p} \sum_{j=1}^n |z_j|^p \sum_{j=1}^n |z_j|^p = |\lambda|^p \|z\|_p \). The proof is concluded by noticing that the triangle inequality is the same as the Minkowski inequality (1.7).
Example 1.15. Let $S \neq \emptyset$ be an otherwise arbitrary set. According to Linear Algebra, the set $\mathcal{F}(S, \mathbb{K})$ of all $\mathbb{K}$-valued functions on $S$ is a vector space over $\mathbb{K}$ if vector addition and scalar multiplication are defined pointwise via

$$(f + g) : S \to \mathbb{K}, \quad (f + g)(x) := f(x) + g(x),$$

$$(\lambda \cdot f) : S \to \mathbb{K}, \quad (\lambda \cdot f)(x) := \lambda \cdot f(x) \quad \text{for each } \lambda \in \mathbb{K}.$$ 

Now consider the subset $B(S, \mathbb{K})$ of $\mathcal{F}(S, \mathbb{K})$, consisting of all bounded $\mathbb{K}$-valued functions on $S$, where we call a $\mathbb{K}$-valued function $f$ bounded if, and only if, the set $\{|f(s)| : s \in S\} \subseteq \mathbb{R}_0^+$ is a bounded subset of $\mathbb{R}$. Define

$$\|f\|_\infty := \|f\|_{\sup} := \sup\{|f(s)| : s \in S\} \in \mathbb{R}_0^+ \quad \text{for each } f \in B(S, \mathbb{K}). \quad (1.9)$$

We will show that $B(S, \mathbb{K})$ constitutes a vector space over $\mathbb{K}$ and $\|\cdot\|_{\sup}$ provides a norm on $B(S, \mathbb{K})$ (i.e. $(B(S, \mathbb{K}), \|\cdot\|_{\sup})$ is a normed vector space). To verify that $B(S, \mathbb{K})$ constitutes a vector space over $\mathbb{K}$, it suffices to show it is a subspace of the vector space $\mathcal{F}(S, \mathbb{K})$, which, is equivalent to showing $f, g \in B(S, \mathbb{K})$ and $\lambda \in \mathbb{K}$ imply $f + g \in B(S, \mathbb{K})$ and $\lambda f \in B(S, \mathbb{K})$.

If $f, g \in B(S, \mathbb{K})$, then

$$\forall s \in S \quad |f(s) + g(s)| \leq |f(s)| + |g(s)| \leq \|f\|_{\sup} + \|g\|_{\sup} \in \mathbb{R}_0^+,$$ 

showing $f + g \in B(S, \mathbb{K})$ and that $\|\cdot\|_{\sup}$ satisfies the triangle inequality

$$\forall f, g \in B(S, \mathbb{K}) \quad \|f + g\|_{\sup} \leq \|f\|_{\sup} + \|g\|_{\sup}. \quad (1.10b)$$

If $f \in B(S, \mathbb{K})$, $\lambda \in \mathbb{K}$, then,

$$\forall s \in S \quad |\lambda f(s)| = |\lambda| |f(s)| \leq |\lambda| \|f\|_{\sup} \in \mathbb{R}_0^+$$ 

implies $\lambda f \in B(S, \mathbb{K})$, completing the proof that $B(S, \mathbb{K})$ is a subspace of $\mathcal{F}(S, \mathbb{K})$. Moreover,

$$\|\lambda f\|_{\sup} = \sup\{|\lambda f(s)| : s \in S\} = \sup\{|\lambda||f(s)| : s \in S\} = |\lambda| \sup\{|f(s)| : s \in S\} = |\lambda|\|f\|_{\sup}, \quad (1.11b)$$

proving $\|\cdot\|_{\sup}$ is homogeneous of degree 1. To see that $\|\cdot\|_{\sup}$ constitutes a norm on $B(S, \mathbb{K})$, it merely remains to show positive definiteness. To this end, we notice that the zero element $f = 0$ of the vector space $B(S, \mathbb{K})$ is the function $f \equiv 0$, which vanishes identically. Thus, $f = 0$ if, and only if, $\|f\|_{\sup} := \sup\{|f(s)| : s \in S\} = 0$, showing $\|\cdot\|_{\sup}$ is positive definite, and completing the proof that $\|\cdot\|_{\sup}$ is a norm, making $B(S, \mathbb{K})$ into a normed vector space.
In generalization of Ex. 1.3(d), every metric (in particular, every norm) induces a topology:

**Theorem 1.16.** Let \((X,d)\) be a metric space. Then \(\mathcal{T} := \{O \subseteq X : O \text{ open}\}\) constitutes a topology on \(X\): One also calls \(\mathcal{T}\) the topology induced by the metric \(d\), making each metric space into a topological space.

**Proof.** Clearly, \(\emptyset \in \mathcal{T}\) and \(X \in \mathcal{T}\). Now consider finitely many open sets \(O_1, \ldots, O_N \in \mathcal{T}\), \(N \in \mathbb{N}\), and let \(O := \bigcap_{j=1}^{N} O_j\). We have to prove that \(O\) is open. Hence, let \(x \in O\). Then \(x \in O_j\) for each \(j \in \{1, \ldots, N\}\). Since each \(O_j\) is open, for each \(j \in \{1, \ldots, N\}\), there is \(\varepsilon_j > 0\) such that \(B_{\varepsilon_j}(x) \subseteq O_j\). If we let \(\varepsilon := \min\{\varepsilon_j : j \in \{1, \ldots, N\}\}\), then \(\varepsilon > 0\) and \(B_{\varepsilon}(x) \subseteq B_{\varepsilon_j}(x) \subseteq O_j\) for each \(j \in \{1, \ldots, N\}\), i.e. \(B_{\varepsilon}(x) \subseteq O\), showing \(O\) is open. Now let \(I\) be an arbitrary index set. For each \(j \in I\), let \(O_j \in \mathcal{T}\). We have to verify that \(O := \bigcup_{j \in I} O_j\) is open. Let \(x \in O\). Then there is \(j \in I\) such that \(x \in O_j\). Since \(O_j\) is open, there is \(\varepsilon > 0\) such that \(B_{\varepsilon}(x) \subseteq O_j \subseteq O\), showing \(O\) to be open. ■

**Definition and Remark 1.17.** A topological space \((X,\mathcal{T})\) is called *metrizable* if, and only if, there exists a metric \(d\) on \(X\) such that \(\mathcal{T}\) is induced by \(d\). Not every topology is metrizable: For example, the indiscrete topology on a set with at least two distinct elements can never be metrizable (see Rem. 1.42(a) below). Other examples of nonmetrizable topologies are the cofinite topology on an uncountable set (see Rem. 1.39(b) below) and the topology of pointwise convergence (see Ex. 1.53(c) below).

**Example 1.18.** Let \(X\) be a set. We will show that the discrete topology on \(X\) (cf. Ex. 1.3(a)) is always metrizable: The corresponding *discrete* metric is defined by

\[
d : X \times X \longrightarrow \{0,1\}, \quad d(x,y) := \begin{cases} 0 & \text{for } x = y, \\ 1 & \text{for } x \neq y. \end{cases} \tag{1.12}
\]

We verify that \(d\) constitutes a metric on \(X\): Since \(d(x,y) = 0\) holds if, and only if, \(x = y\), \(d\) is positive definite. Symmetry, i.e. \(d(x,y) = d(y,x)\) is immediate from (1.12). We still need to show

\[
\forall_{x,y,z \in X} \quad d(x,y) \leq d(x,z) + d(z,y).
\]

For \(x = y\), it is \(d(x,y) = 0\), and we are done. For \(x \neq y\), it is \(d(x,y) = 1\), and we have to show the right-hand cannot be 0. If \(z = x\), then \(z \neq y\) and \(d(y,z) = 1\). If \(z \neq x\), then \(d(x,z) = 1\). Now that we have seen that \(d\) is a metric on \(X\), we still need to prove it induces the discrete topology, i.e. that every subset of \(X\) is open with respect to \(d\). Thus, let \(O \subseteq X\) and \(x \in O\). Then \(B_1(x) = \{x\} \subseteq O\), which already shows \(O\) to be open.

We now want to proceed to introduce convergence on topological spaces (in particular on metric spaces, including \(\mathbb{K}^\mathbb{N}\)). In \(\mathbb{K}\) we were able to characterize continuity of functions as well as the closedness of a set in terms of convergence of sequences. While this will still be possible in metric spaces, in general topological spaces, the convergence of sequences...
no longer suffices for such characterizations. For this reason, we will introduce the more
general (and more powerful) notion of net convergence\(^1\) (also referred to as Moore-Smith
convergence in the literature).

**Definition 1.19. (a)** A directed set \((I, \leq)\) consists of a nonempty set \(I\) and a relation
\(\leq\) on \(I\) that is reflexive, transitive,\(^2\) and has the additional property that every set
consisting of precisely two elements has an upper bound, i.e.
\[
\forall_{i,j \in I} \exists_{M \in I} \left( i \leq M \land j \leq M \right).
\]
(1.13)

(b) Let \(X\) be a set. A net in \(X\) is a family \((x_i)_{i \in I}\) in \(X\) (i.e. a function from \(I\) into \(X\),
cf. [Phi16, Def. 2.15(a)]) indexed by a directed set \(I\). If \(A \subseteq X\), then we say that
the net \((x_i)_{i \in I}\) in \(X\) is eventually in \(A\) if, and only if,
\[
\exists_{i \in I} \forall_{j \geq i} x_j \in A.
\]
(1.14)

**Example 1.20. (a)** If \(I\) is any nonempty set and \(\leq\) is a total order on \(I\), then \((I, \leq)\)
is a directed set. In particular, \(\mathbb{N}\) with its usual order is a directed set, and, thus,
every sequence is a net.

(b) If \(I\) is any nonempty set, then \((I, I \times I)\) is a directed set, but \(I \times I\) is not a partial
order if \(I\) has at least two distinct elements. On the other hand, \(\{(0,0), (1,1)\}\) is a
partial order on \(\{0,1\}\) that does **not** make \(\{0,1\}\) into a directed set.

(c) Let \((X, \mathcal{T})\) be a topological space, \(x \in X\). Then the neighborhood system \(\mathcal{U}(x)\) is
made into a directed set by defining
\[
\forall_{U,V \in \mathcal{U}(x)} \ U \leq V :\iff V \subseteq U : \quad (1.15)
\]
Clearly, \(\leq\) is reflexive and transitive (it is even a partial order), and (1.13) is satisfied,
since \(U, V \in \mathcal{U}(x)\) implies \(U \cap V \in \mathcal{U}(x)\) (one still obtains a directed set when
using \(U \subseteq V\) in (1.15), but this example turns out to be less useful).

(d) If \([a, b] \subseteq \mathbb{R}\) is an interval, \(a, b \in \mathbb{R}\), then the set \(\Pi\) of partitions of \([a, b]\) (cf. [Phi16,
Def. 10.3]) is turned into a directed set by setting \(\Delta \leq \Delta'\) if, and only if, \(\Delta'\) is a
refinement of \(\Delta\) (cf. [Phi16, Def. 10.8(a)]).

The reason one does not require the relation in Def. 1.19(a) to be a partial order is that
it is not necessary – it would clutter the notion of a directed set without any additional
gain for the theory.

---

\(^1\)Alternatively, one can also use the similar, but different, concept of **filter convergence**. In this class,
we only consider net convergence, which seems slightly easier to explain as well as more common in
introductory Analysis texts.

\(^2\)A relation that is reflexive and transitive is sometimes called a **preorder**.
**Definition 1.21.** Let \((X, T)\) be a topological space. The net \((x_i)_{i \in I}\) in \(X\) is said to be *convergent with limit* \(x \in X\) if, and only if, for every neighborhood \(U\) of \(x\), the net is eventually in \(U\), i.e. if, and only if,

\[
\forall \ U \in \mathcal{U}(x) \quad \exists \ i \in I \quad \forall \ j \geq i \quad x_j \in U.
\]  

(1.16)

If \((x_i)_{i \in I}\) converges to \(x\), then we write \(\lim_{i \in I} x_i = x\). If \(I = \mathbb{N}\) (i.e. if the net is a sequence), then we also still write \(\lim_{i \to \infty} x_i = x\) instead of \(\lim_{i \in \mathbb{N}} x_i = x\). A net is called *divergent* if, and only if, it is not convergent.

**Example 1.22.** (a) Let \((X, T)\) be a topological space, where \(T\) is induced by the metric \(d\) on \(X\). Let \((x_i)_{i \in I}\) be a net in \(X\), and \(x \in X\). Since every ball \(B_\epsilon(x)\), \(\epsilon > 0\), is a neighborhood of \(x\) and, conversely, every \(U \in \mathcal{U}(x)\) contains some ball \(B_\epsilon(x) \subseteq U\), \(\epsilon > 0\), we have the equivalence

\[
\lim_{i \in I} x_i = x \iff \forall \ \epsilon \in \mathbb{R}^+ \exists \ i \in I \forall \ j \geq i \ d(x_i, x) < \epsilon.
\]

(1.17a)

In particular, if \(I = \mathbb{N}\) and the net is a sequence, then \((x_i)_{i \in \mathbb{N}}\) converges to \(x\) if, and only if, the real sequence of distances \(\{d(x_i, x)\}_{i \in \mathbb{N}}\) converges to 0 (in the sense of Analysis I):

\[
\lim_{i \to \infty} x_i = x \iff \lim_{i \to \infty} d(x_i, x) = 0.
\]

(1.17b)

(b) Let \((X, T)\) be an arbitrary topological space and \(x \in X\). Consider \(\mathcal{U}(x)\) with the partial order of Ex. 1.20(c). Moreover, let \((x_U)_{U \in \mathcal{U}(x)}\) be a net such that \(x_U \in U\) for each \(U \in \mathcal{U}(x)\). Then, clearly, \(\lim_{U \in \mathcal{U}(x)} x_U = x\).

(c) We consider \(\mathbb{K}^n\), \(n \in \mathbb{N}\), with the 2-norm. Let \((z^k)_{k \in I}\) be a net in \(\mathbb{K}^n\) (for example, a sequence). Here, \(z^k = (z^k_1, \ldots, z^k_n)\), i.e. the \(z^k_j\) are the *coordinates* of the vector \(z^k\). As in the present example, we will subsequently use upper indices for indices in sequences and nets in situations, where we also need to denote coordinates; different coordinates will be referred to via lower indices. For each coordinate \(j \in \{1, \ldots, n\}\), we have the *coordinate net* \((z^k_j)_{k \in I}\) in \(\mathbb{K}\). We now claim that \((z^k)_{k \in I}\) converges with respect to (the topology induced by) the 2-norm to \(a = (a_1, \ldots, a_n) \in \mathbb{K}^n\) if, and only if, each coordinate net \((z^k_j)_{k \in \mathbb{N}}\) converges to \(a_j\) in \(\mathbb{K}\), \(j \in \{1, \ldots, n\}\):

\[
\lim_{k \in I} z^k = a \iff \forall \ j \in \{1, \ldots, n\} \lim_{k \in I} z^k_j = a_j.
\]

(1.18)

In particular, a sequence in \(\mathbb{K}^n\) converges (with respect to the 2-norm) if, and only if, each coordinate sequence converges in \(\mathbb{K}\). Remark: In Th. 1.72 below, we will see that every norm on \(\mathbb{K}^n\) induces the *same* topology, i.e. the validity of (1.18) does actually not depend on the norm. In preparation for the proof of (1.18), we observe that, for each \(z \in \mathbb{K}^n\), one has the following estimates:

\[
\forall \ j \in \{1, \ldots, n\} \quad |z_j| \leq \frac{\|z\|_2}{\sqrt{|z_1|^2 + \cdots + |z_n|^2}} \leq \frac{\|z\|_1}{|z_1| + \cdots + |z_n|}.
\]

(1.19)
Let \( \leq \) denote the relation that makes \( I \) into a directed set. If \( (z^k)_{k \in I} \) converges to \( a \), then, according to (1.17a), given \( \epsilon \in \mathbb{R}^+ \), there is \( N \in I \) such that, for each \( k \geq N \),

\[
\|z^k - a\|_2 < \epsilon.
\]

By (1.19), this implies

\[
\forall j \in \{1, \ldots, n\} \quad |z^k_j - a_j| \leq \|z^k - a\|_2 < \epsilon,
\]

proving \( \lim_{k \in I} z^k_j = a_j \). Conversely, if \( (z^k_j)_{k \in I} \) converges to \( a_j \) for each \( j \in \{1, \ldots, n\} \), then, given \( \epsilon \in \mathbb{R}^+ \), (1.17a) yields \( N \in I \) such that, for each \( k \geq N \),

\[
|z^k_j - a_j| < \frac{\epsilon}{n}.
\]

Since, by (1.19), this implies

\[
\|z^k - a\|_2 \leq \sum_{j=1}^{n} |z^k_j - a_j| < n \frac{\epsilon}{n} = \epsilon,
\]

\( (z^k)_{k \in I} \) converges to \( a \).

(d) We can interpret the function limit \( \lim_{z \rightarrow \zeta} f(z) = \eta \) of [Phi16, Def. 8.17] as a net limit in \( \mathbb{C} \): Let \( M \subseteq \mathbb{C}, f : M \rightarrow \mathbb{C} \), and let \( \zeta \in \mathbb{C} \) be a cluster point of \( M \). We make the set \( I := M \setminus \{\zeta\} \) into a directed set by defining

\[
\forall z,w \in M \quad z \leq w \iff |z - \zeta| \geq |w - \zeta|
\]

(i.e. “bigger” elements are closer to \( \zeta \)). It is then an exercise to show that, for \( \eta \in \mathbb{C} \),

\[
\lim_{z \rightarrow \zeta} f(z) = \eta \iff \lim_{z \in I} f(z) = \eta.
\]

(e) We can interpret the Riemann integral \( \int_a^b f \) of a bounded function \( f : [a, b] \rightarrow \mathbb{R} \), \( a < b \), as a net limit in \( \mathbb{R} \): As in Ex. 1.20(d), we turn the set \( \Pi \) of tagged partitions of \( [a, b] \) into a directed set by setting \( \Delta \leq \Delta' \) if, and only if, \( \Delta' \) is a refinement of \( \Delta \). Then \( \int_a^b f \) exists if, and only if, \( \lim_{\Delta \in \Pi} \rho(\Delta, f) \) exists (where \( \rho(\Delta, f) \) denotes the Riemann sum of [Phi16, (10.7c)]), and, in that case,

\[
\int_a^b f = \lim_{\Delta \in \Pi} \rho(\Delta, f)
\]

(see [Wal02, Sec. 5.6]).

In [Phi16, Sec. 7.1], we studied subsequences and reorderings of sequences (cf. [Phi16, Sec. 7.21]). We found that subsequences and reorderings of convergent sequences in \( \mathbb{C} \) are also convergent with the same limit ([Phi16, Prop. 7.23]). Moreover, in [Phi16, Prop.
7.26], we showed that $z \in \mathbb{K}$ is a cluster point of the sequence $(z_k)_{k \in \mathbb{N}}$ in $\mathbb{K}$ if, and only if, the sequence has a subsequence converging to $z$. We will now study related notions on general nets, where we will find that the mentioned results extend only partially to the most general situation. Reorderings turn out to be much less useful on general nets than on sequences and, while subnets generalize subsequences, the concept of a subnet is somewhat more complicated, such that the concept of a subnet turns out to be noticeably more versatile than that of a subsequence.

**Definition 1.23.** Let $(I, \leq)$ and $(J, \leq)$ be directed sets, let $\phi : J \rightarrow I$, let $X$ be an arbitrary nonempty set, and let $\sigma : I \rightarrow X$ be a net in $X$.

(a) $\phi$ is called final if, and only if,

$$\forall i \in I \quad \exists j_0 \in J \quad \forall j \geq j_0 \quad \phi(j) \geq i.$$ (1.20)

(b) The net $(\sigma \circ \phi) : J \rightarrow X$ is called a subnet of $\sigma$ if, and only if, $\phi$ is final.

(c) For $I = J$ (with the same relation $\leq$), the net $(\sigma \circ \phi) : I \rightarrow X$ is called a reordering of $\sigma$ if, and only if, $\phi$ is bijective.

**Example 1.24.** Using the identity for $\phi$ shows that every net is a subnet of itself. Moreover, every subsequence is a subnet, since, clearly, if $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is strictly increasing, then $\phi$ is final. However, sequences can have subnets that are no subsequences. One reason is that final maps do not need to be monotone: For example, $(2, 1, 4, 3, 6, 5, \ldots)$ is a subnet of $(1, 2, 3, \ldots)$. The other reason is that a final map does not need to be injective. For example, if $\sigma : \mathbb{N} \rightarrow X$ is a sequence in $X$, then $(\sigma \circ \phi) : \mathbb{R} \rightarrow X$,

$$\phi : \mathbb{R} \rightarrow \mathbb{N}, \quad \phi(s) := \min \{k \in \mathbb{N} : k \geq s\},$$

is a subnet of $\sigma$.

**Proposition 1.25.** Let $(X, \mathcal{T})$ be a topological space, $x \in X$.

(a) Let $\sigma : I \rightarrow X$ be a net in $X$. If $\lim_{i \in I} \sigma(i) = x$, then every subnet of $\sigma$ is also convergent with limit $x$.

(b) Let $\sigma : \mathbb{N} \rightarrow X$ be a sequence in $X$. If $\lim_{i \rightarrow \infty} \sigma(i) = x$, then every reordering of $\sigma$ is also convergent with limit $x$.

**Proof.** (a): Let $\sigma \circ \phi$ be a subnet of $\sigma$, where $\phi : J \rightarrow I$ is a final map. Moreover, let $U \in \mathcal{U}(x)$ be a neighborhood of $x$. Since $\lim_{i \in I} \sigma(i) = x$, there exists $i_0 \in I$ such that $\sigma(i) \in U$ for each $i \geq i_0$. As $\phi$ is final, there exists $j_0 \in J$ such that

$$\forall j \geq j_0 \quad \phi(j) \geq i_0.$$

Thus, for each $j \geq j_0$, we have

$$(\sigma \circ \phi)(j) = \sigma(\phi(j)) \in U,$$
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due to $\phi(j) \geq i_0$, proving $\lim_{j \in J}(\sigma \circ \phi)(j) = x$ as desired.

(b): The proof is analogous to the corresponding part of the proof of [Phi16, Prop. 7.23]:
Let $\sigma \circ \phi$ be a reordering of $\sigma$, where $\phi : I \to I$ is bijective. Let $U$ and $i_0$ be as before
(now with $I := \mathbb{N}$). Define

$$M := \max\{\phi^{-1}(i) : i \leq i_0\}.$$ 

As $\phi$ is bijective, it is $\phi(i) > i_0$ for each $i > M$. Then, for each $i > M$, one has

$$(\sigma \circ \phi)(i) = \sigma(\phi(i)) \in U$$
due to $\phi(i) > i_0$, proving $\lim_{i \to \infty}(\sigma \circ \phi)(i) = x$ as desired. ■

Caveat: Reorderings of general convergent nets do not necessarily converge, see Ex. 1.30
below.

**Definition 1.26.** (a) Let $X$ be a set. If $A \subseteq X$, then we say that the net $(x_i)_{i \in I}$ in $X$

is frequently or cofinally in $A$ if, and only if,

$$\forall_{i \in I} \exists_{j \geq i} x_j \in A.$$  \hspace{1cm} (1.21)

(b) Let $(X, T)$ be a topological space, $x \in X$. The net $(x_i)_{i \in I}$ in $X$ is said to have $x$ as

a cluster point (or accumulation point) if, and only if, for every neighborhood $U$ of $x$, the net is frequently in $U$, i.e. if, and only if,

$$\forall_{U \in \mathcal{U}(x)} \forall_{i \in I} \exists_{j \geq i} x_j \in U.$$  \hspace{1cm} (1.22)

**Proposition 1.27.** Let $(X, T)$ be a topological space, $x \in X$. The net $\sigma : I \to X$ in $X$ has $x$ as a cluster point if, and only if, it has a subnet that converges to $x$.

**Proof.** Let $\sigma \circ \phi$ be a subnet of $\sigma$, where $\phi : J \to I$ is a final map. Assume that

$$\lim_{j \in J}(\sigma \circ \phi)(j) = x$$

and let $U \in \mathcal{U}(x)$. If $i \in I$, then, as $\phi$ is final, there exists $j_0 \in J$ such that

$$\forall_{j \geq j_0} \phi(j) \geq i.$$ 

On the other hand, there is $j_1 \in J$ such that

$$\forall_{j \geq j_1} \sigma(\phi(j)) \in U.$$ 

Since $J$ is a directed set, there exists $j_2 \in J$ such that $j_2 \geq j_0$ and $j_2 \geq j_1$, implying

$\phi(j_2) \geq i$ and $\sigma(\phi(j_2)) \in U$. As we have, thus, shown $\sigma$ to be frequently in $U$, $x$ is a cluster point of $\sigma$. Conversely, let $x$ be a cluster point of $\sigma$. We have to construct a subnet that converges to $x$. To this end, we let

$$J := \{(i, U) \in I \times \mathcal{U}(x) : \sigma(i) \in U\}$$
and define
\[ \forall (i,U), (j,V) \in J \quad (i,U) \leq (j,V) \iff (i \leq j \land U \supseteq V). \]

Then \( \leq \) is clearly reflexive and transitive on \( J \). Let \( (i,U), (j,V) \in J \) and note \( U \cap V \in \mathcal{U}(x) \). As \( I \) is a directed set, there is \( M \in I \) such that \( M \geq i \) and \( M \geq j \). Moreover, the assumption that \( \sigma \) is frequently in \( U \cap V \) implies there is \( k \in I \) such that
\[ k \geq M \geq i \land k \geq M \geq j \land \sigma(k) \in U \cap V, \]
and, thus,
\[ (k,U \cap V) \in J \land (k,U \cap V) \geq (i,U) \land (k,U \cap V) \geq (j,V), \]
proving \( (J, \leq) \) to be a directed set. The projection map
\[ \pi : J \rightarrow I, \quad \pi(i,U) := i, \]
is final: Indeed, if \( i \in I \), then \( (i,X) \in J \) and \( (j,U) \geq (i,X) \in J \) implies \( \pi(j,U) = j \geq i \).
In consequence, \( \sigma \circ \pi \) is a subnet of \( \sigma \). It merely remains to show that \( \sigma \circ \pi \) converges to \( x \). Thus, let \( U \in \mathcal{U}(x) \). Since \( \sigma \) is frequently in \( U \), there must be \( i \in I \) with \( \sigma(i) \in U \). Then \( (i,U) \in J \) and if \( (j,V) \geq (i,U) \), then
\[ (\sigma \circ \pi)(j,V) = \sigma(j) \in V \subseteq U, \]
proving \( \lim_{(j,U) \in J} (\sigma \circ \pi)(j,U) = x. \)

In [Phi16, Prop. 7.10(b)], we obtained the result that every convergent sequence in \( \mathbb{C} \) is bounded. We will see in Prop. 1.29(d) below that this result remains true for sequences in metric spaces (but not for nets, see Ex. 1.30). In general topological spaces, however, one no longer has the concept of boundedness.

**Definition 1.28.** Let \( (X,d) \) be a metric space.

(a) The set \( A \subseteq X \) is called **bounded** if, and only if, \( A = \emptyset \) or \( A \neq \emptyset \) and the set \( \{d(x,y) : x,y \in A\} \) is bounded in \( \mathbb{R} \); \( A \subseteq X \) is called **unbounded** if, and only if, \( A \) is not bounded. For each \( A \subseteq X \), the number
\[ \text{diam} \, A := \begin{cases} 0 & \text{for } A = \emptyset, \\ \sup \{d(x,y) : x,y \in A\} & \text{for } \emptyset \neq A \text{ bounded}, \\ \infty & \text{for } A \text{ unbounded}, \end{cases} \] (1.23)
is called the **diameter** of \( A \). Thus, \( \text{diam} \, A \in [0,\infty] := \mathbb{R}_0^+ \cup \{\infty\} \) and \( A \) is bounded if, and only if, \( \text{diam} \, A < \infty \).

(b) The net \( (x_i)_{i \in I} \) in \( X \) is called **bounded** if, and only if, the set \( \{x_i : i \in I\} \) is bounded in the sense of (a).

**Proposition 1.29.** Let \( (X,d) \) be a metric space.
(a) \( A \subseteq X \) is bounded if, and only if, there is \( r > 0 \) and \( x \in X \) such that \( A \subseteq B_r(x) \) (in particular, Def. 1.28(a) is consistent with [Phi16, Def. 7.42(a)]).

(b) Every finite subset of \( X \) is bounded.

(c) The union of two bounded subsets of \( X \) is bounded.

(d) If the sequence \( (x_k)_{k \in \mathbb{N}} \) in \( X \) is convergent, then it is bounded.

Proof. (a): If \( A \) is bounded, then \( \text{diam} \, A < \infty \). Let \( r \) be any real number bigger than \( \text{diam} \, A \), e.g. \( 1 + \text{diam} \, A \). Choose any point \( x \in A \). Then, by the definition of \( \text{diam} \, A \), for each \( y \in A \), it is \( d(x, y) \leq \text{diam} \, A < r \), showing that \( A \subseteq B_r(x) \). Conversely, if \( r > 0 \) and \( x \in X \) such that \( A \subseteq B_r(x) \), then, by the definition of \( B_r(x) \), one has \( d(x, y) < r \) for each \( y \in A \). Now, if \( y, z \in A \), then \( d(z, y) \leq d(z, x) + d(x, y) < 2r \), showing \( \text{diam} \, A \leq 2r < \infty \), i.e. \( A \) is bounded.

(b): Let \( A \) be a finite subset of \( X \) and \( a \in A \). Set \( r := 1 + \max \{d(a, x) : x \in A\} \). Then \( 1 \leq r < \infty \), since \( A \) is finite. Moreover, \( A \subseteq B_r(a) \), showing that \( A \) is bounded.

(c): Let \( A \) and \( B \) be bounded subsets of \( X \). Then there are \( x, y \in X \) and \( r > 0 \) such that \( A \subseteq B_r(x) \) and \( B \subseteq B_r(y) \). Define \( \alpha := d(x, y) \) and \( \epsilon := r + \alpha \). Then \( A \subseteq B_r(x) \subseteq B_{\epsilon}(x) \), and, thus, \( A \cup B \subseteq B_{\epsilon}(x) \), establishing that \( A \cup B \) is bounded.

(d) (cf. the proof for sequences in \( \mathbb{K} \) in [Phi16, Prop. 7.10(b)]): If \( \lim_{k \to \infty} x_k = a \in X \), then there is \( N \in \mathbb{N} \) such that \( x_k \in B_1(a) \) for each \( k > N \), i.e. \( \{x_k : k > N\} \) is bounded. Moreover, the finite set \( \{x_k : k \leq N\} \) is bounded. Therefore, \( \{x_k : k \in \mathbb{N}\} \) is the union of two bounded sets, and, hence, bounded. \( \blacksquare \)

**Example 1.30.** The net \( (e^{-k})_{k \in \mathbb{Z}} \) in \( \mathbb{R} \), which clearly converges to 0, shows that convergent nets do not need to be bounded. Moreover, using the bijective map \( \phi : \mathbb{Z} \to \mathbb{Z} \), \( \phi(k) := -k \), we see that the divergent net \( (e^k)_{k \in \mathbb{Z}} \) is a reordering of \( (e^{-k})_{k \in \mathbb{Z}} \).

We conclude the present section by extending the 1-dimensional Bolzano-Weierstrass theorem of [Phi16, Th. 7.27] to sequences in \( \mathbb{K}^n \):

**Theorem 1.31** (Bolzano-Weierstrass). Let \( (z^k)_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{K}^n \) that is bounded with respect to the 2-norm (in fact, we will see as a consequence of Th. 1.72 below that the property of boundedness in \( \mathbb{K}^n \) does not depend on the chosen norm). Then \( (z^k)_{k \in \mathbb{N}} \) has a subsequence that converges in \( \mathbb{K}^n \).

**Proof.** If \( (z^k)_{k \in \mathbb{N}} \) is bounded with respect to the 2-norm, then, due to (1.19), each coordinate sequence \( (z^k_j)_{k \in \mathbb{N}}, j \in \{1, \ldots, n\} \), is bounded in \( \mathbb{K} \). We prove by induction over \( \{1, \ldots, n\} \) that, for each \( j \in \{1, \ldots, n\} \), there is a subsequence \( (y^{k,j})_{k \in \mathbb{N}} \) of \( (z^k)_{k \in \mathbb{N}} \) such that the coordinate sequences \( (y^{k,j}_\alpha)_{k \in \mathbb{N}} \) converge for each \( \alpha \in \{1, \ldots, j\} \). Base Case \( (j = 1) \): Since \( (z^k_j)_{k \in \mathbb{N}} \) is a bounded sequence in \( \mathbb{K} \), the Bolzano-Weierstrass theorem
for sequences in $\mathbb{K}$ (cf. [Phi16, Prop. 7.26, Th. 7.27]) yields the existence of a convergent subsequence of $(z^k_1)_{k \in \mathbb{N}}$. This provides us with the needed subsequence $(y^{k_1})_{k \in \mathbb{N}}$ of $(z^k)_{k \in \mathbb{N}}$. Now suppose that $1 < j \leq n$. By induction, we already have a subsequence $(y^{k-j_1})_{k \in \mathbb{N}}$ of $(z^k)_{k \in \mathbb{N}}$ such that the coordinate sequences $(y^{k-j_1}_a)_{k \in \mathbb{N}}$ converge for each $\alpha \in \{1, \ldots, j-1\}$. As $(y^{k-j_1}_a)_{k \in \mathbb{N}}$ is a subsequence of the bounded $\mathbb{K}$-valued sequence $(z^k_a)_{k \in \mathbb{N}}$, by the Bolzano-Weierstrass theorem for sequences in $\mathbb{K}$, it has a convergent subsequence. This provides us with the needed subsequence $(y^{k_j})_{k \in \mathbb{N}}$ of $(y^{k-j_1})_{k \in \mathbb{N}}$, which is then also a subsequence of $(z^k)_{k \in \mathbb{N}}$. Moreover, for each $\alpha \in \{1, \ldots, j-1\}$, $(y^{k_j}_a)_{k \in \mathbb{N}}$ is a subsequence of the convergent sequence $(y^{k_j-1}_a)_{k \in \mathbb{N}}$, and, thus, also convergent. In consequence, $(y^{k_j}_a)_{k \in \mathbb{N}}$ converge for each $\alpha \in \{1, \ldots, j\}$ as required. Finally, one observes that $(y^{k_n})_{k \in \mathbb{N}}$ is a subsequence of $(z^k)_{k \in \mathbb{N}}$ such that all coordinate sequences $(y^{k_n}_a)_{k \in \mathbb{N}}$, $\alpha \in \{1, \ldots, n\}$, converge. Let $a_\alpha := \lim_{k \to \infty} y^{k_n}_\alpha$ for each $\alpha \in \{1, \ldots, n\}$. Then, by (1.18), $\lim_{k \to \infty} y^{k_n} = a$, thereby establishing the case. 

\[\Box\]

### 1.2 Open Sets, Closed Sets, and Related Notions

**Definition 1.32.** Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$, and $x \in X$.

(a) $A$ is called open if, and only if, $A \in \mathcal{T}$ (of course, we already defined this in Def. 1.1 – here, it is only repeated for the sake of completeness).

(b) $A$ is called closed if, and only if, $A^c = X \setminus A$ is open (where $A^c = X \setminus A$ denotes the complement of $A$, cf. [Phi16, Def. 1.24(c)])

(c) The point $x$ is called an interior point of $A$ if, and only if, $A \subseteq U(x)$, i.e. if, and only if, there exists $O \in \mathcal{T}$ such that $x \in O \subseteq A$. Note: An interior point of $A$ is always in $A$. The set of all interior points of $A$ is called the interior of $A$. It is denoted by $A^\circ$ or by $\text{int} \ A$.

(d) The point $x$ is called a boundary point of $A$ if, and only if, each $O \in U(x)$ contains at least one point from $A$ and at least one point from $A^c$ ($A \cap O \neq \emptyset$ and $A^c \cap O \neq \emptyset$). Note: A boundary point of $A$ is not necessarily in $A$. The set of all boundary points of $A$ is called the boundary of $A$. It is denoted by $\partial A$. The set $A \cup \partial A$ is called the closure of $A$. It is denoted by $\overline{A}$ or by $\text{cl} \ A$. The set $A$ is called dense in $X$ if, and only if, $\overline{A} = X$; $(X, \mathcal{T})$ is called separable if, and only if, $X$ has a countable dense subset.

(e) The point $x$ is called a cluster point or accumulation point of $A$ if, and only if, every neighborhood of $x$ has a nonempty intersection with $A \setminus \{x\}$, i.e. if, and only if, 

$$\forall U \in U(x) \ U \cap (A \setminus \{x\}) \neq \emptyset$$

(cf. [Phi16, Def. 7.33(a)] and Rem. 1.42(a) below). Note: A cluster point of $A$ is not necessarily in $A$. \[\Box\]
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(f) The point \( x \) is called an *isolated point* of \( A \) if, and only if, it is in \( A \) and not a cluster point of \( A \), i.e. if, and only if, there exists \( U \in \mathcal{U}(x) \) such that \( U \cap A = \{ x \} \) (cf. [Phi16, Def. 7.33(b)]). Note: An isolated point of \( A \) is always in \( A \).

**Example 1.33.** Let \((X, \mathcal{T})\) be a topological space, where \( \mathcal{T} \) is induced by a metric \( d \) on \( X \). Then, given \( x \in X \) and \( r \in \mathbb{R}_+ \), the open ball \( B_r(x) \) is an open set and the closed ball \( \overline{B_r}(x) \) is a closed set: That \( B_r(x) \in \mathcal{T} \) is immediate from the definition of \( \mathcal{T} \). To see that \( \overline{B_r}(x) \) is closed, we need to show \( X \setminus \overline{B_r}(x) \) is open. To this end, let \( y \in X \setminus \overline{B_r}(x) \), \( \epsilon := d(x, y) - r \). If \( z \in B_\epsilon(y) \), then \( d(y, z) < d(x, y) - r \) and \( d(x, y) \leq d(x, z) + d(z, y) \). Thus, \( d(x, z) \geq d(x, y) - d(z, y) > r \), showing \( B_\epsilon(y) \subseteq X \setminus \overline{B_r}(x) \), i.e. \( X \setminus \overline{B_r}(x) \) is open.

**Example 1.34.** Consider \((X, \mathcal{T})\), where \( X = \mathbb{K} \) and \( \mathcal{T} \) is induced by the metric \( d : X \to \mathbb{R}_+ \), \( d(z, w) := |z - w| \).

(a) Let \( A := ]0, 1[ \) and \( \mathbb{K} = \mathbb{R} \). Then \( A^c = ]0, 1[ \), \( \partial A = \{ 0, 1 \} \), \( \overline{A} = [0, 1] \).

(b) Let \( A := ]0, 1[ \) and \( \mathbb{K} = \mathbb{C} \). Then \( A^o = \emptyset \), \( \partial A = \overline{A} = [0, 1] \).

(c) Let \( A := \mathbb{Q} \). In this case, there is no difference between \( \mathbb{K} = \mathbb{R} \) and \( \mathbb{K} = \mathbb{C} \): \( A^o = \emptyset \), \( \partial A = \overline{A} = \mathbb{R} \).

(d) Let \( A := \{ 1/n : n \in \mathbb{N} \} \). Once again, there is no difference between \( \mathbb{K} = \mathbb{R} \) and \( \mathbb{K} = \mathbb{C} \): Every element of \( A \) is an isolated point. In particular \( A^o = \emptyset \). The unique cluster point of \( A \) is 0, and \( \partial A = \overline{A} = A \cup \{ 0 \} \).

**Proposition 1.35.** Let \((X, \mathcal{T})\) be a topological space, \( A \subseteq X \).

(a) \( A \) is open if, and only if, \( A^c \) is closed.

(b) The empty set \( \emptyset \) and the entire space \( X \) are both open and closed. Such sets are sometimes called clopen.

(c) Intersections of arbitrarily many closed sets are closed (cf. [Phi16, Prop. 7.44(b)]). The union of finitely many closed sets is closed (cf. [Phi16, Prop. 7.44(a)]).

(d) \( X \) is the disjoint union of \( A^o \), \( \partial A \), and \( (X \setminus A)^c \).

(e) \( A \) is dense in \( X \) if, and only if, for every nonempty \( O \in \mathcal{T} \), one has \( O \cap A \neq \emptyset \).

**Proof.** (a): According to Def. 1.32(b), \( A^c \) is closed if, and only if, \( (A^c)^c \) is open. However, \( (A^c)^c = X \setminus A^c = X \setminus (X \setminus A) = A \).

(b) is immediate, since \( \emptyset, X \in \mathcal{T} \).

(c): Let \( I \neq \emptyset \) be a (finite or infinite) index set. For each \( j \in I \), let \( C_j \subseteq X \) be closed. We have to verify that \( C := \bigcap_{j \in I} C_j \) is closed. According to the set-theoretic law [Phi16, Prop. 1.39(e)]

\[
C^c = \left( \bigcap_{j \in I} C_j \right)^c \overset{\text{[Phi16, Prop. 1.39(e)]}}{=} \bigcup_{j \in I} C_j^c.
\]
Now, as we know that $C_j$ is closed, we know that $C_j^c$ is open. According to Def. 1.1(iii), that means that $C^c$ is open, showing that $C$ is closed. Similarly, if we consider finitely many closed sets $C_1, \ldots, C_N$, $N \in \mathbb{N}$, and let $C := \bigcup_{j=1}^{N} C_j$, then the set-theoretic law [Phi16, Prop. 1.39(f)] yields

$$C^c = \left( \bigcup_{j=1}^{N} C_j \right)^c = \bigcap_{j=1}^{N} C_j^c.$$  

Since $C_j$ is closed, $C_j^c$ is open, and, by Def. 1.1(ii), $C^c$ is open, hence $C$ closed.

(d): One has to show four parts: $X = A^c \cup \partial A \cup (X \setminus A)^\circ$, $A^c \cap \partial A = \emptyset$, $\partial A \cap (X \setminus A)^c = \emptyset$, and $A^c \cap (X \setminus A)^\circ = \emptyset$. Suppose $x \in X \setminus (A^c \cup \partial A)$. Since $x \notin \partial A$, there exists $U \in \mathcal{U}(x)$ such that $U \subseteq A$ or $U \subseteq X \setminus A$. As $x \notin A^c$, it must be $U \subseteq X \setminus A$, i.e. $x \in (X \setminus A)^\circ$. $A^c \cap \partial A = \emptyset$: If $x \in A^c$, then there is $U \in \mathcal{U}(x)$ such that $U \subseteq A$, thus, $x \notin \partial A$. $\partial A \cap (X \setminus A)^c = \emptyset$: Since $\partial A = \partial (X \setminus A)$, this follows from $A^c \cap \partial A = \emptyset$. $A^c \cap (X \setminus A)^\circ = \emptyset$ holds as $A^c \subseteq A$, $(X \setminus A)^\circ \subseteq X \setminus A$, and $A \setminus (X \setminus A) = \emptyset$.

(e): If $A$ is dense in $X$, then $X = A \cup \partial A$. Let $x \in O$. If $x \in A$, then $x \in O \cap A$. If $x \in \partial A$, then $O \cap A \neq \emptyset$ also holds. Conversely, suppose $O \cap A \neq \emptyset$ for each nonempty $O \in \mathcal{T}$. Then, for $x \in X \setminus A$ and $x \in O \in \mathcal{T}$, both $O \cap A \neq \emptyset$ and $O \cap A^c \neq \emptyset$, showing $x \in \partial A$.

**Example 1.36.** Due to Prop. 1.35(e), $\mathbb{Q}$ is dense in $\mathbb{R}$ (endowed with the topology given by $| \cdot |$). More generally, $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, $n \in \mathbb{N}$, and

$$A := \{(x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n : x_j, y_j \in \mathbb{Q}, j \in \{1, \ldots, n\}\}$$

is dense in $\mathbb{C}^n$ (endowed with the topology given by $\| \cdot \|_2$). As $\mathbb{Q}^n$ and $A$ are countable, $\mathbb{R}^n$ and $\mathbb{C}^n$ are separable in the norm topology. On the other hand, if $X$ is uncountable and $(X, \mathcal{T})$ is discrete, then the space can never be separable (since $A = \overline{A}$ for each $A \subseteq X$). As every discrete space is metrizable (by the discrete metric), this shows that not every metric space is separable.

**Theorem 1.37.** Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$.

(a) The interior $A^\circ$ is the union of all open subsets of $A$. In particular, $A^\circ$ is open. In other words, $A^\circ$ is the largest open set contained in $A$.

(b) The closure $\overline{A}$ is the intersection of all closed supersets of $A$. In particular, $\overline{A}$ is closed. In other words, $\overline{A}$ is the smallest closed set containing $A$.

(c) The boundary $\partial A$ is closed.

**Proof.** (a): Let $O$ be the union of all open subsets of $A$. Then $O$ is open by Def. 1.1(iii). If $x \in A^c$, then $x$ is an interior point of $A$, i.e. there is $U \in \mathcal{U}(x)$ and $O_1 \in \mathcal{T}$ such that $x \in O_1 \subseteq U \subseteq A$ (cf. Def. 1.1). Since $O_1$ is open, $x \in O$. Conversely, if $x \in O$, then, as $O$ is open, $O \in \mathcal{U}(x)$, showing that $x$ is an interior point of $A$, i.e. $x \in A^c$. 


(b): According to (a), \((A^c)^\circ\) is the union of all open subsets of \(A^c\), i.e.
\[
(A^c)^\circ = \bigcup_{O \in \{S \subseteq A^c : S \text{ open}\}} O,
\]
then
\[
((A^c)^\circ)^c = \bigcap_{O \in \{S \subseteq A^c : S \text{ open}\}} O^c = \bigcap_{C \in \{S \supseteq A : S \text{ closed}\}} C
\]
is the intersection of all closed supersets of \(A\) (note that \(C\) is a closed superset of \(A\) if, and only if, \(C^c\) is an open subset of \(A^c\)). As, by Prop. 1.35(d),
\[
((A^c)^\circ)^c = \partial(A^c) \cup A^o = \partial A \cup A^o = \partial A \cup A = \overline{A},
\]
\(\overline{A}\) is the intersection of all closed supersets of \(A\) as claimed.

(c): According to Prop. 1.35(d), it is \(\partial A = X \setminus (A^o \cup (X \setminus A)^o)\). Since \(A^o\) and \((X \setminus A)^o\) are open, \(\partial A\) is closed. ■

We will now proceed to study some relations between cluster points of a set \(A\), the closure of a set \(A\), and convergent nets in \(A\). Sequences suffice instead of nets for topological spaces that are first countable, a notion provided by the following definition:

**Definition 1.38.** Let \((X, \mathcal{T})\) be a topological space, \(x \in X\). A set \(B \subseteq U(x)\) is called a local base at \(x\) or a neighborhood base at \(x\) if, and only if, every \(U \in U(x)\) contains some \(B \in B\), i.e. if, and only if,
\[
\forall U \in U(x) \quad \exists B \in B \quad B \subseteq U.
\]
Moreover, \(X\) is called first countable (sometimes also called a \(C_1\)-space) if, and only if, at every \(x \in X\) there exists a countable local base.

**Remark 1.39.** (a) Indiscrete spaces as well as metric spaces, are first countable: If \((X, \mathcal{T})\) is indiscrete, then \(U(x) = \{X\}\) for every \(x \in X\). If \((X, \mathcal{T})\) is metric (i.e. \(\mathcal{T}\) is induced by a metric \(d\) on \(X\)), then, clearly, for each \(x \in X\),
\[
\mathcal{B}(x) := \{B_\varepsilon(x) : \varepsilon \in \mathbb{Q}^+\}
\]
constitutes a countable local base at \(x\).

(b) Let \(X\) be a set endowed with the cofinite topology \(\mathcal{T}\) of Ex. 1.3(c). The space is first countable if, and only if, \(X\) is countable: If \(X\) is countable, then the set of finite subsets of \(X\) is countable, i.e. \(\mathcal{T}\) is countable and, for each \(x \in X\), \(\mathcal{T} \cap U(x)\) is a countable local base at \(x\). Conversely, assume there exists a countable local base \(\mathcal{B}\) at \(x \in X\). It holds that
\[
\bigcap_{B \in \mathcal{B}} B = \bigcap_{U \in U(x)} U = \{x\}:
\]
(1.24a)
The inclusions \(\supseteq\) in (1.24a) are immediate; \(\subseteq\) at the first equality holds, as every \(U \in U(x)\) contains a \(B\) from \(\mathcal{B}\) as a subset; \(\subseteq\) at the second equality holds, since,
for each $y \neq x$, one has $\{y\}^c \in \mathcal{U}(x)$, i.e. $y \notin \bigcap_{U \in \mathcal{U}(x)} U$. Taking complements in (1.24a), we have

$$X \setminus \{x\} = \bigcup_{B \in \mathcal{B}} B^c.$$  

(1.24b)

Since $\mathcal{B}$ is countable and each $B^c$ is finite (as $B$ contains an open set), (1.24b) proves $X$ to be countable. In particular, for uncountable $X$, $(X, T)$ is not metrizable.

Another example of a topology that is not first countable is given by the topology of pointwise convergence, see Ex. 1.53(c) below.

First countable spaces are typically much easier to handle due to the following result:

**Proposition 1.40.** Let $(X, T)$ be a topological space, $x \in X$. Assume there exists a countable local base at $x$. Then each net $(x_i)_{i \in I}$ in $X$, converging to $x$, contains a sequence $(x_{i_k})_{k \in \mathbb{N}}$ (which is not necessarily a subnet!) such that $\lim_{k \to \infty} x_{i_k} = x$ (in particular, for each $A \subseteq X$, there is a net in $A$ converging to $x$ if, and only if, there is a sequence in $A$ converging to $x$).

**Proof.** Assume there exists a countable local base $\mathcal{B}$ at $x$. Without loss of generality, we may assume the elements of $\mathcal{B}$ to be open. Let $B_1, B_2, \ldots$ be an enumeration of the elements of $\mathcal{B}$ (not necessarily injective). Define, for each $k \in \mathbb{N}$,

$$B_{0k} := \bigcap_{j=1}^{k} B_j.$$  

Then, since $x \in B_{0k} \subseteq B_k$ and each $B_{0k}$ is open, $B_0 := \{B_{0k} : k \in \mathbb{N}\}$ is still a countable local base at $x$, but with the additional property that $B_{0,k+1} \subseteq B_{0k}$ for each $k \in \mathbb{N}$. Now suppose $(x_i)_{i \in I}$ is a net in $X$, converging to $x$. We have to that the net contains a sequence that converges to $x$ as well. Since $\lim_{i \in I} x_i = x$, given $k \in \mathbb{N}$, there exists $i_k \in I$ such that $x_{i_k} \in B_{0k}$. We claim $\lim_{k \to \infty} x_{i_k} = x$. Indeed, if $U \in \mathcal{U}(x)$, then there is $N \in \mathbb{N}$ with $B_{0N} \subseteq U$. Thus,

$$\forall_{k \geq N} x_{i_k} \in B_{0k} \subseteq B_{0N} \subseteq U,$$

proving $\lim_{k \to \infty} x_{i_k} = x$ as desired.  

In Ex. 1.53(c) below, we will see that, if there does not exist a local base at $x$, then it can happen that there is a net in some set $A$, converging to $x$, but there is no sequence in $A$ converging to $x$.

**Lemma 1.41.** Let $(X, T)$ be a topological space, $A \subseteq X$. Then $x \in X$ is a cluster point of $A$ if, and only if, there is a net $(a_i)_{i \in I}$ in $A \setminus \{x\}$ such that $\lim_{i \in I} a_i = x$ (by Prop. 1.40, we may replace “net” by “sequence” if $X$ is first countable).
Proof. Let \((a_i)_{i \in I}\) be a net in \(A \setminus \{x\}\) such that \(\lim_{i \in I} a_i = x\). If \(U \in \mathcal{U}(x)\), then there must be \(i \in I\) with \(a_i \in U\). Since \(a_i \in A\) and \(a_i \neq x\), this already shows \(U \cap (A \setminus \{x\}) \neq \emptyset\), i.e. \(x\) is a cluster point of \(A\). Conversely, if \(x\) is a cluster point of \(A\), for each \(U \in \mathcal{U}(x)\), there exists \(x_U \in U \cap (A \setminus \{x\})\). Then \((x_U)_{U \in \mathcal{U}(x)}\) is a net in \(A \setminus \{x\}\) such that \(\lim_{U \in \mathcal{U}(x)} x_U = x\).

**Remark 1.42.** Let \((X, \mathcal{T})\) be a topological space, \(A \subseteq X, x \in X\).

(a) If \(\mathcal{T}\) is induced by a metric \(d\) on \(X\), then \(x\) is a cluster point of \(A\) if, and only if, every \(U \in \mathcal{U}(x)\) contains infinitely many distinct points from \(A\) (exercise) (thus, our new definition of cluster point is consistent with the definition in [Phi16, Def. 7.33(a)]). On the other hand, we now consider a set \(X\) with at least two distinct elements and with the indiscrete topology. Let \(x, y \in X, x \neq y, A := \{y\}\). Then \(x\) is a cluster point of \(A\), even though \(X\) (the only neighborhood of \(x\)) contains only one point of \(A\) distinct from \(x\). In particular, we see that \(X\) with the indiscrete topology cannot be metrizable.

(b) If \(x\) is an isolated point of \(A\) and \((a_i)_{i \in I}\) is a net in \(A\) converging to \(x\), then the net must be finally constant with value \(x\), i.e. \(\exists N \in I \forall i \geq N \ a_i = x\) \hspace{1cm} (1.25)

Indeed, since there exists some \(U \in \mathcal{U}(x)\) with \(U \cap A = \{x\}\) and \((a_i)_{i \in I}\) converges to \(x\), (1.25) must hold.

**Theorem 1.43.** Let \((X, \mathcal{T})\) be a topological space, \(A \subseteq X\). Let \(H(A)\) denote the set of cluster points of \(A\), and let \(L(A)\) denote the set of limits of nets in \(A\), i.e. \(L(A)\) consists of all \(x \in X\) such that there is a net \((x_i)_{i \in I}\) in \(A\) satisfying \(\lim_{i \in I} x_i = x\) (by Prop. 1.40, \(L(A)\) consists of all limits of sequences in \(A\) if \(X\) is first countable).

It then holds that \(\overline{A} = L(A) = A \cup H(A)\).

**Proof.** It suffices to show that \(L(A) \subseteq \overline{A} \subseteq A \cup H(A) \subseteq L(A)\).

“\(L(A) \subseteq \overline{A}\)”\: Suppose \(x \notin A\). Since \(X \setminus \overline{A}\) is open, \(X \setminus \overline{A} \in \mathcal{U}(x)\), showing that no net in \(A\) can converge to \(x\), i.e. \(x \notin L(A)\).

“\(\overline{A} \subseteq A \cup H(A)\)”\: Let \(x \in \overline{A} \setminus A\). We need to show that \(x \in H(A)\). As \(\overline{A} = A \cup \partial A\) and \(x \notin A\), we have \(x \in \partial A\). Thus, if \(U \in \mathcal{U}(x)\), then there must exist \(x_U \in A \cap U\). Since \(x \notin \overline{A}\), we have \(x_U \neq x\), showing \(x\) to be a cluster point of \(A\).

“\(A \cup H(A) \subseteq L(A)\)”\: If \(a \in A\), then the constant sequence \((a, a, \ldots)\) converges to \(a\), implying \(a \in L(A)\). If \(a \in H(A)\), then \(a \in L(A)\) according to Lem. 1.41.

**Corollary 1.44.** Let \((X, \mathcal{T})\) be a topological space, \(A \subseteq X\). Then the following statements are equivalent:

(i) \(A\) is closed.

(ii) \(A = \overline{A}\).
(iii) $A$ contains all cluster points of $A$.

(iv) $A$ contains all limits of nets in $A$ that are convergent in $X$ (by Prop. 1.40, we may replace “nets” by “sequences” if $X$ is first countable\(^3\), also cf. [Phi16, Def. 7.42(b)])

In particular, if $A$ does not have any cluster points, then $A$ is closed.

Proof. The equivalence of (i) and (ii) is due to Th. 1.37(b) ($\overline{A}$ is the smallest closed set containing $A$). The equivalences of (ii), (iii), and (iv) are due to Th. 1.43: Using the notation $L(A)$ and $H(A)$ from Th. 1.43, one has that $A = \overline{A}$ implies $A = A \cup H(A)$, i.e. $H(A) \subseteq A$, i.e. (ii) implies (iii). If $H(A) \subseteq A$, then $L(A) = A \cup H(A) = A$, i.e. (iii) implies (iv). If $A = L(A)$, then $A = \overline{A}$, i.e. (iv) implies (ii).

Example 1.45. Let $p, q \in \mathbb{N}$ and consider the metric spaces given by $\mathbb{K}^p$, $\mathbb{K}^q$, $\mathbb{K}^{p+q}$, each endowed with the topology induced by the respective 2-norm (cf. Ex. 1.22(c)). Let $A \subseteq \mathbb{K}^p$, $B \subseteq \mathbb{K}^q$.

(a) If $A$ and $B$ are closed, then $A \times B$ is closed in $\mathbb{K}^{p+q} = \mathbb{K}^p \times \mathbb{K}^q$: Let $(c^k)_{k \in \mathbb{N}}$ be a convergent sequence in $A \times B$ with $\lim_{k \to \infty} c^k = c \in \mathbb{K}^{p+q}$. Then, for each $k \in \mathbb{N}$, $c^k = (a^k, b^k)$ with $a^k \in \mathbb{K}^p$, $b^k \in \mathbb{K}^q$. Moreover, $c = (a, b)$ with $a \in \mathbb{K}^p$ and $b \in \mathbb{K}^q$. According to (1.18), one has $a = \lim_{k \to \infty} a^k$ and $b = \lim_{k \to \infty} b^k$. Since $A$ and $B$ are closed, from Cor. 1.44(iv), we know that $a \in A$ and $b \in B$, i.e. $c = (a, b) \in A \times B$, showing that $A \times B$ is closed.

(b) If $A$ and $B$ are open, then $A \times B$ is open in $\mathbb{K}^{p+q} = \mathbb{K}^p \times \mathbb{K}^q$: It suffices to show that $(A \times B)^c = (A^c \times \mathbb{K}^q) \cup (\mathbb{K}^p \times B^c)$.

For a point $(z, w) \in \mathbb{K}^p \times \mathbb{K}^q = \mathbb{K}^{p+q}$, one reasons as follows:

$$(z, w) \in (A \times B)^c \iff (z, w) \notin A \times B$$

$$\iff (z \notin A \text{ and } w \in \mathbb{K}^q) \text{ or } (z \in \mathbb{K}^p \text{ and } w \notin B)$$

$$\iff (z, w) \in A^c \times \mathbb{K}^q \text{ or } (z, w) \in \mathbb{K}^p \times B^c$$

$$\iff (z, w) \in (A^c \times \mathbb{K}^q) \cup (\mathbb{K}^p \times B^c),$$

thereby proving (1.26). One now observes that $A^c$ and $B^c$ are closed, as $A$ and $B$ are open. As $\mathbb{K}^p$ and $\mathbb{K}^q$ are also closed, by (a), $A^c \times \mathbb{K}^q$ and $\mathbb{K}^p \times B^c$ are closed, and, thus, by (1.26), so is $(A \times B)^c$. In consequence, $A \times B$ is open as claimed.

(c) Intervals in $\mathbb{R}^n$, $n \in \mathbb{N}$: A subset $I$ of $\mathbb{R}^n$ is called an $n$-dimensional interval if, and only if, $I$ has the form $I = I_1 \times \cdots \times I_n$, where $I_1, \ldots, I_n$ are intervals in $\mathbb{R}$. The lengths $|I_1|, \ldots, |I_n| \in \mathbb{R}_0^+ \cup \{\infty\}$ are called the edge lengths of $I$. An interval $I$ is

\(^3\)One calls $A$ sequentially closed if, and only if, $A$ contains all limits of sequences in $A$. Thus, for first countable $X$, $A$ is closed if, and only if, $A$ is sequentially closed.
called a (hyper)cube if, and only if, all its edge lengths are equal. For \( x, y \in \mathbb{R}^n \), we write \( x < y \) (resp. \( x \leq y \)) if, and only if, \( x_j < y_j \) (resp. \( x_j \leq y_j \)) for each \( j \in \{1, \ldots, n\} \) (clearly, \( \leq \) is a partial order on \( \mathbb{R}^n \), but not a total order for \( n > 1 \), for example, neither \((0, 1) \leq (1, 0)\), nor \((1, 0) \leq (0, 1)\)). If \( x, y \in \mathbb{R}^n, x < y \), then we define the following intervals

\[
\begin{align*}
  [x, y] &:= \{ z \in \mathbb{R}^n : x \leq z \leq y \} = [x_1, y_1] \times \cdots \times [x_n, y_n] & \text{closed interval} \\
  ]x, y[ := \{ z \in \mathbb{R}^n : x < z < y \} &= [x_1, y_1] \times \cdots \times [x_n, y_n] & \text{open interval} \\
  [x, y[ := \{ z \in \mathbb{R}^n : x \leq z < y \} &= [x_1, y_1] \times \cdots \times [x_n, y_n] & \text{halfopen interval} \\
  ]x, y] := \{ z \in \mathbb{R}^n : x < z \leq y \} &= [x_1, y_1] \times \cdots \times [x_n, y_n] & \text{halfopen interval}
\end{align*}
\]

Due to (a),(b) open intervals in \( \mathbb{R}^n \) are open sets and closed intervals in \( \mathbb{R}^n \) are closed sets (with respect to the 2-norm).

### 1.3 Construction of Topological Spaces

#### 1.3.1 Bases, Subbases

One can often build a topology \( \mathcal{T} \) on \( X \) from smaller sets \( \mathcal{B} \subseteq \mathcal{T} \) by taking unions (then \( \mathcal{B} \) is called a base of \( \mathcal{T} \)) or from \( \mathcal{S} \subseteq \mathcal{T} \) by taking unions of finite intersections (then \( \mathcal{S} \) is called a subbase of \( \mathcal{T} \)). This can be useful, since one might be able to prove something about \( \mathcal{T} \) by merely proving it for the (smaller) base or subbase (cf. Cor. 1.50 below).

**Definition 1.46.** Let \((X, \mathcal{T})\) be a topological space.

(a) A set \( \mathcal{B} \subseteq \mathcal{T} \) is called a base of the topology \( \mathcal{T} \) if, and only if, every \( O \in \mathcal{T} \) is a union of sets from \( \mathcal{B} \), i.e. if, and only if, for each \( O \in \mathcal{T} \), there exists an index set \( I \) (where \( I \) can be empty, finite, or infinite) and sets \( O_i \in \mathcal{B}, i \in I \), such that

\[
O = \bigcup_{i \in I} O_i.
\]

\( X \) is called second countable (sometimes also called a \( C_2 \)-space) if, and only if, there exists a countable base of \( \mathcal{T} \) (recall the definition of first countable from Def. 1.38).

(b) A set \( \mathcal{S} \subseteq \mathcal{T} \) is called a subbase of \( \mathcal{T} \) if, and only if, \( \{X\} \) united with all finite intersections of sets from \( \mathcal{S} \) forms a base of \( \mathcal{T} \), i.e. if, and only if,

\[
\mathcal{B} := \beta(\mathcal{S}) := \{X\} \cup \left\{ \bigcap_{i=1}^n O_i : O_1, \ldots, O_n \in \mathcal{S}, n \in \mathbb{N} \right\}
\]

forms a base of \( \mathcal{T} \).\(^4\)

\(^4\)Some authors require a subbase to have the additional property of being a cover of \( X \) (i.e. every \( x \in X \) must be in some element of \( \mathcal{S} \)), which leads to a similar, but nonequivalent, notion.
Lemma 1.47. Let \((X, \mathcal{T})\) be a topological space. Then \(B \subseteq \mathcal{T}\) is a base of \(\mathcal{T}\) if, and only if, for each \(x \in X\), \(B(x) := \{B \in B : x \in B\}\) is a local base at \(x\) (cf. Def. 1.38).

Proof. Let \(B \subseteq \mathcal{T}\). If \(B\) is a base of \(\mathcal{T}\), \(x \in X\), \(U \in \mathcal{U}(x)\), then there is \(O \in \mathcal{T}\) with \(x \in O \subseteq U\). Since \(B\) is a base, there must be \(B \in B\) with \(x \in B \subseteq O \subseteq U\), showing \(B(x)\) to be a local base at \(x\). Conversely, assume, for each \(x \in X\), \(B(x)\) is a local base at \(x\). In general, not every set of subsets of \(X\) is the base of some topology on \(X\). One has the following result:

Proposition 1.48. Let \(X\) be a set and \(B \subseteq \mathcal{P}(X)\). Then \(B\) is the base of some topology \(\mathcal{T}\) on \(X\) if, and only if, \(B\) satisfies the following two conditions:

(i) \(B\) is a cover of \(X\), i.e.
\[
X = \bigcup_{B \in B} B.
\]

(ii) For each \(B_1, B_2 \in B\) and each \(x \in B_1 \cap B_2\), there exists \(B_3 \in B\) such that
\[
x \in B_3 \subseteq B_1 \cap B_2.
\] (1.27)

Moreover, if \(B\) satisfies (i) and (ii), then
\[
\mathcal{T} = \left\{ \bigcup_{i \in I} O_i : O_i \in B, \text{ I some index set} \right\}.
\] (1.28)

In particular, the topology \(\mathcal{T}\) is uniquely determined by \(B\); it is the coarsest topology on \(X\) containing \(B\), i.e.
\[
\mathcal{T} = \min \left\{ \mathcal{M} \subseteq \mathcal{P}(X) : B \subseteq \mathcal{M} \land \mathcal{M} \text{ is topology on } X \right\}
\] (1.29)

where \(\mathcal{P}(\mathcal{P}(X))\) is endowed with the partial order given by \(\subseteq\).

Proof. Suppose \(B\) satisfies (i) and (ii). We now define \(\mathcal{T}\) by (1.28) and show it constitutes a topology on \(X\): Coosing \(I := \emptyset\), shows \(\emptyset \in \mathcal{T}\). As an immediate consequence of (i), \(X \in \mathcal{T}\) also holds. Now let \(O_1, O_2 \in \mathcal{T}\) and assume \(O_1 \cap O_2 \neq \emptyset\). If \(x \in O_1 \cap O_2\), then, by (1.28), there exist sets \(B_1, B_2 \in B\) such that \(B_1 \subseteq O_1, B_2 \subseteq O_2\), and \(x \in B_1 \cap B_2\). According to (ii), there exists \(B_x \in B\) such that \(x \in B_x \subseteq B_1 \cap B_2 \subseteq O_1 \cap O_2\). Thus, setting \(I := O_1 \cap O_2\),
\[
O_1 \cap O_2 = \bigcup_{x \in I} B_x \in \mathcal{T},
\]
showing \(\mathcal{T}\) is closed under finite intersections. To see that \(\mathcal{T}\) is closed under arbitrary unions, let \(I\) be an index set and \(O_i \in \mathcal{T}\) for each \(i \in I\). Then
\[
\forall i \in I \quad O_i = \bigcup_{j \in J_i} B_j
\]
with some index sets $J_i$ and each $B_j \in \mathcal{B}$. Thus, letting $J := \bigcup_{i \in I} J_i$,

$$\bigcup_{i \in I} O_i = \bigcup_{j \in J} B_j \in \mathcal{T},$$

showing $\mathcal{T}$ to be closed under arbitrary unions. It merely remains to note that it is immediate from the definition of $\mathcal{T}$ that $\mathcal{B}$ is a base of $\mathcal{T}$.

Conversely, assume $\mathcal{T}$ to be an arbitrary topology on $X$ such that $\mathcal{B}$ is a base. Since $X \in \mathcal{T}$, (i) must hold. If $B_1, B_2 \in \mathcal{B}$, then there must be $B_3 \in \mathcal{B}$ satisfying $x \in B_3 \subseteq B_1 \cap B_2$, proving (ii) to hold. Now let $\tilde{T}$ denote the right-hand side of (1.28). Then $\tilde{T} \subseteq \mathcal{T}$ since we also know $\tilde{T}$ to be a topology on $X$ (with $\mathcal{B}$ as a base), we obtain $\tilde{T} = \mathcal{T}$. Similarly, if we let

$$\mathfrak{T} := \left\{ \mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{B} \subseteq \mathcal{M} \land \mathcal{M} \text{ is topology on } X \right\},$$

then $\mathcal{T} \in \mathfrak{T}$ and Prop. 1.4 implies

$$\tau := \bigcap_{\mathcal{M} \in \mathfrak{T}} \mathcal{M}$$

to be a topology. Then $\tau \subseteq \mathcal{T}$ and (1.28) implies $\mathcal{T} \subseteq \tau$. \hfill \blacksquare

**Proposition 1.49.** Let $X$ be a set. Then every $\mathcal{S} \subseteq \mathcal{P}(X)$ forms a subbase of some topology on $X$. More precisely, the set $\mathcal{B} := \beta(\mathcal{S})$ of Def. 1.46(b) forms a base of a topology on $X$. According to Prop. 1.48, this topology must be unique; we denote it by $\tau(\mathcal{S})$ and call it the topology generated by $\mathcal{S}$. Then $\tau(\mathcal{S})$ is the coarsest topology containing $\mathcal{S}$.

**Proof.** One merely needs to check that $\mathcal{B}$ satisfies (i) and (ii) of Prop. 1.48. However, (i) holds, as $X \in \beta(\mathcal{S})$; and (ii) holds, as $\beta(\mathcal{S})$ is clearly closed under finite intersections. \hfill \blacksquare

**Corollary 1.50.** Let $(X, \mathcal{T})$ be a topological space, $x \in X$. Let $\mathcal{B}$ be a base, $\mathcal{S}$ be a subbase for $\mathcal{T}$, and define $\mathcal{B}(x) := \{B \in \mathcal{B} : x \in B\}$, $\mathcal{S}(x) := \{S \in \mathcal{S} : x \in S\}$.

(a) A net $(x_i)_{i \in I}$ in $X$ converges to $x \in X$ if, and only if,

$$\forall S \in \mathcal{S}(x) \exists i \in I \forall j \geq i \ x_j \in S. \quad (1.30)$$

(b) The point $x \in X$ is a cluster point of $A \subseteq X$ if, and only if,

$$\forall B \in \mathcal{B}(x) \ B \cap (A \setminus \{x\}) \neq \emptyset.$$

(c) The point $x \in X$ is a closure point of $A \subseteq X$ (i.e. $x \in \overline{A}$) if, and only if,

$$\forall B \in \mathcal{B}(x) \ B \cap A \neq \emptyset.$$
(d) A set $A \subseteq X$ is dense if, and only if, for every nonempty $B \in \mathcal{B}$, one has $B \cap A \neq \emptyset$.

Proof. (a): Since $S(x) \subseteq U(x)$, $\lim_{i \in I} x_i = x$ implies (1.30). Conversely, assume (1.30) and let $U \in \mathcal{U}(x)$. Then there exists $n \in \mathbb{N}$ and $S_1, \ldots, S_n \in S(x)$ such that $x \in \bigcap_{k=1}^{n} S_k \subseteq U$. Due to (1.30), for each $k \in \{1, \ldots, n\}$, there is $i_k \in I$ such that, for each $j \geq i_k$, $x_j \in S_k$. Since $I$ is a directed set, there is $i \in I$ with $i \geq i_k$ for each $k \in \{1, \ldots, n\}$. Then, for each $j \geq i$, one has $x_j \in \bigcap_{k=1}^{n} S_k \subseteq U$, proving $\lim_{i \in I} x_i = x$.

(b): Since $\mathcal{B}(x) \subseteq \mathcal{U}(x)$, if $x$ is a cluster point of $A$, then the condition in (b) holds by Def. 1.32(e). Conversely, assume for every $x \in X$ and every $B \in \mathcal{B}(x)$, one has $B \cap (A \setminus \{x\}) = \emptyset$, and let $U \in \mathcal{U}(x)$. Since there is $B \in \mathcal{B}(x)$ with $B \subseteq U$ and $B \cap (A \setminus \{x\}) \neq \emptyset$, we have $U \cap (A \setminus \{x\}) \neq \emptyset$ as well, showing $x$ to be a cluster point of $A$.

(c) follows from (b), since, according to Th. 1.43, $\overline{A}$ is the union of $A$ with its cluster points.

(d) is now immediate from (c), since $A \subseteq X$ is dense if, and only if, $\overline{A} = X$. ■

Caveat: At the end of Ex. 1.53(c) below, we will see that, in general, one can not replace the base $\mathcal{B}$ in Cor. 1.50(b),(c),(d) by the subbase $S$.

**Proposition 1.51.** Let $(X, \mathcal{T})$ be a topological space.

(a) If $X$ is second countable, then it is separable.

(b) If $\mathcal{T}$ is induced by a metric $d$ on $X$, then $X$ is second countable if, and only if, it is separable.

Proof. (a): Suppose, $X$ is second countable. Then there is a countable base $\mathcal{B}$ for $\mathcal{T}$. For each $B \in \mathcal{B}$, let $x_B \in B$. Then $A := \{x_B : B \in \mathcal{B}\}$ is countable. We claim that $A$ is also dense in $X$. Indeed, if $x \in X$ and $U \in \mathcal{U}(x)$, then there is $B \in \mathcal{B}$ with $x \in B \subseteq U$. Then $x_B \in A \cap U$, i.e. $A$ is dense by Prop. 1.35(e).

(b): We need to show that if $X$ is separable, then it has a countable base. Thus, let $A \subseteq X$ be a countable dense set. Then $\mathcal{B} := \{B_r(a) : r \in \mathbb{Q}, a \in A\}$ is countable. We claim $\mathcal{B}$ is also a base of $\mathcal{T}$. Indeed, let $O \in \mathcal{T}$ and $x \in O$. Then there exists $r \in \mathbb{Q}$ such that $B_r(x) \subseteq O$. Moreover, since $A$ is dense, there exists some $a \in A \cap B_r(x)$. Consider $B := B_r(a)$. Then $B \in \mathcal{B}$, $x \in B$, and, due to

$$\forall y \in B \quad d(y, x) \leq d(y, a) + d(a, x) < \frac{r}{2} + \frac{r}{2} = r,$$

also $B \subseteq B_r(x) \subseteq O$, proving $\mathcal{B}$ to be a countable base of $\mathcal{T}$. ■

**Example 1.52.** (a) The set of all open $n$-dimensional intervals (cf. Ex. 1.45(c)) is a base for $(\mathbb{R}^n, \mathcal{T})$, $n \in \mathbb{N}$, if $\mathcal{T}$ is induced by the 2-norm; the set of all open $n$-dimensional intervals having only endpoints with rational coordinates is a countable base of $\mathcal{T}$. Moreover, the set all unbounded open intervals is a subbase of $\mathcal{T}$ (since every bounded open interval is a finite intersection of unbounded open intervals).
(b) Let $X$ be a nonempty set with a total order $\leq$. As on $\mathbb{R}$, one can define open intervals on $X$ by letting, for $a, b \in X$ with $a < b$, $I_{a,b} := \{x \in X : a < x < b\}$, $I_{a,b} := \{x \in X : x < b\}$, $I_{a} := \{x \in X : a < x\}$. We consider $X$ itself to be an open interval as well. Then, clearly, the set $\mathcal{B}$ of all open intervals satisfies conditions (i) and (ii) of Prop. 1.48 and, thus, forms a base for a topology $\mathcal{T}$ on $X$, called the *order topology* induced by $\leq$ on $X$. The set $\mathcal{S} := \{I_{a,b} : b \in \mathbb{X}\} \cup \{I_{a} : a \in \mathbb{X}\}$ is a subbase for $\mathcal{T}$. On $\mathbb{R}$, the order topology is just the normal topology. Another example is given by $X := \mathbb{R} := \mathbb{R} \cup \{-\infty, \infty\}$, letting

$$
\forall_{x,y \in X} \quad x \leq y \Leftrightarrow \begin{cases} 
    x = -\infty & \text{or} \\
    y = \infty & \text{or} \\
    x, y \in \mathbb{R}, x \leq y.
\end{cases}
$$

In $X$, $I_{a,b} = [-\infty, b]$ and $I_{a} = [a, \infty]$. In contrast to $\mathbb{R}$, $\{I_{a,b} : a, b \in \mathbb{R}, a < b\}$ is *not* a base for the order topology on $X$. Also note that a sequence (even a net, actually) in $\mathbb{R}$ converges in $X$ if, and only if, it converges to $x \in \mathbb{R}$ or it diverges to either $\infty$ or $-\infty$.

(c) If $(X, \mathcal{T})$ is a discrete topological space, then $\mathcal{B} := \{\{x\} : x \in X\}$ is a base for $\mathcal{T}$ and no strict subset of $\mathcal{B}$ is a base for $\mathcal{T}$.

(d) Consider $\mathbb{R}$ and $\mathcal{B} := \{[a,b] : a, b \in \mathbb{R}, a < b\}$. Then, clearly, $\mathcal{B}$ satisfies conditions (i) and (ii) of Prop. 1.48 and, thus, forms a base for a topology $\mathcal{T}$ on $\mathbb{R}$. This topology is called the *Sorgenfrey topology* or the *right half-open interval topology* on $\mathbb{R}$. This topology has a number of interesting properties: It is strictly finer than the usual topology $\mathcal{T}_0$ on $\mathbb{R}$ (the one generated by the open intervals), i.e. $\mathcal{T}_0 \subset \mathcal{T}$: Intervals of the form $[a,b]$, $a < b$, are not in $\mathcal{T}_0$, but

$$
[a,b] = \bigcup_{n \in \mathbb{N}}[a + 1/n, b] \in \mathcal{T}.
$$

Moreover, $\mathcal{T}$ is an example of a topology that is separable and first countable, but *not* second countable (thus, by Prop. 1.51, $\mathcal{T}$ cannot be metrizable): $\mathbb{Q}$ is dense, since every $[a,b]$, $a < b$, contains a rational number. $\mathcal{T}$ is first countable, since, for each $x \in \mathbb{R}$, $\mathcal{B}(x) := \{[x, x + \frac{1}{n}] : n \in \mathbb{N}\}$ is a countable local base at $x$. Now let $\mathcal{C}$ be an arbitrary base for $\mathcal{T}$, $x \in \mathbb{R}$. Since $x \in O_x := [x, x + 1]$ and $O_x$ is open, there exists $B_x \in \mathcal{C}$ such that $x \in B_x \subseteq O_x$, implying $x = \min B_x$. In particular, if $x, y \in \mathbb{R}, x \neq y$, then $B_x \neq B_y$, showing that $\mathcal{C}$ cannot be countable.

**Example 1.53.** Let $I$ be a nonempty index set and, for each $i \in I$, let $(X_i, \mathcal{T}_i)$ be a topological space. Moreover, for each $i \in I$, let $\mathcal{B}_i$ be a base of $\mathcal{T}_i$ and let $\mathcal{S}_i$ be a subbase of $\mathcal{T}_i$. We consider the Cartesian product $X := \prod_{i \in I} X_i$ (cf. [Phi16, Def. 2.15(c)]). If $x = (x_i)_{i \in I} \in X$, then we call $x_i$ the *ith coordinate* of $x$. Moreover, for each $j \in I$, we call the map

$$
\pi_j : X \longrightarrow X_j, \quad \pi_j((x_i)_{i \in I}) := x_j, \quad (1.31)
$$

the *projection* on the $j$th coordinate.
(a) Consider the set
\[ B_p := \left\{ \prod_{i \in I} O_i : \left( \forall_{i \in I} O_i \in T_i \right) \land \#\{i \in I : O_i \neq X_i\} < \infty \right\} \]
\[ = \left\{ \bigcap_{j \in J} \pi_j^{-1}(O_j) : J \subseteq I, 0 < \#J < \infty, \forall_{j \in J} O_j \in T_j \right\}. \]

Then \( B_p \) satisfies conditions (i) and (ii) of Prop. 1.48 (since \( X \in B_p \) and since \( B_p \) is closed under finite intersections by (A.1) of Appendix A and Def. 1.1(iii)) and, thus, forms a base for a topology \( T_p \) on \( X \), called the product topology on \( X \). Another base for \( T_p \) is
\[ B^*_p := \left\{ \prod_{i \in I} B_i : \left( \forall_{i \in I} B_i \in B_i \cup \{X_i\} \right) \land \#\{i \in I : B_i \neq X_i\} < \infty \right\} \]
\[ = \left\{ \bigcap_{j \in J} \pi_j^{-1}(B_j) : J \subseteq I, 0 < \#J < \infty, \forall_{j \in J} B_j \in B_j \right\} \cup \{X\}. \]

Let \( \emptyset \neq J \subseteq I \) be finite, \( O_j \in T_j \). Since \( B_j \) is a base for \( T_j \), there exist \( I_j \) and \( B_{ji} \in B_j, i \in I_j \), such that \( O_j = \bigcup_{i \in I_j} B_{ji} \). Let \( K := \prod_{j \in J} I_j \). Then
\[ \prod_{j \in J} O_j = \bigcup_{j \in K} \prod_{j \in J} B_{j,i(j)}, \]
showing that every element of \( B_p \) is the union of elements from \( B^*_p \). A subbase for \( T_p \) is
\[ S_p := \left\{ \prod_{i \in I} O_i : \left( \forall_{i \in I} O_i \in T_i \right) \land \#\{i \in I : O_i \neq X_i\} \leq 1 \right\} \]
\[ = \left\{ \pi_i^{-1}(O_i) : i \in I, O_i \in T_i \right\}. \]

Another subbase for \( T_p \) is
\[ S^*_p := \left\{ \prod_{i \in I} S_i : \left( \forall_{i \in I} S_i \in S_i \cup \{X_i\} \right) \land \#\{i \in I : S_i \neq X_i\} \leq 1 \right\} \]
\[ = \left\{ \pi_i^{-1}(S_i) : i \in I, S_i \in S_i \right\} \cup \{X\} \]
(clearly, since \( S_i \) is a subbase for \( T_i \), each element of \( S_p \) is a union of finite inter-
sections of elements from \( S^*_p \)). It is an exercise to show that, if \( I = \{i_1, i_2, \ldots\} \) is 
countable, and \( T_{i_k} \) is induced by the metric \( d_k \) on \( X_{i_k} \), then the product topology 
on \( X \) is metrizable via the metric
\[ d : X \times X \to \mathbb{R}_0^+, \quad d((x_{i_k})_{k \in \mathbb{N}}, (y_{i_k})_{k \in \mathbb{N}}) := \sum_{k=1}^{\infty} \frac{d_k(x_{i_k}, y_{i_k})}{2^k(1 + d_k(x_{i_k}, y_{i_k}))} \quad (1.32) \]
(though, we will see in (c) below that, e.g., the product topology on \( \mathbb{K}^\mathbb{R} \) is not 
metrizable).
(b) An important special case of the product topology of (a) one obtains for \( I = \{1, \ldots, n\}, n \in \mathbb{N}, \) and \( X_i = \mathbb{K}, \) in which case \( X = \mathbb{K}^n. \) If we endow \( \mathbb{K} \) with the usual topology (given by \(| \cdot |\)), then, we claim that the product topology on \( \mathbb{K}^n \) is precisely the one induced by the 2-norm (and, by Th. 1.72 below, this is the same topology induced by any other norm on \( \mathbb{K}^n\)): Let \( \mathcal{T}_2 \) be the topology induced by the 2-norm on \( \mathbb{K}^n. \) We have to show \( \mathcal{T}_2 = \mathcal{T}_p. \) Let \( O \in \mathcal{T}_p, \) \( z = (z_1, \ldots, z_n) \in O. \) Then there is \( \epsilon > 0 \) such that \( B := \prod_{j=1}^n B_{\epsilon/2}(z_j) \subseteq O. \)

According to (1.19), we obtain \( B_{\frac{\epsilon}{\sqrt{n}}}(z) \subseteq B, \) showing \( O \in \mathcal{T}_2 \) and \( \mathcal{T}_2 \supseteq \mathcal{T}_p. \) For the remaining inclusion, let \( z = (z_1, \ldots, z_n) \in \mathbb{K}^n \) and \( \epsilon > 0. \) We need to find \( O \in \mathcal{T}_p \) such that \( z \in O \subseteq B_{\frac{\epsilon}{\sqrt{n}}}(z). \) However, again, due to (1.19), we can simply choose

\[
O := \prod_{j=1}^n B_{\epsilon/n}(z_j).
\]

This shows \( \mathcal{T}_2 \subseteq \mathcal{T}_p \) and, thus, \( \mathcal{T}_2 = \mathcal{T}_p \) as desired.

(c) Another important special case of the product topology of (a) one obtains for \( I = \mathbb{R} \) and \( X_i = \mathbb{K} \) for each \( i \in I \) (as in (b) with each \( \mathcal{T}_i \) given by \(| \cdot |\) on \( \mathbb{K} \)). Then

\[
X = \prod_{i \in I} X_i = \mathbb{K}^\mathbb{R} = \mathcal{F}(\mathbb{R}, \mathbb{K})
\]

is the set of functions from \( \mathbb{R} \) into \( \mathbb{K}. \) Then the product topology \( \mathcal{T}_p \) is the so-called topology of pointwise convergence. Indeed, we can show that a net \((f_j)_{j \in J}\) in \( X\) converges to \( f \in X\) pointwise, i.e.

\[
\forall s \in \mathbb{R} \quad \lim_{j \to J} f_j(s) = f(s), \quad (1.33)
\]

if, and only if, \( \lim_{j \in J} f_j = f \) with respect to \( \mathcal{T}_p. \) Suppose, (1.33) holds. If \( f \in S \in \mathcal{S}_p \) (as defined in (a)), then there exists \( s_0 \in \mathbb{R} \) and an open set \( O \subseteq \mathbb{K} \) such that \( f(s_0) \in O, \) \( S = \pi_{s_0}^{-1}(O). \) Since \( \lim_{j \in J} f_j(s_0) = f(s_0), \)

\[
\exists j \in J, \quad f_j(s_0) \in O, \quad (1.34)
\]

implying \( f_j \in S \) for each \( j \geq i. \) Thus, using Cor. 1.50(a), \( \lim_{j \in J} f_j = f \) with respect to \( \mathcal{T}_p. \) Conversely, assume \( \lim_{j \in J} f_j = f \) with respect to \( \mathcal{T}_p \) and fix \( s_0 \in \mathbb{R}. \) Let \( O \subseteq \mathbb{K} \) be open and such that \( f(s_0) \in O. \) Defining \( S = \pi_{s_0}^{-1}(O) \) as above, we see that (1.34) must hold again, proving (1.33), i.e. pointwise convergence.

Next, we will see that \( \mathcal{T}_p \) is not first countable (in particular, not metrizable): Define

\[
A := \left\{ f : \mathbb{R} \to \mathbb{K} : \exists J \in \mathbb{R} \text{ finite } f(s) = \begin{cases} 0 & \text{for } s \in J, \\ 1 & \text{for } s \notin J \end{cases} \right\} \subseteq X,
\]
$f_0 \in X; \ f_0 = 0$. Then $f_0 \in \overline{A}$: Indeed, if $f_0 \in B \in \mathcal{B}_p$, then $B = \bigcap_{s \in J} \pi_s^{-1}(O_s)$, $J \subseteq \mathbb{R}$ finite, for each $s \in J$ is $0 \in O_s \subseteq \mathbb{K}$ open. Then

$$f \in X, \ f(s) = \begin{cases} 0 & \text{for } s \in J, \\ 1 & \text{for } s \notin J \end{cases}$$

is in $A \cap B$, showing $A \cap B \neq \emptyset$, proving $f_0 \in \overline{A}$. However, no sequence in $A$ converges to $f_0$: If $(f_k)_{k \in \mathbb{N}}$ is a sequence in $A$, then $J := \{ s \in \mathbb{R} : f_k(s) = 0 \text{ for some } k \in \mathbb{N} \}$ is countable, i.e., for $s \notin J$, $f_k(s) \to 1 \neq 0$, showing $f_k \not\to f_0$. Thus, according to Th. 1.43 and Prop. 1.40, $f_0$ can not have a countable local base (and an analogous argument actually shows that no $f \in X$ can have a countable local base).

Somewhat surprisingly, $\mathcal{T}_p$ turns out to be separable: Let $(d_1, d_2, \ldots)$ be an enumeration of a dense subset of $\mathbb{K}$ (e.g. of $\mathbb{Q}$ for $\mathbb{K} = \mathbb{R}$ and of $\mathbb{Q} \times \mathbb{Q}$ for $\mathbb{K} = \mathbb{C}$), and let

$$K := \left\{ (I_1, \ldots, I_k, n_1, \ldots, n_k) : n_\alpha \in \mathbb{N}; I_\alpha = [r_{\alpha_1}, r_{\alpha_2}]; r_{\alpha_1}, r_{\alpha_2} \in \mathbb{Q}; I_\alpha \cap I_\beta = \emptyset \text{ for } \alpha \neq \beta; k \in \mathbb{N} \right\}$$

($K$ is the set of pairs of $k$-tuples, where the first $k$-tuple consists of disjoint closed intervals with rational endpoints and the second $k$-tuple consists of natural numbers). Then, clearly, $K$ is countable (since $\mathbb{Q}$ and $\mathbb{N}$ are countable, and countable unions of countable sets are countable). For each $\tau := (I_1, \ldots, I_k, n_1, \ldots, n_k) \in K$ define $f_\tau \in X$ by setting

$$f_\tau : \mathbb{R} \to \mathbb{K}, \ f_\tau(s) := \begin{cases} d_{n_\alpha} & \text{if } s \in I_\alpha, \\ d_1 & \text{otherwise.} \end{cases}$$

As $K$ is countable, so is $A := \{ f_\tau : \tau \in K \} \subseteq X$. We show $A$ to be dense as well: Let $\emptyset \neq B \subseteq \mathcal{B}_p$. Then there exists $\{ s_1, \ldots, s_k \} \subseteq \mathbb{R}$, $k \in \mathbb{N}$, and $\emptyset \neq O_\alpha \subseteq \mathbb{K}$ open, $\alpha \in \{ 1, \ldots, k \}$, such that

$$B = \bigcap_{\alpha=1}^k \pi_{s_\alpha}^{-1}(O_\alpha).$$

For each $\alpha \in \{ 1, \ldots, k \}$, there is $d_{n_\alpha} \in O_\alpha$, $n_\alpha \in \mathbb{N}$. Then there is

$$\tau := (I_1, \ldots, I_k, n_1, \ldots, n_k) \in K$$

such that $s_\alpha \in I_\alpha$ for each $\alpha \in \{ 1, \ldots, k \}$. Then $f_\tau \in B$, since $f_\tau(s_\alpha) = d_{n_\alpha} \in O_\alpha$ for each $\alpha \in \{ 1, \ldots, k \}$, proving $A$ to be dense and $\mathcal{T}_p$ to be separable.

Finally, we can use the present example to show that, in Cor. 1.50(b),(c),(d), one can, in general, not replace the base $\mathcal{B}$ by the subbase $\mathcal{S}$: Define

$$A_1 := \left\{ (f : \mathbb{R} \to \mathbb{K}) : \exists_{s_0 \in \mathbb{R}} f(s) = \begin{cases} 0 & \text{for } s = s_0, \\ 1 & \text{for } s \neq s_0 \end{cases} \right\} \subseteq X,$$
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\( f_0 \in X, f_0 \equiv 0. \) Then \( f_0 \) is not a cluster point of \( A_1 \) (and also \( f_0 \notin \overline{A}_1 \), as \( f_0 \notin A_1 \)): Let \( U := B_2(0) \subseteq K. \) Then \( O := \pi_0^{-1}(U) \cap \pi_1^{-1}(U) \) is an open neighborhood of \( f_0, \) but \( A_1 \cap O = \emptyset. \) On the other hand, if \( f_0 \in S \in S_p, \) then, clearly, \( S \cap A_1 \neq \emptyset, \) showing the base \( \mathcal{B} \) in Cor. 1.50(b),(c) cannot be replaced by the subbase \( \mathcal{S}. \) To see that the same holds for Cor. 1.50(d), one observes that \( A_1 \) is not dense in \( Y := A_1 \cup \{f_0\} \) if \( Y \) is considered as a subspace of \( X \) (cf. Prop. 1.54(a) below).

If, in Ex. 1.53, instead of using the base \( \mathcal{B}_p \) to obtain the product topology on \( X, \) one uses the base

\[
\mathcal{B}_b := \left\{ \prod_{i \in I} O_i : \left( \forall i \in I \right) O_i \in \mathcal{T}_i \right\} = \left\{ \bigcap_{i \in I} \pi_i^{-1}(O_i) : \forall i \in I O_i \in \mathcal{T}_i \right\},
\]

then one obtains the so-called box topology on \( X \) (see Appendix B). In general, it has different properties that turn out to be less useful.

1.3.2 Subspaces

**Proposition 1.54.** Let \((X, \mathcal{T})\) be a topological space, \( M \subseteq X. \)

(a) We call

\[
\mathcal{T}_M := \left\{ A \subseteq M : \exists O \in \mathcal{T} \ A = O \cap M \right\}
\]

(1.35)

the relative topology on \( M. \) It is also called the subspace or trace topology on \( M. \) It is, indeed, a topology. In this context, we also call \( O \in \mathcal{T} \ X-\)open and \( O \in \mathcal{T}_M \ M-\)open or relatively open.

(b) A set \( A \subseteq M \) is \( M-\)closed (i.e. closed with respect to \( \mathcal{T}_M, \) also called relatively closed) if, and only if, there exists an \( X-\)closed \( C \subseteq X \) such that \( A = C \cap M. \)

(c) If \( \mathcal{B} \) is a base for \( \mathcal{T}, \) \( \mathcal{S} \) is a subbase for \( \mathcal{T}, \) and

\[
\mathcal{B}_M := \left\{ A \subseteq M : \exists B \in \mathcal{B} \ A = B \cap M \right\}, \quad (1.36a)
\]

\[
\mathcal{S}_M := \left\{ A \subseteq M : \exists S \in \mathcal{S} \ A = S \cap M \right\}, \quad (1.36b)
\]

then \( \mathcal{B}_M \) is a base for \( \mathcal{T}_M \) and \( \mathcal{S}_M \) is a subbase for \( \mathcal{T}_M. \) Moreover, if \( a \in M \) and \( \mathcal{B}(a) \) is a local base at \( a \) for \( \mathcal{T}, \) then

\[
\mathcal{B}_M(a) := \left\{ A \subseteq M : \exists B \in \mathcal{B}(a) \ A = B \cap M \right\}, \quad (1.36c)
\]

is a local base at \( a \) for \( \mathcal{T}_M. \)
Proof. (a): We have $\emptyset = M \cap \emptyset \in \mathcal{T}_M$, $M = M \cap X \in \mathcal{T}_M$. If $A_1, A_2 \in \mathcal{T}_M$, then $A_1 = O_1 \cap M$, $A_2 = O_2 \cap M$ with $O_1, O_2 \in T$. Then $A_1 \cap A_2 = (O_1 \cap O_2) \cap M$, showing $A_1 \cap A_2 \in \mathcal{T}_M$. If $I$ is an index set and $A_i \in \mathcal{T}_M$ for each $i \in I$, then there are $O_i \in \mathcal{T}$ such that $A_i = O_i \cap M$. Then

$$A := \bigcup_{i \in I} A_i = \bigcup_{i \in I} (O_i \cap M) \stackrel{[\text{Ref. 1.39(d)}]}{=} \left( \bigcup_{i \in I} O_i \right) \cap M,$$

showing $A \in \mathcal{T}_M$, i.e. $\mathcal{T}_M$ is a topology.

(b): Let $A \subseteq M$. If $A$ is $M$-closed, then $M \setminus A$ is $M$-open, i.e. there is an $X$-open set $O \subseteq X$ such that $M \setminus A = M \cap O$. Then $C := X \setminus O$ is an $X$-closed set and $M \cap C = M \cap (X \setminus O) = M \setminus (M \cap O) = M \setminus (M \setminus A) = A$. Conversely, if there is an $X$-closed set $C \subseteq X$ with $A = C \cap M$, then $O := X \setminus C$ is an $X$-open set satisfying $O \cap M = M \cap (X \setminus C) = M \setminus (C \cap M) = M \setminus A$, showing $M \setminus A$ is $M$-open, i.e. $A$ is $M$-closed.

(c): If $A \in \mathcal{T}_M$, then $A = O \cap M$ with $O \in \mathcal{T}$. Thus, $O = \bigcup_{i \in I} B_i$ with each $B_i \in B$. In consequence,

$$A = \left( \bigcup_{i \in I} B_i \right) \cap M \stackrel{[\text{Ref. 1.39(d)}]}{=} \bigcup_{i \in I} (B_i \cap M).$$

Since each $B_i \cap M \in B_M$, this shows $B_M$ to be a base for $\mathcal{T}_M$. We now assume the base $B$ to be the set of finite intersections of sets from $S$. If $A \in B_M$, then $A = B \cap M$ with $B \in B$. Moreover, $B = \bigcap_{i=1}^n S_i$, $n \in \mathbb{N}$, each $S_i \in S$. Thus,

$$A = \left( \bigcap_{i=1}^n S_i \right) \cap M \stackrel{[\text{Ref. 1.39(a)}]}{=} \bigcap_{i=1}^n (S_i \cap M).$$

Since each $S_i \cap M \in S_M$, this shows $S_M$ to be a subbase for $\mathcal{T}_M$. Now let $U \subseteq M$ be a $\mathcal{T}_M$-neighborhood of $a$. Then there exists $O \in \mathcal{T}$ such that $a \in O \cap M \subseteq U$. Thus, there exists $B \in B(a)$ such that $a \in B \subseteq O$. Then $a \in B \cap M \subseteq O \cap M \subseteq U$, and, since $B \cap M \in B_M(a)$, this shows $B_M(a)$ to be a local base at $a$ for $\mathcal{T}_M$. \qed

One has to use care when working with a subspace $(M, \mathcal{T}_M)$ of a topological space $(X, \mathcal{T})$: The following Prop. 1.55 already shows that some properties of $(X, \mathcal{T})$ are inherited by the subspace (e.g. first and second countable, metrizable), but others are not (e.g. separable, complete metric (to be studied later), compact (to be studied later as well). Moreover, a set that is $M$-open might not be $X$-open and a set that is $M$-closed might not be $X$-closed (see Ex. 1.56 below).

**Proposition 1.55.** Let $(X, \mathcal{T})$ be a topological space and $(M, \mathcal{T}_M)$ a subspace.

(a) If $(a_i)_{i \in I}$ is a net in $M$, $a \in M$, then the net converges to $a$ with respect to $\mathcal{T}_M$ if, and only if, it converges to $a$ with respect to $\mathcal{T}$ (caveat: $(\frac{1}{k})_{k \in \mathbb{N}}$ converges to $0$ in $\mathbb{R}$, but does not converge in $]0,1]$; $(k)_{k \in \mathbb{N}}$ converges to $\infty$ in $\mathbb{R} \cup \{\infty\}$, but does not converge in $\mathbb{R}$).
(b) If \( a \in M \) and there is a countable local base at \( a \) with respect to \( T \), then there is a countable local base at \( a \) with respect to \( T_M \). In particular, if \( T \) is first countable, then so is \( T_M \).

(c) If \( T \) is second countable, then so is \( T_M \).

(d) If \( T \) is metrizable by the metric \( d : X \times X \to \mathbb{R}_+^+ \), then \( T_M \) is metrizable by \( d|_{M \times M} \) (we then call \((M,d)\) a metric subspace of \((X,d)\)).

(e) Somewhat surprisingly, every topological space \((X,T)\) is the subspace of some separable topological space \((Y,T_Y)\), where \( Y \setminus X \) is even countable (in particular, in general, not every subspace of a separable space is separable).

Proof. (a): Suppose, \((a_i)_{i \in I}\) converges to \( a \) with respect to \( T \), and let \( U \subseteq M \) be a \( T_M \)-neighborhood of \( a \). Then there is \( O \in T \) such that \( a \in O \cap M \subseteq U \). Since \( O \) is a \( T \)-neighborhood of \( a \),

\[
\exists_{i \in I} \forall_{j \geq i} a_j \in O.
\]

Then \( a_j \in O \cap M \subseteq U \), showing \((a_i)_{i \in I}\) converges to \( a \) with respect to \( T_M \). Conversely, assume \( \lim_{i \in I} a_i = a \) holds with respect to \( T_M \), and let \( U \subseteq X \) be a \( T \)-neighborhood of \( a \). Again there is \( O \in T \) such that \( a \in O \cap M \subseteq U \). Since \( O \cap M \) is an \( M \)-neighborhood of \( a \),

\[
\exists_{i \in I} \forall_{j \geq i} a_j \in O \cap M \subseteq U,
\]

showing \((a_i)_{i \in I}\) converges to \( a \) with respect to \( T \).

(b) follows directly from the local base part of Prop. 1.54(c).

(c) follows directly from the base part of Prop. 1.54(c).

(d): We note that \( d|_{M \times M} \) is, indeed, a metric on \( M \) (since \( d \) satisfies the laws (i) – (iii) from Def. 1.5 for all \( x,y,z \in X \), in particular, \( d \) satisfies the same laws for all \( x,y,z \in M \subseteq X \)). Since \( T \) is metrizable, \( B := \{B_r(x) : x \in X, r \in \mathbb{R}_+^+\} \) is a base of \( T \). We have to show that \( C := \{B_{r,M}(a) : a \in M, r \in \mathbb{R}_+^+\} \), where

\[
\forall_{a \in M} \forall_{r \in \mathbb{R}_+^+} B_{r,M}(a) := M \cap B_r(a) = \{x \in M : d(a,x) < r\} \tag{1.37}
\]

are the open balls with respect to \( d|_{M \times M} \), constitutes a base for \( T_M \). If \( A \in T_M \), then \( A = O \cap M \) with \( O \in T \). Thus,

\[
\forall_{a \in A} \exists_{r \in \mathbb{R}_+^+} B_r(a) \subseteq O.
\]

Intersecting with \( M \) yields \( B_{r,M}(a) = B_r(a) \cap M \subseteq O \cap M = A \), proving \( C \) to be a base for \( T_M \) as desired.

(e): See [Sie52, p. 49].

\[\square\]

Example 1.56. (a) If \((M,T_M)\) is a topological subspace of \((X,T)\), then, \( M \) is always both \( M \)-open and \( M \)-closed (irrespective of \( M \) being \( X \)-open or \( X \)-closed).
(b) Let \( X = \mathbb{R} \) with the usual metric, i.e. \( d(x, y) = |x - y| \) for each \( x, y \in \mathbb{R} \). Let \( M = [0, 1] \). According to (a), \( M \) is both \( M \)-closed and \( M \)-open, even though \([0, 1]\) is not open in \( X \). If \( M = ]0, 1[ \), then, again, \( M \) is both \( M \)-closed and \( M \)-open, even though \( ]0, 1[ \) is neither closed nor open in \( X \). Moreover, \( ]0, 1[ \) is \( M \)-closed (but not \( X \)-closed) and \( ]1/2, 1[ \) is \( M \)-open (but not \( X \)-open).

1.4 Topics Particular to Metric and Normed Spaces

1.4.1 Basic Inequalities

**Lemma 1.57.** (a) The following law holds in every metric space \((X, d)\):

\[
|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y') \quad \text{for each } x, x', y, y' \in X. \tag{1.38a}
\]

(b) The following law holds in every normed vector space \((X, \| \cdot \|)\):

\[
\|\|x\| - \|y\|\| \leq \|x - y\| \quad \text{for each } x, y \in X. \tag{1.38b}
\]

This law is sometimes referred to as the inverse triangle inequality.

**Proof.** (a): First, note \( d(x, y) \leq d(x, x') + d(x', y') + d(y', y) \), i.e.

\[
d(x, y) - d(x', y') \leq d(x, x') + d(y', y). \tag{1.39a}
\]

Second, \( d(x', y') \leq d(x', x) + d(x, y) + d(y, y') \), i.e.

\[
d(x', y') - d(x, y) \leq d(x', x) + d(y, y'). \tag{1.39b}
\]

Taken together, (1.39a) and (1.39b) complete the proof of (1.38a).

(b): Let \( d(x, y) := \|x - y\| \) be the induced metric on \( X \). Applying (a) to \( d \) yields the estimate

\[
\|\|x\| - \|y\|\| = |d(x, 0) - d(y, 0)| \leq d(x, y) + d(0, 0) = \|x - y\|,
\]

which establishes the case. ■

1.4.2 Completeness

**Definition 1.58.** Let \((X, d)\) be a metric space. The sequence \((x_k)_{k \in \mathbb{N}}\) in \( X \) is said to be a Cauchy sequence if, and only if, for each \( \epsilon > 0 \), there is \( N \in \mathbb{N} \) such that, \( d(x_k, x_l) < \epsilon \) for each \( k, l > N \).

**Proposition 1.59.** Let \((X, d)\) be a metric space and let \((x_k)_{k \in \mathbb{N}}\) be a sequence in \( X \).

(a) If \((x_k)_{k \in \mathbb{N}} \) is convergent, then it is a Cauchy sequence.
(b) If \( X = \mathbb{K}^n \), \( n \in \mathbb{N} \), and \( d \) is induced by the 2-norm on \( \mathbb{K}^n \), then the converse of (a) also holds: If \((x^k)_{k \in \mathbb{N}}\) is a Cauchy sequence, then it is convergent in \( \mathbb{K}^n \).

**Proof.** (a) (cf. the proof for sequences in \( \mathbb{K} \) in [Phi16, Th. 7.29]): If \( \lim_{k \to \infty} x^k = a \in X \), then, given \( \epsilon > 0 \), there is \( N \in \mathbb{N} \) such that \( x^k \in B_2(a) \) for each \( k > N \). If \( k, l > N \), then \( d(x^k, x^l) \leq d(x^k, a) + d(a, x^l) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \), establishing that \((x^k)_{k \to \infty}\) is a Cauchy sequence.

(b): As \((x^k)_{k \in \mathbb{N}}\) is Cauchy, given \( \epsilon \in \mathbb{R}^+ \), there is \( N \in \mathbb{N} \) such that, for each \( k, l > N \),

\[
\|x^k - x^l\|_2 < \epsilon.
\]

Then, by (1.19), for each \( k, l > N \),

\[
\forall j \in \{1, \ldots, n\} \quad |x^k_j - x^l_j| \leq \|x^k - x^l\|_2 < \epsilon,
\]

implying, according to [Phi16, Def. 7.28], \((x^k_j)_{k \in \mathbb{N}}\) is a Cauchy sequence in \( \mathbb{K} \) for each \( j \in \{1, \ldots, n\} \). Then [Phi16, Th. 7.29] yields each \((x^k_j)_{k \in \mathbb{N}}\) to be convergent in \( \mathbb{K} \) with some limit \( a_j \in \mathbb{K} \). Thus, finally, \((x^k)_{k \in \mathbb{N}}\) converges to \( a := (a_1, \ldots, a_n) \in \mathbb{K}^n \) by Ex. 1.22(c).

While we just saw that, in \( \mathbb{K}^n \), each Cauchy sequence converges, this is not true for all metric spaces, as simple examples show. Take, e.g., \( X = \mathbb{Q} \) or \( X = [0, 1] \) with \( d \) being given by the absolute value. A less trivial example is the following:

**Example 1.60.** Let \( X \) be the vector space over \( \mathbb{K} \) of sequences in \( \mathbb{K} \) that are finally constant and equal to 0. Thus, the sequence \( z = (z_n)_{n \in \mathbb{N}} \), \( z_n \in \mathbb{K} \) for each \( n \in \mathbb{N} \), is in \( X \) if, and only if, there exists \( N \in \mathbb{N} \) such that \( z_n = 0 \) for each \( n \geq N \). Clearly, \( X \) endowed with the norm \( \| \cdot \|_{\sup} \) is a subspace of the normed vector space \( B(S, \mathbb{K}) \) of Example 1.15 with \( S := \mathbb{N} \). Defining, for each \( n, k \in \mathbb{N} \),

\[
z_n^k := \begin{cases} 1/n & \text{for } 1 \leq n \leq k, \\ 0 & \text{for } n > k, \end{cases}
\]

one sees that \((z^k)_{k \in \mathbb{N}}\) is a Cauchy sequence in \( X \) (i.e. with respect to \( \| \cdot \|_{\sup} \)), but it is not convergent in \( X \) (its limit, the sequence \((1/n)_{n \in \mathbb{N}}\) is not finally constant and, thus, not in \( X \)).

**Definition 1.61.** A metric space \((X, d)\) and its metric \( d \) are both called complete if, and only if, every Cauchy sequence in \( X \) converges. A normed space is called a Banach space if, and only if, the metric induced by the norm is complete. In that case, one also says that the normed space and the norm itself are complete. We call the complete metric space \((Y, d_Y)\) a completion of \((X, d)\) if, and only if, there exists an injective map \( \phi : X \to Y \) that is isometric, i.e.

\[
\forall_{x_1, x_2 \in X} \quad d_Y(\phi(x_1), \phi(x_2)) = d(x_1, x_2),
\]

and \( \phi(X) \) is dense in \( Y \) (according to Th. 2.35 below, a completion always exists and, in a certain sense, it is even unique).
Prop. 1.59(b) means that $\mathbb{K}^n$ is a Banach space.

1.4.3 Inner Products and Hilbert Space

**Definition 1.62.** Let $X$ be a vector space over $\mathbb{K}$. A function $\langle \cdot , \cdot \rangle : X \times X \to \mathbb{K}$ is called an *inner product* or a *scalar product* on $X$ if, and only if, the following three conditions are satisfied:

(i) $\langle x, x \rangle \in \mathbb{R}^+$ for each $0 \neq x \in X$.

(ii) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ for each $x, y, z \in X$ and each $\lambda, \mu \in \mathbb{K}$ (i.e. an inner product is $\mathbb{K}$-linear in its first argument).

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for each $x, y \in X$ (i.e. an inner product is conjugate-symmetric, even symmetric for $\mathbb{K} = \mathbb{R}$).

**Lemma 1.63.** For each inner product $\langle \cdot , \cdot \rangle$ on a vector space $X$ over $\mathbb{K}$, the following formulas are valid:

(a) $\langle x, \lambda y + \mu z \rangle = \bar{\lambda} \langle x, y \rangle + \bar{\mu} \langle x, z \rangle$ for each $x, y, z \in X$ and each $\lambda, \mu \in \mathbb{K}$, i.e. $\langle \cdot , \cdot \rangle$ is conjugate-linear (also called antilinear) in its second argument, even linear for $\mathbb{K} = \mathbb{R}$. Together with Def. 1.62(ii), this means that $\langle \cdot , \cdot \rangle$ is a sesquilinear form, even a bilinear form for $\mathbb{K} = \mathbb{R}$.

(b) $\langle 0, x \rangle = \langle x, 0 \rangle = 0$ for each $x \in X$.

**Proof.** (a): One computes, for each $x, y, z \in X$ and each $\lambda, \mu \in \mathbb{K}$,

$$\langle x, \lambda y + \mu z \rangle = \bar{\lambda} \langle x, y \rangle + \bar{\mu} \langle x, z \rangle$$

(b): One computes, for each $x \in X$,

$$\overline{\langle x, 0 \rangle} = \overline{\langle 0, x \rangle} = \langle 0, x \rangle \overset{\text{Def. 1.62(ii)}}{=} \langle 0, x \rangle = 0$$

thereby completing the proof of the lemma.  

**Theorem 1.64.** The following Cauchy-Schwarz inequality (1.41) holds for each inner product $\langle \cdot , \cdot \rangle$ on a vector space $X$ over $\mathbb{K}$:

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad \text{for each} \quad x, y \in X, \quad (1.41)$$

where

$$\|x\| := \sqrt{\langle x, x \rangle}, \quad \|y\| := \sqrt{\langle y, y \rangle}. \quad (1.42)$$

Moreover, equality in (1.41) holds if, and only if, $x$ and $y$ are linearly dependent, i.e. if, and only if, $y = 0$ or there exists $\lambda \in \mathbb{K}$ such that $x = \lambda y$. 


Proof. If $y = 0$, then it is immediate that both sides of (1.41) vanish. If $x = \lambda y$ with $\lambda \in \mathbb{K}$, then $|\langle x, y \rangle| = |\lambda \langle y, y \rangle| = |\lambda||y|^2 = \sqrt{\lambda \lambda \langle y, y \rangle}||y|| = ||x|||y||$, showing that (1.41) holds with equality. If $x$ and $y$ are not linearly independent, then $y \neq 0$ and $x - \lambda y \neq 0$ for each $\lambda \in \mathbb{K}$, i.e.

$$0 < \langle x - \lambda y, x - \lambda y \rangle = \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle$$

Let $\lambda = \langle x, y \rangle/||y||^2$ (using $y \neq 0$) to get

$$0 < ||x||^2 - 2\langle x, y \rangle \overline{\langle x, y \rangle} + \frac{\langle x, y \rangle \overline{\langle x, y \rangle}}{||y||^2} = \frac{||x||^2||y||^2 - \langle x, y \rangle \overline{\langle x, y \rangle}}{||y||^2}$$

or $\langle x, y \rangle \overline{\langle x, y \rangle} < ||x||^2||y||^2$. Finally, taking the square root on both sides shows that (1.41) holds with strict inequality. ■

**Proposition 1.65.** If $X$ is a vector space over $\mathbb{K}$ with an inner product $\langle \cdot, \cdot \rangle$, then the map

$$|| \cdot || : X \rightarrow \mathbb{R}^+_{\geq}, \quad ||x|| := \sqrt{\langle x, x \rangle},$$

defines a norm on $X$. One calls this the norm induced by the inner product.

**Proof.** If $x = 0$, then $\langle x, x \rangle = 0$ and $||x|| = 0$ as well. Conversely, if $x \neq 0$, then $\langle x, x \rangle > 0$ and $||x|| > 0$ as well, showing that $|| \cdot ||$ is positive definite. For $\lambda \in \mathbb{K}$ and $x \in X$, one has $||\lambda x|| = \sqrt{\lambda \lambda \langle x, x \rangle} = \sqrt{||x||^2} = ||x||$, showing that $|| \cdot ||$ is homogeneous of degree 1. Finally, if $x, y \in X$, then

$$||x + y||^2 = \langle x + y, x + y \rangle = ||x||^2 + \langle x, y \rangle + \langle y, x \rangle + ||y||^2$$

$$\leq ||x||^2 + 2||x|| ||y|| + ||y||^2 = (||x|| + ||y||)^2,$$

establishing that $|| \cdot ||$ satisfies the triangle inequality. In conclusion, we have shown that $|| \cdot ||$ constitutes a norm on $X$. ■

**Definition 1.66.** Let $X$ be a vector space over $\mathbb{K}$. If $\langle \cdot, \cdot \rangle$ is an inner product on $X$, then $(X, \langle \cdot, \cdot \rangle)$ is called an inner product space or a pre-Hilbert space. An inner product space is called a Hilbert space if, and only if, $(X, || \cdot ||)$ is a Banach space, where $|| \cdot ||$ is the induced norm, i.e. $||x|| := \sqrt{\langle x, x \rangle}$. Frequently, the inner product on $X$ is understood and $X$ itself is referred to as an inner product space or Hilbert space.

**Example 1.67.** On the space $\mathbb{K}^n$, $n \in \mathbb{N}$, we define an inner product by letting, for each $z = (z_1, \ldots, z_n) \in \mathbb{K}^n$, $w = (w_1, \ldots, w_n) \in \mathbb{K}^n$:

$$z \cdot w := \sum_{j=1}^n z_j \bar{w}_j$$

(called the Euclidean inner product for $\mathbb{K} = \mathbb{R}$). Let us verify that (1.46), indeed, defines an inner product in the sense of Def. 1.62: If $z \neq 0$, then there is $j_0 \in \{1, \ldots, n\}$ such
that \( z_{j_0} \neq 0 \). Thus, \( z \cdot z = \sum_{j=1}^{n} |z_j|^2 \geq |z_{j_0}|^2 > 0 \), i.e. Def. 1.62(i) is satisfied. Next, let \( z, w, u \in \mathbb{K}^n \) and \( \lambda, \mu \in \mathbb{K} \). One computes
\[
(\lambda z + \mu w) \cdot u = \sum_{j=1}^{n} (\lambda z_j + \mu w_j) \bar{u}_j = \sum_{j=1}^{n} \lambda z_j \bar{u}_j + \sum_{j=1}^{n} \mu w_j \bar{u}_j = \lambda (z \cdot u) + \mu (w \cdot u),
\]
i.e. Def. 1.62(ii) is satisfied. For Def. 1.62(iii), merely note that
\[
z \cdot w = \sum_{j=1}^{n} z_j w_j = \sum_{j=1}^{n} \bar{w}_j z_j = \bar{w} \cdot \bar{z}.
\]
Hence, we have shown that (1.46) defines an inner product according to Def. 1.62. Since the 2-norm of Def. 1.11 is the same as the norm induced by the inner product of (1.46), this also proves that the 2-norm satisfies the Cauchy-Schwarz inequality (1.41). Due to Prop. 1.59(b), the 2-norm is complete, i.e. \( \mathbb{K}^n \) with the inner product of (1.46) is a Hilbert space.

**Definition 1.68.** If \((X, \langle \cdot, \cdot \rangle)\) is an inner product space, then \(x, y \in X\) are called orthogonal or perpendicular (denoted \( x \perp y \)) if, and only if, \( \langle x, y \rangle = 0 \). A unit vector is \( x \in X \) such that \( \|x\| = 1 \), where \( \| \cdot \| \) is the induced norm. An orthogonal system is a family \((x_i)_{i \in I}, x_i \in X, I\) being some index set, such that \( \langle x_i, x_j \rangle = 0 \) for each \( i, j \in I \) with \( i \neq j \). An orthogonal system is called an orthonormal system if, and only if, it consists entirely of unit vectors.

**Remark 1.69.** If \((X, \langle \cdot, \cdot \rangle)\) is an inner product space, then one has Pythagoras’ theorem, namely that for each \( x, y \in X \) with \( x \perp y \):
\[
\|x + y\|^2 = \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 = \|x\|^2 + \|y\|^2.
\]

**1.4.4 Equivalence of Metrics and Equivalence of Norms**

The \( p \)-norms of Def. 1.11 provide an uncountable number of different norms on \( \mathbb{K}^n \). It is an important result that they all generate the same open sets (i.e. the same topology – the so-called norm topology) on \( \mathbb{K}^n \) – one says that all norms on \( \mathbb{K}^n \) are equivalent. Before we will prove this result in Th. 1.72 below, we introduce the notion of equivalence for metrics and norms. We will also see that, even though all norms on \( \mathbb{K}^n \) are equivalent, norms on other normed vector spaces are not necessarily equivalent (see Example 1.75 below).

**Definition 1.70.** (a) Let \( d_1 \) and \( d_2 \) be metrics on a set \( X \). Then \( d_1 \) and \( d_2 \) are said to be equivalent if, and only if, the topologies \( T_1 \) and \( T_2 \) on \( X \), induced by \( d_1 \) and \( d_2 \), respectively, are identical, i.e. if, and only if, for each \( A \subseteq X \), the following holds:
\[
A \text{ is } d_1\text{-open} \iff A \text{ is } d_2\text{-open}.
\]
(b) Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be norms on a vector space $X$ over $\mathbb{K}$. Then $\| \cdot \|_1$ and $\| \cdot \|_2$ are said to be equivalent if, and only if, there exist positive constants $\alpha, \beta \in \mathbb{R}^+$ such that

$$\alpha \| x \|_1 \leq \| x \|_2 \leq \beta \| x \|_1 \quad \text{for each } x \in X. \quad (1.49)$$

**Proposition 1.71.** Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be norms on a vector space $X$ over $\mathbb{K}$, and let $d_1$ and $d_2$ be the respective induced metrics on $X$. Then $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent norms if, and only if, $d_1$ and $d_2$ are equivalent metrics.

**Proof.** If $X = \{0\}$, then there is nothing to show. Thus, assume that there exists some $v \in X \setminus \{0\}$. Let $\mathcal{T}_1, \mathcal{T}_2$ denote the topologies induced by $d_1, d_2$, respectively. First, assume (1.49) holds, i.e. the norms are equivalent. If $A \in \mathcal{T}_1$ and $x \in A$, then there exists $\epsilon > 0$ such that $B_{\epsilon,d_1}(x) \subseteq A$. Thus, for each $y \in B_{\delta,d_2}(x)$ satisfying $\delta := \alpha \epsilon$, one obtains

$$d_1(x, y) \leq \frac{1}{\alpha} \| x - y \|_2 < \frac{\delta}{\alpha} = \epsilon,$$

showing $B_{\delta,d_2}(x) \subseteq B_{\epsilon,d_1}(x) \subseteq A$ and that $A \in \mathcal{T}_2$. Now assume $A \in \mathcal{T}_2$. If $x \in A$, then there exists $\epsilon > 0$ such that $B_{\epsilon,d_2}(x) \subseteq A$. Then, for each $y \in B_{\delta,d_1}(x)$ with $\delta := \epsilon/\beta$, it holds that

$$d_2(x, y) \leq \beta \| x - y \|_1 < \beta \delta = \epsilon,$$

showing $B_{\delta,d_1}(x) \subseteq B_{\epsilon,d_2}(x) \subseteq A$. Hence, $A \in \mathcal{T}_1$. So far, we have proved that the validity of (1.49) implies $\mathcal{T}_1 = \mathcal{T}_2$ (i.e. $d_1$ and $d_2$ are equivalent).

Conversely, assume that the induced metrics $d_1$ and $d_2$ are equivalent. According to Def. 1.70(a), $0 \in X$ has to be a $d_1$-interior point of both the open $d_1$-ball $B_{d_1}(0)$ and the open $d_2$-ball $B_{d_2}(0)$. Moreover, $0$ also has to be a $d_2$-interior point of both open balls. We claim that the set $M := \{ \| x \|_2 : \| x \|_1 = 1 \} \subseteq \mathbb{R}^+_0$ is bounded. Proceeding by contraposition, assume that $M$ is unbounded (from above, as it is always bounded from below by $0$). Then there exists a sequence $(x^k)_{k \in \mathbb{N}}$ in $X$ such that $\| x^k \|_1 = 1$ for each $k \in \mathbb{N}$ and $\lim_{k \to \infty} \| x^k \|_2 = \infty$. Define $\eta^k := \| x^k \|_2$ and $y^k := x^k/\eta^k$ (note $\eta^k \neq 0$, since $\| x^k \|_1 = 1$). Then $\| y^k \|_2 = 1$ for each $k \in \mathbb{N}$. Moreover, $\| y^k \|_1 = 1/\eta^k$, showing $\lim_{k \to \infty} d_1(0, y^k) = \lim_{k \to \infty} \| y^k \|_1 = 0$. Thus, for each $\epsilon > 0$, $B_{d_1}(0)$ contains elements $y^k$ with $\| y^k \|_2 = 1$, i.e. $0$ is not a $d_1$-interior point of $B_{1,d_2}(0)$. Thus, if $0$ is a $d_1$-interior point of $B_{d_2}(0)$, then $M$ must be bounded. Letting

$$\beta := \sup \{ \| x \|_2 : \| x \|_1 = 1 \} \in \mathbb{R}^+ \quad \text{(indeed, } \beta > 0, \text{ as } \| x \|_1 = 1 \text{ implies } x \neq 0 \text{ and } \| x \|_2 > 0),$$

one has

$$\forall x \in X \setminus \{0\} \quad \| x \|_2 = \left\| x \left\|_1 \left\| x \|_1 \right\|_1 \right\|_2 \leq \beta \| x \|_1.$$

We have therefore found a constant $\beta > 0$ such that the corresponding part of (1.49) is satisfied. One can now proceed completely analogously to show that the hypothesis of $0$ being a $d_2$-interior point of $B_{1,d_1}(0)$ implies that the set $\{ \| x \|_1 : \| x \|_2 = 1 \}$ is bounded and

$$\gamma := \sup \{ \| x \|_1 : \| x \|_2 = 1 \} \in \mathbb{R}^+ \quad \text{and} \quad \delta := \sup \{ \| x \|_2 : \| x \|_1 = 1 \} \in \mathbb{R}^+.$$
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satisfies \( \|x\|_1 \leq \gamma \|x\|_2 \) for each \( x \in X \). Finally, letting \( \alpha := \gamma^{-1} \) completes the proof of the equivalence of \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \).

**Theorem 1.72.** All norms on \( \mathbb{K}^n \), \( n \in \mathbb{N} \), are equivalent.

**Proof.** It suffices to show that every norm on \( \mathbb{K}^n \) is equivalent to the 2-norm on \( \mathbb{K}^n \). So let \( \| \cdot \|_2 \) denote the 2-norm on \( \mathbb{K}^n \) and let \( \| \cdot \| \) denote an arbitrary norm on \( \mathbb{K}^n \). Clearly, every \( z \in \mathbb{K}^n \) can be written as \( z = \sum_{j=1}^n z_je_j \), where

\[
e_1 := (1, 0, \ldots, 0), \quad e_2 := (0, 1, \ldots, 0), \quad \ldots, \quad e_n := (0, \ldots, 0, 1)
\]

are the standard unit vectors of \( \mathbb{K}^n \). Moreover, the 2-norm satisfies the Cauchy-Schwarz inequality (1.41) (cf. Ex. 1.67), which can be exploited to get

\[
\|z\| = \left\| \sum_{j=1}^n z_je_j \right\| \leq \sum_{j=1}^n |z_j| \|e_j\| = (|z_1|, \ldots, |z_n|) \cdot (\|e_1\|, \ldots, \|e_n\|) \leq \|z\|_2 \|\|e_1\|, \ldots, \|e_n\|\|_2,
\]

that means, with \( \beta := \sqrt{\sum_{j=1}^n \|e_j\|^2} > 0 \),

\[
\|z\| \leq \beta \|z\|_2 \quad \text{for each } z \in \mathbb{K}^n.
\]

We claim that there is also \( \alpha > 0 \) such that

\[
\alpha \|z\|_2 \leq \|z\| \quad \text{for each } z \in \mathbb{K}^n.
\]

Seeking a contradiction, assume that there is no \( \alpha > 0 \) satisfying (1.52). Then there is a sequence \( (z^k)_{k \in \mathbb{N}} \) in \( \mathbb{K}^n \) such that, for each \( k \in \mathbb{N} \), \( \frac{1}{k} \|z^k\|_2 > \|z^k\| \). Letting \( w^k := z^k/\|z^k\|_2 \), one gets \( \frac{1}{k} \|w^k\|_2 > \|w^k\| \) and \( \|w^k\|_2 = 1 \) for each \( k \in \mathbb{N} \). The Bolzano-Weierstrass Th. 1.31 yields a subsequence \( (w^k)_{k \in \mathbb{N}} \) of \( (w^k)_{k \in \mathbb{N}} \) that converges with respect to \( \| \cdot \|_2 \) to some \( u \in \mathbb{K}^n \). We use the inverse triangle inequality to obtain

\[
\left| \|u\|_2 - 1 \right| = \left| \|u\|_2 - \|u^k\|_2 \right| \leq \|u - u^k\|_2 \rightarrow 0 \quad \text{for } k \rightarrow \infty,
\]

implying \( \|u\|_2 = 1 \), and, in particular, \( u \neq 0 \). On the other hand, using (1.51), one has

\[
\left| \|u\| - \|u^k\| \right| \leq \left| \|u - u^k\| \right| \leq \beta \|u^k - u\|_2 \rightarrow 0 \quad \text{for } k \rightarrow \infty,
\]

implying \( \|u\| = \lim_{k \rightarrow \infty} \|u^k\| \leq \lim_{k \rightarrow \infty} \frac{1}{k} \|u^k\|_2 = \lim_{k \rightarrow \infty} \frac{1}{k} = 0 \), i.e. \( u = 0 \) in a contradiction to \( u \neq 0 \). Thus, the assumption that there is no \( \alpha > 0 \) satisfying (1.52) must have been wrong, i.e. (1.52) must hold for some \( \alpha > 0 \). The proof is concluded by the observation that (1.51) together with (1.52) is precisely the statement that \( \| \cdot \|_2 \) and \( \| \cdot \| \) are equivalent.

**Definition 1.73.** According to Th. 1.72, there exists a unique topology on \( \mathbb{K}^n \), \( n \in \mathbb{N} \), that is induced by the norms on \( \mathbb{K}^n \). We call this topology the norm topology on \( \mathbb{K}^n \).
Caveat 1.74. Even though it follows from Th. 1.72 and Prop. 1.71 that all metrics on \( \mathbb{K}^n \) induced by norms on \( \mathbb{K}^n \) are equivalent, there exist nonequivalent metrics on \( \mathbb{K}^n \) (examples?).

The following Ex. 1.75 shows that, in general, there can be norms on a real vector space \( X \) that are not equivalent.

Example 1.75. As in Ex. 1.60 before, let \( X \) be the vector space over \( \mathbb{K} \), consisting of the sequences in \( \mathbb{K} \) that are finally constant and equal to zero. Then

\[
\|(z_n)_{n\in\mathbb{N}}\|_1 := \sum_{n=1}^{\infty} |z_n| \quad \text{and} \\
\|(z_n)_{n\in\mathbb{N}}\|_{\text{sup}} := \max \{|z_n| : n \in \mathbb{N}\}
\]

(1.53a) (1.53b)

define norms on \( X \) (\( \|\cdot\|_{\text{sup}} \) is the same norm that was considered in the earlier example). Clearly, the sequence \((z^k)_{k\in\mathbb{N}}\) in \( X \) defined by

\[
z^k_n := \begin{cases} 1/k & \text{for } 1 \leq n \leq k, \\ 0 & \text{for } n > k,
\end{cases}
\]

(1.54)

converges to \((0,0,\ldots) \in X\) with respect to \( \|\cdot\|_{\text{sup}} \); however, the sequence does not converge in \( X \) with respect to \( \|\cdot\|_1 \) (exercise), proving \( \|\cdot\|_1 \) and \( \|\cdot\|_{\text{sup}} \) are not equivalent.

We remarked before that boundedness, which is a useful concept in metric spaces, is not a topological concept. To emphasize this further, we show in the following Prop. 1.76 that every metric is equivalent to a bounded metric.

Proposition 1.76. Let \((X,d)\) be a metric space. Then

\[d_1 : X \times X \longrightarrow \mathbb{R}_0^+, \quad d_1(x,y) := \min\{1, d(x,y)\}\]

(1.55)

defines a metric on \( X \) that is equivalent to \( d \) (in particular, every metric is equivalent to a bounded metric).

Proof. We verify that \( d_1 \) is a metric. Let \( x, y, z \in X \). We have

\[d_1(x,y) = 0 \iff d(x,y) = 0 \iff x = y,
\]

showing \( d_1 \) to be positive definite. The symmetry of \( d \) immediately implies the symmetry of \( d_1 \). For the triangle inequality, we estimate

\[d_1(x,z) = \min\{1, d(x,z)\} \leq L := \min\{1, d(x,y) + d(y,z)\}
\]

\[\leq d_1(x,y) + d_1(y,z) = \min\{1, d(x,y)\} + \min\{1, d(y,z)\} =: R,
\]

(1.56)
where we still need to prove the inequality at $(\ast)$. If $d(x, y) \leq 1$ and $d(y, z) \leq 1$, then $R = d(x, y) + d(y, z)$ and $L \leq R$ is clear. If $d(x, y) > 1$ or $d(y, z) > 1$, then $L = 1 \leq R$, finishing the proof of $(\ast)$. Thus, $d_1$ is a metric, and it remains to show that $d$ and $d_1$ are equivalent. Let $O \subseteq X$. If $O$ is $d$-open and $x \in O$, then there is $r > 0$ such that $B_r(x) \subseteq O$. Let $\epsilon := \min\{r, 1\}$. Then $B_{\epsilon,d}(x) = B_{\epsilon,d}(x) \subseteq B_{r,d}(x) \subseteq O$, showing $O$ to be $d_1$-open. Similarly, if $O$ is $d_1$-open and $x \in O$, then there is $1 \geq r > 0$ such that $B_{r,d_1}(x) \subseteq O$. Since $r \leq 1$, $B_{r,d}(x) = B_{r,d_1}(x) \subseteq O$, showing $O$ to be $d$-open. 

\section{Limits and Continuity of Functions}

\subsection{Definitions and Properties}

In the following Definitions 2.1(a) and 2.3, we will generalize the notion of continuity [Phi16, Def. 7.31] to topological spaces, the notions of uniform continuity [Phi16, (10.39)] and Lipschitz continuity [Phi16, Def. and Rem. 10.17] to metric spaces. Even though it is less obvious, we will also see that the limit definition of Def. 2.1(b) is a generalization of [Phi16, Def. 8.17].

\textbf{Definition 2.1.} Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, $M \subseteq X$, $f : M \rightarrow Y$.

\begin{enumerate}[(a)]
    \item $f$ is called \textit{continuous} in $\xi \in M$ if, and only if,
    \begin{equation}
        \forall \ U \in \mathcal{U}(f(\xi)) \exists \ V \in \mathcal{U}(\xi) \quad f(V \cap M) \subseteq U. \tag{2.1}
    \end{equation}
    \item If $\xi \in X$ is a cluster point of $M$, then $f$ is said to tend to $\eta \in Y$ (or to have the limit $\eta \in Y$) for $x \rightarrow \xi$ (denoted by $\lim_{x \rightarrow \xi} f(x) = \eta$) if, and only if,
    \begin{equation}
        \forall \ U \in \mathcal{U}(\eta) \exists \ V \in \mathcal{U}(\xi) \quad f((V \cap M) \setminus \{\xi\}) \subseteq U. \tag{2.2}
    \end{equation}
\end{enumerate}

\textbf{Remark 2.2. (a)} The reason that $\xi$ is removed from $V$ in (2.2) is that one wants to allow the situation $f(\xi) \neq \lim_{x \rightarrow \xi} f(x)$, i.e. the value of a function in $\xi$ is allowed to differ from the functions limit for $x \rightarrow \xi$. Thus, for a cluster point $\xi$ of $M$ with $\xi \in M$, one of three distinct cases will always occur: (i) $\lim_{x \rightarrow \xi} f(x)$ does not exist, (ii) $f(\xi) \neq \lim_{x \rightarrow \xi} f(x)$, (iii) $f(\xi) = \lim_{x \rightarrow \xi} f(x)$.

\textbf{(b)} If $\xi \in M$ is a cluster point of $M$, then it is an immediate consequence of (2.1) and (2.2) that $f$ is continuous in $\xi$ if, and only if, $f(\xi) = \lim_{x \rightarrow \xi} f(x)$.

\textbf{(c)} According to Def. 2.1(a), $f : M \rightarrow Y$ is continuous in $\xi \in M$ with respect to the topology $\mathcal{T}_X$ on $X$ if, and only if, $f$ is continuous in $\xi$ with respect to the topology $\mathcal{T}_M$ on $M$, where $\mathcal{T}_M$ is the relative topology on $M$ induced by $\mathcal{T}_X$ (and analogously for Def. 2.1(b)).
Recall that, if the topology $\mathcal{T}$ is induced by the metric $d$, then $U \subseteq \mathcal{U}(x)$ if, and only if, there exist $\varepsilon > 0$ with $x \in B_\varepsilon(x) \subseteq U$. Thus, if $\mathcal{T}_X$ and $\mathcal{T}_Y$ are induced by metrics $d_X$ on $X$ and $d_Y$ on $Y$, respectively, then (2.1) is equivalent to

$$\forall \varepsilon \in \mathbb{R}^+ \quad \exists \delta \in \mathbb{R}^+ \quad f(B_\delta(\xi) \cap M) \subseteq B_\varepsilon(f(\xi)), \quad (2.3a)$$

and (2.2) is equivalent to

$$\forall \varepsilon \in \mathbb{R}^+ \quad \exists \delta \in \mathbb{R}^+ \quad f((B_\delta(\xi) \cap M) \setminus \{\xi\}) \subseteq B_\varepsilon(\eta). \quad (2.3b)$$

**Definition 2.3.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, $M \subseteq X$, $f : M \rightarrow Y$.

(a) $f$ is called continuous in $M$ if, and only if, $f$ is continuous in each $\xi \in M$. The set of all continuous functions from $M$ into $Y$ is denoted by $C(M, Y)$.

(b) Suppose $\mathcal{T}_X$ and $\mathcal{T}_Y$ are induced by metrics $d_X$ on $X$ and $d_Y$ on $Y$, respectively. Then $f$ is called uniformly continuous in $M$ if, and only if, for each $\epsilon > 0$, there is $\delta > 0$ such that:

$$\forall_{x,y \in M} \quad d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon \quad (2.4)$$

(note that, here, $\delta$ must not depend on $x$ and $y$). Moreover, $f$ is called Lipschitz continuous in $M$ with Lipschitz constant $L \in \mathbb{R}_0^+$ if, and only if,

$$\forall_{x,y \in M} \quad d_Y(f(x), f(y)) \leq L \, d_X(x, y). \quad (2.5)$$

Of course, if continuity is used in a metric space, it is meant with respect to the induced topology; if uniform continuity or Lipschitz continuity are used in a normed space, then they are meant with respect to the induced metric.

**Lemma 2.4.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, $M \subseteq X$, $f : M \rightarrow Y$. If $f$ is Lipschitz continuous in $M$, then $f$ is uniformly continuous in $M$. If $f$ is uniformly continuous in $M$, then $f$ is continuous in $M$.

**Proof.** If $f$ is Lipschitz continuous, then there is $L \in \mathbb{R}_0^+$ such that $d_Y(f(x), f(y)) \leq L \, d_X(x, y)$ for each $x, y \in M$. Thus, given $\epsilon > 0$, choose $\delta := \epsilon$ for $L = 0$ and $\delta := \epsilon/L$ for $L > 0$. Let $x, y \in M$ such that $d_X(x, y) < \delta$. If $L = 0$, then $d_Y(f(x), f(y)) = 0 < \epsilon$. If $L > 0$, then $d_Y(f(x), f(y)) \leq L \, d_X(x, y) < L\epsilon/L = \epsilon$, showing that $f$ is uniformly continuous. If $f$ is uniformly continuous, then, given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that, for each $x, y \in M$ with $d_X(x, y) < \delta(\epsilon)$, it is $d_Y(f(x), f(y)) < \epsilon$. Fix $x \in M$. Since $f(B_{\delta(\epsilon)}(x) \cap M) \subseteq B_\varepsilon(f(x))$, $f$ is continuous in $x$. As $x$ was arbitrary, $f$ is continuous. 

**Example 2.5.** Consider $X = \mathbb{R}$ with the usual metric given by the absolute value function, $M := \mathbb{R}^+$. 

(a) \(f : M \to \mathbb{R}, f(x) := 1/x\), is continuous, but not uniformly continuous: For each \(\xi \in \mathbb{R}^+\) and each \(\delta > 0\), one has

\[
f(\xi) - f(\xi + \delta) = \frac{1}{\xi} - \frac{1}{\xi + \delta} = \frac{\delta}{\xi(\xi + \delta)}.
\]

Thus, for a fixed \(\delta > 0\) and \(\epsilon > 0\), one has, for each \(\xi \in \mathbb{R}^+\) that is chosen smaller than \(\delta/2\) and also smaller than \(1/(2\epsilon)\),

\[
f(\xi) - f(\xi + \delta/2) = \frac{\delta}{2\xi(\xi + \delta/2)} > \frac{1}{2\xi} > \epsilon,
\]

i.e. \(x := \xi\) and \(y := \xi + \delta/2\) are points such that \(|x - y| = \delta/2 < \delta\), but

\[
|f(x) - f(y)| = \frac{\delta}{2\xi(\xi + \delta/2)} > \epsilon,
\]

showing \(f\) is not uniformly continuous.

(b) \(g : M \to \mathbb{R}, g(x) := x^2\), is continuous, but not uniformly continuous: For each \(\xi \in \mathbb{R}^+\) and each \(\delta > 0\), one has

\[
g(\xi + \delta) - g(\xi) = (\xi + \delta)^2 - \xi^2 = 2\xi\delta + \delta^2.
\]

Thus, for a fixed \(\delta > 0\) and \(\epsilon > 0\), one has, for each \(\xi \in \mathbb{R}^+\) that is chosen bigger than \(\epsilon/\delta\),

\[
g(\xi + \delta/2) - g(\xi) = \xi\delta + \delta^2/4 > \xi\delta > \epsilon,
\]

i.e. \(x := \xi\) and \(y := \xi + \delta/2\) are points such that \(|x - y| = \delta/2 < \delta\), but

\[
|g(x) - g(y)| = \xi\delta + \frac{\delta^2}{4} > \epsilon,
\]

showing \(f\) is not uniformly continuous.

(c) \(h : M \to \mathbb{R}, h(x) := \sqrt{x}\), is uniformly continuous, but not Lipschitz continuous: To show that \(h\) is uniformly continuous is left as an exercise. If \(h\) were Lipschitz continuous, then there needed to be \(L \geq 0\) such that

\[
\sqrt{\xi + \delta} - \sqrt{\xi} \leq L\delta
\]

for each \(\xi \in \mathbb{R}^+, \delta > 0\). However, since

\[
\frac{\sqrt{\xi + \delta} - \sqrt{\xi}}{\delta} = \frac{\delta}{\delta(\sqrt{\xi + \delta} + \sqrt{\xi})} = \frac{1}{\sqrt{\xi + \delta} + \sqrt{\xi}},
\]

by choosing \(\xi\) and \(\delta\) sufficiently small, one can always make the expression in (2.7) larger than any given \(L\), showing that \(h\) is not Lipschitz continuous.
Example 2.6. (a) According to Lem. 1.57(b), the norm $\| \cdot \|$ on a normed vector space $X$ satisfies the inverse triangle inequality
\[
\|x\| - \|y\| \leq \|x - y\| \text{ for each } x, y \in X,
\]
i.e. the norm is Lipschitz continuous with Lipschitz constant 1.

(b) Let $(X, d)$ be a metric space. According to Lem. 1.57(a), we have
\[
\forall x, x', y, y' \in X \quad |d(x, y) - d(x', y')| \leq d(x, x') + d(y, y').
\]
In particular, this yields the Lipschitz continuity of $d : X^2 \to \mathbb{R}_0^+$ (with Lipschitz constant 1) with respect to the metric $d_1$ on $X^2$ defined by
\[
d_1 : X^2 \times X^2 \to \mathbb{R}_0^+, \quad d_1((x, y), (x', y')) = d(x, x') + d(y, y'). \tag{2.8}
\]
Further consequences are the continuity and even uniform continuity of $d$, and also the continuity of $d$ in both components. If $X$ is nonempty, then, for each $\emptyset \neq A, B \subseteq X$, we define the distance between $A$ and $B$ by
\[
dist(A, B) := \inf \{d(a, b) : a \in A, b \in B\} \in [0, \infty[,
\tag{2.9}
\]
and
\[
\forall x \in X \quad \text{dist}(x, B) := \text{dist}(\{x\}, B), \quad \text{dist}(A, x) := \text{dist}(A, \{x\}). \tag{2.10}
\]
It is an exercise to show that, if $A \subseteq X$ and $A \neq \emptyset$, then the functions
\[
\delta, \tilde{\delta} : X \to \mathbb{R}_0^+, \quad \delta(x) := \text{dist}(x, A), \quad \tilde{\delta}(x) := \text{dist}(A, x), \tag{2.11}
\]
are both Lipschitz continuous with Lipschitz constant 1 (in particular, they are both continuous and even uniformly continuous).

Theorem 2.7. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces (for example, metric or normed spaces), $f : X \to Y$. Let $\mathcal{S}_Y$ be a subbase for $\mathcal{T}_Y$. Then the following four statements are equivalent:

(i) $f$ is continuous.

(ii) For each open set $O \subseteq Y$ (i.e. each $O \in \mathcal{T}_Y$), the preimage $f^{-1}(O) = \{x \in X : f(x) \in O\}$ is open in $X$ (i.e. $f^{-1}(O) \in \mathcal{T}_X$).

(iii) For each $S \in \mathcal{S}_Y$, the preimage $f^{-1}(S)$ is open in $X$ (i.e. $f^{-1}(S) \in \mathcal{T}_X$).

(iv) For each closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in $X$ (i.e. $X \setminus f^{-1}(C) \in \mathcal{T}_X$).
Proof. “(i) ⇒ (ii)”: Assume \( f \) is continuous and consider \( O \subseteq Y \) open. Let \( \xi \in f^{-1}(O) \) and \( \eta := f(\xi) \). As \( f \) is continuous in \( \xi \) and \( O \) is a neighborhood of \( \eta \), there is \( O_\xi \in \mathcal{T}_X \) such that \( \xi \in O_\xi \) and \( f(O_\xi) \subseteq O \), i.e. \( O_\xi \subseteq f^{-1}(O) \). Thus, \( f^{-1}(O) = \bigcup_{\xi \in f^{-1}(O)} O_\xi \) is a union of open sets and, hence, open.

“(ii) ⇒ (i)”: Assume that, for each open set \( O \subseteq Y \), \( f^{-1}(O) \) is open in \( X \). Let \( \xi \in X \) and, once again, write \( \eta := f(\xi) \). Consider \( O \in \mathcal{T}_Y \) with \( \eta \in O \). We have to find \( U \in \mathcal{T}_X \) such that \( \xi \in U \) and \( f(U) \subseteq O \). However, since \( f^{-1}(O) \in \mathcal{T}_X \), we simply choose \( U := f^{-1}(O) \), showing the continuity of \( f \) in \( \xi \). As \( \xi \) was arbitrary, \( f \) is continuous.

Proving “(ii) ⇔ (iii)” and “(ii) ⇔ (iv)” is left as an exercise.

As we have seen in [Phi16, Sec. 7.2.2] in the one-dimensional context, it is often more convenient to use sequences rather than neighborhoods in order to check if functions have limits or are continuous. For functions between metric spaces (in particular, between normed spaces), it is possible to generalize [Phi16, (8.31)] and [Phi16, Th. 7.37] to use sequences in that way; in general topological spaces, one has to use nets rather than sequences.

**Theorem 2.8.** Let \( (X, \mathcal{T}_X) \) and \( (Y, \mathcal{T}_Y) \) be topological spaces, \( M \subseteq X \), \( f : M \rightarrow Y \), \( \xi \in M \). Then the following statements (i) and (ii) are equivalent:

(i) \( f \) is continuous in \( \xi \).

(ii) For each net \( (x_i)_{i \in I} \) in \( M \) with \( \lim_{i \in I} x_i = \xi \), the net \( (f(x_i))_{i \in I} \) in \( Y \) converges to \( f(\xi) \), i.e.

\[
\lim_{i \in I} x_i = \xi \quad \Rightarrow \quad \lim_{i \in I} f(x_i) = f(\xi).
\]  

(2.12)

If \( (X, \mathcal{T}_X) \) has a countable local base at \( \xi \) (e.g. if \( (X, \mathcal{T}_X) \) is metrizable), then (i) and (ii) are also equivalent to the following statement:

(iii) For each sequence \( (x_k)_{k \in \mathbb{N}} \) in \( M \) with \( \lim_{k \to \infty} x_k = \xi \), the sequence \( (f(x_k))_{k \in \mathbb{N}} \) in \( Y \) converges to \( f(\xi) \), i.e.

\[
\lim_{k \to \infty} x_k = \xi \quad \Rightarrow \quad \lim_{k \to \infty} f(x_k) = f(\xi).
\]  

(2.13)

Proof. “(i) ⇔ (ii)” (the proof is analogous to the proof of [Phi16, Th. 7.37]): If \( \xi \in M \) is not a cluster point of \( M \), then there is \( U \in \mathcal{U}(\xi) \) such that \( M \cap U = \{\xi\} \) (i.e. \( \xi \) is an isolated point of \( M \)). Then every \( f : M \rightarrow Y \) is continuous in \( \xi \). On the other hand, every net \( (x_i)_{i \in I} \) in \( M \) converging to \( \xi \) must be eventually equal to \( \xi \), in the sense that

\[
\exists_{i_0 \in I} \forall_{i \geq i_0} \quad x_i = \xi.
\]  

(2.14)

Thus, (2.12) is trivially valid for every \( f : M \rightarrow Y \), i.e. the assertion of the theorem holds if \( \xi \in M \) is not a cluster point of \( M \). Now let \( \xi \in M \) be a cluster point of \( M \).
Assume that $f$ is continuous in $\xi$ and $(x_i)_{i \in I}$ is a net in $M$ with $\lim_{i \in I} x_i = \xi$. As $f$ is continuous in $\xi$, (2.1) holds, i.e.
\[
\forall \ U \in \mathcal{U}(f(\xi)) \quad \exists \ V \in \mathcal{U}(\xi) \quad f(V \cap M) \subseteq U.
\]
Since $\lim_{i \in I} x_i = \xi$, there is also $i \in I$ such that, for each $j \geq i$, $x_j \in V \cap M$. Thus, for each $j \geq i$, $f(x_j) \in U$, proving $\lim_{i \in I} f(x_i) = f(\xi)$. Conversely, assume that $f$ is not continuous in $\xi$. We have to construct a net $(x_i)_{i \in I}$ in $M$ with $\lim_{i \in I} x_i = \xi$, but $(f(x_i))_{i \in I}$ does not converge to $f(\xi)$. Since $f$ is not continuous in $\xi$, there must be some $U \in \mathcal{U}(f(\xi))$ such that, for each $V \in \mathcal{U}(\xi)$, there exists at least one $x_V \in M \cap V$ with $f(x_V) \notin U$. As in Ex. 1.20(c) and Ex. 1.22(b), we let $I := \mathcal{U}(\xi)$ with
\[
\forall \ V_1, V_2 \in \mathcal{U}(\xi) \quad V_1 \subseteq V_2 :\iff V_2 \subseteq V_1.
\]
Then, clearly, $\lim_{V \in \mathcal{U}(\xi)} x_V = \xi$, but $(f(x_V))_{V \in \mathcal{U}(\xi)}$ does not converge to $f(\xi)$.

Now let $(X, \mathcal{T}_X)$ have a countable local base at $\xi$. Since every sequence is a net, it is immediate that (ii) implies (iii). If $(X, \mathcal{T}_X)$ has a countable local base at $\xi$, then (iii) implies (i): We have shown above that, if $f$ is not continuous in $\xi$, then there is a net $(x_i)_{i \in I}$ in $M$ with $\lim_{i \in I} x_i = \xi$, but $(f(x_i))_{i \in I}$ does not converge to $f(\xi)$, as there is $U \in \mathcal{U}(f(\xi))$ such that $f(x_i) \notin U$ for each $i \in I$. Now, if $(X, \mathcal{T}_X)$ has a countable local base at $\xi$, then, by Prop. 1.40, $(x_i)_{i \in I}$ contains a sequence $(x_{i_k})_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} x_{i_k} = \xi$. But then $(f(x_{i_k}))_{k \in \mathbb{N}}$ still does not converge to $f(\xi)$.

**Corollary 2.9.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, $M \subseteq X$, $f : M \to Y$, $\eta : Y \to Y$. Let $\xi \in X$ be a cluster point of $M$ and let $\eta \in Y$. Then the following statements (i) and (ii) are equivalent:

(i) $\lim_{x \to \xi} f(x) = \eta$.

(ii) For each net $(x_i)_{i \in I}$ in $M$ with $\lim_{i \in I} x_i = \xi$, the net $(f(x_i))_{i \in I}$ in $Y$ converges to $\eta$, i.e.
\[
\lim_{i \in I} x_i = \xi \Rightarrow \lim_{i \in I} f(x_i) = \eta.
\]

If $(X, \mathcal{T}_X)$ has a countable local base at $\xi$ (e.g. if $(X, \mathcal{T}_X)$ is metrizable), then (i) and (ii) are also equivalent to the following statement:

(iii) For each sequence $(x_k)_{k \in \mathbb{N}}$ in $M$ with $\lim_{k \to \infty} x_k = \xi$, the sequence $(f(x_k))_{k \in \mathbb{N}}$ in $Y$ converges to $\eta$, i.e.
\[
\lim_{k \to \infty} x_k = \xi \Rightarrow \lim_{k \to \infty} f(x_k) = \eta.
\]

**Proof.** According to Rem. 2.2(b), the map
\[
g : M \cup \{\xi\} \to Y, \quad g(x) := \begin{cases} f(x) & \text{for } x \neq \xi, \\ \eta & \text{for } x = \xi, \end{cases}
\]
is continuous in $\xi$ if, and only if (i) holds. Thus, everything is an immediate consequence of Th. 2.8.  \[\blacksquare\]
2 LIMITS AND CONTINUITY OF FUNCTIONS

Caveat 2.10. Consider the situation of Th. 2.8. If \((X, \mathcal{T})\) is not first countable, then it can happen that \(f\) satisfies Th. 2.8(iii), but \(f\) is not continuous in \(\xi\): The following construction is quite general: Let \(A \subseteq X\) and let \(\xi \in \overline{A}\) be such that there is no sequence in \(A\) converging to \(\xi\) (such a \(\xi\) can only exist if \((X, \mathcal{T})\) is not first countable, cf. Cor. 1.44(iv); a concrete example was given in Ex. 1.53(c) and we will come back to that shortly). Now let \(M := A \cup \{\xi\}\) and define

\[
f : M \to \mathbb{R}, \quad f(x) := \begin{cases} 0 & \text{for } x \in A, \\ 1 & \text{for } x = \xi. \end{cases}
\]

Then \(f\) satisfies Th. 2.8(iii) (trivially, since every sequence in \(M\), converging to \(\xi\), must be finally constant with value \(\xi\)). However, there exists a net \((a_i)_{i \in I}\) in \(A\), converging to \(\xi\). Then \((f(a_i))_{i \in I}\) is constant and equal to 0, i.e. \(\lim_{i \in I} f(a_i) = 0 \neq 1 = f(\xi)\), showing that \(f\) is not continuous in \(\xi\). For a concrete example, recall that, in Ex. 1.53(c), we considered \(X := \mathcal{F}(\mathbb{R}, \mathbb{K}) = \mathbb{K}^\mathbb{R}\) with the product topology (i.e. with the topology of pointwise convergence) and the subset

\[
A := \left\{ (x : \mathbb{R} \to \mathbb{K}) : \exists \text{ } J \subseteq \mathbb{R} \text{ finite } x(s) = \begin{cases} 0 & \text{for } s \in J, \\ 1 & \text{for } s \notin J. \end{cases} \right\}.
\]

It was shown in Ex. 1.53(c) that \(\xi := x_0\) with \(x_0(s) = 0\) for each \(s \in \mathbb{R}\) is in \(\overline{A}\), but no sequence in \(A\) converges to \(\xi\).

Theorem 2.11. Let \((X, \mathcal{T}_X), (Y, \mathcal{T}_Y), (Z, \mathcal{T}_Z)\) be topological spaces, \(D_f \subseteq X\), \(f : D_f \to Y\), \(D_g \subseteq Y\), \(g : D_g \to Z\), \(f(D_f) \subseteq D_g\). If \(f\) is continuous in \(\xi \in D_f\) and \(g\) is continuous in \(f(\xi) \in D_g\), then \(g \circ f : D_f \to Z\) is continuous in \(\xi\). In consequence, if \(f\) and \(g\) are both continuous, then the composition \(g \circ f\) is also continuous.

Proof. Let \(\xi \in D_f\) and assume that \(f\) is continuous in \(\xi\) and \(g\) is continuous in \(f(\xi)\). If \((x_i)_{i \in I}\) is a net in \(D_f\) such that \(\lim_{i \in I} x_i = \xi\), then the continuity of \(f\) in \(\xi\) implies that \(\lim_{i \in I} f(x_i) = f(\xi)\). Then the continuity of \(g\) in \(f(\xi)\) implies that \(\lim_{i \in I} g(f(x_i)) = g(f(\xi))\), thereby establishing the continuity of \(g \circ f\) in \(\xi\). \(\blacksquare\)

Example 2.12. (a) Constant functions \(f : X \to Y\) are always continuous, since \(X\) and \(\emptyset\) are the only preimages, and both are always open.

(b) As in Ex. 1.53, consider topological spaces \((X_i, \mathcal{T}_i), i \in I\), and \(X := \prod_{i \in I} X_i\) with the product topology \(\mathcal{T}\) \((X = \mathbb{K}^n, n \in \mathbb{N}, \text{ with the norm topology is, of course, a particularly simple special case})\). Recall the projections

\[
\forall j \in I, \quad \pi_j : X \to X_j, \quad \pi_j((x_i)_{i \in I}) := x_j.
\]

(i) Each projection \(\pi_j\) is continuous, since \(\pi_j^{-1}(O) \in \mathcal{T}\) if \(O \in \mathcal{T}_j\) by the definition of the product topology.

\(^5\)A function that satisfies Th. 2.8(iii) is sometimes called \textit{sequentially continuous} in \(\xi\) – thus, in general, a function can be sequentially continuous without being continuous.
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(ii) Each projection $\pi_j$ is a so-called open map, i.e.

$$\forall O \in T \quad \pi_j(O) \in T_j :$$

(2.15)

Let $x_j \in \pi_j(O)$. Then there is $x = (x_i)_{i \in I} \in O$ such that $\pi_j(x) = x_j$. Since $O \in T$, there is $j \in J \subseteq I$ finite and $O_i \in T_i$ for each $i \in J$ such that $x \in B := \cap_{i \in J} \pi_i^{-1}(O_i) \subseteq O$. Then $x_j = \pi_j(x) \in O_j = \pi_j(B) \subseteq \pi_j(O)$, which already proves $\pi_j(O)$ to be open.

(c) Let $X$ be as in (b). If $Y$ is a set and $f : Y \to X$, then the functions $f_i := \pi_i \circ f$, $i \in I$, are called the coordinate functions of $f$. Then

$$f_i : Y \to X_i, \quad f_i(y) = \pi_i(f(y)), \quad f(y) = (f_i(y))_{i \in I}.$$  

(2.16)

If, as in (b), $T$ is the product topology on $X$, $T_Y$ is a topology on $Y$, and $y \in Y$, then the following statements are equivalent:

(i) $f$ is continuous in $y$.

(ii) For each $i \in I$, the coordinate function $f_i$ is continuous in $y$.

Indeed, if $f$ is continuous in $y$, then each $f_i$, $i \in I$, is continuous in $y$ by (b) and Th. 2.11. For the converse, assume each $f_i$, $i \in I$, to be continuous in $y$. Let $(y_j)_{j \in J}$ be a net in $Y$ such that $\lim_{j \in J} y_j = y$. Let $O \in T_i$, $i \in I$, such that $f(y) \in \pi_i^{-1}(O)$, i.e. such that $f_i(y) \in O$. Then the continuity of $f_i$ in $y$ implies $\lim_{j \in J} f_i(y_j) = f_i(y)$.

Thus,

$$\exists j_0 \in J \quad \forall j \geq j_0 \quad (f_i(y_j) \in O \quad \text{i.e.} \quad f(y_j) \in \pi_i^{-1}(O)),$$

implying $\lim_{j \in J} f(y_j) = f(y)$ by Cor. 1.50(a) and, thus, the continuity of $f$ in $y$.

(d) We are staying in the setting of (c) with the additional assumption that $X_i = \mathbb{C}$ for each $i \in I$. Then $f_i$ is continuous in $y$ if, and only if, both $\Re f_i$ and $\Im f_i$ are continuous in $y$: This is actually merely a corollary of (c), since $\mathbb{C} = \mathbb{R}^2$, $| \cdot |$ on $\mathbb{C}$ is precisely $\| \cdot \|_2$ on $\mathbb{R}^2$, and $\Re f_i$ and $\Im f_i$ are just the coordinate functions of $f_i : Y \to \mathbb{R}^2$.

Remark 2.13. Let $X \neq \emptyset$ be an arbitrary nonempty set, $f, g : X \to \mathbb{K}$, and $\lambda \in \mathbb{K}$. In [Phil16, Not. 6.2], we defined the functions $f + g$, $\lambda f$, $fg$, $f/g$, $|f|$, and, for $\mathbb{K} = \mathbb{R}$, also $\max(f, g)$, $\min(f, g)$, $f^+$, $f^-$. If $Y$ is an arbitrary vector space over $\mathbb{K}$ and $f, g : X \to Y$, then we can generalize the definition of $f + g$ and $\lambda f$ by letting

$$
(f + g) : X \to Y, \quad (f + g)(x) := f(x) + g(x), \quad (2.17a)
$$

$$
(\lambda f) : X \to Y, \quad (\lambda f)(x) := \lambda f(x). \quad (2.17b)
$$

It turns out that this makes the set of functions from $X$ into $Y$, $\mathcal{F}(X, Y)$, into a vector space over $\mathbb{K}$ with zero element $f \equiv 0$ (exercise). Finally, for $f : X \to \mathbb{C}^n$, $f(x) = (f_1(x), \ldots, f_n(x))$, $n \in \mathbb{N}$, we define

$$
\Re f : X \to \mathbb{R}^n, \quad \Re f(x) := (\Re f_1(x), \ldots, \Re f_n(x)), \quad (2.18a)
$$

$$
\Im f : X \to \mathbb{R}^n, \quad \Im f(x) := (\Im f_1(x), \ldots, \Im f_n(x)), \quad (2.18b)
$$

$$
\bar{f} : X \to \mathbb{C}^n, \quad \bar{f}(x) := (\bar{f}_1(x), \ldots, \bar{f}_n(x)), \quad (2.18c)
$$
such that

\[ f = \text{Re } f + i \text{ Im } f, \quad f = \text{Re } f - i \text{ Im } f. \]  

(2.19a)

\[ \overset{\text{(2.19b)}}{\text{(2.19b)}} \]

**Lemma 2.14.** Let \((X, \| \cdot \|)\) be a normed vector space, and let \((x^k)_{k \in \mathbb{N}}\) and \((y^k)_{k \in \mathbb{N}}\) be sequences in \(X\) with \(\lim_{k \to \infty} x^k = x \in X\) and \(\lim_{k \to \infty} y^k = y \in X\). Then the following holds:

\[
\begin{align*}
\lim_{k \to \infty} (x^k + y^k) &= x + y, \quad \text{for each } \lambda \in \mathbb{K}, \\
\lim_{k \to \infty} (\lambda x^k) &= \lambda x
\end{align*}
\]

(2.20a)

(2.20b)

**Proof.** Since \(\lim_{k \to \infty} \|x^k - x\| = 0\) and \(\lim_{k \to \infty} \|y^k - y\| = 0\), it follows from \(\|x^k + y^k - x - y\| \leq \|x^k - x\| + \|y^k - y\|\) that also \(\lim_{k \to \infty} \|x^k + y^k - x - y\| = 0\). For each \(\lambda \in \mathbb{K}\), one has \(\lim_{k \to \infty} \|\lambda x^k - \lambda x\| = \lim_{k \to \infty} (|\lambda| \|x^k - x\|) = |\lambda| \lim_{k \to \infty} \|x^k - x\| = 0\) \(\blacksquare\)

**Theorem 2.15.** Let \(X\) be a metric space (e.g. a normed space), let \(Y\) be a normed vector space, and assume that \(f, g : X \to Y\) are continuous in \(\xi \in X\). Then \(f + g\) and \(\lambda f\) are continuous in \(\xi\) for each \(\lambda \in \mathbb{K}\) (in particular, \(C(X, Y)\) constitutes a subspace of the vector space \(\mathcal{F}(X, Y)\) over \(\mathbb{K}\)). Moreover, if \(Y = \mathbb{C}^n\), then \(\text{Re } f, \text{ Im } f, \text{ f}^+\), and \(\text{f}^-\) are all continuous in \(\xi\); if \(Y = \mathbb{R}\), then \(\max(f, g), \min(f, g), f^+, \text{ and } f^-\) are all continuous in \(\xi\) as well.

**Proof.** Let \((x^k)_{k \in \mathbb{N}}\) be a sequence in \(X\) such that \(\lim_{k \to \infty} x^k = \xi\). Then the continuity of \(f\) and \(g\) in \(\xi\) yields \(\lim_{k \to \infty} f(x^k) = f(\xi)\) and \(\lim_{k \to \infty} g(x^k) = g(\xi)\). Lemma 2.14 then yields \(\lim_{k \to \infty} (f + g)(x^k) = (f + g)(\xi)\) and \(\lim_{k \to \infty} (\lambda f)(x^k) = (\lambda f)(\xi)\). For \(Y = \mathbb{C}^n\), \(n \in \mathbb{N}\), (1.18) together with [Phi16, (7.2)] and [Phi16, (7.11)] shows \(\lim_{k \to \infty} \text{Re } f(x^k) = \text{Re } f(\xi)\), \(\lim_{k \to \infty} \text{Im } f(x^k) = \text{Im } f(\xi)\), and \(\lim_{k \to \infty} f(x^k) = f(\xi)\), providing the continuity of \(\text{Re } f\), \(\text{Im } f\), and \(f\) at \(\xi\). For \(Y = \mathbb{K}\), the rules for the limits of sequences in \(\mathbb{K}\) [Phi16, Th. 7.13(a)] yield \(\lim_{k \to \infty} (f/g)(x^k) = (f/g)(\xi)\), \(\lim_{k \to \infty} (f/g)(x^k) = (f/g)(\xi)\) for \(g \neq 0\), and \(\lim_{k \to \infty} |f(x^k)| = |f(\xi)|\). This provides the continuity of \(f + g\), \(\lambda f\), \(fg\), \(f/g\), and \(|f|\) at \(\xi\). Moreover, for \(Y = \mathbb{R}\), [Phi16, Th. 7.13(b)] implies \(\lim_{k \to \infty} \max(f, g)(x^k) = \max(f, g)(\xi)\) and \(\lim_{k \to \infty} \min(f, g)(x^k) = \min(f, g)(\xi)\), proving the continuity of \(\max(f, g), \min(f, g), f^+, \text{ and } f^-\) at \(\xi\). \(\blacksquare\)

**Example 2.16.** Each \(\mathbb{K}\)-linear function \(A : \mathbb{K}^n \to \mathbb{K}^m\), \((n, m) \in \mathbb{N}^2\), is continuous: Using the standard unit vectors \(e_j\), for each \(z \in \mathbb{K}^n\), one has \(A(z) = A(\sum_{j=1}^n z_j e_j) = \sum_{j=1}^n z_j A(e_j)\). Thus, one can build \(A\) by summing the functions \(A_j : \mathbb{K}^n \to \mathbb{K}^m\), \(A_j(z) := z_j A(e_j)\) for each \(j \in \{1, \ldots, n\}\). Since \(\lim_{k \to \infty} z^k = z\) implies \(\lim_{k \to \infty} z_j^k = z_j\), which implies \(\lim_{k \to \infty} z_j^k A(e_j) = z_j A(e_j)\), all \(A_j\) are continuous, and, thus \(A\) is continuous by Th. 2.15.

**Example 2.17.** The function \(f : \mathbb{R}^+ \times \mathbb{C} \to \mathbb{C}\), \(f(x, z) := x^z = \exp(z \ln x)\), is continuous: With the projections \(\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}\), we can write \(f = \exp \circ (\pi_2(\ln \circ \pi_1))\) (note \(\pi_1\) is \(\mathbb{R}^+\)-valued on \(\mathbb{R}^+ \times \mathbb{C}\)). Since \(\pi_1\) and \(\pi_2\) are continuous by Example 2.12(b),
\[ \ln \circ \pi_1 \] is continuous by Th. 2.11 and \( \pi_2(\ln \circ \pi_1) \) is continuous by Th. 2.15. Finally, \( f = \exp \circ (\pi_2(\ln \circ \pi_1)) \) is continuous by Th. 2.11.

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In [Phi16, Ex. 7.40(b),(c)], we had shown that 1-dimensional polynomials and rational functions are continuous (where they are defined). We will now extend [Phi16, Ex. 7.40(b),(c)] to \( n \)-dimensional polynomials and rational functions:

**Definition 2.18.** Let \( n \in \mathbb{N} \). An element \( p = (p_1, \ldots, p_n) \in (\mathbb{N}_0)^n \) is called a multi-index; \( |p| := p_1 + \cdots + p_n \) is called the degree of the multi-index. If \( x = (x_1, \ldots, x_n) \in \mathbb{K}^n \) and \( p = (p_1, \ldots, p_n) \) is a multi-index, then we define

\[
    x^p := x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}. \tag{2.21}
\]

Each function from \( \mathbb{K}^n \) into \( \mathbb{K} \), \( x \mapsto x^p \), is called a monomial; the degree of \( p \) is called the degree of the monomial. A function \( P \) from \( \mathbb{K}^n \) into \( \mathbb{K} \) is called a polynomial if, and only if, it is a linear combination of monomials, i.e., if, and only if \( P \) has the form

\[
P : \mathbb{K}^n \longrightarrow \mathbb{K}, \quad P(x) = \sum_{|p| \leq k} a_p x^p, \quad k \in \mathbb{N}_0, \quad a_p \in \mathbb{K}. \tag{2.22}
\]

The degree of \( P \), still denoted \( \deg(P) \), is the largest number \( d \leq k \) such that there is \( p \) with \( |p| = d \) and \( a_p \neq 0 \). If all \( a_p = 0 \), i.e., if \( P \equiv 0 \), then \( P \) is the \((n\text{-dimensional}) \) zero polynomial and, as for \( n = 1 \), its degree is defined to be \(-1\). A rational function is once again a quotient of two polynomials.

**Example 2.19.** Writing \( x, y, z \) instead of \( x_1, x_2, x_3, xy^3z, x^2y^2, x^2y, x^2, y, 1 \) are examples of monomials of degree 5, 4, 3, 2, 1, and 0, respectively, \( P(x, y) := 5x^2y - 3x^2 + y - 1 \) and \( Q(x, y, z) := xy^3z - 2x^2y^2 + 1 \) are polynomials of degree 3 and 5, respectively, and \( P(x, y)/Q(x, y, z) \) is a rational function defined for each \( (x, y, z) \in \mathbb{K}^3 \) such that \( Q(x, y, z) \neq 0 \).

**Theorem 2.20.** Each polynomial \( P : \mathbb{K}^n \longrightarrow \mathbb{K}, \ n \in \mathbb{N}, \) is continuous and each rational function \( P/Q \) is continuous at each \( z \in \mathbb{K}^n \) such that \( Q(z) \neq 0 \).

**Proof.** Let

\[
P : \mathbb{K}^n \longrightarrow \mathbb{K}, \quad P(z) = \sum_{|p| \leq k} a_p z^p, \quad k \in \mathbb{N}_0, \quad p = (p_1, \ldots, p_n) \in (\mathbb{N}_0)^n, \quad |p| = p_1 + \cdots + p_n, \quad z^p = z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}, \quad a_p \in \mathbb{K}.
\]

First, from Ex. 2.12(b), we know that the projections \( \pi_j : \mathbb{K}^n \longrightarrow \mathbb{K}, \ \pi_j(z) := z_j, \ j \in \{1, \ldots, n\}, \) are continuous. An induction and Th. 2.15 then show the monomials \( z \mapsto a_p z^p \) to be continuous, and another induction then shows \( P \) to be continuous. Applying Th. 2.15 once more finally shows that each rational function \( P/Q \) is continuous at each \( z \in \mathbb{K}^n \) such that \( Q(z) \neq 0 \).
Example 2.21. For \( n \in \mathbb{N} \), let \( \mathcal{M}(n, \mathbb{K}) \) denote the set of \( n \times n \) matrices over \( \mathbb{K} \), which is the same as \( \mathbb{K}^{n^2} \) and, thus, can be considered as a normed vector space in the usual way. From Linear Algebra, recall the determinant function \( \det: \mathcal{M}(n, \mathbb{K}) \rightarrow \mathbb{K} \).

(a) From Linear Algebra, we know that the determinant \( \det \) is a polynomial on \( \mathcal{M}(n, \mathbb{K}) \) (i.e. on \( \mathbb{K}^{n^2} \)), i.e. \( \det \) is continuous as a consequence of Th. 2.20.

(b) From Linear Algebra, we also know that \( A \in \mathcal{M}(n, \mathbb{K}) \) is invertible if, and only if, \( \det(A) \neq 0 \). Using (a) and Th. 2.7(ii), this implies that \( \text{GL}(n, \mathbb{K}) := \det^{-1}(\mathbb{K} \setminus \{0\}) \) is an open subset of \( \mathcal{M}(n, \mathbb{K}) \) (in Linear Algebra, \( \text{GL}(n, \mathbb{K}) \) is known as the general linear group of degree \( n \) over \( \mathbb{K} \)). Moreover, we claim the map \( \text{inv}: \text{GL}(n, \mathbb{K}) \rightarrow \text{GL}(n, \mathbb{K}), \quad \text{inv}(A) := A^{-1}, \) is continuous: Indeed, according to another Linear Algebra result, all the coordinate maps \( \text{inv}_{kl} \) (i.e. the entries of the inverse matrix) are rational functions on \( \mathcal{M}(n, \mathbb{K}) \) (i.e. on \( \mathbb{K}^{n^2} \)), i.e. they are continuous as a consequence of Th. 2.20, i.e. \( \text{inv} \) is continuous by Ex. 2.12(c).

Theorem 2.22. For a \( \mathbb{K} \)-linear function \( A: X \rightarrow Y \) between normed vector spaces \( (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \) over \( \mathbb{K} \), the following statements are equivalent:

(i) \( A \) is continuous.

(ii) There exists \( \xi \in X \) such that \( A \) is continuous in \( \xi \).

(iii) \( A \) is Lipschitz continuous.

Proof. Exercise. \( \blacksquare \)

We will now see two examples that show that, in contrast to linear maps between finite-dimensional spaces as considered in Example 2.16 above, linear maps between infinite-dimensional spaces can be discontinuous.

Example 2.23. (a) Once again, consider the space \( X \) from Example 1.60, consisting of all sequences in \( \mathbb{K} \) that are finally constant and equal to zero, endowed with the norm \( \| \cdot \|_{\sup} \). The function

\[
A: X \rightarrow \mathbb{K}, \quad A((z_n)_{n \in \mathbb{N}}) := \sum_{n=1}^{\infty} z_n, \quad (2.23)
\]

is clearly linear. However, we will see that \( A \) is not continuous: The sequence \( (z^k)_{k \in \mathbb{N}} \) defined by

\[
z_n^k := \begin{cases} 
1/k & \text{for } 1 \leq n \leq k, \\
0 & \text{for } n > k,
\end{cases} \quad (2.24)
\]

converges to \( 0 = (0, 0, \ldots) \in X \) with respect to \( \| \cdot \|_{\sup} \). However, for each \( k \in \mathbb{N} \), \( A(z^k) = \sum_{n=1}^{k} (1/k) = 1 \), i.e. \( \lim_{k \rightarrow \infty} A(z^k) = 1 \neq 0 = A(0) \), showing that \( A \) is not continuous at \( 0 \).
(b) Let $X$ be the normed vector space consisting of all bounded and differentiable functions $f : \mathbb{R} \to \mathbb{R}$, endowed with the sup-norm. Then the function $d : X \to \mathbb{R}$, $d(f) := f'(0)$, is linear, but not continuous (exercise).

A notion related to, but different from, continuity is componentwise continuity (see Def. 2.24). Both notions have to be distinguished carefully, as componentwise continuity does not imply continuity (see Example 2.26).

**Definition 2.24.** Let $(Y, T)$ be a topological space and let $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{K}^n$, $n \in \mathbb{N}$. A function $f : \mathbb{K}^n \to Y$ is called continuous in $\zeta$ with respect to the $j$th component, $j \in \{1, \ldots, n\}$, if, and only if, the function

$$\phi : \mathbb{K} \to Y, \quad \phi(\alpha) := f(\zeta_1, \ldots, \zeta_{j-1}, \alpha, \zeta_{j+1}, \ldots, \zeta_n),$$

is continuous in $\alpha = \zeta_j$.

**Lemma 2.25.** Let $(Y, T)$ be a topological space and let $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{K}^n$, $n \in \mathbb{N}$. If $f$ is continuous in $\zeta$, then $f$ is continuous in $\zeta$ with respect to all components.

**Proof.** Let $j \in \{1, \ldots, n\}$ and let $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{K}$ with $\lim_{k \to \infty} \alpha_k = \zeta_j$. Then $(z_k)_{k \in \mathbb{N}}$ with $z_k := (\zeta_1, \ldots, \zeta_{j-1}, \alpha_k, \zeta_{j+1}, \ldots, \zeta_n)$ is a sequence in $\mathbb{K}^n$ with $\lim_{k \to \infty} z_k = \zeta$. Thus, the continuity of $f$ yields $\lim_{k \to \infty} f(z_k) = f(\zeta)$. If $\phi$ is defined as in (2.25), then $\phi(\alpha_k) = f(z_k)$, showing $\lim_{k \to \infty} \phi(\alpha_k) = f(\zeta) = \phi(\zeta_j)$, i.e. $\phi$ is continuous in $\zeta_j$. We have, hence, shown, for each $j \in \{1, \ldots, n\}$, that $f$ is continuous in $\zeta$ with respect to the $j$th component. ■

**Example 2.26.** A function can be continuous with respect to all components at a point $\zeta$ without being continuous at $\zeta$: Consider the function

$$f : \mathbb{K}^2 \to \mathbb{K}, \quad f(z, w) := \begin{cases} 0 & \text{for } zw = 0, \\ 1 & \text{for } zw \neq 0. \end{cases}$$

Let $\phi_1, \phi_2 : \mathbb{K} \to \mathbb{K}$, $\phi_1(\alpha) := f(\alpha, 0)$, $\phi_2(\alpha) := f(0, \alpha)$. Then both $\phi_1$ and $\phi_2$ are identically 0 and, in particular, continuous at $\alpha = 0$. However, $f$ is not continuous at $(0, 0)$, since, for example,

$$(z_k, w_k) := \begin{cases} (1/k, 0) & \text{for } k \text{ even,} \\ (1/k, 1/k) & \text{for } k \text{ odd} \end{cases}$$

yields a sequence that converges to $(0, 0)$, but $f(z_k, w_k) = 0$ if $k$ is even and $f(z_k, w_k) = 1$ if $k$ is odd, i.e. the sequence $(f(z_k, w_k))_{k \in \mathbb{N}}$ does not converge.
2.2 Banach Fixed Point Theorem a.k.a. Contraction Mapping Principle

Definition 2.27. Let $\emptyset \neq A$ be a subset of a metric space $(X, d)$, $\varphi : A \rightarrow A$.

(a) The map $\varphi$ is called a contraction if, and only if, there exists $0 \leq L < 1$ satisfying
$$d(\varphi(x), \varphi(y)) \leq L d(x, y) \quad \text{for each } x, y \in A. \quad (2.28)$$

(b) $x_* \in A$ is called a fixed point of $\varphi$ if, and only if, $\varphi(x_*) = x_*$.

Remark 2.28. According to Def. 2.27, $\varphi : A \rightarrow A$ is a contraction if, and only if, $\varphi$ is Lipschitz continuous with Lipschitz constant $L < 1$.

The following Th. 2.29 constitutes the Banach fixed point theorem. It is also known as the contraction mapping principle. Its proof is surprisingly simple, e.g. about an order of magnitude easier than the proof of the Brouwer fixed point theorem.

Theorem 2.29 (Banach Fixed Point Theorem). Let $\emptyset \neq A$ be a closed subset of a complete metric space $(X, d)$ (for example, a Banach space). If $\varphi : A \rightarrow A$ is a contraction with Lipschitz constant $0 \leq L < 1$, then $\varphi$ has a unique fixed point $x_* \in A$. Moreover, for each initial value $x_0 \in A$, the sequence $(x_n)_{n \in \mathbb{N}_0}$, defined by
$$x_{n+1} := \varphi(x_n) \quad \text{for each } n \in \mathbb{N}_0, \quad (2.29)$$
converges to $x_*: \quad \lim_{n \to \infty} \varphi^n(x_0) = x_*$. \quad (2.30)

Furthermore, for each such sequence, we have the error estimate
$$d(x_n, x_*) \leq \frac{L^n}{1-L} d(x_1, x_0) \quad (2.31)$$
for each $n \in \mathbb{N}$.

Proof. We start with uniqueness: Let $x_*, x_{**} \in A$ be fixed points of $\varphi$. Then
$$d(x_*, x_{**}) = d(\varphi(x_*), \varphi(x_{**})) \leq L d(x_*, x_{**}), \quad (2.32)$$
which implies $1 \leq L$ for $d(x_*, x_{**}) > 0$. Thus, $L < 1$ implies $d(x_*, x_{**}) = 0$ and $x_* = x_{**}$.

Next, we turn to existence. A simple induction on $m - n$ shows
$$d(x_{m+n}, x_m) \leq L d(x_m, x_{m-1}) \leq L^{m-n} d(x_{n+1}, x_n) \quad (2.33)$$
for each $m, n \in \mathbb{N}_0, m > n$.

This, in turn, allows us to estimate, for each $n, k \in \mathbb{N}_0$:
$$d(x_{n+k}, x_n) \leq \sum_{m=n}^{n+k-1} d(x_{m+1}, x_m) \leq \sum_{m=n}^{n+k-1} L^{m-n} d(x_{n+1}, x_n) \leq \frac{1}{1-L} d(x_{n+1}, x_n) \leq \frac{L^n}{1-L} d(x_1, x_0) \to 0 \quad \text{for } n \to \infty. \quad (2.34)$$
establishing that \((x_n)_{n \in \mathbb{N}_0}\) constitutes a Cauchy sequence. Since \(X\) is complete, this Cauchy sequence must have a limit \(x_* \in X\), and since the sequence is in \(A\) and \(A\) is closed, \(x_* \in A\). The continuity of \(\varphi\) allows to take limits in (2.29), resulting in \(x_* = \varphi(x_*),\) showing that \(x_*\) is a fixed point and proving existence.

Finally, the error estimate (2.31) follows from (2.34) by fixing \(n\) and taking the limit for \(k \to \infty\). \(\blacksquare\)

**Example 2.30.** Suppose, we are looking for a fixed point of the map \(\varphi(x) = \cos x\) (or, equivalently, for a zero of \(f(x) = \cos x - x\)). To apply the Banach fixed point theorem, we need to restrict \(\varphi\) to a set \(A\) such that \(\varphi(A) \subseteq A\). This is the case for \(A := [0,1]\). Moreover, \(\varphi : A \to A\) is a contraction, due to \(\sin 1 < 1\) and the mean value theorem providing \(\tau \in ]0,1[\), satisfying

\[
|\varphi(x) - \varphi(y)| = |\varphi'(\tau)| |x - y| < (\sin 1)|x - y|
\]  
(2.35)

for each \(x,y \in A\). Since \(\mathbb{R}\) is complete and \(A\) is closed in \(\mathbb{R}\), the Banach fixed point theorem yields the existence of a unique fixed point \(x_* \in [0,1]\) and \(\lim \varphi^n(x_0) = x_*\) for each \(x_0 \in [0,1]\).

### 2.3 Homeomorphisms, Norm-Preserving and Isometric Maps, Embeddings

**Definition 2.31.** (a) Given topological spaces \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\), a function \(f : X \to Y\) is called homeomorphism (and \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) are called homeomorphic) if, and only if, \(f\) is bijective and both \(f\) and \(f^{-1}\) are continuous. If \(f\) is injective, then it is called an embedding of (the topological space) \(X\) into \(Y\) if, and only if, \(f\) is a homeomorphism onto its image \((f(X), \mathcal{T}_{f(X)})\).

(b) Given metric spaces \((X, d_X)\) and \((Y, d_Y)\), a function \(f : X \to Y\) is called distance-preserving or isometric or an isometry if, and only if,

\[
d_Y(f(x), f(y)) = d_X(x, y) \quad \text{for each } x, y \in X.
\]  
(2.36)

(c) Given normed vector spaces \((X, \|\cdot\|_X)\) and \((Y, \|\cdot\|_Y)\) over \(\mathbb{K}\), a function \(f : X \to Y\) is called norm-preserving if, and only if,

\[
\|f(x)\|_Y = \|x\|_X \quad \text{for each } x \in X.
\]  
(2.37)

Moreover, \(f\) is called an embedding of (the normed space) \(X\) into \(Y\) if, and only if, it is norm-preserving and linear.

In general, a structure-preserving bijective map is called an isomorphism. Thus, a homeomorphism is an isomorphism of topological spaces, a bijective isometry is an isomorphism of metric spaces, and a norm-preserving linear isomorphism is an isomorphism of normed spaces. Properties preserved by homeomorphisms are called topological invariants (e.g., separability, being first or second countable etc. – see Prop. I.1 of the Appendix for a more extensive list).
Lemma 2.32. (a) If \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) are topological spaces, then \(f : X \to Y\) is an embedding if, and only if, \(f\) is injective and both \(f\) and \(f^{-1} : f(X) \to X\) are continuous.

(b) Each isometric map between metric spaces is continuous.

(c) Each isometric map between metric spaces is injective.

(d) If \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) are normed vector spaces over \(\mathbb{K}\), then a \(\mathbb{K}\)-linear function \(f : X \to Y\) is norm-preserving if, and only if, it is isometric with respect to the induced metrics.

Proof. Exercise. \[\square\]

The following Ex. 2.33(a) shows that the assertion of Lem. 2.32(d) becomes false if the word “linear” is omitted: In general, a norm-preserving map is not isometric and not even injective or continuous. On the other hand, Ex. 2.33(b) shows that an isometric map does not need to be norm-preserving, and Ex. 2.33(c) shows that a homeomorphism is not necessarily isometric.

Example 2.33. (a) Let \((X, \| \cdot \|_X)\) be a normed vector space over \(\mathbb{K}\), and \(f : X \to \mathbb{K}\), \(f(x) := \|x\|_X\). If we take \(\| \cdot \|_Y\) to be the usual norm on \(\mathbb{K}\), i.e. \(\|y\|_Y := |y|\), then, for each \(x \in X\), \(\|f(x)\|_Y = \|\|x\|_X\| = \|x\|_X\), i.e. \(f\) is norm-preserving. However, if \(\dim X > 0\) (i.e. if \(X \neq \{0\}\)), then \(f\) is not isometric with respect to the induced metrics: Take any \(0 \neq x \in X\). One computes \(\|f(x) - f(x)\|_Y = \|\|x\|_X - \|x\|_X\| = 0 \neq \|x - (-x)\|_X = 2\|x\|_X\). Moreover, for \(x \neq 0\), one has \(x \neq -x\), but \(f(x) = \|x\|_X = f(-x)\), i.e. \(f\) is not injective. Similarly, if \(y \in X\), \(y \neq 0\), then \(g : X \to X\), \(g(x) := x\) for \(x \neq y\), \(g(y) := -y\) is norm-preserving, but not continuous in \(y\) (also not injective, since \(g(y) = g(-y)\)). The map \(h : \mathbb{R} \to \mathbb{R}\), \(h(x) = x\) for \(x \in \mathbb{Q}\), \(h(x) = -x\) for \(x \notin \mathbb{Q}\), is norm-preserving, but nowhere continuous, except in 0 (this example was pointed out by Charlotte Dietze).

(b) Consider \((X, \| \cdot \|_X)\), \((Y, \| \cdot \|_Y)\), where \(X = Y = \mathbb{K}\) and \(\|x\|_X = \|x\|_Y = |x|\) for each \(x \in \mathbb{K}\). Then \(f : X \to Y\), \(f(x) := 1 + x\), is isometric due to \(|f(x) - f(y)| = |1 + x - (1 + y)| = |x - y|\), but \(f\) is not norm-preserving, since \(0 = 0 \neq |f(0)| = 1\).

(c) We know that all norms on \(\mathbb{K}^n\), \(n \in \mathbb{N}\), are equivalent, i.e. they all induce the same topology on \(\mathbb{K}^n\). Thus, if we consider \(\mathbb{K}^n\) with two distinct norms, then the identity \(\text{Id} : \mathbb{K}^n \to \mathbb{K}^n\) is a homeomorphism, but neither norm-preserving nor isometric.

Remark 2.34. If \((X, \| \cdot \|)\) is a normed space, \(d\) is the induced metric, and \(M \subseteq X\), then \((M, d)\) can be considered as the metric subspace of \((X, d)\) according to Prop. 1.55(d). Thus, every subset of a normed space is turned into a metric space in a natural way. It is quite remarkable that actually every metric space arises in this way. That means, given any metric space \((M, d)\), there exists a normed space \((X, \| \cdot \|)\) and an isometric (in particular, injective) function \(f : M \to X\): One can choose \(X\) as the vector space over \(\mathbb{R}\) of bounded functions from \(M\) into \(\mathbb{R}\) with the sup-norm (for \(F \in X\), define...
We conclude this section by showing that each metric space has a completion:

**Theorem 2.35.** Let \((X, d_X)\) be a metric space. Then \(X\) has a completion in the sense of Def. 1.61, i.e. there exists a complete metric space \((Y, d_Y)\) and an isometry \(\phi : X \rightarrow Y\) such that \(\phi(X)\) is dense in \(Y\). Moreover, the completion is unique in the sense that if \((Z, d_Z)\) is another completion of \(X\), where \(\psi : X \rightarrow Z\) is an isometry and \(\psi(X)\) is dense in \(Z\), then there exists a bijective isometry \(f : Y \rightarrow Z\) such that \(f \circ \phi = \psi\).

**Proof.** The idea is to construct \(Y\) as a set of equivalence classes of Cauchy sequences in \(X\) (analogously to the construction that yields \(\mathbb{R}\) from \(\mathbb{Q}\)). In a first step, let \(X'\) be the set of Cauchy sequences in \(X\), and define

\[
d' : X' \times X' \rightarrow \mathbb{R}_0^+, \quad d'((x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}) := \lim_{k \to \infty} d_X(x_k, y_k).
\]

To see that \(d'\) is well-defined, we need to show that the limit in its definition exists: To this end, it suffices to show that \((d_X(x_k, y_k))_{k \in \mathbb{N}}\) is a Cauchy sequence in \(\mathbb{R}\). Indeed, given \(\epsilon > 0\), let \(N \in \mathbb{N}\) be such that

\[
\forall_{k,l > N} \left( d_X(x_k, x_l) < \frac{\epsilon}{2} \text{ and } d_X(y_k, y_l) < \frac{\epsilon}{2} \right).
\]

Then

\[
\forall_{k,l > N} \left| d_X(x_k, y_k) - d_X(x_l, y_l) \right| \leq d_X(x_k, x_l) + d_X(y_k, y_l) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

thereby establishing the case. If \(\tau = \sigma\), then, clearly, \(d'(\tau, \sigma) = 0\); \(d'\) is symmetric, since, clearly, \(d'(\tau, \sigma) = d'(\sigma, \tau)\); and \(d'\) also satisfies the triangle inequality:

\[
d'((x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}) = \lim_{k \to \infty} d_X(x_k, y_k) \leq \lim_{k \to \infty} d_X(x_k, z_k) + \lim_{k \to \infty} d_X(z_k, y_k) = d'((x_k)_{k \in \mathbb{N}}, (z_k)_{k \in \mathbb{N}}) + d'((z_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}).
\]

Altogether, we have shown that \(d'\) constitutes a so-called pseudometric on \(X'\) (cf. Def. E.1 of the Appendix). Unfortunately, one can not expect \(d'\) to be a metric on \(X'\), since it can happen that \(d'(\tau, \sigma) = 0\), even though \(\tau \neq \sigma\). However, according to Th. E.8,

\[
\sigma \sim \tau \iff d'(\sigma, \tau) = 0
\]

defines an equivalence relation on \(X'\) and, if one lets \(Y := \{[\sigma] : \sigma \in X'\}\) be the set of the corresponding equivalence classes, then

\[
d_Y : Y \times Y \rightarrow \mathbb{R}_0^+, \quad d_Y([\sigma], [\tau]) := d'(\sigma, \tau),
\]
defines a metric on $Y$. Define
\[
\phi : X \rightarrow Y, \quad \phi(x) := [(x)_{k \in \mathbb{N}}].
\]
Then, for each $x, y \in X$,
\[
d_Y(\phi(x), \phi(y)) = \phi([x]_{k \in \mathbb{N}}) = \phi([y]_{k \in \mathbb{N}}) = d_X(x, y),
\]
showing $\phi$ to be an isometry. Next, we show that, for each Cauchy sequence $(x_k)_{k \in \mathbb{N}}$ in $X$, we have
\[
\lim_{l \to \infty} \phi(x_l) = [(x_k)_{k \in \mathbb{N}}] \quad \text{(i.e. } \lim_{l \to \infty} d_Y(\phi(x_l), [(x_k)_{k \in \mathbb{N}}]) = 0)\]
(in particular, this implies $\phi(X)$ to be dense in $Y$). Let $N \in \mathbb{N}$ be such that, for each $k, l > N$, we have $d_X(x_k, x_l) < \frac{\epsilon}{2}$. Then
\[
\forall_{l > N} d_Y(\phi(x_{N+1}), [(x_k)_{k \in \mathbb{N}}]) = \lim_{k \to \infty} d_X(x_{N+1}, x_k) \leq \frac{\epsilon}{2} < \epsilon,
\]
proving $\lim_{n \to \infty} \phi(x_{n+1}) = [(x_k)_{k \in \mathbb{N}}]$. We can now show that $(Y, d_Y)$ is complete: Let $(y_{n(k)})_{k \in \mathbb{N}}$ be a Cauchy sequence in $Y$. As $\phi(X)$ is dense in $Y$, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $d_Y(y_{n(k)}, \phi(x_n)) < \frac{1}{n}$. Then $(x_{n(k)})_{k \in \mathbb{N}}$ is a Cauchy sequence: Given $\epsilon > 0$, choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\epsilon}{3}$ and $d_Y(y_{n(k)}, y_{m(k)}) < \frac{\epsilon}{3}$ for each $n, m > k$. Then
\[
\forall_{n, m > k} d_X(x_n, x_m) = d_Y(\phi(x_n), \phi(x_m)) \leq d_Y(\phi(x_n), y_{n(k)}) + d_Y(y_{n(k)}, y_{m(k)}) + d_Y(y_{m(k)}, \phi(x_m)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\]
showing $(x_n)_{n \in \mathbb{N}}$ is Cauchy and, as we have shown above, $\lim_{n \to \infty} \phi(x_n) = y := [(x_k)_{k \in \mathbb{N}}]$. Then $\lim_{n \to \infty} y_{n(k)} = y$ as well: Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$ and $d_Y(y, \phi(x_n)) < \frac{\epsilon}{2}$ for each $n > N$. Then
\[
\forall_{n > N} d_Y(y, y_{n(k)}) \leq d_Y(y, \phi(x_n)) + d_Y(\phi(x_n), y_{n(k)}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]
showing $\lim_{n \to \infty} y_{n(k)} = y$ and completing the proof that $(Y, d_Y)$ is complete. Finally, let $(Z, d_Z)$ and $\psi : X \rightarrow Z$ be as in the statement of the theorem. Define $f : Y \rightarrow Z$ as follows: Given $y \in Y$, there is a sequence $(x_{n(k)})_{k \in \mathbb{N}}$ in $X$ such that $y = \lim_{k \to \infty} \phi(x_{n(k)})$. Set $f(y) := \lim_{k \to \infty} \psi(x_{n(k)})$. We need to verify that $f$ is well-defined. Since $y = \lim_{k \to \infty} \phi(x_{n(k)})$ and $\phi$ is an isometry, $(x_{k})_{k \in \mathbb{N}}$ must be a Cauchy sequence, and $(\psi(x_{k}))_{k \in \mathbb{N}}$ must, indeed, converge in $Z$. If $(a_k)_{k \in \mathbb{N}}$ is another sequence in $X$ with $y = \lim_{k \to \infty} \phi(a_k)$, then
\[
\lim_{k \to \infty} d_Z(\psi(a_k), \psi(x_k)) = \lim_{k \to \infty} d_X(a_k, x_k) = \lim_{k \to \infty} d_Y(\phi(a_k), \phi(x_k)) = 0.
\]
Letting $z_1 := \lim_{k \to \infty} \psi(x_k)$, $z_2 := \lim_{k \to \infty} \psi(a_k)$, this, together with
\[
\forall_{k \in \mathbb{N}} d_Z(z_1, z_2) \leq d_Z(z_1, \psi(x_k)) + d_Z(\psi(x_k), \psi(a_k)) + d_Z(\psi(a_k), z_2) \rightarrow 0
\]
shows $z_1 = z_2$ and the independence of $f(y)$ from the chosen sequence. Next, for each $x \in X$, $f(\phi(x)) = \psi(x)$, i.e. $f \circ \phi = \psi$ as desired. To see that $f$ is isometric, let $y_1, y_2 \in Y$ and let $(x_k)_{k \in \mathbb{N}}, (a_k)_{k \in \mathbb{N}}$ be sequences with $y_1 = \lim_{k \to \infty} \phi(x_k)$, $y_2 = \lim_{k \to \infty} \phi(a_k)$. Then $f(y_1) = \lim_{k \to \infty} \psi(x_k)$, $f(y_2) = \lim_{k \to \infty} \psi(a_k)$, and we need to show $d_Z(f(y_1), f(y_2)) = d_Y(y_1, y_2)$.

Since
\[
\forall k \in \mathbb{N} \quad |d_Y(y_1, y_2) - d_Y(\phi(x_k), \phi(a_k))| \leq d_Y(y_1, \phi(x_k)) + d_Y(y_2, \phi(a_k)),
\]
and
\[
|d_Z(f(y_1), f(y_2)) - d_Z(\psi(x_k), \psi(a_k))| \leq d_Z(f(y_1), \psi(x_k)) + d_Z(f(y_2), \psi(a_k)),
\]
we know
\[
d_Z(f(y_1), f(y_2)) = \lim_{k \to \infty} d_Z(\psi(x_k), \psi(a_k)) = \lim_{k \to \infty} d_X(x_k, a_k)
\]
\[
= \lim_{k \to \infty} d_Y(\phi(x_k), \phi(a_k)) = d_Y(y_1, y_2)
\]
as required. Finally, $f$ is also surjective: If $z \in Z$, then, as $\psi(X)$ is dense in $Z$, there is a sequence $(x_k)_{k \in \mathbb{N}}$ in $X$ such that $z = \lim_{k \to \infty} \psi(x_k)$. Then $(x_k)_{k \in \mathbb{N}}$ is Cauchy and we can let $y := \lim_{k \to \infty} \phi(x_k)$. Then $f(y) = z$, showing $f$ to be surjective. 

\section{Further Topologic Properties}

\subsection{Separation}

Separation properties (also called separation axioms) are important properties of topological spaces that are closely related to the uniqueness of limits and to the existence of continuous maps. In general, separation properties are rather subtle and there exist an abundance of different such properties in the literature. We only consider some of the most important ones.

\textbf{Definition 3.1.} Let $(X, T)$ be a topological space.

(a) $(X, T)$ is called a $T_1$ \textit{space} if, and only if, points can be \textit{separated}, i.e. if, and only if,
\[
\forall x, y \in X \quad (x \neq y \Rightarrow \exists_{O_x \in T} (y \in O_y \land x \notin O_y)),
\]
i.e. if, and only if, for each $x \in X$, the set $\{x\}$ is closed (somewhat lax, one also says that $T_1$ means “points are closed”).

(b) $(X, T)$ is called a $T_2$ \textit{space} or a Hausdorff \textit{space} if, and only if, points can be separated by \textit{disjoint} open sets, i.e. if, and only if,
\[
\forall x, y \in X \quad (x \neq y \Rightarrow \exists_{O_x, O_y \in T} (x \in O_x \land y \in O_y \land O_x \cap O_y = \emptyset)).
\]
(c) \((X, \mathcal{T})\) is called a \(T_3\) space if, and only if, points and closed sets can be separated by disjoint open sets, i.e. if, and only if,

\[
\forall x \in X \quad \forall C \subseteq X, C \text{ closed} \quad \left( x \notin C \implies \exists O_x, O_C \in \mathcal{T} \quad \left( x \in O_x \land C \subseteq O_C \land O_x \cap O_C = \emptyset \right) \right).
\]

\((X, \mathcal{T})\) is called regular if, and only if, it is both \(T_1\) and \(T_3\).\(^6\)

(d) \((X, \mathcal{T})\) is called a \(T_4\) space if, and only if, disjoint closed sets can be separated by disjoint open sets, i.e. if, and only if,

\[
\forall C_1, C_2 \subseteq X, C_1, C_2 \text{ closed} \quad \left( C_1 \cap C_2 = \emptyset \implies \exists O_1, O_2 \in \mathcal{T} \quad \left( C_1 \subseteq O_1 \land C_2 \subseteq O_2 \land O_1 \cap O_2 = \emptyset \right) \right).
\]

\((X, \mathcal{T})\) is called normal if, and only if, it is both \(T_1\) and \(T_4\).\(^7\)

**Lemma 3.2.** Let \((X, \mathcal{T})\) be a topological space. Then we have the following implications of separation properties:

(a) \(T_2\) implies \(T_1\).

(b) Regular implies \(T_1\), \(T_2\), \(T_3\).

(c) Normal implies \(T_1\), \(T_2\), \(T_3\), \(T_4\).

*Proof.* (a) is immediate and so are (b) and (c) (since points are closed in regular spaces as well as in normal spaces). \(\blacksquare\)

**Proposition 3.3.** Let \((X, \mathcal{T})\) be a topological space. Then the following statements are equivalent:

(i) \((X, \mathcal{T})\) is a \(T_2\) space.

(ii) Limits of nets in \(X\) are unique, i.e. for each net \((x_i)_{i \in I}\) in \(X\) it holds that if \((x_i)_{i \in I}\) converges to both \(x \in X\) and \(y \in X\), then \(x = y\).

*Proof.* Exercise. \(\blacksquare\)

**Example 3.4.** (a) Clearly, indiscrete spaces \((X, \mathcal{T})\) with at least two distinct points are not \(T_1\). However, every indiscrete space is both \(T_3\) and \(T_4\) (since \(\emptyset\) and \(X\) are the only closed subsets of \(X\), the conditions of Def. 3.1(c),(d) are trivially satisfied).

(b) Let \((X, \mathcal{T})\) be a cofinite space. Then it is always \(T_1\), since, for each \(x \in X\), \(X \setminus \{x\}\) is open, hence \(\{x\}\) is closed. However, if \(X\) is infinite, then \((X, \mathcal{T})\) is not \(T_2\) (exercise).

\(^6\)Caveat: Unfortunately, about half the literature switches the meaning of regular and \(T_3\).

\(^7\)Caveat: Unfortunately, about half the literature switches the meaning of normal and \(T_4\).
(c) Every metric space is normal (in particular, as a consequence of Prop. 3.3, limits in metric spaces are unique): If \((X, T)\) is a topological space, where \(T\) is induced by the metric \(d\) on \(X\), then \((X, T)\) is \(T_1\) and \(T_4\): Let \(x, y \in X\) with \(x \neq y\). Then \(r := d(x, y) > 0\) and \(y \notin B_r(x), x \notin B_r(y)\), showing that \((X, T)\) is \(T_1\). To show that \((X, T)\) is \(T_4\), let \(A, B \subseteq X\) be closed with \(A \cap B = \emptyset\). Recalling the continuous distance functions from Ex. 2.6(b), define \(d_A := \text{dist}(\cdot, A), d_B := \text{dist}(\cdot, B)\),
\[O_A := \{x \in X : d_A(x) < d_B(x)\}, \quad O_B := \{x \in X : d_B(x) < d_A(x)\}.
\]
We claim that \(O_A\) and \(O_B\) are open sets that separate \(A\) and \(B\): Indeed, \(O_A \cap O_B = \emptyset\) is immediate. Suppose \(x \in A^c\). Since \(A\) is closed, \(A^c\) is open and \(B_r(x) \subseteq A^c\) for some \(r > 0\). Thus, \(d_A(x) \geq r > 0\). In consequence, \(A \subseteq O_A\) and \(B \subseteq O_B\). It remains to show that \(O_A, O_B \in T\). We can write
\[O_A = \bigcup_{s \in \mathbb{R}^+} A_s, \quad O_B = \bigcup_{s \in \mathbb{R}^+} B_s,
\]
where
\[A_s := \{x \in X : d_A(x) < s < d_B(x)\} = d_A^{-1}([-\infty, s]) \cap d_B^{-1}(s, \infty], \quad B_s := \{x \in X : d_B(x) < s < d_A(x)\} = d_B^{-1}([-\infty, s]) \cap d_A^{-1}(s, \infty].
\]
Due to the continuity of \(d_A\) and \(d_B\), we have \(A_s, B_s \in T\) for each \(s \in \mathbb{R}^+\), proving \(O_A, O_B \in T\) as well.

Further counterexamples regarding implications for separation properties are provided in Appendix Sec. G.

**Proposition 3.5.** (a) \(T_1, T_2, T_3\) are inherited by subspaces (but cf. Ex. G.1(d)): Let \((X, T)\) be a topological space, \(M \subseteq X\). Let \(T_M\) denote the relative topology on \(M\). If \((X, T)\) is \(T_n\), where \(n \in \{1, 2, 3\}\), then \((M, T_M)\) is \(T_n\) as well.

(b) \(T_1, T_2, T_3\) are inherited by product spaces (but cf. Ex. G.1(e)): Consider nonempty topological spaces \((X_i, T_i), i \in I, X := \prod_{i \in I} X_i\) with the product topology \(T\) (cf. Ex. 1.53). Then \((X, T)\) is \(T_n\), where \(n \in \{1, 2, 3\}\), if, and only if, each \((X_i, T_i), i \in I, is T_n.

**Proof.** (a): Let \((X, T)\) be \(T_1\) (resp. \(T_2\)), \(x, y \in M\) with \(x \neq y\). Then there are \(O_x, O_y \in T\) such that \(x \in O_x, y \in O_y\), and \(x \notin O_y, y \notin O_x\) (resp. \(O_x \cap O_y = \emptyset\)). Let \(M_x := M \cap O_x\), \(M_y := M \cap O_y\). Then \(M_x, M_y \in T_M, x \in M_x, y \in M_y, and x \notin M_y, y \notin M_x\) (resp. \(M_x \cap M_y = \emptyset\)), showing \((M, T_M)\) is \(T_1\) (resp. \(T_2\)). Now assume \((X, T)\) to be \(T_3\), let \(x \in M\) and let \(A \subseteq X\) be closed, satisfying \(x \notin A, A \subseteq M \cap A \subseteq M\). Then there are \(O_x, O_A \in T\) such that \(x \in O_x, A \subseteq O_A, O_x \cap O_A = \emptyset\). Let \(M_x := M \cap O_x\), \(M_A := M \cap O_A\). Then \(M_x, M_A \in T_M, x \in M_x, M_A \subseteq M_A\), and \(M_x \cap M_A = \emptyset\), showing \((M, T_M)\) is \(T_3\).

(b): Since \(X \neq \emptyset\), there exists \(x = (x_i)_{i \in I} \in X\). Fix \(j \in I\). Let \(M := \prod_{i \in I} A_i\), where \(A_i := \{x_i\} for i \neq j and A_j := X_j\). Then \((M, T_M)\) is homeomorphic to \((X_j, T_j)\): If \(τ :
$M \rightarrow X$ is the identity inclusion map and $\pi_j : X \rightarrow X_j$ is the projection on $X_j$, then, clearly, $f := \pi_j \circ \iota$ is continuous and bijective. Moreover, since $\{M \cap \pi_j^{-1}(O) : O \in T_j\}$ is a subbase of $T_M$ and, for each $O \in T_j$, $f(M \cap \pi_j^{-1}(O)) = O$, showing $f^{-1}$ to be continuous and $f$ to be a homeomorphism. Thus, it $(X, \mathcal{T})$ is $T_n$, $n \in \{1, 2, 3\}$, then $(X_j, T_j)$ is $T_n$ by (a) and Prop. 1.1(h). Let each $(X_i, T_i)$ be $T_1$ (resp. $T_2$), $x, y \in X$ with $x \neq y$. Then there exists $j \in I$ with $x_j \neq y_j$, and $O_{j,x}, O_{j,y} \in T_j$ such that $x_j \in O_{j,x}$, $y_j \in O_{j,y}$, and $x \notin O_{j,y}$, $y \notin O_{j,x}$ (resp. $O_{j,x} \cap O_{j,y} = \emptyset$). Let $O_x := \pi_j^{-1}(O_{j,x})$, $O_y := \pi_j^{-1}(O_{j,y})$. Then $O_x, O_y \in \mathcal{T}$, $x \in O_x$, $y \in O_y$, and $x \notin O_y$, $y \notin O_x$ (resp. $O_x \cap O_y = \emptyset$), showing $(X, \mathcal{T})$ is $T_1$ (resp. $T_2$). Now assume each $(X_i, T_i)$ to be $T_3$, let $x \in X$ and let $A \subseteq X$ be closed, satisfying $x \notin A$. Since $x \in O := X \setminus A$ and $O \in \mathcal{T}$, there is a finite $J \subseteq I$ such that $x \in B := \bigcap_{j \in J} \pi_j^{-1}(O_j) \subseteq O$, each $O_j \in T_j$. Then, for each $j \in J$, $x_j \notin A_j := X_j \setminus O_j$ and each $A_j$ is closed. As each $(X_j, T_j)$ is $T_3$, there are $O_{j,x}, O_{j,A} \in T_j$ such that $x_j \in O_{j,x}$, $A_j \subseteq O_{j,A}$, $O_{j,x} \cap O_{j,A} = \emptyset$. Let $O_x := \bigcap_{j \in J} \pi_j^{-1}(O_{j,x})$, $O_A := \bigcup_{j \in J} \pi_j^{-1}(O_{j,A})$. Then $O_x, O_A \in \mathcal{T}$, $x \in O_x$, $A \subseteq O_A$ ($a \in A$ implies $a \notin B$, which implies $a_{j_0} \in A_{j_0}$ for some $j_0 \in J$, which implies $a \notin O_A$), and $O_x \cap O_A = \emptyset$, showing $(X, \mathcal{T})$ is $T_3$. 

**Theorem 3.6 (Tietze-Urysohn).** Let $(X, \mathcal{T})$ be a topological space. Then the following statements are equivalent:

(i) $(X, \mathcal{T})$ is $T_4$.

(ii) If $\emptyset \neq A, B \subseteq X$ are arbitrary closed sets with $A \cap B = \emptyset$, and if $a, b \in \mathbb{R}$, $a < b$, then there exists a continuous function $f : X \rightarrow [a, b]$ such that $f |_{A} \equiv a$ and $f |_{B} \equiv b$.

(iii) If $\emptyset \neq A \subseteq X$ is an arbitrary closed set, $a, b \in \mathbb{R}$, $a < b$, and $f : A \rightarrow [a, b]$ is continuous, then $f$ can be continuously extended to $X$, i.e. there exists a continuous $g : X \rightarrow [a, b]$ such that $g |_{A} = f$.

**Proof.** See, e.g., [Pre75, Th. 4.5.2, Th. 4.5.4] or [RF10, Sec. 12.1].

**3.2 Compactness**

In [Phi16, Def. 7.42(c)], we defined a subset $C$ of $\mathbb{K}$ to be compact if, and only if, $C$ was closed and bounded; and we saw in [Phi16, Th. 7.48] that compactness of $C$ was equivalent to every sequence in $C$ having a convergent subsequence. The appropriate definition of compactness in general topological spaces looks quite different, at least at first glance (see Def. 3.7 below). It does turn out to be equivalent to our old definition in $\mathbb{K}$ with its standard topology. Even in $\mathbb{K}^n$, a set is still compact if, and only if, it is closed and bounded (see Cor. 3.16(c)). However, in infinite-dimensional normed vector spaces, this is no longer true (see Th. 3.18). In general metric spaces, it is at least still true that a set $C$ is compact if, and only if, every sequence in $C$ has a convergent subsequence (see Th. 3.14). In general topological spaces this still remains true if one replaces sequences with nets (see Th. 3.8) – however, sequences, in general, no longer suffice (see Caveat 3.15).
Definition 3.7. Let \((X, \mathcal{T})\) be a topological space, \(C \subseteq X\). We call a family of open sets \((O_i)_{i \in I}\), \(O_i \in \mathcal{T}\), an open cover of \(C\) if, and only if,
\[
C \subseteq \bigcup_{i \in I} O_i. \tag{3.1}
\]
We call \(C\) compact if, and only if, every open cover of \(C\) has a finite subcover, i.e. if \((O_i)_{i \in I}\) is an open cover of \(C\), then there exist \(i_1, \ldots, i_N \in I\), \(N \in \mathbb{N}\), such that \(C \subseteq \bigcup_{k=1}^N O_{i_k}\).

Theorem 3.8. Let \((X, \mathcal{T})\) be a topological space, \(C \subseteq X\). Then the following statements are equivalent:

(i) \(C\) is compact.

(ii) \(C\) has the finite intersection property, i.e. if \((A_i)_{i \in I}\) is a family of closed subsets of \(X\) such that \(C \cap \bigcap_{i \in I} A_i = \emptyset\), then there exist \(i_1, \ldots, i_N \in I\), \(N \in \mathbb{N}\), such that \(C \cap \bigcap_{k=1}^N A_{i_k} = \emptyset\).

(iii) Every net in \(C\) has a subnet that converges in \(C\).

Proof. We show “(i) \(\iff\) (ii)”, “(ii) \(\Rightarrow\) (iii)”, and “(iii) \(\Rightarrow\) (i)”.

“(i) \(\iff\) (ii)”: \((O_i)_{i \in I}\) is an open cover of \(C\) if, and only if, \(A_i := X \setminus O_i\) are closed sets satisfying
\[
X \setminus C \supseteq \bigcap_{i \in I} A_i \iff C \cap \bigcap_{i \in I} A_i = \emptyset.
\]
Thus, the \((O_i)_{i \in I}\) have a finite subcover \((O_{i_1}, \ldots, O_{i_N})\) of \(C\) if, and only if, there are \(i_1, \ldots, i_N \in I\) such that \(C \cap \bigcap_{k=1}^N A_{i_k} = \emptyset\).

“(ii) \(\Rightarrow\) (iii)”: Let \((c_i)_{i \in I}\) be a net in \(C\). For each \(i \in I\), let \(A_i := \text{cl}\{c_j : j \geq i\}\). Consider \(i_1, \ldots, i_N \in I\), \(N \in \mathbb{N}\). Then, since \(I\) is a directed set, there is \(i_0 \in I\) satisfying \(i_0 \geq i_k\) for each \(k = 1, \ldots, N\), implying
\[
c_{i_0} \in C \cap \bigcap_{k=1}^N A_{i_k}, \quad \text{i.e.} \quad C \cap \bigcap_{k=1}^N A_{i_k} \neq \emptyset. \tag{3.2}
\]
As we assume (ii), this now implies there exists \(c \in C \cap \bigcap_{i \in I} A_i\).

We proceed to construct a subnet of \((c_i)_{i \in I}\) that converges to \(c\). We define \(J := \{(U, i) : U \in \mathcal{U}(c), c_i \in U\}\). Due to (3.2), for each \(i \in I\), \(c\) is in the closure of \(\{c_j : j \geq i\}\), implying, for each \(U \in \mathcal{U}(c)\), the existence of \(c_U \in U \cap \{c_j : j \geq i\}\), also showing \(J \neq \emptyset\).

We make \(J\) into a directed set by letting
\[
(U, i) \leq (V, j) \iff (U \supseteq V \land i \leq j):
\]
Clearly, \( \leq \) is reflexive and transitive on \( J \). Given \((U, i), (V, j) \in J\), let \( M \in I \) be such that \( M \geq i, j \). Then there is \( M_0 \geq M \) such that \( \phi_{M_0} \in U \cap V \cap \{ c_j : j \geq M \} \). Thus, \((U \cap V, M_0) \in J\) and \((U \cap V, M_0) \geq (U, i), (V, j)\), proving \( J \) to be a directed set. Then \( \phi : J \rightarrow I\), \( \phi(U, i) := i \), is final, since, for each \( i \in I \) and \( U \in \mathcal{U}(c)\), there exists \( i_0 \geq i \) such that \((U, i_0) \in J\). Then

\[
\forall (V, j) \geq (U, i_0) \quad \phi(V, j) = j \geq i_0 \geq i,
\]

proving \( \phi \) to be final. Thus \( (\phi(U, i))_{(U, i) \in J} \) is a subnet of \( (c_i)_{i \in I} \). It merely remains to verify \( \lim_{(U, i) \in J} c_{\phi(U, i)} = c \). However, if \( U \in \mathcal{U}(c) \) and \((U, i) \in J\), then, for each \((V, j) \in J\) with \((V, j) \geq (U, i)\), one has \( c_{\phi(V, j)} = c_j \in V \subseteq U\), which establishes the case.

“(iii)\(\Rightarrow\)(i)”: Seeking a contradiction, let \((O_i)_{i \in I} \) be an open cover of \( C \) that does not admit a finite subcover. If \( J \) denotes the set of all finite subsets of \( I \), then \( J \) is directed by \( \subseteq \). If, for each \( K \in J \), we choose \( c_K \in C \setminus \bigcup_{i \in K} O_i \), then \( (c_K)_{K \in J} \) defines a net in \( C \). According to (iii), \( (c_K)_{K \in J} \) has a subnet \( (c_{\phi(l)})_{l \in L} \), where \( \phi : L \rightarrow J \) is final, and \( \lim_{l \in L} c_{\phi(l)} = c \in C \). Since \((O_i)_{i \in I} \) is an open cover of \( C \), there exists \( i_0 \in I \) such that \( O_{i_0} \in \mathcal{U}(c) \). Then, for each \( l \in L \) such that \( \phi(l) \geq \{ i_0 \} \), one has \( c_{\phi(l)} \notin O_{i_0} \), in contradiction to \( \lim_{l \in L} c_{\phi(l)} = c \).

**Proposition 3.9.** Let \((X, \mathcal{T})\) be a topological space, \( C \subseteq X \).

(a) If \( C \) is compact and \( A \subseteq C \) is closed, then \( A \) is compact.

(b) If \( C \) is compact and \( X \) is \( T_2 \), then \( C \) is closed.

*Proof.* (a): If \((x_i)_{i \in I} \) is a net in \( A \), then \((x_i)_{i \in I} \) is a net in \( C \). Since \( C \) is compact, it must have a subnet that converges to some \( c \in C \). However, as \( A \) is closed, \( c \) must be in \( A \), showing that \((x_i)_{i \in I} \) has a subnet that converges to some \( c \in A \), i.e. \( A \) is compact.

(b): Let \((x_i)_{i \in I} \) be a net in \( C \) that converges in \( X \), i.e. \( \lim_{i \in I} x_i = x \in X \). Since \( C \) is compact, \((x_i)_{i \in I} \) must have a subnet that converges to some \( c \in C \). Since the subnet also converges to \( x \) and since limits are unique in \( T_2 \) spaces by Prop. 3.3, \( x = c \in C \), showing \( C \) is closed.

**Example 3.10.** (a) Clearly, finite sets are always compact.

(b) If \((X, \mathcal{T})\) is a cofinite topological space, then every \( C \subseteq X \) is compact (if \( C \) is infinite, then it is not closed, showing that a compact subset of a \( T_1 \) space does not need to be closed): Let \((O_i)_{i \in I} \) be an open cover of \( C \), \( i_0 \in I \). Then \( A := C \setminus O_{i_0} \) is finite. For each \( a \in A \), there is \( i_a \in I \) such that \( a \in O_{i_a} \). Thus, letting \( J := \{ i_0 \} \cup \{ i_a : a \in A \} \), \((O_i)_{i \in J} \) is a finite subcover of \((O_i)_{i \in I} \).

**Proposition 3.11.** Let \((X, \mathcal{T})\) be a topological space.

(a) Unions of finitely many compact subsets of \( X \) are compact.

(b) If \( X \) is \( T_2 \), then arbitrary intersections of compact subsets of \( X \) are compact.
Proof. (a): It suffices to consider two compact sets $C_1, C_2 \subseteq X$ (then the general case follows by induction). Let $(O_i)_{i \in I}$ be an open cover of $C := C_1 \cup C_2$. Then $(O_i)_{i \in I}$ constitutes an open cover of both $C_1$ and $C_2$. As $C_1, C_2$ are compact, there are $K, L \subseteq I$ finite, such that $(O_i)_{i \in K}$ still covers $C_1$ and $(O_i)_{i \in L}$ still covers $C_2$. Then the $(O_i)_{i \in K \cup L}$ forms a finite cover of $C$.

(b): If $(C_i)_{i \in I}$, $I \neq \emptyset$, is a family of compact subsets of a $T_2$ space, then each $C_i$ is closed by Prop. 3.9(b). Thus, $C := \bigcap_{i \in I} C_i$ is a closed subset of a compact set and, thus, compact by Prop. 3.9(a). ■

$\mathbb{N}$ (with the discrete topology) already shows that infinite unions of compact sets need not be compact. If $(X, T)$ is not $T_2$, then, in general, not even intersections of two compact sets need to be compact (see Ex. H.1 of the Appendix).

**Theorem 3.12.** If $(X, T_X)$ and $(Y, T_Y)$ are topological spaces, $C \subseteq X$ is compact, and $f : C \to Y$ is continuous, then $f(C)$ is compact.

Proof. If $(y_i)_{i \in I}$ is a net in $f(C)$, then, for each $i \in I$, there is some $x_i \in C$ such that $f(x_i) = y_i$. As $C$ is compact, there is a subnet $(a_j)_{j \in J}$ of $(x_i)_{i \in I}$ with $\lim_{j \in J} a_j = a$ for some $a \in C$. Then $(f(a_j))_{j \in J}$ is a subnet of $(y_i)_{i \in I}$ and the continuity of $f$ yields $\lim_{j \in J} f(a_j) = f(a) \in f(C)$, showing that $(y_i)_{i \in I}$ has a convergent subnet with limit in $f(C)$. We have therefore established that $f(C)$ is compact. ■

**Definition 3.13.** A subset $A$ of a metric space $(X, d)$ is called **precompact** or **totally bounded** if, and only if, for each $\epsilon > 0$, $A$ can be covered by finitely many $\epsilon$-balls, i.e. if, and only if, there exist finitely many points $a_1, \ldots, a_N \in A$, $N \in \mathbb{N}$, such that

$$A \subseteq \bigcup_{j=1}^{N} B_{\epsilon}(a_j).$$ (3.3)

**Theorem 3.14.** For a subset $C$ of a metric space $(X, d)$, the following statements are equivalent:

(i) $C$ is compact as defined in Def. 3.7.

(ii) Every sequence in $C$ has a subsequence that converges to some limit $c \in C$.

(iii) $C$ is precompact (i.e. totally bounded) as defined in Def. 3.13 and complete, i.e. every Cauchy sequence in $C$ converges to a limit in $C$.

Proof. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

“(i) $\Rightarrow$ (ii)”: Assume $C$ is compact. Seeking a contradiction, assume there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in $C$ such that no subsequence of $(c_n)_{n \in \mathbb{N}}$ converges to a limit in $C$. According to Lem. 1.41, no $c \in C$ can be a cluster point of $A := \{c_n : n \in \mathbb{N}\}$. Thus, by Rem. 1.42(a), for each $c \in C$, there exists $\epsilon_c > 0$ such that $B_{\epsilon_c}(c)$ contains only finitely many of the $c_n$. Since $C \subseteq \bigcup_{c \in C} B_{\epsilon_c}(c)$, the family $(B_{\epsilon_c}(c))_{c \in C}$ constitutes an open cover
of $C$. As $C$ is compact, there exist finitely many points $a_1, \ldots, a_N \in C$, $N \in \mathbb{N}$, such that $C \subseteq \bigcup_{j=1}^{N} B_{\varepsilon_j}(a_j)$, i.e. $C$ contains only finitely many of the $c_n$, in contradiction to $(c_n)_{n \in \mathbb{N}}$ being a sequence in $C$.

"(ii) ⇒ (iii)" : Let $(c_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $C$. By (ii), $(c_n)_{n \in \mathbb{N}}$ has a subsequence $(c_{n_j})_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} c_{n_j} = c \in C$. Given $\varepsilon > 0$ choose $K \in \mathbb{N}$ such that, for each $m, n \geq K$, $d(c_m, c_n) < \frac{\varepsilon}{2}$, and such that, for each $n_j \geq K$, $d(c_{n_j}, c) < \frac{\varepsilon}{2}$. Then, fixing some $n_j \geq K$,

$$\forall n \geq K \quad d(c_n, c) \leq d(c_n, c_{n_j}) + d(c_{n_j}, c) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

showing $\lim_{n \to \infty} c_n = c$ and the completeness of $C$. We now show $C$ to be also totally bounded. We proceed by contraposition and assume $C$ not to be totally bounded, i.e. there exists $\varepsilon > 0$ such that $C$ is not contained in any finite union of $\varepsilon$-balls. Inductively, we construct a sequence $(c_n)_{n \in \mathbb{N}}$ in $C$ such that

$$\forall m, n \in \mathbb{N}, \quad m \neq n \quad d(c_m, c_n) \geq \varepsilon : \quad (3.4)$$

To start with, we note $C \neq \emptyset$ and choose some arbitrary $c_1 \in C$. Assuming $c_1, \ldots, c_k \in C$, $k \in \mathbb{N}$, have already been constructed such that $d(c_m, c_n) \geq \varepsilon$ holds for each $m, n \in \{1, \ldots, k\}$, there must be

$$c \in C \setminus \bigcup_{j=1}^{k} B_{\varepsilon}(c_j). \quad (3.5)$$

Choosing $c_{k+1} := c$, (3.5) guarantees (3.4) now holds for each $m, n \in \{1, \ldots, k+1\}$. Due to (3.4), no subsequence of $(c_n)_{n \in \mathbb{N}}$ can be a Cauchy sequence, i.e. $(c_n)_{n \in \mathbb{N}}$ does not have a convergent subsequence, proving (ii) does not hold.

"(iii) ⇒ (i)" : Assume $C$ to be precompact and complete. For each $k \in \mathbb{N}$, the precompactness yields points $c_1^k, \ldots, c_{N_k}^k \in C$, $N_k \in \mathbb{N}$, such that

$$C \subseteq \bigcup_{j=1}^{N_k} B_{\frac{1}{k}}(c_j^k). \quad (3.6)$$

Seeking a contradiction, assume there exists an open cover $(O_j)_{j \in I}$ of $C$ which does not have a finite subcover. Inductively, we construct a decreasing sequence of subsets $C_k$ of $C$, $C \supseteq C_1 \supseteq C_2 \supseteq \ldots$, such that no $C_k$ can be covered by a finite subcover of $(O_j)_{j \in I}$ and such that

$$\forall k \in \mathbb{N} \quad \exists j \in \{1, \ldots, N_k\} \quad C_k \subseteq B_{\frac{1}{k}}(c_j^k) :$$

To start out, we note that (3.6) implies at least one of the finitely many sets $C \cap B_{\frac{1}{k}}(c_1^1), \ldots, C \cap B_{\frac{1}{k}}(c_{N_k}^1)$ can not be covered by a finite subcover of $(O_j)_{j \in I}$, say, $C \cap B_{\frac{1}{k}}(c_{j_1}^1)$. Define $C_1 := C \cap B_{\frac{1}{k}}(c_{j_1}^1)$. Then, given $C_1, \ldots, C_k$ have already been constructed for some $k \in \mathbb{N}$, since $C_k$ can not be covered by a finite subcover of $(O_j)_{j \in I}$ and

$$C_k \subseteq C \subseteq \bigcup_{j=1}^{N_{k+1}} B_{\frac{1}{k+1}}(c_{j}^{k+1}),$$
there exists \( j_{k+1} \in \{1, \ldots, N_k+1 \} \) such that \( C_k \cap B_{\frac{1}{k+1}}(c_{j_{k+1}}) \) can not be covered by a finite subcover of \((O_j)_{j \in I}\), either. Define \( C_{k+1} := C_k \cap B_{\frac{1}{k+1}}(c_{j_{k+1}}) \). For each \( k \in \mathbb{N} \), choose some \( s_k \in C_k \) (note \( C_k \neq \emptyset \), as it can not be covered by finitely many \( O_j \)). Given \( \epsilon > 0 \), there is \( K \in \mathbb{N} \) such that \( \frac{2}{K} < \epsilon \). If \( k, l \geq K \), then \( s_k, s_l \in C_K \subseteq B_{\frac{1}{K}}(c_K) \) for some suitable \( j \in \{1, \ldots, N_K \} \). In particular, \( d(s_k, s_l) < \frac{2}{K} < \epsilon \), showing \((s_k)_{k \in \mathbb{N}}\) is a Cauchy sequence. As \((s_k)_{k \in \mathbb{N}}\) is a Cauchy sequence in \( C \) and \( C \) is complete, there exists \( c \in C \) such that \( \lim_{k \to \infty} s_k = c \). However, then there must exist some \( j \in I \) such that \( c \in O_j \) and, since \( O_j \) is open, there is \( \epsilon > 0 \) with \( B_{\epsilon}(c) \subseteq O_j \), and \( B_{\epsilon}(c) \) must contain almost all of the \( s_k \). Choose \( k \) sufficiently large such that \( \frac{1}{k} < \frac{\epsilon}{4} \) and \( d(s_k, c) < \frac{\epsilon}{2} \). Then, since \( s_k \in C_k \subseteq B_{\frac{1}{k}}(c_{j_k}) \),

One has

\[
\forall \ x \in B_{\frac{1}{k}}(c_{j_k}) \quad d(x, c) \leq d(x, s_k) + d(s_k, c) < \frac{2}{k} + \frac{\epsilon}{2} < \frac{2\epsilon}{4} + \frac{\epsilon}{2} = \epsilon,
\]

showing \( C_k \subseteq B_{\frac{1}{k}}(c_{j_k}) \subseteq B_{\epsilon}(c) \subseteq O_j \), in contradiction to \( C_k \) not being coverable by finitely many \( O_j \).

**Caveat 3.15.** A subset \( C \) of a topological space is defined to be **sequentially compact** if, and only if, every sequence in \( C \) has a convergent subsequence. Using this terminology, one can rephrase the equivalence between (ii) and (i) in Th. 3.14 by stating that a metric space is sequentially compact if, and only if, it is compact. However, in general topological spaces, neither implication remains true ((iii) of Th. 3.14 does not even make sense in general topological spaces, as the concepts of boundedness, total boundedness, and Cauchy sequences are, in general, not available): For an example of a topological space that is compact, but not sequentially compact, see, e.g. [Pre75, 7.2.10(a)]; for an example of a topological space that is sequentially compact, but not compact, see, e.g. [Pre75, 7.2.10(c)].

**Corollary 3.16.** (a) Let \((X, d)\) be a metric space, \( C, A \subseteq X \). If \( C \) is compact, \( A \) is closed, and \( A \cap C = \emptyset \), then \( \text{dist}(C, A) > 0 \).

(b) A compact subset \( C \) of a metric space is closed and bounded.

(c) Heine-Borel Theorem: A subset \( C \) of \( \mathbb{R}^n \), \( n \in \mathbb{N} \), is compact if, and only if, \( C \) is closed and bounded.

**Proof.** (a): Proceeding by contraposition, we show that \( \text{dist}(C, A) = 0 \) implies \( A \cap C \neq \emptyset \). If \( \text{dist}(C, A) = 0 \), then there exists a sequence \(((c_k, a_k))_{k \in \mathbb{N}}\) in \( C \times A \) such that \( \lim_{k \to \infty} d(c_k, a_k) = 0 \). As \( C \) is compact, we may assume \( \lim_{k \to \infty} c_k = c \in C \), also implying

\[
\lim_{k \to \infty} a_k = c, \quad \text{since} \quad \forall \ k \in \mathbb{N} \quad d(a_k, c) \leq d(a_k, c_k) + d(c_k, c). \quad (3.7)
\]

Since \( A \) is closed, (3.7) yields \( c \in A \), i.e. \( c \in A \cap C \).

(b): Let \( C \) be a compact subset of a metric space \( X \). Since \( X \) is \( T_2 \) by Ex. 3.4(c), \( C \) is closed by Prop. 3.9(b). Since \( C \) is totally bounded by Th. 3.14(iii), clearly, \( C \) is also bounded.
(c): If $C$ is compact, then it is closed and bounded by (b). If $C$ is closed and bounded, and $(x^k)_{k \in \mathbb{N}}$ is a sequence in $C$, then the boundedness and the Bolzano-Weierstrass Th. 1.31 yield a subsequence that converges to some $x \in \mathbb{R}^n$. However, since $C$ is closed, $x \in C$, showing that $C$ is compact.

The following Ex. 3.17 and Th. 3.18 show that, in general, sets in metric spaces can be closed and bounded without being compact.

**Example 3.17.** If $(X, d)$ is a noncomplete metric space, then it contains a Cauchy sequence that does not converge. It is not hard to see that such a sequence can not have a convergent subsequence, either. This shows that no noncomplete metric space can be compact. Moreover, the closure of every bounded subset of $X$ that contains such a nonconvergent Cauchy sequence is an example of a closed and bounded set that is noncompact. Concrete examples are given by $\mathbb{Q} \cap [a, b]$ for each $a, b \in \mathbb{R}$ with $a < b$ (these sets are $\mathbb{Q}$-closed, but not $\mathbb{R}$-closed!) and $]a, b[$ for each $a, b \in \mathbb{R}$ with $a < b$, in each case endowed with the usual metric $d(x, y) := |x - y|$.

In the previous example, the compactness of the closed and bounded sets failed due to noncompleteness. However, even in Banach spaces, there can be closed and bounded sets that are noncompact. In fact, according to the following Th. 3.18, the closed unit ball in a normed vector space $X$ is compact if, and only if, $X$ is finite-dimensional.

**Theorem 3.18.** A normed vector space $(X, \| \cdot \|)$ over $\mathbb{K}$ is finite-dimensional if, and only if, its closed unit ball $\overline{B}_1(0)$ is compact.

**Proof.** The proof is provided in Sec. H.2 of the Appendix.

**Theorem 3.19.** If $(X, \mathcal{T})$ is a topological space, $C \subseteq X$ is compact, and $f : C \rightarrow \mathbb{R}$ is continuous, then $f$ assumes its max and its min, i.e. there are $x_m \in C$ and $x_M \in C$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for each $x \in C$.

**Proof.** Since $C$ is compact and $f$ is continuous, $f(C) \subseteq \mathbb{R}$ is compact according to Th. 3.12. Then, by [Phi16, Lem. 7.53], $f(C)$ contains a smallest element $m$ and a largest element $M$. This, in turn, implies that there are $x_m, x_M \in C$ such that $f(x_m) = m$ and $f(x_M) = M$.

**Theorem 3.20.** If $(X, d_X)$ and $(Y, d_Y)$ are metric spaces, $C \subseteq X$ is compact, and $f : C \rightarrow Y$ is continuous, then $f$ is uniformly continuous.

**Proof.** If $f$ is not uniformly continuous, then there must be some $\epsilon > 0$ such that, for each $k \in \mathbb{N}$, there exist $x^k, y^k \in C$ satisfying $d_X(x^k, y^k) < 1/k$ and $d_Y(f(x^k), f(y^k)) \geq \epsilon$. Since $C$ is compact, there is $a \in C$ and a subsequence $(a^k)_{k \in \mathbb{N}}$ of $(x^k)_{k \in \mathbb{N}}$ such that $a = \lim_{k \to \infty} a^k$. Then there is a corresponding subsequence $(b^k)_{k \in \mathbb{N}}$ of $(y^k)_{k \in \mathbb{N}}$ such that $d_X(a^k, b^k) < 1/k$ and $d_Y(f(a^k), f(b^k)) \geq \epsilon$ for all $k \in \mathbb{N}$. Using the compactness of $C$ again, there is $b \in C$ and a subsequence $(v^k)_{k \in \mathbb{N}}$ of $(b^k)_{k \in \mathbb{N}}$ such that $b = \lim_{k \to \infty} v^k$. 
Now there is a corresponding subsequence \((u^k)_{k \in \mathbb{N}}\) of \((a^k)_{k \in \mathbb{N}}\) such that \(d_X(u^k, v^k) < 1/k\) and \(d_Y(f(u^k), f(v^k)) \geq \epsilon\) for all \(k \in \mathbb{N}\). Note that we still have \(a = \lim_{k \to \infty} v^k\).

Given \(\alpha > 0\), there is \(N \in \mathbb{N}\) such that, for each \(k > N\), one has \(d_X(a, u^k) < \alpha/3\), \(d_X(b, v^k) < \alpha/3\), and \(d_X(u^k, v^k) < 1/k < \alpha/3\). Thus, \(d_X(a, b) < d_X(a, u^k) + d_X(u^k, v^k) + d_X(b, v^k) < \alpha\), implying \(d(a, b) = 0\) and \(a = b\). Finally, the continuity of \(f\) implies \(f(a) = \lim_{k \to \infty} f(u^k) = \lim_{k \to \infty} f(v^k)\) in contradiction to \(d_Y(f(u^k), f(v^k)) \geq \epsilon\). ■

**Theorem 3.21 (Lebesgue Number).** Let \((X, d)\) be a metric space and \(C \subseteq X\). If \(C\) is compact and \((O_j)_{j \in I}\) is an open cover of \(C\), then there exists a Lebesgue number \(\delta\) for the open cover, i.e., some \(\delta > 0\) such that, for each \(A \subseteq C\) with \(\text{diam}(A) < \delta\), there exists \(j_0 \in I\), where \(A \subseteq O_{j_0}\).

**Proof.** Seeking a contradiction, assume there is no Lebesgue number for the open cover \((O_j)_{j \in I}\). Then there are sequences \((x_k)_{k \in \mathbb{N}}\) in \(C\) and \((A_k)_{k \in \mathbb{N}}\) in \(\mathcal{P}(C)\) such that

\[
\forall \ k \in \mathbb{N} \quad \left( x_k \in A_k, \ \text{diam} A_k < \frac{1}{k}, \ \text{and} \ \forall \ j \in I \ A_k \nsubseteq O_j \right). \tag{3.8}
\]

As \(C\) is compact, we may assume that \(\lim_{k \to \infty} x_k = c \in C\). Then there must be \(O_j\) such that \(c \in O_j\) and \(\epsilon > 0\) such that \(B_{\epsilon}(c) \subseteq O_j\). If \(k \in \mathbb{N}\) is such that \(\frac{1}{k} < \frac{\epsilon}{2}\) and \(d(x_k, c) < \frac{\epsilon}{2}\), then, for each \(a \in A_k\), we have \(d(a, c) \leq d(a, x_k) + d(x_k, c) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\), implying the contradiction \(A_k \nsubseteq B_{\epsilon}(c) \subseteq O_j\). ■

**Theorem 3.22.** If \((X, T_X)\) and \((Y, T_Y)\) are topological spaces, \(C \subseteq X\) is compact, \((Y, T_Y)\) is \(T_2\), and \(f : C \to Y\) is continuous and one-to-one, then \(f^{-1} : f(C) \to C\) is continuous (i.e., \(f : C \to f(C)\) is a homeomorphism).

**Proof.** By Th. 2.7(iv), it suffices to show \(f(A)\) is closed in \(f(C)\) for each \(C\)-closed \(A \subseteq C\). If \(A \subseteq C\) is closed in \(C\), then \(A\) is compact by Prop. 3.9(a). Then \(f(A)\) is compact by Th. 3.12 and, thus, closed in \(f(C)\) by Prop. 3.9(b) (since \(Y\) and, hence, \(f(C)\) are \(T_2\)). ■

**Example 3.23. (a)** The following example shows that the statement of Th. 3.22 does not hold without the hypothesis that \(C\) be compact: Let \(C := [0, 2\pi[\) and \(Y := \mathbb{C}\) with the usual topologies, \(f : C \to Y, f(t) := e^{it}\). Then \(f\) is continuous due to the continuity of the exponential function. Moreover, \(f\) is injective with \(f(C) = S_1(0)\) by [Phil16, Cor. 8.30] However, \(f^{-1} : f(C) \to C\) is not continuous: Consider the sequence \((z_n)_{n \in \mathbb{N}}\), where

\[
z_n := \begin{cases} 
e^{\frac{1}{n}} & \text{for } n \text{ even,} \\ e^{i(2\pi - \frac{1}{n})} & \text{for } n \text{ odd.} \end{cases}
\]

Then, clearly, \((z_n)_{n \in \mathbb{N}}\) is a sequence in \(f(C)\) with \(\lim_{n \to \infty} z_n = 1\). On the other hand,

\[
f^{-1}(z_n) = \begin{cases} \frac{1}{n} & \text{for } n \text{ even,} \\ 2\pi - \frac{1}{n} & \text{for } n \text{ odd,} \end{cases}
\]

showing that \((f^{-1}(z_n))_{n \in \mathbb{N}}\) does not converge.
The following example shows that the statement of Th. 3.22 does not hold without the hypothesis that \((Y, T_Y)\) be \(T_2\): Let \(X := Y := \lbrack 0, 1\rbrack\), let \(T_X\) be the usual (metric) topology, and let \(T_Y\) be the cofinite topology. Then \(T_Y \subseteq T_X\) (since finite sets are \(T_X\)-closed, but, e.g., \(]0, \frac{1}{2}\lbrack \not\in T_Y\)). Then \((X, T_X)\) is compact, \(\text{Id} : X \to Y\) is bijective and continuous, but \(\text{Id} : Y \to X\) is not continuous.

**Proposition 3.24.** Let \((X, T)\) be a topological space. If \((X, T)\) is both compact and \(T_2\), then it is normal.

*Proof.* Exercise.

**Theorem 3.25** (Tychonoff). Let \((X_i, T_i)\) be topological spaces, \(i \in I\). If \(X = \prod_{i \in I} X_i\) is endowed with the product topology \(T\) and each \(X_i\) is compact, then \(X\) is compact.

*Proof.* The proof is provided in Sec. H.3 of the Appendix.

### 3.3 Connectedness

**Definition 3.26.** Let \((X, T)\) be a topological space \(C \subseteq X\).

(a) \(X\) is called *connected* if, and only if, there are no nonempty disjoint open sets \(O_1, O_2 \in T\) such that \(X = O_1 \cup O_2\). The subset \(C\) is called *connected* if, and only if, it is connected with respect to the subspace topology \(T_C\).

(b) A *path* in \(X\) is a continuous map \(\phi : \lbrack 0, 1\rbrack \to X\). We say that \(x, y \in X\) are connected by the path \(\phi\) if, and only if, \(\phi(0) = x\) and \(\phi(1) = y\). We call \(C\) path-connected if, and only if, for each \(x, y \in C\), there exists a path in \(C\), connecting \(x\) and \(y\).

**Lemma 3.27.** For a topological space \((X, T)\), the following statements are equivalent:

(i) \((X, T)\) is connected.

(ii) The only clopen (i.e. closed and open) subsets of \(X\) are \(X\) and \(\emptyset\).

(iii) If \(X = A_1 \cup A_2\), \(A_1 \cap A_2 = \emptyset\), and \(A_1\) and \(A_2\) are both open (or both closed), then \(A_1 = \emptyset\) or \(A_2 = \emptyset\).

(iv) If \(f : X \to \{0, 1\}\) is continuous (with respect to the discrete topology on \(\{0, 1\}\)), then \(f\) is constant.

*Proof.* The equivalences of (i) – (iii) are immediate from Def. 3.26(a).

(ii) \(\Rightarrow\) (iv): If \(f : X \to \{0, 1\}\) is continuous and nonconstant, then \(O_1 := f^{-1}\{\{0\}\}\) and \(O_1 := f^{-1}\{\{1\}\}\) are disjoint nonempty clopen subsets of \(X\).

(iv) \(\Rightarrow\) (i): If \(X = O_1 \cup O_2\) with both \(O_1, O_2\) nonempty and open, then \(f : X \to \{0, 1\}\), \(f(x) := 0\) for \(x \in O_1\), \(f(x) := 1\) for \(x \in O_2\), defines a continuous nonconstant map.
Example 3.28. It is immediate that a discrete space with at least two points is never connected and that an indiscrete space is always connected.

Proposition 3.29. Let \((X, \mathcal{T})\) be a topological space.

(a) If \(A, B \subseteq X\) such that \(A \subseteq B \subseteq \overline{A}\), then \(A\) connected implies \(B\) connected.

(b) Let \((A_i)_{i \in I}, I \neq \emptyset\), be a family of subsets of \(X\) such that \(B := \bigcap_{i \in I} A_i \neq \emptyset\). If each \(A_i\) is connected (resp. path-connected), then \(C := \bigcup_{i \in I} A_i\) is connected (resp. path-connected).

Proof. (a): Let \(M_1, M_2\) be closed subsets of \(X\) such that \(B = (M_1 \cap B) \cup (M_2 \cap B)\). Since \(A\) is connected and \(A = (M_1 \cap A) \cup (M_2 \cap A)\), we have \(M_1 \cap A = \emptyset\) or \(M_2 \cap A = \emptyset\), say \(M_1 \cap A = \emptyset\). Then \(A \subseteq M_2\). Since \(M_2\) is closed, this implies \(\overline{A} \subseteq M_2\) and \(B \subseteq M_2\).

Thus, \(B = M_2 \cap B\) and \(M_1 \cap B = \emptyset\), proving \(B\) to be connected.

(b): Let \(U, V\) be disjoint subsets of \(C\) that are open in \(C\) and satisfy \(C = U \cup V\). Moreover, let \(x \in B\). Then either \(x \in U\) or \(x \in V\). Then, for each \(i \in I\), \(x \in U \cap A_i\). Since \(A_i = (U \cap A_i) \cup (V \cap A_i)\) and both \(U \cap A_i\) and \(V \cap A_i\) are open in \(A_i\), then connectedness of \(A_i\) implies \(V \cap A_i = \emptyset\). As this holds for each \(i \in I\), we have \(V \cap C = \emptyset\), showing that \(C\) is connected. The path-connected version is left as an exercise.

Example 3.34(b) below shows that Prop. 3.29(a) does not hold with “connected” replaced by “path-connected”.

Definition and Remark 3.30. Let \((X, \mathcal{T})\) be a topological space.

(a) Proposition 3.29(b) allows to define, for each \(x \in X\), the connected component \(C_x\) (resp. the path-component \(P_x\)) of \(x\) as the union of all connected (resp. path-connected) subsets of \(X\) that contain \(x\). Then these components are the largest connected (resp. path-connected) subsets of \(X\) that contain \(x\). Then, clearly, if one defines \(x \sim y\) if, and only if, \(y \in C_x\) (resp. \(y \in P_x\)), then \(\sim\) constitutes an equivalence relation on \(X\). The equivalence classes of \(\sim\) are called the connected components (resp. the path-components) of \(X\). Thus, each topological space is the disjoint union of its connected components and also the disjoint union of its path-components.

(b) As a consequence of Prop. 3.29(a), connected components are always closed, whereas Ex. 3.34(b) below shows that path-components are not necessarily closed.

Theorem 3.31. If \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) are topological spaces, \(X\) is connected (resp. path-connected), and \(f : X \rightarrow Y\) is continuous, then \(f(X)\) is connected (resp. path-connected).

Proof. If \(B := f(X)\) is not connected, then there are nonempty sets \(O_1, O_2 \subseteq B\) that are open in \(B\), such that \(B = O_1 \cup O_2\). Then \(U_1 := f^{-1}(O_1)\), \(U_2 := f^{-1}(O_2)\) are disjoint open subsets of \(X\) such that \(X = U_1 \cup U_2\), showing \(X\) is not connected. Now, if \(y_1, y_2 \in B\), then there are \(x_1, x_2 \in X\) such that \(f(x_1) = y_1\) and \(f(x_2) = y_2\). If \(\phi : [0, 1] \rightarrow X\), is a path in \(X\) connecting \(x_1\) and \(x_2\), then \(f \circ \phi\) is a path in \(B\) connecting \(y_1\) and \(y_2\).
Theorem 3.32. If \( A \subseteq \mathbb{R} \), then \( A \) is connected if, and only if, \( A \) is an interval.

Proof. If \( A \) is not an interval, then there are \( a, b, c \in \mathbb{R} \) such that \( a < c < b \), \( a, b \in A \), \( c \notin A \). Then \( O_1 := ] - \infty, c[ \) and \( O_2 := ]c, \infty[ \) are disjoint open subsets of \( \mathbb{R} \) such that \( A = (O_1 \cap A) \cup (O_2 \cap A) \) and both \( O_1 \cap A \) and \( O_2 \cap A \) are nonempty. It remains to show that, if \( A \) is an interval, then \( A \) is connected. We may assume that \( A \) consists of more than one point. Moreover, let \( O_1, O_2 \) be disjoint subsets of \( A \) that are open in \( A \) and such that \( A = O_1 \cup O_2 \). Seeking a contradiction, assume \( O_1 \neq \emptyset \) and \( O_2 \neq \emptyset \). Thus, there are \( s \in O_1 \), \( t \in O_2 \), and, without loss of generality, \( s < t \) (otherwise, switch the names of \( O_1 \) and \( O_2 \)). Let \( a := \sup \{x \in O_1 : x < t \} \). Then \( a \in O_1 \) since \( O_1 \) is closed in \( A \) (note \( a \in A \) since \( a \leq t \) and \( A \) is an interval). In particular \( a < t \). However, since \( O_1 \) is also open in the interval \( A \), there must be an open interval \( I \) such that \( a \in I, I \subseteq O_1 \), showing that \( O_1 \) contains elements between \( a \) and \( t \), in contradiction to the definition of \( a \).

Corollary 3.33. If the topological space \((X, T)\) is path-connected, then it is connected.

Proof. Assume \((X, T)\) is path-connected and let \( x \in X \). If \( y \in X \), then there is a path \( \phi_y : [0, 1] \to X \), connecting \( x \) and \( y \). Since \([0, 1]\) is connected by Th. 3.32, so is \( A_y := \phi_y([0, 1]) \). Since \( X = \bigcup_{y \in X} A_y \), \( X \) is connected by Prop. 3.29(b).

Example 3.34. (a) In general, neither unions nor intersections of (even just two) connected sets are connected: Let \((X, T)\) be a topological space. Then, for each \( x \in X \), \( \{x\} \) is connected (even path-connected), but, if \((X, T)\) is \( T_2 \) and \( x, y \in X \) with \( x \neq y \), then \( \{x, y\} \) is not connected. Now we consider

\[
C_1 := \{e^{it} : t \in [0, \pi]\}, \quad C_2 := \{e^{it} : t \in [\pi, 2\pi]\}.
\]

Then, as the continuous images of intervals, both \( C_1 \) and \( C_2 \) are connected (even path-connected) subsets of \( \mathbb{C} \). However, \( C_1 \cap C_2 = \{-1, 1\} \) is not a connected subset of \( \mathbb{C} \).

(b) The following example shows that a connected set is not necessarily path-connected, and that the closure of a path-connected set is not necessarily path-connected: Let \( X := \mathbb{R}^2 \) with the norm topology. Let

\[
G := \left\{(t, \sin \left(\frac{1}{t}\right)) : t \in \mathbb{R}^+\right\}, \quad A := \left(\{0\} \times [-1, 1]\right) \cup G.
\]

Then \( A \) is connected but not path-connected: As the image of the connected set \( \mathbb{R}^+ \) under the continuous map \( t \mapsto (t, \sin(\frac{1}{t})) \), the set \( G \) is connected. We claim \( A = \overline{G} \): Let \( s \in [-1, 1] \). By the intermediate value theorem, there is a sequence \((t_n)_{n \in \mathbb{N}}\) in \( \mathbb{R}^+ \) such that \( \lim_{n \to \infty} t_n = 0 \) and \( s = \sin(\frac{1}{t_n}) \) for each \( n \in \mathbb{N} \). Then \((t_n, s)_{n \in \mathbb{N}}\) is a sequence in \( G \) with \( \lim_{n \to \infty} (t_n, s) = (0, 0) \), showing \( A \subseteq \overline{G} \). On the other hand, let \((t_n, s_n)_{n \in \mathbb{N}}\) be an arbitrary sequence in \( G \) with \( \lim_{n \to \infty} (t_n, s_n) = (t, s) \). If \( t \in \mathbb{R}^+ \), then \( s = \sin(\frac{1}{t}) \) and \((t, s) \in G \). Otherwise, \( \lim_{n \to \infty} t_n = 0 \) and \( s \in [-1, 1] \), since each \( s_n \in [-1, 1] \) and \([-1, 1] \) is closed, showing \( A \supseteq \overline{G} \).
Now $A = \overline{G}$ implies $A$ is connected by Prop. 3.29(a). It remains to verify $A$ is not path-connected. Proceeding by contradiction, assume there were a path $\lambda : [0, 1] \rightarrow A$ connecting $(0, 0)$ and $(1, \sin 1)$. Since $\lambda([0, 1])$ must be connected, one obtains $\{(t, \sin(t)) : t \in [0, 1]\} \subseteq \lambda([0, 1])$. Let $s := \sup \{t \in [0, 1] : \lambda(t) \in \{0\} \times [0, 1]\}$. Then $\lambda(s) \in \{0\} \times [0, 1]$, since $\{0\} \times [0, 1]$ is closed. In particular, $s < 1$. Moreover, with respect to the max-norm on $\mathbb{R}^2$, there is $\epsilon \in \mathbb{R}^+$ such that $\lambda(B(s)) \subseteq B_{\frac{\epsilon}{2}}(\lambda(s))$. Let $s_+ := \min \{s + \frac{\epsilon}{2}, \frac{s+1}{2}\}$. By the definition of $s$ there is $t_0 \in [0, 1]$ such that $\lambda(s_+) = (t_0, \sin(\frac{1}{t_0}))$. As $\lambda(B(s))$ must be connected, one has $C := \{(t, \sin(t)) : t \in [0, t_0]\} \subseteq \lambda(B(s))$, which is in contradiction to $\lambda(B(s)) \subseteq B_{\frac{\epsilon}{2}}(\lambda(s))$ (since $\text{diam}(B_{\frac{\epsilon}{2}}(\lambda(s))) \leq 1$ and $\text{diam}(C) \geq |1 - (-1)| = 2$. This finishes the proof that $A$ is not path-connected.

**Theorem 3.35.** An open subset of a normed vector space (e.g. an open subset of $\mathbb{R}^n$) is connected if, and only if, it is path-connected.

**Proof.** See, e.g., [Heu08, Th. 161.4]. ■

**Theorem 3.36.** Let $(X_i, \mathcal{T}_i)$ be topological spaces, $i \in I$. If $X = \prod_{i \in I} X_i$ is endowed with the product topology $\mathcal{T}$ and each $X_i$ is connected (resp. path-connected), then $X$ is connected (resp. path-connected).

**Proof.** Exercise. ■

### 4 Differential Calculus

#### 4.1 Partial Derivatives and Gradients

The goal of the following is to generalize the notion of derivative from one-dimensional functions to functions $f : G \rightarrow \mathbb{K}$, where $G \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$. Later we will also allow functions with values in $\mathbb{K}^m$. For $\xi \in G$, $G \subseteq \mathbb{R}^n$, we will define a function $f : G \rightarrow \mathbb{K}$ to have a so-called partial derivative (or just partial for short) at $\xi$ with respect to the variable $x_j$ if, and only if, the one-dimensional function that results from keeping all but the $j$th variable fixed, namely

$$x_j \mapsto \phi(x_j) := f(\xi_1, \ldots, \xi_{j-1}, x_j, \xi_{j+1}, \ldots, \xi_n),$$

is differentiable at $x_j = \xi_j$ in the usual sense for one-dimensional functions. The partial derivative of $f$ at $\xi$ with respect to $x_j$ is then identified with $\phi'(\xi_j)$. This leads to the following definition:

**Definition 4.1.** Let $G \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $f : G \rightarrow \mathbb{K}$, $\xi \in G$, $j \in \{1, \ldots, n\}$. If there is $\epsilon > 0$ such that $\xi + he_j \in G$ for each $h \in ]-\epsilon, \epsilon[$ (this condition is trivially satisfied if $\xi$ is an interior point of $G$), then $f$ is said to have a **partial derivative** at $\xi$ with respect to the variable $x_j$ (or a $j$th partial for short) if, and only if, the limit

$$\lim_{h \rightarrow 0} \frac{f(\xi + he_j) - f(\xi)}{h} \quad (0 \neq h \in ]-\epsilon, \epsilon[) \quad (4.1)$$

is finite.
exists in $\mathbb{K}$. In that case, the limit is defined to the $j$th partial of $f$ at $\xi$ and it is denoted with one of the symbols

$$\partial_j f(\xi), \partial_{x_j} f(\xi), \frac{\partial f(\xi)}{\partial x_j}, f_{x_j}(\xi), D_j f(\xi).$$

If $\xi$ is a boundary point of $G$ and there is $\varepsilon > 0$ such that, for each $h \in [0, \varepsilon], \xi + he_j \in G$ and $\xi - he_j \notin G$ (resp. $\xi - he_j \in G$ and $\xi + he_j \notin G$), then, instead of the limit in (4.1), one uses the one-sided limit

$$\lim_{h \downarrow 0} \frac{f(\xi + he_j) - f(\xi)}{h} \quad \text{(resp. } \lim_{h \uparrow 0} \frac{f(\xi + he_j) - f(\xi)}{h})$$

in the above definition of the $j$th partial at $\xi$. If all the partials of $f$ exist in $\xi$, then the vector

$$\nabla f(\xi) := (\partial_1 f(\xi), \ldots, \partial_n f(\xi))$$

is called the gradient of $f$ at $\xi$ (the symbol $\nabla$ is called nabla, the corresponding operator is sometimes called del). It is customary to consider the gradient as a row vector. If the $j$th partial $\partial_j f(\xi)$ exists for each $\xi \in G$, then the function

$$\partial_j f : G \rightarrow \mathbb{K}, \; \xi \mapsto \partial_j f(\xi),$$

is also called the $j$th partial of $f$.

**Example 4.2.** The following example shows that, in general, the existence of partial derivatives does not imply continuity: Consider the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \; f(x, y) := \begin{cases} \frac{xy}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

Using the quotient rule for $(x, y) \neq (0, 0)$ and the fact that $f(x, 0) = f(0, y) = 0$ for all $(x, y) \in \mathbb{R}^2$, one obtains

$$\nabla f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \; \nabla f(x, y) = \begin{cases} \left( \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}, \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \right) & \text{for } (x, y) \neq (0, 0), \\ (0, 0) & \text{for } (x, y) = (0, 0). \end{cases}$$

In particular, both partials $\partial_x f$ and $\partial_y f$ exist everywhere in $\mathbb{R}^2$. However, $f$ is not continuous in $(0, 0)$: For $k \in \mathbb{N}$, let $x_k := (1/k), y_k := (1/k)$. Then $\lim_{k \rightarrow \infty} (x_k, y_k) = (0, 0)$, but

$$f(x_k, y_k) = \frac{1}{\frac{1}{k^2} + \frac{1}{k^2}} = \frac{1}{2}$$

for each $k \in \mathbb{N}$. In particular, $\lim_{k \rightarrow \infty} f(x_k, y_k) = \frac{1}{2} \neq 0 = f(0, 0)$, showing that $f$ is not continuous in $(0, 0)$.

**Remark 4.3.** The problem in Example 4.2 is the discontinuity of the partials in $(0, 0)$. We will see in Th. 4.29 below that, if all partials of $f$ exist and are continuous in some neighborhood of a point $\xi$, then $f$ is continuous (and even differentiable) in $\xi$. 
4.2 The Jacobian

If \( f : G \longrightarrow \mathbb{K}^m \), where \( G \subseteq \mathbb{R}^n \), then we can compute partials for each of the coordinate functions \( f_j \) of \( f \) (provided the partials exist).

**Definition 4.4.** Let \( G \subseteq \mathbb{R}^n \), \( f : G \longrightarrow \mathbb{K}^m \), \((n,m) \in \mathbb{N}^2\), \( \xi \in G \). If, for each \( l \in \{1, \ldots, m\} \), the coordinate function \( f_l = \pi_l \circ f \) (recall that \( f = (f_1, \ldots, f_m) \)) has all partials \( \partial_k f_l \) at \( \xi \), then these \( m \cdot n \) partials form an \( m \times n \) matrix, namely

\[
J_f(\xi) := \frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}(\xi) := \begin{pmatrix}
\partial_1 f_1(\xi) & \cdots & \partial_n f_1(\xi) \\
\vdots & & \vdots \\
\partial_1 f_m(\xi) & \cdots & \partial_n f_m(\xi)
\end{pmatrix} = \begin{pmatrix}
\nabla f_1(\xi) \\
\vdots \\
\nabla f_m(\xi)
\end{pmatrix},
\]

(4.5)
called the Jacobian matrix of \( f \) at \( \xi \). In the case that \( m = n \), the Jacobian matrix \( J_f(\xi) \) is quadratic and one can compute its determinant \( \det J_f(\xi) \). This determinant is then called the Jacobian determinant of \( f \) at \( \xi \). Both the Jacobi matrix and the Jacobian determinant are sometimes referred to as the Jacobian. One then has to determine from the context which of the two is meant.

**Remark 4.5.** In many situations, it does not matter if you interpret \( z \in \mathbb{K}^n \) as a column vector or a row vector, and the same is true for the gradient. However, in the context of matrix multiplications, it is important to work with a consistent interpretation of such vectors. We will therefore adhere to the following agreement: In the context of matrix multiplications, we always interpret \( x \in \mathbb{R}^n \) and \( f(x) \in \mathbb{K}^m \) for \( \mathbb{K}^m \)-valued functions \( f \) as column vectors, whereas we always interpret the gradients \( \nabla g(x) \) of \( \mathbb{K} \)-valued functions \( g \) as row vectors.

**Example 4.6.** (a) Let \( A \) be an \( m \times n \) matrix over \( \mathbb{K} \),

\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{pmatrix}.
\]

Then the map \( x \mapsto Ax \), \( A : \mathbb{R}^n \longrightarrow \mathbb{K}^m \), is \( \mathbb{K} \)-linear for \( \mathbb{K} = \mathbb{R} \), and it is the restriction to \( \mathbb{R}^n \) of the \( \mathbb{C} \)-linear map \( A \) on \( \mathbb{C}^n \) for \( \mathbb{K} = \mathbb{C} \) (note that, due to the agreement from Rem. 4.5, \( Ax \) can be interpreted as a matrix multiplication in the usual way). Thus, if we denote the coordinate functions \( \pi_l \circ A \) by \( A_l \), \( l \in \{1, \ldots, m\} \), then \( A_l(x) = \sum_k^n a_{lk}x_k \) and \( \partial_l A_l(x) = \frac{\partial A_l(x)}{\partial x_k} = a_{lk} \). Thus, \( J_A(x) = A \) for each \( x \in \mathbb{R}^n \).

(b) Consider \( (f,g) : \mathbb{R}^3 \longrightarrow \mathbb{C}^2 \), \( (f(x,y,z),g(x,y,z)) := (ixyz^2, ix + yz) \). Then one computes the following Jacobian:

\[
J_{(f,g)}(x,y,z) = \begin{pmatrix}
\nabla f(x,y,z) \\
\nabla g(x,y,z)
\end{pmatrix} = \begin{pmatrix}
\frac{\partial f(x,y,z)}{\partial x} & \frac{\partial f(x,y,z)}{\partial y} & \frac{\partial f(x,y,z)}{\partial z} \\
\frac{\partial g(x,y,z)}{\partial x} & \frac{\partial g(x,y,z)}{\partial y} & \frac{\partial g(x,y,z)}{\partial z}
\end{pmatrix} = \begin{pmatrix}
ixz^2 & 2ixyz & iy^2 \\
ixz^2 & 2ixyz & iy^2
\end{pmatrix}.
\]
Consider \((f, g) : \mathbb{R}^2 \rightarrow \mathbb{C}^2, (f(x, y), g(x, y)) := (e^{ixy}, x + 2y)\). Then one computes the following Jacobian determinant:

\[
\det J_{(f,g)}(x, y) = \begin{vmatrix}
  iy e^{ixy} & ix e^{ixy} \\
  1 & 2
\end{vmatrix} = ie^{ixy}(2y - x).
\]

**Remark 4.7.** The linearity of forming the derivative of one-dimensional functions directly implies the linearity of forming partial derivatives, gradients, and Jacobians (provided they exist). More precisely, if \(G \subseteq \mathbb{R}^n, f, g : G \rightarrow \mathbb{K}^m, (n, m) \in \mathbb{N}^2, \xi \in G, \) and \(\lambda \in \mathbb{K}, \) then, for each \((l, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\}, \)

\[
\partial_k(f + g)_l(\xi) = \partial_k f_l(\xi) + \partial_k g_l(\xi), \quad \partial_k(\lambda f)_l(\xi) = \lambda \partial_k f_l(\xi),
\]

\[
\nabla (f + g)_l(\xi) = \nabla f_l(\xi) + \nabla g_l(\xi), \quad \nabla (\lambda f)_l(\xi) = \lambda \nabla f_l(\xi),
\]

\[
J_{f+g}(\xi) = J_f(\xi) + J_g(\xi), \quad J_{\lambda f}(\xi) = \lambda J_f(\xi),
\]

(4.6a)

where, in each case, the assumed existence of the objects on the right-hand side of the equation implies the existence of the object on the left-hand side.

### 4.3 Higher Order Partials and the Spaces \(C^k\)

Partial derivatives can, in turn, have partial derivatives themselves and so on. For example, a function \(f : \mathbb{R}^3 \rightarrow \mathbb{K}\) might have the following partial derivative of 6th order: \(\partial_1 \partial_2 \partial_3 \partial_1 \partial_2 \partial_3 f.\) We will see that, in general, it is important in which order the different partial derivatives are carried out (see Example 4.9). If all partial derivatives are continuous, then the situation is much better and the result is the same, no matter what order is used for the partial derivatives (continuous partials commute, see Th. 4.12). We start with the definition of higher order partials:

**Definition 4.8.** Let \(G \subseteq \mathbb{R}^n, f : G \rightarrow \mathbb{K}, \xi \in G.\) Fix \(k \in \mathbb{N}.\) For each element \(p = (p_1, \ldots, p_k) \in \{1, \ldots, n\}^k,\) define the following partial derivative of \(k\)th order provided that it exists:

\[
\partial_p f(\xi) := \frac{\partial^k f(\xi)}{\partial x_{p_1} \cdots \partial x_{p_k}} := \partial_{p_1} \cdots \partial_{p_k} f(\xi).
\]

(4.7)

One also defines \(f\) itself to be its own partial derivative of order 0. Analogous to Def. 4.4, if \(f : G \rightarrow \mathbb{K}^m, m \in \mathbb{N},\) then one defines the higher order partials for each coordinate function \(f_l, l = 1, \ldots, m,\) i.e. one uses \(f_l\) instead of \(f\) in (4.7).

**Example 4.9.** The following example shows that, in general, partial derivatives do not commute: Consider the function

\[
f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) := \begin{cases} 
  \frac{-xy^3}{x^2+y^2} & \text{for } (x, y) \neq (0, 0), \\
  0 & \text{for } (x, y) = (0, 0).
\end{cases}
\]

Analogous to Example 4.2, using the quotient rule for \((x, y) \neq (0, 0)\) and the fact that
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\[ f(x,0) = f(0,y) = 0 \text{ for all } (x,y) \in \mathbb{R}^2, \] one obtains

\[ \nabla f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \nabla f(x,y) = (\partial_1 f(x,y), \partial_2 f(x,y)) = (\partial_x f(x,y), \partial_y f(x,y)) \]

\[ = \begin{cases} 
\left( \frac{y^3(x^2-y^2)}{(x^2+y^2)^2}, \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2} \right) & \text{for } (x,y) \neq (0,0), \\
(0,0) & \text{for } (x,y) = (0,0).
\end{cases} \]

In particular, we have \( \partial_1 f(0,y) = \partial_x f(0,y) = y \) for each \( y \in \mathbb{R} \) and \( \partial_2 f(x,0) = \partial_y f(x,0) = 0 \) for each \( x \in \mathbb{R} \). Thus, \( \partial_y \partial_x f(0,y) \equiv 1 \) and \( \partial_x \partial_y f(x,0) \equiv 0 \). Evaluating at \((0,0)\) yields \( \partial_2 \partial_1 f(0,0) = \partial_y \partial_x f(0,0) = 1 \neq 0 = \partial_x \partial_y f(0,0) = \partial_1 \partial_2 f(0,0) \).

---

As in Ex. 4.2, the problem in Ex. 4.9 lies in the discontinuity of the partials in \((0,0)\). As mentioned above, if all partials are continuous, then they do commute. To prove this result is our next goal. We will accomplish this in several steps. We start with a preparatory lemma that provides a variant of the mean value theorem in two dimensions.

**Lemma 4.10.** Let \( a, \tilde{a}, b, \tilde{b} \in \mathbb{R}, a \neq \tilde{a}, b \neq \tilde{b} \), and consider the square \( I = [a, \tilde{a}] \times [b, \tilde{b}] \) (which constitutes a closed interval in \( \mathbb{R}^2 \)). Suppose \( f : I \rightarrow \mathbb{R}, (x,y) \mapsto f(x,y) \), and set

\[ \Delta_I(f) := f(\tilde{a}, \tilde{b}) + f(a,b) - f(a,\tilde{b}) - f(\tilde{a},b). \]

If \( \partial_x f \) and \( \partial_y \partial_x f \) exist everywhere in \( I \), then there is some point \((\xi, \eta) \in I^o \) (i.e. with \( \xi \in ]a, \tilde{a}[ \) and \( \eta \in ]b, \tilde{b}[ \), satisfying

\[ \Delta_I(f) = (\tilde{a} - a)(\tilde{b} - b)\partial_y \partial_x f(\xi, \eta). \]

**Proof.** Since the function \( g : [a, \tilde{a}] \rightarrow \mathbb{R}, g(x) := f(x, \tilde{b}) - f(x, b) \), is differentiable, the one-dimensional mean value theorem [Phi16, Th. 9.18] yields the existence of some \( \xi \in ]a, \tilde{a}[ \) satisfying

\[ \Delta_I(f) = g(\tilde{a}) - g(a) = (\tilde{a} - a)g'(\xi) = (\tilde{a} - a)(\partial_x f(\xi, \tilde{b}) - \partial_x f(\xi, b)). \quad (4.8a) \]

Since the function \( G : [b, \tilde{b}] \rightarrow \mathbb{R}, G(y) := \partial_x f(\xi, y) \), is differentiable, the one-dimensional mean value theorem [Phi16, Th. 9.18] yields the existence of some \( \eta \in ]b, \tilde{b}[ \) satisfying

\[ \partial_x f(\xi, \tilde{b}) - \partial_x f(\xi, \eta) = G(\tilde{b}) - G(b) = (\tilde{b} - b)G'(\eta) = (\tilde{b} - b)\partial_y \partial_x f(\xi, \eta). \quad (4.8b) \]

Combining (4.8a) and (4.8b) proves the lemma.

**Theorem 4.11** (Schwarz). Let \( G \) be an open subset of \( \mathbb{R}^2 \). Suppose that \( f : G \rightarrow \mathbb{K}, (x,y) \mapsto f(x,y) \), has partial derivatives \( \partial_x f, \partial_y f, \) and \( \partial_y \partial_x f \) everywhere in \( G \). If \( \partial_y \partial_x f \) is continuous in \((a,b) \in G\), then \( \partial_x \partial_y f(a,b) \) exists and \( \partial_x \partial_y f(a,b) = \partial_y \partial_x f(a,b) \) (in particular, \( \partial_y \partial_x f = \partial_x \partial_y f \) if all the functions \( f, \partial_x f, \partial_y f, \partial_y \partial_x f \) are continuous).
Proof. We first note that it suffices to prove the theorem for \( \mathbb{K} = \mathbb{R} \), as one can then apply the result to both \( \text{Re} \, f \) and \( \text{Im} \, f \) to obtain the case \( \mathbb{K} = \mathbb{C} \). Thus, for the remainder of the proof, we assume \( f \) to be \( \mathbb{R} \)-valued. Given \( \epsilon > 0 \), since \( \partial_y \partial_x f \) is continuous in \((a, b)\) and since \( G \) is open, there exists \( \delta > 0 \) such that \( I := [a-\delta, a+\delta] \times [b-\delta, b+\delta] \subseteq G \) and

\[
\forall (x,y) \in I \quad \| \partial_y \partial_x f(x, y) - \partial_y \partial_x f(a, b) \| < \epsilon.
\] (4.9)

Let \( (h, k) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) with \( 0 < |h|, |k| < \delta \). Since

\[
\frac{f(a+h, b+k) + f(a, b) - f(a, b+k) - f(a+h, b)}{hk} = \frac{1}{h} \left( \frac{f(a+h, b+k) - f(a+h, b)}{k} - \frac{f(a, b+k) - f(a, b)}{k} \right) - \partial_y \partial_x f(a, b).
\] (4.10)

Lem. 4.10 together with (4.9) implies

\[
\left| \frac{1}{h} \left( \frac{f(a+h, b+k) - f(a+h, b)}{k} - \frac{f(a, b+k) - f(a, b)}{k} \right) - \partial_y \partial_x f(a, b) \right| < \epsilon.
\] (4.11)

Taking the limit for \( k \to 0 \) in (4.11) yields

\[
\left| \frac{\partial_y f(a+h, b) - \partial_y f(a, b)}{h} - \partial_y \partial_x f(a, b) \right| \leq \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we have shown \( \partial_x \partial_y f(a, b) = \partial_y \partial_x f(a, b) \) as desired. \( \blacksquare \)

Using the combinatorial result that one can achieve an arbitrary permutation by a finite sequence of permutations of precisely two juxtaposed elements (cf. [Phi16, Th. B.7(b)]) one can easily extend Th. 4.11 to partial derivatives of order \( k > 2 \).

**Theorem 4.12.** Let \( G \) be an open subset of \( \mathbb{R}^n \), \( n \in \mathbb{N} \), and let \( k \in \mathbb{N} \). Suppose that for \( f : G \to \mathbb{K} \) all partial derivatives of order less than or equal to \( k \) exist in \( G \) and are continuous in \( \xi \in G \). Than the value of each partial derivative of \( f \) of order \( k \) in \( \xi \) is independent of the order in which the individual partial derivatives are carried out. In other words, if \( p = (p_1, \ldots, p_k) \in \{1, \ldots, n\}^k \) and \( q = (q_1, \ldots, q_k) \in \{1, \ldots, n\}^k \) such that there exists a permutation (i.e. a bijective map) \( \pi : \{1, \ldots, k\} \to \{1, \ldots, k\} \) satisfying \( q = (p_{\pi(1)}, \ldots, p_{\pi(k)}) \), then \( \partial_p f(\xi) = \partial_q f(\xi) \). If \( f : G \to \mathbb{K}^m \), \( m \in \mathbb{N} \), then the same holds with respect to each coordinate function \( f_j \) of \( f \), \( j \in \{1, \ldots, m\} \).

**Proof.** For \( k = 1 \), there is nothing to prove. So let \( k > 1 \). For \( l \in 1, \ldots, k-1 \), let \( \tau_l : \{1, \ldots, k\} \to \{1, \ldots, k\} \) be the transposition that interchanges \( l \) and \( l+1 \) and leaves all other elements fixed (i.e. \( \tau_l(l) = l+1, \tau_l(l+1) = l \), \( \tau_l(\alpha) = \alpha \) for each \( \alpha \in \{1, \ldots, k\} \setminus \{l, l+1\} \)) and let \( T := \{\tau_1, \ldots, \tau_{k-1}\} \). Then Th. 4.11 directly implies that the theorem holds for \( \pi = \tau \) for each \( \tau \in T \). For a general permutation \( \pi : \{1, \ldots, k\} \to \{1, \ldots, k\} \), the abovementioned combinatorial result provides a finite sequence \( (\tau^1, \ldots, \tau^N) \), \( N \in \mathbb{N} \), of elements of \( T \) such that \( \pi = \tau^N \circ \cdots \circ \tau^1 \). Thus, as we already know that the theorem holds for \( N = 1 \), the case \( N > 1 \) follows by induction. \( \blacksquare \)
Now that we have seen that functions with continuous partials are particularly benign, we introduce some special notation dedicated to such functions:

**Definition 4.13.** Let $G \subseteq \mathbb{R}^n$, $f : G \rightarrow \mathbb{K}$, $k \in \mathbb{N}_0$. If all partials of $f$ up to order $k$ exist everywhere in $G$, and if $f$ and all its partials up to order $k$ are continuous on $G$, then $f$ is said to be of class $C^k$ (one also says that $f$ has continuous partials up to order $k$). The set of all $\mathbb{K}$-valued functions of class $C^k$ is denoted by $C^k(G, \mathbb{K})$ (in particular, $C^0(G, \mathbb{K}) = C(G, \mathbb{K})$). If $f$ has continuous partials of all orders, than $f$ is said to be of class $C^\infty$, i.e. $C^\infty(G, \mathbb{K}) := \bigcap_{k=0}^\infty C^k(G, \mathbb{K})$. For $\mathbb{R}$-valued functions, we introduce the shorter notation $C^k(G) := C^k(G, \mathbb{R})$ for each $k \in \mathbb{N}_0 \cup \{\infty\}$. Finally, for $f : G \rightarrow \mathbb{K}^m$, we say that $f$ is of class $C^k$ if, and only if, each coordinate function $f_j$, $j \in \{1, \ldots, m\}$, is of class $C^k$. The set of all such functions is denoted by $C^k(G, \mathbb{K}^m)$.

**Notation 4.14.** For two vectors $u = (u_1, u_2, u_3) \in \mathbb{K}^3$, $v = (v_1, v_2, v_3) \in \mathbb{K}^3$, the cross product is an element of $\mathbb{K}^3$ defined as follows:

$$ u \times v := (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1). \tag{4.12} $$

**Definition 4.15.** Let $G \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $\xi \in G$.

(a) If $f : G \rightarrow \mathbb{K}^n$ and the partials $\partial_j f_j(\xi)$ exist for each $j \in \{1, \ldots, n\}$, then the divergence of $f$ in $\xi$ is defined as

$$ \text{div } f(\xi) := \sum_{j=1}^n \partial_j f_j(\xi) = \frac{\partial f_1(\xi)}{\partial x_1} + \cdots + \frac{\partial f_n(\xi)}{\partial x_n}. \tag{4.13} $$

If $\text{div } f(\xi)$ exists for all $\xi \in G$, then $\text{div } f : G \rightarrow \mathbb{K}$. Sometimes, one defines the del operator $\nabla = (\partial_1, \ldots, \partial_n)$ and then writes $\text{div } f = \nabla \cdot f$, using the analogue between (4.13) and the definition of the Euclidean scalar product. Also note that $\text{div } f(\xi)$ is precisely the trace of the corresponding Jacobi matrix, $\text{div } f(\xi) = \text{tr } J_f(\xi)$.

(b) If $f : G \rightarrow \mathbb{K}$ has second-order partials at $\xi$, then one defines the Laplacian (also known as the Laplace operator) of $f$ in $\xi$ by

$$ \Delta f(\xi) := \text{div } \nabla f(\xi) = \sum_{j=1}^n \partial_j \partial_j f(\xi) = \partial_1^2 f(\xi) + \cdots + \partial_n^2 f(\xi). \tag{4.14} $$

If $\Delta f(\xi)$ exists for all $\xi \in G$, then $\Delta : G \rightarrow \mathbb{K}$.

(c) If $n = 3$ and $f : G \rightarrow \mathbb{K}^3$ has first-order partials at $\xi$, then one defines the curl of $f$ in $\xi$ by

$$ \text{curl } f(\xi) := (\partial_3 f_2(\xi) - \partial_2 f_3(\xi), \partial_3 f_1(\xi) - \partial_1 f_3(\xi), \partial_1 f_2(\xi) - \partial_2 f_1(\xi)) = \left( \frac{\partial f_3(\xi)}{\partial x_2} - \frac{\partial f_2(\xi)}{\partial x_3}, \frac{\partial f_1(\xi)}{\partial x_3} - \frac{\partial f_3(\xi)}{\partial x_1}, \frac{\partial f_2(\xi)}{\partial x_1} - \frac{\partial f_1(\xi)}{\partial x_2} \right). \tag{4.15} $$

If $\text{curl } f(\xi)$ exists for all $\xi \in G$, then $\text{curl } f : G \rightarrow \mathbb{K}^3$. Again, one sometimes defines the del operator $\nabla = (\partial_1, \partial_2, \partial_3)$ and then writes $\text{curl } f = \nabla \times f$, using the analogue between (4.15) and the definition of the cross product or two vectors in $\mathbb{K}^3$. 

\[ \text{4 DIFFERENTIAL CALCULUS} \]
Proposition 4.16. Let $G \subseteq \mathbb{R}^3$, let $f : G \rightarrow \mathbb{K}$ be a scalar-valued function and let $v : G \rightarrow \mathbb{K}^3$ be a vector-valued function.

(a) If $\xi \in G$ is such that $f$ and $v$ have all partials of first order at $\xi$, then

$$\text{curl}(fv)(\xi) = f(\xi)\text{curl}v(\xi) + \nabla f(\xi) \times v(\xi).$$

(b) If $G$ is open and $f \in C^2(G, \mathbb{K})$, then $\text{curl} \nabla f$ vanishes identically on $G$, i.e.

$$\text{curl} \nabla f \equiv 0.$$

(c) If $G$ is open and $v \in C^2(G, \mathbb{K}^3)$, then $\text{div} \text{curl} v$ vanishes identically on $G$, i.e.

$$\text{div} \text{curl} v \equiv 0.$$

Proof. Exercise. \[\blacksquare\]

4.4 Interlude: Graphical Representation in Two Dimensions

In this section, we will briefly address the problem of drawing graphs of functions $f : D_f \rightarrow \mathbb{R}$ with $D_f \subseteq \mathbb{R}^2$. If the function $f$ is sufficiently benign (for example, if $f \in C^1(\mathbb{R}^2)$), then the graph of $f$, namely the set $\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D_f, z = f(x,y)\} \subseteq \mathbb{R}^3$ will represent a two-dimensional surface in the three-dimensional space $\mathbb{R}^3$. The two most important methods for depicting the graph of $f$ as a picture in a two-dimensional plane (such as a sheet of paper or a board) are:

(a) The use of perspective.

(b) The use of level sets, in particular, level curves (also known as contour lines).

The Use of Perspective

Nowadays, this is most effectively accomplished by the use of computer graphics software. Widely used programs include commercial software such as MATLAB and Mathematica as well as the noncommercial software Gnuplot.

The Use of Level Sets

By a level set or an isolevel, we mean a set of the form $f^{-1}\{C\} = \{(x,y) \in D_f : f(x,y) = C\}$ with $C \in \mathbb{R}$. If $f^{-1}\{C\}$ constitutes a curve in $\mathbb{R}^2$, then we speak of a level curve or a contour line. Representation of functions depending on two variables by contour lines is well-known from everyday live. For example, contour lines are used to depict the height above sea level on hiking maps; on meteorological maps, isobars and isotherms are used to depict levels of equal pressure and equal temperature, respectively. Determining
level sets and contour lines can be difficult, and the appropriate method depends on the function under consideration. In some cases, it is possible to determine the contour line corresponding to the level \( C \in f(D_f) \) by solving the equation \( C = f(x, y) \) for \( y \) (the difficulty is that an explicit solution of this equation can not always be found). The following Example 4.17 provides some cases, where \( C = f(x, y) \) can be solved explicitly:

**Example 4.17.** (a) For \( f : \mathbb{R}^2 \to \mathbb{R} \), \( f(x, y) := x^2 + y^2 \), and \( C \in \mathbb{R}^+ \), one has
\[
|y| = \sqrt{C - x^2} \quad \text{for} \quad -\sqrt{C} \leq x \leq \sqrt{C}.
\]

(b) For \( f : \mathbb{R}^2 \to \mathbb{R} \), \( f(x, y) := xy \), and \( C \in \mathbb{R} \), one has
\[
y = \frac{C}{x} \quad \text{for} \ x \neq 0.
\]

For \( C = 0 \), one actually gets \( x = 0 \) or \( y = 0 \), which provides one additional contour line.

In some cases, it helps to write \( C = f(x, y) \) in different coordinates (e.g., polar coordinates). In general, the question if \( C = f(x, y) \) can be solved for \( y \) (or \( x \)) is related to the implicit function Th. 4.49 below.

### 4.5 The Total Derivative and the Notion of Differentiability

Roughly, a function \( f : G \to \mathbb{K}^m \), \( G \subseteq \mathbb{K}^n \), will be called differentiable if, locally, it can be approximated by a \( \mathbb{K} \)-affine function, i.e., if, for each \( \zeta \in G \), there exists an \( \mathbb{K} \)-linear function \( L(\zeta) \) such that \( f(\zeta + h) \approx f(\zeta) + L(h) \) for sufficiently small \( h \in \mathbb{K}^n \). Analogous to the treatment in the one-dimensional situation in [Phi16, Sec. 9], we will call a function \( f : G \to \mathbb{C}^m \), \( G \subseteq \mathbb{R}^n \), \( \mathbb{R} \)-differentiable if, and only if, both \( \mathbb{R}^m \)-valued functions \( \text{Re} \ f \) and \( \text{Im} \ f \) are \( \mathbb{R}^m \)-differentiable.

**Definition 4.18.** Let \( G \) be an open subset of \( \mathbb{K}^n \), \( n \in \mathbb{N} \), \( f : G \to \mathbb{K}^m \), \( m \in \mathbb{N} \), \( \zeta \in G \). Then \( f \) is called \( \mathbb{K} \)-differentiable (or just differentiable if the field \( \mathbb{K} \) is understood) in \( \zeta \) if, and only if, there exists a \( \mathbb{K} \)-linear map \( L : \mathbb{K}^n \to \mathbb{K}^m \) such that
\[
\lim_{h \to 0} \frac{f(\zeta + h) - f(\zeta) - L(h)}{\|h\|_2} = 0. \quad (4.16a)
\]

Note that, in general, \( L \) will depend on \( \zeta \). If \( f \) is differentiable in \( \zeta \), then \( L \) is called the total derivative or the total differential of \( f \) in \( \zeta \). In that case, one writes \( Df(\zeta) \) instead of \( L \). For \( G \subseteq \mathbb{R}^n \), we call \( f : G \to \mathbb{C}^m \) to be \( \mathbb{R} \)-differentiable in \( \zeta \in G \) if, and only if, both \( \text{Re} \ f \) and \( \text{Im} \ f \) are \( \mathbb{R} \)-differentiable in \( \xi \) in the above sense. If \( f \) is \( \mathbb{R} \)-differentiable in \( \xi \), define \( Df(\xi) := D \text{Re} \ f(\xi) + iD \text{Im} \ f(\xi) \) to be the total derivative or the total differential of \( f \) in \( \xi \). It is then an easy exercise to show
\[
\lim_{h \to 0} \frac{f(\xi + h) - f(\xi) - Df(\xi)(h)}{\|h\|_2} = 0. \quad (4.16b)
\]
Finally, $f$ is called $\mathbb{K}$-differentiable if, and only if, $f$ is $\mathbb{K}$-differentiable in every $\zeta \in G$ (a $\mathbb{C}$-differentiable function is also called holomorphic – holomorphic functions are the central topic of the field of Complex Analysis).

**Remark 4.19.** (a) As the set $G \subseteq \mathbb{K}^n$ in Def. 4.18 is open, it is guaranteed that $\zeta + h \in G$ for $\|h\|_2$ sufficiently small: There exists $\epsilon > 0$ such that $\|h\|_2 < \epsilon$ implies $\zeta + h \in G$.

(b) As all norms on $\mathbb{K}^n$ are equivalent, instead of the Euclidean norm $\| \cdot \|_2$, one can use any other norm on $\mathbb{K}^n$ in (4.16a) without changing the definition.

(c) Since $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and since addition in $\mathbb{C}^n$ is precisely addition in $\mathbb{R}^{2n}$, multiplication by $\lambda \in \mathbb{R}$ in $\mathbb{C}^n$ is precisely multiplication by $\lambda$ in $\mathbb{R}^{2n}$, if $G \subseteq \mathbb{C}^n$ and $f : G \rightarrow \mathbb{C}^m$ in Def. 4.18 is $\mathbb{C}$-differentiable in $\zeta \in G$, then it is also $\mathbb{R}$-differentiable in $\zeta \in G$ (if $L$ is $\mathbb{C}$-linear as a map from $\mathbb{C}^n$ to $\mathbb{C}^m$, then $L$ is also $\mathbb{R}$-linear as a map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$). However, $\mathbb{C}$-differentiability is a much stronger condition than $\mathbb{R}$-differentiability (cf. Rem. 4.23(b) and Rem. 4.27 below).

**Lemma 4.20.** Let $G$ be an open subset of $\mathbb{K}^n$, $n \in \mathbb{N}$, $\zeta \in G$. Then $f : G \rightarrow \mathbb{K}^m$, $m \in \mathbb{N}$, is $\mathbb{K}$-differentiable in $\zeta$ if, and only if, there exists a $\mathbb{K}$-linear map $L : \mathbb{K}^n \rightarrow \mathbb{K}^m$ and another (not necessarily linear) map $r : \mathbb{K}^n \rightarrow \mathbb{K}^m$ such that

$$f(\zeta + h) - f(\zeta) = L(h) + r(h) \quad (4.17a)$$

for each $h \in \mathbb{K}^n$ with sufficiently small $\|h\|_2$, and

$$\lim_{h \rightarrow 0} \frac{r(h)}{\|h\|_2} = 0. \quad (4.17b)$$

**Proof.** Suppose $L, r$ are as above and satisfy (4.17). Then, for each $0 \neq h \in \mathbb{K}^n$ with sufficiently small $\|h\|_2$, it holds that

$$\frac{f(\zeta + h) - f(\zeta) - L(h)}{\|h\|_2} = \frac{r(h)}{\|h\|_2}. \quad (4.18)$$

Thus, (4.17b) implies (4.16a), showing that $f$ is differentiable. Conversely, if $f$ is differentiable in $\zeta$, then there exists a $\mathbb{K}$-linear map $L : \mathbb{K}^n \rightarrow \mathbb{K}^m$ satisfying (4.16a). Choose $\epsilon > 0$ such that $B_{\epsilon\|\cdot\|_2}(\zeta) \subseteq G$ and define

$$r : \mathbb{K}^n \rightarrow \mathbb{K}^m, \quad r(h) := \begin{cases} f(\zeta + h) - f(\zeta) - L(h) & \text{for } h \in B_{\epsilon\|\cdot\|_2}(\zeta), \\ 0 & \text{otherwise}. \end{cases} \quad (4.19)$$

Then (4.17a) is immediate. Since (4.18) also holds, (4.16a) implies (4.17b). ■

In the following Th. 4.21 and Cor. 4.22, we consider $\mathbb{R}$-differentiability, then coming back to $\mathbb{C}$-differentiability in Rem. 4.23 and Cor. 4.24.
Theorem 4.21. Let $G$ be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $\xi \in G$. If $f : G \rightarrow \mathbb{K}$ is $\mathbb{R}$-differentiable in $\xi$, then $f$ is continuous in $\xi$, all partials at $\xi$, i.e. $\partial_j f(\xi)$, $j \in \{1, \ldots, n\}$, exist, and $Df(\xi) = \nabla f(\xi)$ (that means, for each $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$, one has $Df(\xi)(h) = \nabla f(\xi)h = \sum_{j=1}^n \partial_j f(\xi)h_j$). In particular, $Df(\xi)$ is unique and, hence, well-defined.

Proof. Assume $f$ is $\mathbb{R}$-differentiable in $\xi$. We first consider the case $\mathbb{K} = \mathbb{R}$. Let the linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}$ and $r : \mathbb{R}^n \rightarrow \mathbb{R}$ be as in Lem. 4.20. We already know from Example 2.16 that each linear map from $\mathbb{R}^n$ into $\mathbb{R}$ is continuous. In particular, $L$ must be continuous. Now let $(x^k)_{k \in \mathbb{N}}$ be a sequence in $G$ that converges to $\xi$, i.e. $\lim_{k \to \infty} \|x^k - \xi\|_2 = 0$. Then $(h^k)_{k \in \mathbb{N}}$ with $h^k := x^k - \xi$ constitutes a sequence in $\mathbb{R}^n$ such that $\lim_{k \to \infty} \|h^k\|_2 = 0$. Note that (4.17b) implies that $0 \leq |r(h)| < \|h\|_2$ for $\|h\|_2$ sufficiently small. Thus, $\lim_{k \to \infty} \|h^k\|_2 = 0$ implies $\lim_{k \to \infty} |r(h^k)| = 0$. As the continuity of $L$ also yields $\lim_{k \to \infty} |L(h^k)| = 0$, (4.17a) provides

$$
\lim_{k \to \infty} |f(x^k) - f(\xi)| = \lim_{k \to \infty} |f(\xi + h^k) - f(\xi)| = \lim_{k \to \infty} |L(h^k)| + \lim_{k \to \infty} |r(h^k)| = 0,
$$

(4.20)

establishing the continuity of $f$ in $\xi$. To see that the partials exist and that $L$ is given by the gradient, set $l_j := L(e_j)$ for each $j \in \{1, \ldots, n\}$. If $h = te_j$ with $t \in \mathbb{R}$ sufficiently close to 0, than (4.17a) yields

$$
f(\xi + te_j) - f(\xi) = tl_j + r(te_j). \quad (4.21)
$$

For $t \neq 0$, we can divide by $t$. Letting $t \to 0$, we see from (4.17b) that the right-hand side converges to $l_j$. But this means that the left-hand side must converge as well, and comparing with (4.1), we see that its limit is precisely $\partial_j f(\xi)$, thereby proving $l_j = \partial_j f(\xi)$ as claimed. We now consider the case $\mathbb{K} = \mathbb{C}$. From the case $\mathbb{K} = \mathbb{R}$, we know Re $f$ and Im $f$ are both continuous at $\xi$, such that, by Ex. 2.12(d), $f$ must be continuous at $\xi$ as well. Moreover, from the case $\mathbb{K} = \mathbb{R}$, we know $\partial_j \text{Re} f(\xi)$ and $\partial_j \text{Im} f(\xi)$ exist for each $j \in \{1, \ldots, n\}$. Thus, $\partial_j f(\xi) = \partial_j \text{Re} f(\xi) + i \partial_j \text{Im} f(\xi)$ exist as well by [Phi16, Rem. 9.2].

By applying Th. 4.21 to coordinate functions, we can immediately extend it to $\mathbb{K}^m$-valued functions:

Corollary 4.22. Let $G$ be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $\xi \in G$. If $f : G \rightarrow \mathbb{K}^m$ is $\mathbb{R}$-differentiable in $\xi$, then $f$ is continuous in $\xi$, all partials at $\xi$, i.e. $\partial_k f_l(\xi)$, $k \in \{1, \ldots, n\}$, $l \in \{1, \ldots, m\}$, exist, and $Df(\xi) = J_f(\xi)$: For each $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$, one has

$$
Df(\xi)(h) = J_f(\xi) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \nabla f_1(\xi)(h) \\ \vdots \\ \nabla f_m(\xi)(h) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n \partial_k f_1(\xi)h_k \\ \vdots \\ \sum_{k=1}^n \partial_k f_m(\xi)h_k \end{pmatrix}.
$$

In particular, $Df(\xi)$ is unique and, hence, well-defined.
We now proceed to further investigate the relation between \( \mathbb{R} \)-differentiability and \( \mathbb{C} \)-differentiability.

**Remark 4.23.** (a) If \( L : \mathbb{C} \to \mathbb{C} \) is a \( \mathbb{C} \)-linear map, then there exists \( a \in \mathbb{C} \) such that \( L(z) = az \). As \( \mathbb{C} = \mathbb{R}^2 \), using the definition of complex multiplication and letting \( a = \alpha + i\beta \), \( z = x + iy \), we can write this in matrix form as

\[
L(z) = az = \begin{pmatrix} \alpha x - \beta y \\ \alpha y + \beta x \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

Thus, a map \( L : \mathbb{R}^{2n} \to \mathbb{R}^{2m} \) can be interpreted as a \( \mathbb{C} \)-linear map \( L : \mathbb{C}^n \to \mathbb{C}^m \) if, and only if, it is \( \mathbb{R} \)-linear and each \( 2 \times 2 \) block in its matrix representation has the form of (4.22).

(b) If \( G \subseteq \mathbb{C} \) is open, \( f : G \to \mathbb{C}, \zeta \in G \), then combining (a) with Cor. 4.22 implies \( f \) to be \( \mathbb{C} \)-differentiable in \( \zeta \) if, and only if, \( f \) is \( \mathbb{R} \)-differentiable in \( \zeta \) with

\[
Df(\zeta) = \begin{pmatrix} \partial_1 \text{Re} f(\zeta) & \partial_2 \text{Re} f(\zeta) \\ \partial_1 \text{Im} f(\zeta) & \partial_2 \text{Im} f(\zeta) \end{pmatrix},
\]

being \( \mathbb{C} \)-linear. According to (4.22), this means

\[
\partial_1 \text{Re} f(\zeta) = \partial_2 \text{Im} f(\zeta), \quad \partial_1 \text{Im} f(\zeta) = -\partial_2 \text{Re} f(\zeta).
\]

The equations of (4.23) are known as the Cauchy-Riemann differential equations (they are partial differential equations, as the involve partial derivatives).

**Corollary 4.24.** Let \( G \) be an open subset of \( \mathbb{C}^n = \mathbb{R}^{2n}, n \in \mathbb{N}, \zeta \in G \). Then \( f : G \to \mathbb{C}^m = \mathbb{R}^{2m} \) is \( \mathbb{C} \)-differentiable in \( \zeta \) if, and only if, \( f \) is \( \mathbb{R} \)-differentiable in \( \zeta \) and each \( 2 \times 2 \) block of the real \( (2m) \times (2n) \) matrix \( J_f(\zeta) \) has the form of (4.22) (i.e. each \( \mathbb{C} \)-valued component function \( f_1 : G \to \mathbb{C} \) satisfies a set of Cauchy-Riemann equations as in (4.23) for each of its \( n \) complex input arguments). Thus, if \( f \) is \( \mathbb{C} \)-differentiable in \( \zeta \), then each entry of the “complex Jacobian”, i.e. of the complex \( m \times n \) matrix representing \( Df(\zeta) : \mathbb{C}^n \to \mathbb{C}^m, \) uniquely corresponds to a \( 2 \times 2 \) block of partials in \( J_f(\zeta) \). Each entry of the complex Jacobian can be seen as a complex partial \( \partial_k f_1 \), uniquely determined by the corresponding four real partials in \( J_f(\zeta) \). In particular, \( f \) is continuous in \( \zeta \) and \( Df(\zeta) \) is unique (and, hence, well-defined).

**Proof.** The corollary merely combines Cor. 4.22 with Rem. 4.23.

**Example 4.25.** (a) If \( G \subseteq \mathbb{K}^n, n \in \mathbb{N}, \) is open and \( f : G \to \mathbb{K}^m \) is constant (i.e. there is \( c \in \mathbb{K}^m \) such that \( f(x) = c \) for each \( x \in G \)), than \( f \) is \( \mathbb{K} \)-differentiable with \( Df \equiv 0 \): It suffices to notice that, for a constant \( f \) and \( L \equiv 0 \), the numerator in (4.16a) vanishes identically.

(b) If \( A : \mathbb{K}^n \to \mathbb{K}^m \) is \( \mathbb{K} \)-linear, then \( A \) is \( \mathbb{K} \)-differentiable with \( DA(\zeta) = A \) for each \( \zeta \in \mathbb{K}^n \): If \( \zeta, h \in \mathbb{K}^n \), then \( A(\zeta + h) - A(\zeta) - A(h) = 0 \), showing that, as in (a), the numerator in (4.16a) (with \( f = L = A \)) vanishes identically.
Example 4.26. We will show that if \( r \) is the power series radius of convergence where \( a \) always follows by noticing that the definitions of \( \xi \) and \( \eta \) are identical. Showing that \( f \) is \( R \)-differentiable in \( \xi \) in the sense of Def. 4.18, then, according to Th. 4.21, \( \partial_1 f(\xi) \) exists and \( Df(\xi)(h) = \partial_1 f(\xi)h \). Thus, the one-dimensional differentiability of \( f \) at \( \xi \) as well as (4.24) follow by noticing that the definitions of \( \partial_1 f(\xi) \) and of \( f'(\xi) \) are identical.

Example 4.26. We will show that if \( f : B_r(0) \to \mathbb{C} \) is represented by a power series, where \( r \in [0, \infty] \) is its radius of convergence, \( B_r(0) \subseteq \mathbb{C} \) (cf. [Phi16, Th. 8.9]), then \( f \) is always \( \mathbb{C} \)-differentiable (i.e. holomorphic) and also \( C^\infty \). Then, in particular, this holds on all of \( \mathbb{C} \) for each polynomial and for the functions \( \exp, \sin, \text{and} \cos \) (as they all have radius of convergence \( r = \infty \)). Thus, let

\[
f : B_r(0) \to \mathbb{C}, \quad f(z) = \sum_{j=0}^\infty a_j z^j,
\]

where \( a_j \in \mathbb{C} \) and \( r \in [0, \infty] \) is the radius of convergence of the power series. Consider the power series

\[
g : B_r(0) \to \mathbb{C}, \quad g(z) = \sum_{j=1}^\infty j a_j z^{j-1}.
\]
Note that the radius of convergence of $g$ is the same as for $f$, as
\[
\limsup_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\limsup_{n \to \infty} |a_n|} = 1 \limsup_{n \to \infty} \sqrt[n]{|a_n|} = r^{-1}.
\]
We have to show that
\[
\forall \quad z \in B_r(0) \quad Df(z) : \mathbb{C} \to \mathbb{C}, \quad Df(z)(h) = g(z)h.
\]
Thus, for each $z \in B_r(0)$, we need to prove
\[
\lim_{h \to 0} \frac{f(z+h) - f(z) - g(z)h}{|h|} = 0.
\] (4.27)
To this end, given $z \in B_r(0)$, choose $\rho \in [\|z\|, r]$ and define, for sufficiently small $h \neq 0$,
\[
\delta(h) := \frac{f(z+h) - f(z) - g(z)h}{h} = \frac{\sum_{j=0}^{\infty} a_j ((z+h)^j - z^j)}{h} - \sum_{j=1}^{\infty} a_j z^{j-1}
\]
\[
= \sum_{j=1}^{\infty} a_j \left( \frac{(z+h)^j - z^j}{h} - j z^{j-1} \right) = \sum_{j=1}^{\infty} a_j w_j,
\]
where
\[
\forall \quad j \in \mathbb{N} \quad w_j := \frac{(z+h)^j - z^j}{h} - j z^{j-1}.
\]
Then $w_1 = 0$ and, for each $j \geq 2$,
\[
w_j = h \sum_{k=1}^{j-1} k z^{k-1} (z+h)^{j-k-1},
\]
as can be verified via induction over $j$ (exercise). Thus, for each $h \neq 0$ such that $|z+h| < \rho$, one has
\[
\forall \quad j \geq 2 \quad |w_j| < |h| \sum_{k=1}^{j-1} k \rho^{k-1} \rho^{j-k-1} \leq |h| \frac{j (j-1)}{2} \rho^{j-2},
\]
implying
\[
|\delta(h)| \leq |h| \sum_{j=2}^{\infty} j^2 |a_j| \rho^{j-2}.
\] (4.28)
As $\rho < r$, the series in (4.28) converges to some finite (nonnegative real) number, showing
\[
\lim_{h \to 0} \delta(h) = 0 \quad \text{and} \quad g(z) = Df(z).
\]

**Remark 4.27.** It is an important result of Complex Analysis that the converse of Ex. 4.26 is also true: If $G \subseteq \mathbb{C}$ is open and $f : G \to \mathbb{C}$ is holomorphic, then $f$ is **analytic**, i.e. locally representable as a power series. More precisely, for each $a \in G$, there exists $r > 0$ and a sequence $(c_j)_{j \in \mathbb{N}}$ in $\mathbb{C}$ such that $B_r(a) \subseteq G$ and $f(z) = \sum_{j=0}^{\infty} c_j (z-a)^j$ for each $z \in B_r(a)$ (see, e.g., [Kön04, Sec. 6.2] or [Rud87, Th. 10.16]). As a consequence, Ex. 4.26 implies that every holomorphic function is automatically $C^\infty$.  


Proposition 4.28. Forming the total derivative is a linear operation: Let G be an open subset of $\mathbb{K}^n$, $n \in \mathbb{N}$, $\zeta \in G$.

(a) If $f, g : G \to \mathbb{K}^m$, $m \in \mathbb{N}$, are both $\mathbb{K}$-differentiable at $\zeta$, then $f + g$ is $\mathbb{K}$-differentiable at $\zeta$ and $D(f + g)(\zeta) = Df(\zeta) + Dg(\zeta)$.

(b) If $f : G \to \mathbb{K}^m$, $m \in \mathbb{N}$, is $\mathbb{K}$-differentiable at $\zeta$ and $\lambda \in \mathbb{K}$, then $\lambda f$ is $\mathbb{K}$-differentiable at $\zeta$ and $D(\lambda f)(\zeta) = \lambda Df(\zeta)$.

Proof. (a): We note that, for each $h \in \mathbb{K}^n$ with $0 \neq \|h\|_2$ sufficiently small,

$$
(f + g)(\zeta + h) - (f + g)(\zeta) - Df(\zeta)(h) - Dg(\zeta)(h)
$$

$$
= \frac{f(\zeta + h) - f(\zeta) - Df(\zeta)(h)}{\|h\|_2} + \frac{g(\zeta + h) - g(\zeta) - Dg(\zeta)(h)}{\|h\|_2}.
$$

Thus, if the limit $\lim_{h \to 0}$ exists and equals 0 for both summands on the right-hand side, then the same must be true for the left-hand side.

(b): For $\lambda \in \mathbb{K}$, one computes

$$
\lim_{h \to 0} \frac{(\lambda f)(\zeta + h) - (\lambda f)(\zeta) - \lambda Df(\zeta)(h)}{\|h\|_2} = \lambda \lim_{h \to 0} \frac{f(\zeta + h) - f(\zeta) - Df(\zeta)(h)}{\|h\|_2} = 0,
$$

thereby establishing the case. ■

Even though we have seen in Example 4.2 that the existence of all partial derivatives does not even imply continuity, let alone differentiability, the next theorem and its corollary will show that if all partial derivatives exist and are continuous, then that does, indeed, imply differentiability.

Theorem 4.29. Let $G$ be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $\xi \in G$, and $f : G \to \mathbb{K}$. If all partials $\partial_j f$, $j \in \{1, \ldots, n\}$ exist everywhere in $G$ and are continuous in $\xi$, then $f$ is $\mathbb{R}$-differentiable in $\zeta$, and, in particular, $f$ is continuous in $\xi$.

Proof. As usual, the case $\mathbb{K} = \mathbb{C}$ follows by applying the case $\mathbb{K} = \mathbb{R}$ to $\text{Re} f$ and $\text{Im} f$.

We, therefore proceed to treat the case $\mathbb{K} = \mathbb{R}$. We first consider the special case where $\partial_j f(\xi) = 0$ for each $j \in \{1, \ldots, n\}$. In that case, we need to show

$$
\lim_{h \to 0} \frac{f(\xi + h) - f(\xi)}{\|h\|_1} = 0
$$

(4.29)

(noting that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on $\mathbb{R}^n$). Since $G$ is open and since the $\partial_j f$ are continuous in $\xi$, given $\epsilon > 0$, there is $\delta > 0$ such that, for each $h \in \mathbb{R}^n$ with $\|h\|_1 < \delta$, one has $\xi + h \in G$ and $|\partial_j f(\xi + h)| < \epsilon$ for every $j \in \{1, \ldots, n\}$. Fix $h \in \mathbb{R}^n$ with
\[ ||h||_1 < \delta. \] Then
\[
\begin{align*}
  f(\xi + h) - f(\xi) &= f(\xi_1 + h_1, \ldots, \xi_{n-1} + h_{n-1}, \xi_n + h_n) - f(\xi_1 + h_1, \ldots, \xi_{n-1} + h_{n-1}, \xi_n) \\
  &+ f(\xi_1 + h_1, \ldots, \xi_{n-1} + h_{n-1}, \xi_n) - f(\xi_1 + h_1, \ldots, \xi_{n-1}, \xi_n) \\
  &+ \cdots + f(\xi_1 + h_1, \xi_2, \ldots, \xi_n) - f(\xi_1, \xi_2, \ldots, \xi_n)
\end{align*}
\]
\[
= f(\xi + h) - f\left(\xi + \sum_{k=1}^{n-1} h_k e_k\right)
+ f\left(\xi + \sum_{k=1}^{n-1} h_k e_k\right) - f\left(\xi + \sum_{k=1}^{n-2} h_k e_k\right) + \cdots + f(\xi + h_{1e_1}) - f(\xi)
= \sum_{j=0}^{n-1} \left( f\left(\xi + \sum_{k=1}^{n-j} h_k e_k\right) - f\left(\xi + \sum_{k=1}^{n-(j+1)} h_k e_k\right) \right)
= \sum_{j=0}^{n-1} (\phi_j(h_{n-j}) - \phi_j(0)), \quad (4.30)
\]
where, for each \( j \in \{0, \ldots, n-1\}, \)
\[
\phi_j : [0, h_{n-j}] \rightarrow \mathbb{R}, \quad \phi_j(t) := f\left(\xi + te_{n-j} + \sum_{k=1}^{n-(j+1)} h_k e_k\right).
\]
If \( h_{n-j} = 0, \) then set \( \theta_j := 0. \) Otherwise, apply the one-dimensional mean value theorem [Phi16, Th. 9.18] to the one-dimensional function \( \phi_j \) to get numbers \( \theta_j \in [0, h_{n-j}] \) such that
\[
\phi_j(h_{n-j}) - \phi_j(0) = h_{n-j} \phi_j'(\theta_j) = h_{n-j} \partial_{n-j} f\left(\xi + \theta_j e_{n-j} + \sum_{k=1}^{n-(j+1)} h_k e_k\right). \quad (4.31)
\]
Combining (4.30) with (4.31) yields
\[
\begin{align*}
  f(\xi + h) - f(\xi) &= \sum_{j=0}^{n-1} h_{n-j} \partial_{n-j} f\left(\xi + \theta_j e_{n-j} + \sum_{k=1}^{n-(j+1)} h_k e_k\right).
\end{align*}
\]
Noting that \( ||h||_1 < \delta \) implies \( ||\theta_j e_{n-j} + \sum_{k=1}^{n-(j+1)} h_k e_k||_1 < \delta, \) we obtain from (4.32) that, for \( 0 \neq h \) with \( ||h||_1 < \delta, \)
\[
\frac{|f(\xi + h) - f(\xi)|}{||h||_1} < \frac{1}{||h||_1} \sum_{j=0}^{n-1} |h_{n-j}| \epsilon = \epsilon,
\]
thereby proving (4.29) and establishing the case. It remains to consider a general \( f : G \rightarrow \mathbb{R}, \) without the restriction of a vanishing gradient. For such a general \( f, \) consider
the modified function \( g : G \to \mathbb{R}, g(x) := f(x) - \nabla f(\xi)(x) = f(x) - \sum_{j=1}^{n} \partial_j f(\xi)x_j. \)

For \( g \), we then get \( \partial_j g(x) = \partial_j f(x) - \partial_j f(\xi) \) for each \( x \in G \). In particular, the \( \partial_j g \) exist in \( G \), are continuous at \( x = \xi \), and vanish at \( x = \xi \). Thus, the first part of the proof applies to \( g \), showing that \( g \) is \( \mathbb{R} \)-differentiable at \( \xi \). Since \( f = g + \nabla f(\xi) \) and both \( g \) and the linear map \( \nabla f(\xi) \) are \( \mathbb{R} \)-differentiable at \( \xi \), so is \( f \) by Prop. 4.28(a). \( \blacksquare \)

**Corollary 4.30.** Let \( G \) be an open subset of \( \mathbb{R}^n, n \in \mathbb{N}, \xi \in G \), and \( f : G \to \mathbb{K}^m, m \in \mathbb{N} \). If all partials \( \partial_k f_l, k \in \{1, \ldots, n\}, l \in \{1, \ldots, m\}, \) exist everywhere in \( G \) and are continuous in \( \xi \), then \( f \) is \( \mathbb{R} \)-differentiable in \( \xi \), and, in particular, \( f \) is continuous in \( \xi \).

**Proof.** Applying Th. 4.29 to the coordinate functions \( f_l, l \in \{1, \ldots, m\} \), yields that each \( f_l \) is \( \mathbb{R} \)-differentiable at \( \xi \). However, since a \( \mathbb{K}^m \)-valued function converges if, and only if, each of its coordinate functions converges, \( f \) must also be \( \mathbb{R} \)-differentiable at \( \xi \). \( \blacksquare \)

### 4.6 Higher Order Total Derivatives as Multilinear Maps

Let \( G \) be an open subset of \( \mathbb{K}^n, n \in \mathbb{N}, f : G \to \mathbb{K}^m, m \in \mathbb{N} \). If \( f \) is \( \mathbb{K} \)-differentiable in \( G \), then \( Df : G \to \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m) \), where \( \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m) \cong \mathbb{K}^{nm} \) denotes the vector space over \( \mathbb{K} \) of all \( \mathbb{K} \)-linear maps from \( \mathbb{K}^n \) into \( \mathbb{K}^m \). The coordinate functions are the partial derivatives \( \partial_j f_l \) (as mentioned in Cor. 4.24, this also makes sense for \( \mathbb{K} = \mathbb{C} \)). If this function \( Df \) is \( \mathbb{K} \)-differentiable in \( \zeta \in G \), then we call its derivative the *second total derivative* of \( f \) in \( \zeta \), denoted by \( D^2 f(\zeta) \). It is an element of \( \mathcal{L}(\mathbb{K}^n, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)) \cong \mathbb{K}^{nm^2} \).

Fortunately, \( \mathcal{L}(\mathbb{K}^n, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)) \cong \mathcal{L}^2(\mathbb{K}^n, \mathbb{K}^m) \), where \( \mathcal{L}^2(\mathbb{K}^n, \mathbb{K}^m) \) is the vector space over \( \mathbb{K} \) of all bilinear maps from \( \mathbb{K}^n \times \mathbb{K}^n \) into \( \mathbb{K}^m \) (cf. Sec. J, Th. J.3, in the Appendix). Thus,

\[
D^2 f(\zeta) : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^m, \quad D^2 f(\zeta)(h, k)_l = \sum_{j_1, j_2=1}^{n} \partial_{j_1} \partial_{j_2} f_l(\zeta) h_{j_1} k_{j_2}, \quad l \in \{1, \ldots, m\}.
\]

If \( f \) is twice \( \mathbb{K} \)-differentiable in \( G \), then \( D^2 f : G \to \mathcal{L}^2(\mathbb{K}^n, \mathbb{K}^m) \) and its coordinate functions are the second partials of \( f \).

Inductively, one obtains that, if \( f \) is \( \alpha \) times \( \mathbb{K} \)-differentiable in \( G \), \( \alpha \in \mathbb{N} \), then \( D^\alpha f : G \to \mathcal{L}^\alpha(\mathbb{K}^n, \mathbb{K}^m) \cong \mathcal{L}(\mathbb{K}^n, \mathcal{L}^{\alpha-1}(\mathbb{K}^n, \mathbb{K}^m)) \cong \mathbb{K}^{n \cdot m^\alpha} \), where \( \mathcal{L}^\alpha(\mathbb{K}^n, \mathbb{K}^m) \) is the vector space over \( \mathbb{K} \) of all \( \alpha \)-times linear maps from \( (\mathbb{K}^n)^\alpha \) into \( \mathbb{K}^m \), \( \mathcal{L}^0(\mathbb{K}^n, \mathbb{K}^m) := \mathbb{K}^m \) (again, cf. Sec. J, Th. J.3). For each \( \zeta \in G \),

\[
D^\alpha f(\zeta) : (\mathbb{K}^n)^\alpha \to \mathbb{K}^m,
\]

\[
D^\alpha f(\zeta)(h^1, \ldots, h^\alpha)_l = \sum_{j_1, \ldots, j_\alpha=1}^{n} \partial_{j_1} \cdots \partial_{j_\alpha} f_l(\zeta) h_{j_1}^1 \cdots h_{j_\alpha}^\alpha, \quad l \in \{1, \ldots, m\}, \quad (433)
\]

i.e. the coordinate functions of \( D^\alpha f \) are precisely the partials of order \( \alpha \) of \( f \).
4.7 The Chain Rule

As for one-dimensional differentiable functions, one can also prove a chain rule for vector-valued differentiable functions:

**Theorem 4.31.** Let \( m, n, p \in \mathbb{N} \). Let \( G_f \subseteq \mathbb{K}^n \) be open, \( f : G_f \rightarrow \mathbb{K}^m \), let \( G_g \subseteq \mathbb{K}^m \) be open, \( g : G_g \rightarrow \mathbb{K}^p \), \( f(G_f) \subseteq G_g \). If \( f \) is \( \mathbb{K} \)-differentiable at \( \zeta \in G_f \) and \( g \) is \( \mathbb{K} \)-differentiable at \( f(\zeta) \in G_g \), then \( g \circ f : G_f \rightarrow \mathbb{K}^p \) is \( \mathbb{K} \)-differentiable at \( \zeta \) and, for the \( \mathbb{K} \)-linear maps \( D(g \circ f)(\zeta) : \mathbb{K}^n \rightarrow \mathbb{K}^p \), \( Df(\zeta) : \mathbb{K}^n \rightarrow \mathbb{K}^m \), and \( Dg(f(\zeta)) : \mathbb{K}^m \rightarrow \mathbb{K}^p \), the following chain rule holds:

\[
D(g \circ f)(\zeta) = Dg(f(\zeta)) \circ Df(\zeta). \tag{4.34}
\]

In particular, if both \( f \) and \( g \) are \( \mathbb{K} \)-differentiable, then \( g \circ f \) is \( \mathbb{K} \)-differentiable.

**Proof.** Since \( f \) is \( \mathbb{K} \)-differentiable at \( \zeta \) and \( g \) is \( \mathbb{K} \)-differentiable at \( f(\zeta) \), according to Lem. 4.20, there are functions \( r_f : \mathbb{K}^n \rightarrow \mathbb{K}^m \) and \( r_g : \mathbb{K}^m \rightarrow \mathbb{K}^p \) satisfying

\[
\begin{align*}
      r_f(h) &= f(\zeta + h) - f(\zeta) - Df(\zeta)(h), \tag{4.35a} \\
      r_g(h) &= g(f(\zeta + h)) - g(f(\zeta)) - Dg(f(\zeta))(h) \tag{4.35b}
\end{align*}
\]

for each \( h \in \mathbb{K}^n \) (resp. each \( h \in \mathbb{K}^m \)) such that \( \|h\|_2 \) is sufficiently small, as well as

\[
\lim_{h \to 0} \frac{r_f(h)}{\|h\|_2} = 0, \quad \lim_{h \to 0} \frac{r_g(h)}{\|h\|_2} = 0. \tag{4.36}
\]

Defining \( r_{g \circ f} : \mathbb{K}^n \rightarrow \mathbb{K}^p \) by

\[
r_{g \circ f}(h) := \begin{cases} (g \circ f)(\zeta + h) - (g \circ f)(\zeta) - \left( Dg(f(\zeta)) \circ Df(\zeta) \right)(h) & \text{for } \zeta + h \in G_f, \\ 0 & \text{otherwise}, \end{cases} \tag{4.37}
\]

it remains to show

\[
\lim_{h \to 0} \frac{r_{g \circ f}(h)}{\|h\|_2} = 0. \tag{4.38}
\]

For each \( h \in \mathbb{K}^n \) with \( \|h\|_2 \) sufficiently small, we use (4.35) to compute

\[
(g \circ f)(\zeta + h) = g \left( f(\zeta) + Df(\zeta)(h) + r_f(h) \right) = g(f(\zeta)) + Dg(f(\zeta)) \left( Df(\zeta)(h) + r_f(h) \right) + r_g \left( Df(\zeta)(h) + r_f(h) \right),
\]

implying

\[
r_{g \circ f}(h) = Dg(f(\zeta)) \left( r_f(h) \right) + r_g \left( Df(\zeta)(h) + r_f(h) \right).
\]

From Th. 2.22, we know that \( Dg(f(\zeta)) \) is Lipschitz continuous with some Lipschitz constant \( L_g \in \mathbb{R}^+ \). Thus, for each \( 0 \neq h \in \mathbb{K}^n \),

\[
0 \leq \frac{\| Dg(f(\zeta)) \left( r_f(h) \right) \|_2}{\| h \|_2} \leq L_g \frac{\| r_f(h) \|_2}{\| h \|_2},
\]
implying
\[
\lim_{h \to 0} \frac{\|Dg(f(\xi))(r_f(h))\|_2}{\|h\|_2} = 0
\]
due to (4.36). Thus, to prove (4.38), it merely remains to show
\[
\lim_{h \to 0} \frac{\|r_g(Df(\xi)(h) + r_f(h))\|_2}{\|h\|_2} = 0. \tag{4.39}
\]
To that end, we rewrite, for \(Df(\xi)(h) + r_f(h) \neq 0\),
\[
\frac{\|r_g(Df(\xi)(h) + r_f(h))\|_2}{\|h\|_2} = \frac{\|Df(\xi)(h) + r_f(h)\|_2}{\|h\|_2} \frac{\|r_g(Df(\xi)(h) + r_f(h))\|_2}{\|Df(\xi)(h) + r_f(h)\|_2}. \tag{4.40a}
\]
Next, note
\[
\lim_{h \to 0} \|Df(\xi)(h) + r_f(h)\|_2 = 0 \quad \Rightarrow \quad \lim_{h \to 0} \frac{\|r_g(Df(\xi)(h) + r_f(h))\|_2}{\|Df(\xi)(h) + r_f(h)\|_2} = 0. \tag{4.40b}
\]
Once again, from Th. 2.22, we know that \(Df(\xi)\) is Lipschitz continuous with some Lipschitz constant \(L_f \in \mathbb{R}^+_0\), implying
\[
\frac{\|Df(\xi)(h) + r_f(h)\|_2}{\|h\|_2} \leq \frac{\|Df(\xi)(h)\|_2 + \|r_f(h)\|_2}{\|h\|_2} \leq L_f + 1 \tag{4.40c}
\]
for \(0 \neq \|h\|_2\) sufficiently small. Combining (4.40a) – (4.40c) proves (4.39) and, thus, (4.38). Together with (4.37) and Lem. 4.20, this shows that \(g \circ f\) is differentiable at \(\xi\) with \(D(g \circ f)(\xi) = Dg(f(\xi)) \circ Df(\xi)\).

**Example 4.32.** In the setting of the chain rule of Th. 4.31, we consider the special case \(n = p = 1\). Thus, we have an open subset \(G_f\) of \(\mathbb{R}\) and \(f : G_f \to \mathbb{R}^m\). The map \(g\) maps \(G_f\) into \(\mathbb{K}\) and for \(h := g \circ f : G_f \to \mathbb{K}\), we have \(h(t) = g(f_1(t), \ldots, f_m(t))\).

In this case, one computes the one-dimensional function \(h\) by making a detour through the \(m\)-dimensional space \(\mathbb{R}^m\). If \(f\) is \(\mathbb{R}\)-differentiable at \(\xi \in G_f\) and \(g\) is \(\mathbb{R}\)-differentiable at \(f(\xi) \in G_g\), the chain rule (4.34) now reads
\[
Dh(\xi) = D(g \circ f)(\xi) = Dg(f(\xi)) \circ Df(\xi) = \nabla g(f(\xi)) J_f(\xi) = \sum_{j=1}^m \partial_j g(f(\xi)) \partial_1 f_j(\xi), \tag{4.41}
\]
where, for \(\mathbb{K} = \mathbb{C}\), we applied the case \(\mathbb{K} = \mathbb{R}\) of the chain rule to \(\text{Re} g\) and \(\text{Im} g\). Recall from Example 4.25(c) that, for one-dimensional functions such as \(h\), the function \(Dh(\xi) : \mathbb{R} \to \mathbb{K}\) corresponds to the number \(h'(\xi) \in \mathbb{K}\) via (4.24). Also recall that, for one-dimensional functions such as \(f_j\), the partial derivative \(\partial_1 f_j\) coincides with the one-dimensional derivative \(f'_j\). Thus, (4.41) implies
\[
h'(\xi) = \sum_{j=1}^m \partial_j g(f(\xi)) f'_j(\xi). \tag{4.42}
\]
**Definition 4.33.** Let $G \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$. A differentiable path is an $\mathbb{R}$-differentiable function $\phi : [a, b[ \rightarrow G$, $a, b \in \mathbb{R}$, $a < b$. The set $G$ is called connected by differentiable paths if, and only if, for each $x, y \in G$, there exists some differentiable path $\phi : [a, b[ \rightarrow G$ such that $\phi(s) = x$ and $\phi(t) = y$ for suitable $s, t \in ]a, b[$.

**Proposition 4.34.** Let $G \subseteq \mathbb{R}^m$ be open, $m \in \mathbb{N}$. If $G$ is connected by differentiable paths and $f : G \rightarrow \mathbb{K}$ is $\mathbb{R}$-differentiable with $\nabla f \equiv 0$, then $f$ is constant.

**Proof.** Let $x, y \in G$, and let $\phi : [a, b[ \rightarrow G$ be a differentiable path connecting $x$ and $y$, i.e. $\phi(s) = x$ and $\phi(t) = y$ for suitable $s, t \in ]a, b[$. Define the auxiliary function $h : [a, b[ \rightarrow \mathbb{K}$, $h = f \circ \phi$. By the chain rule of Th. 4.31, $h$ is $\mathbb{R}$-differentiable and, using (4.42) and $\partial_j f \equiv 0$ for each $j \in \{1, \ldots, m\}$,

$$h'(\xi) = \sum_{j=1}^{m} \partial_j f(\phi(\xi)) \phi'(\xi)(\alpha) = 0 \quad \text{for each } \xi \in ]a, b[.$$  

As a one-dimensional function on an open interval with vanishing derivative, $h$ must be constant (as both Re$h$ and Im$h$ must be constant by [Phi16, Cor. 9.19(b)]), implying $f(x) = f(\phi(s)) = h(s) = h(t) = f(\phi(t)) = f(y)$, showing that $f$ is constant as well.  

### 4.8 The Mean Value Theorem

Another application of the chain rule in several variables is the mean value theorem in several variables:

**Theorem 4.35.** Let $G \subseteq \mathbb{R}^n$ be open, $n \in \mathbb{N}$, $f : G \rightarrow \mathbb{R}$. If $f$ is $\mathbb{R}$-differentiable on $G$ and $x, y \in G$ are such that the entire line segment connecting $x$ and $y$ is also contained in $G$, i.e. $S_{x,y} := \{x + t(y - x) : 0 < t < 1\} \subseteq G$, then there is $\xi \in S_{x,y}$ satisfying

$$f(y) - f(x) = Df(\xi)(y - x) = \nabla f(\xi)(y - x) = \sum_{j=1}^{n} \partial_j f(\xi)(y_j - x_j). \quad (4.43)$$

**Proof.** We merely need to combine the one-dimensional mean value theorem [Phi16, Th. 9.18] with the chain rule of Th. 4.31. A small problem arises from the fact that, in Th. 4.31, we required $G_f$ to be open. We therefore note that the openness of $G$ allows us to find some $\varepsilon > 0$ such that the small extension $S_{x,y,\varepsilon} := \{x + t(y - x) : -\varepsilon < t < 1 + \varepsilon\}$ is still contained in $G$: $S_{x,y,\varepsilon} \subseteq G$. Consider the auxiliary functions

$$\phi : ] - \varepsilon, 1 + \varepsilon [ \rightarrow \mathbb{R}^n, \quad \phi(t) := x + t(y - x) \quad \text{and} \quad h : ] - \varepsilon, 1 + \varepsilon [ \rightarrow \mathbb{R}, \quad h(t) := (f \circ \phi)(t) = f(x + t(y - x)).$$

As the sum of a constant function and a linear function, $\phi$ is differentiable, and $D\phi(t) : \mathbb{R} \rightarrow \mathbb{R}^n$, $D\phi(t) = y - x$ (that means, for each $\alpha \in \mathbb{R}$, one has $D\phi(t)(\alpha) = \alpha(y - x)$). Thus, according to Th. 4.31, $h$ is differentiable, and, using (4.41),

$$Dh(t) = Df(\phi(t)) \circ D\phi(t) = \nabla f(\phi(t))(y - x). \quad (4.44)$$
The one-dimensional mean value theorem \[\text{[Phi16, Th. 9.18]}\] provides \(\theta \in ]0, 1[\) such that

\[f(y) - f(x) = h(1) - h(0) = h'(\theta).\]

As in Example 4.32, we recall from (4.24) that the real number \(h'(\theta)\) represents the linear map \(Dh(\theta)\) such that we can combine (4.44) and (4.45) to obtain

\[f(y) - f(x) = h'(\theta) = \nabla f(\phi(\theta))(y-x) = \nabla f(\xi)(y-x)\]

with \(\xi := \phi(\theta) = x + \theta(y-x) \in S_{x,y}\), concluding the proof of (4.43).

**Caveat 4.36.** Unlike many other results of this class, Th. 4.35 does not extend to \(\mathbb{C}\)-valued functions – actually, even the one-dimensional mean value theorem does not extend to \(\mathbb{C}\)-valued functions. It is an exercise to find an explicit counterexample of a differentiable function \(f : \mathbb{R} \to \mathbb{C}\) and \(x,y \in \mathbb{R}\), \(x < y\), such that there does not exist \(\xi \in ]x, y[\) satisfying \(f(y) - f(x) = f'(\xi)(y-x)\).}

As an application of Th. 4.35, let us prove that differentiable maps with bounded partials are Lipschitz continuous on convex sets.

**Definition 4.37.** A set \(G \subseteq \mathbb{R}^n\), \(n \in \mathbb{N}\), is called convex if, and only if, for each \(x, y \in G\), one has \(S_{x,y} := \{x + t(y-x) : 0 < t < 1\} \subseteq G\).

**Theorem 4.38.** Let \(m,n \in \mathbb{N}\), let \(G \subseteq \mathbb{R}^n\) be open, and let \(f : G \to \mathbb{R}^m\) be \(\mathbb{R}\)-differentiable. Suppose there exists \(M \in \mathbb{R}_+^m\) such that \(|\partial_j f_i(\xi)| \leq M\) for each \(j \in \{1, \ldots, n\}\), each \(i \in \{1, \ldots, m\}\), and each \(\xi \in G\). If \(G\) is convex, then \(f\) is Lipschitz continuous with Lipschitz constant \(L := mM\) with respect to the 1-norms on \(\mathbb{R}^n\) and \(\mathbb{R}^m\) and with Lipschitz constant \(cL\), \(c > 0\), with respect to arbitrary norms on \(\mathbb{R}^n\) and \(\mathbb{R}^m\).

**Proof.** Fix \(l \in \{1, \ldots, m\}\). We first show that \(f_l\) is \(M\)-Lipschitz with respect to the 1-norm on \(\mathbb{R}^n\): Since \(f_l\) is differentiable and \(G\) is convex, given \(x,y \in G\), we can apply Th. 4.35 to obtain \(\xi \in G\) such that

\[|f_l(y) - f_l(x)| \leq \sum_{j=1}^n |f_j f_l(\xi_j)| |y_j - x_j| \leq M \|y - x\|_1,
\]

showing that, with respect to the 1-norm, \(f_l\) is Lipschitz continuous with Lipschitz constant \(M\). In consequence, we obtain, for each \(x,y \in G\),

\[\|f(y) - f(x)\|_1 = \sum_{i=1}^m |f_i(y) - f_i(x)| \leq m M \|y - x\|_1,
\]

showing that, with respect to the 1-norms on \(\mathbb{R}^n\) and \(\mathbb{R}^m\), \(f\) is Lipschitz continuous with Lipschitz constant \(m M\). Since all norms on \(\mathbb{R}^n\) and \(\mathbb{R}^m\) are equivalent, we also get that \(f\) is Lipschitz continuous with Lipschitz constant \(cL\), \(c > 0\), with respect to all other norms on \(\mathbb{R}^n\) and \(\mathbb{R}^m\).
4.9 Directional Derivatives

Given a real-valued function $f$, the partial derivatives $\partial_j f$ (if they exist) describe the local change of $f$ in the direction of the standard unit vector $e_j$. We would now like to generalize the notion of partial derivative in such a way that it allows us to study the change of $f$ in an arbitrary direction $e \in \mathbb{R}^n$. This leads to the following notion of directional derivatives.

**Definition 4.39.** Let $G \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $f : G \rightarrow \mathbb{K}$, $\xi \in G$, $e \in \mathbb{R}^n$. If there is $\epsilon > 0$ such that $\xi + he \in G$ for each $h \in [0, \epsilon]$ (this condition is trivially satisfied if $\xi$ is an interior point of $G$), then $f$ is said to have a **directional derivative** at $\xi$ in the direction $e$ if, and only if, the limit

$$
\lim_{h \downarrow 0} \frac{f(\xi + he) - f(\xi)}{h}
$$

exists in $\mathbb{K}$. In that case, this limit is identified with the corresponding directional derivative and denoted by $\frac{\partial f}{\partial e}(\xi)$ or by $\delta f(\xi, e)$. If the directional derivative of $f$ in the direction $e$ exists for each $\xi \in G$, then the function

$$
\frac{\partial f}{\partial e} : G \rightarrow \mathbb{K}, \quad \xi \mapsto \frac{\partial f}{\partial e}(\xi),
$$

is also called the directional derivative of $f$ in the direction $e$.

**Remark 4.40.** Consider the setting of Def. 4.39 and suppose $e = e_j$ for some $j \in \{1, \ldots, n\}$. If $\xi$ is an interior point of $G$, then the directional derivative $\frac{\partial f}{\partial e_j}(\xi)$ coincides with the partial derivative $\partial_j f(\xi)$ of Def. 4.1 if, and only if, both $\frac{\partial f}{\partial e_j}(\xi)$ and $\frac{\partial f}{\partial (-e)}(\xi)$ exist and $\frac{\partial f}{\partial e_j}(\xi) = -\frac{\partial f}{\partial (-e)}(\xi)$: If $\partial_j f(\xi)$ exists, then

$$
\partial_j f(\xi) = \lim_{h \rightarrow 0} \frac{f(\xi + he) - f(\xi)}{h} = \lim_{h \downarrow 0} \frac{f(\xi + he_j) - f(\xi)}{h} = \frac{\partial f}{\partial e_j}(\xi) = \frac{\partial f}{\partial e}(\xi) = \lim_{h \uparrow 0} \frac{f(\xi + he_j) - f(\xi)}{h} = \lim_{h \downarrow 0} \frac{f(\xi - he_j) - f(\xi)}{-h} = -\frac{\partial f}{\partial (-e)}(\xi).
$$

On the other hand, if both $\frac{\partial f}{\partial e_j}(\xi)$ and $\frac{\partial f}{\partial (-e)}(\xi)$ exist and $\frac{\partial f}{\partial e_j}(\xi) = -\frac{\partial f}{\partial (-e)}(\xi)$, then the corresponding equalities in (4.48) show that both one-sided partials exist at $\xi$ and that their values agree, showing that $\partial_j f(\xi) = \frac{\partial f}{\partial e_j}(\xi)$ exists.

We can now generalize Th. 4.21:

**Theorem 4.41.** Let $G$ be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $\xi \in G$. If $f : G \rightarrow \mathbb{K}$ is $\mathbb{R}$-differentiable in $\xi$, then, for each $e = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n$, the directional derivative $\frac{\partial f}{\partial e}(\xi)$ exists and

$$
\frac{\partial f}{\partial e}(\xi) = \nabla f(\xi) \cdot e = \sum_{j=1}^{n} \epsilon_j \partial_j f(\xi).
$$
Moreover, if we consider \( \mathbb{K} = \mathbb{R} \) and only allow normalized \( e \in \mathbb{R}^n \) with \( \|e\|_2 = 1 \), then the directional derivatives can take only values between \( \alpha := \|\nabla f(\xi)\|_2 \) and \(-\alpha\), where the largest value (i.e. \( \alpha \)) is attained in the direction \( e_{\text{max}} := \nabla f(\xi)/\alpha \) and the smallest value (i.e. \(-\alpha\)) is attained in the direction \( e_{\text{min}} := -e_{\text{max}} \). For \( n = 1 \), \( e = \pm 1 \) are the only possible directions, yielding precisely the values \( \alpha \) and \(-\alpha\). For \( n \geq 2 \), all values in \([-\alpha, \alpha]\) are attained.

**Proof.** Since \( G \) is open, there is \( \epsilon > 0 \) such that \( \xi + \epsilon e \in G \) for each \( h \in ]-\epsilon, \epsilon[ \). Similarly to the proof of Th. 4.35, consider auxiliary functions

\[
\phi : ]-\epsilon, \epsilon[ \rightarrow \mathbb{R}^n, \quad \phi(h) := \xi + \epsilon h, \\
g : ]-\epsilon, \epsilon[ \rightarrow \mathbb{K}, \quad g(h) := (f \circ \phi)(h) = f(\xi + \epsilon h).
\]

Theorem 4.31 yields the \( \mathbb{R} \)-differentiability of \( g \), and, as \( D\phi \equiv e \) (i.e., for each \( h \in ]-\epsilon, \epsilon[ \) and each \( \alpha \in \mathbb{R} \), it is \( D\phi(h)(\alpha) = \alpha e \)), by (4.41), we have

\[
\forall h \in ]-\epsilon, \epsilon[ \quad g'(h) = D(f \circ \phi)(h) = Df(\phi(h)) \circ D\phi(h) = \nabla f(\xi + \epsilon h) \cdot e
\]

and

\[
\frac{\partial f}{\partial e}(\xi) = g'(0) = \nabla f(\xi) \cdot e,
\]

proving (4.49). Applying the Cauchy-Schwarz inequality (1.41) to (4.49) yields

\[
\left| \frac{\partial f}{\partial e}(\xi) \right| = \left| \nabla f(\xi) \cdot e \right| \leq \left\| \nabla f(\xi) \right\|_2 \left\| e \right\|_2 = \alpha \|e\|_2.
\]

(4.50)

Thus, for \( \mathbb{K} = \mathbb{R} \) and \( e \in \mathbb{R}^n \) with \( \|e\|_2 = 1 \), we have \(-\alpha \leq \frac{\partial f}{\partial e}(\xi) \leq \alpha \). It remains to show that, for \( \mathbb{K} = \mathbb{R} \) and \( n \geq 2 \), the map

\[
D : S_1(0) \longrightarrow [-\alpha, \alpha], \quad D(e) := \nabla f(\xi) \cdot e = \sum_{j=1}^{n} \epsilon_j \partial_j f(\xi),
\]

is surjective. The details are bit tedious and are carried out in App. K.2.

The following example shows that the existence of all directional derivatives does not imply continuity, let alone differentiability.

**Example 4.42.** Consider the function

\[
f : \mathbb{R}^2 \longrightarrow \mathbb{K}, \quad f(x, y) := \begin{cases} 
1 & \text{for } 0 < y < x^2, \\
0 & \text{otherwise}.
\end{cases}
\]

The function is not continuous in \((0,0)\): Let \( x_n := 1/n \) and \( y_n := 1/n^3 \). Then \( \lim_{n \to \infty} (x_n, y_n) = (0,0) \). However, since \( y_n = 1/n^3 < 1/n^2 = x_n^2 \) for \( n > 1 \), one has

\[
\lim_{n \to \infty} f(x_n, y_n) = 1 \neq 0 = f(0,0).
\]
We now claim that, for each \( e = (\epsilon_x, \epsilon_y) \in \mathbb{R}^2 \), the directional derivative \( \frac{\partial f}{\partial e}(0, 0) \) exists and \( \frac{\partial f}{\partial e}(0, 0) = 0 \). For \( \epsilon_y \leq 0 \) this is immediate since, for each \( h \in \mathbb{R}^+ \), \( f((0, 0) + h(\epsilon_x, \epsilon_y)) = f(h\epsilon_x, h\epsilon_y) = 0 \). Now assume \( \epsilon_y > 0 \). If \( \epsilon_x = 0 \), then \( f(h\epsilon_x, h\epsilon_y) = 0 \) for each \( h \in \mathbb{R}^+ \), showing \( \frac{\partial f}{\partial e}(0, 0) = 0 \). It remains the case, where \( \epsilon_y > 0 \) and \( \epsilon_x \neq 0 \). In that case, one obtains \( h^2 \epsilon_x^2 < h\epsilon_y \) for each \( 0 < h < \frac{\epsilon_y}{\epsilon_x} \). Thus, for such \( h \), \( f(h\epsilon_x, h\epsilon_y) = 0 \), once again proving \( \frac{\partial f}{\partial e}(0, 0) = 0 \).

4.10 Taylor’s Theorem

We will now extend Taylor’s theorem to higher dimensions by means of the chain rule. First, we need to introduce some notation.

**Notation 4.43.** In the context of Taylor’s theorem, we need to consider directional derivatives of higher order. In this context, one often uses a slightly different notation than the one we used earlier. Let \( n \in \mathbb{N} \) and \( h = (h_1, \ldots, h_n) \in \mathbb{R}^n \). If \( G \subseteq \mathbb{R}^n \) is open and \( f : G \rightarrow \mathbb{K} \) is differentiable at some \( \xi \in G \), then, according to (4.49), we can compute the directional derivative

\[
(h \nabla)(f)(\xi) := \frac{\partial f}{\partial h}(\xi) = \sum_{j=1}^{n} h_j \partial_j f(\xi) = h_1 \partial_1 f(\xi) + \cdots + h_n \partial_n f(\xi). \tag{4.51}
\]

The object \( h \nabla \) is also called a **differential operator**. If \( f \) has all partials of second order at \( \xi \), then we can apply \( h \nabla \) again to the function in (4.51), obtaining

\[
(h \nabla)^2(f)(\xi) := (h \nabla)(h \nabla)(f)(\xi) = \sum_{j=1}^{n} (h \nabla)(h_j \partial_j f)(\xi) = \sum_{j,k=1}^{n} h_k h_j \partial_k \partial_j f(\xi). \tag{4.52}
\]

Thus, if \( f \) has all partials of order \( k \) at \( \xi, k \in \mathbb{N} \), then an induction yields

\[
(h \nabla)^k(f)(\xi) = \sum_{j_1, \ldots, j_k=1}^{n} h_{j_k} \cdots h_{j_1} \partial_{j_k} \cdots \partial_{j_1} f(\xi). \tag{4.53a}
\]

If \( f \) is \( k \) times \( \mathbb{R} \)-differentiable, then comparing (4.53a) with (4.33) (for \( h^1 = \cdots = h^k = h \in \mathbb{R}^n \)) yields

\[
(h \nabla)^k(f)(\xi) = D^k f(\xi)(\underbrace{h, \ldots, h}_{k \text{ times}}). \tag{4.53b}
\]

Finally, it is also useful to define

\[
D^0 f(\xi) := (h \nabla)^0(f)(\xi) := f(\xi). \tag{4.54}
\]

**Theorem 4.44** (Taylor’s Theorem). Let \( G \subseteq \mathbb{R}^n \) be open, \( n \in \mathbb{N} \), and \( f \in C^{m+1}(G, \mathbb{K}) \) for some \( m \in \mathbb{N}_0 \) (i.e. \( f : G \rightarrow \mathbb{K} \) and \( f \) has continuous partials up to order \( m + 1 \)).
Let \( \xi \in G \) and \( h \in \mathbb{R}^n \) such that the line segment \( S_{\xi, \xi + h} \) between \( \xi \) and \( \xi + h \) is a subset of \( G \). Then the following formula, also known as Taylor’s formula, holds:

\[
f(\xi + h) = \sum_{k=0}^{m} \frac{(h \nabla)^k(f)(\xi)}{k!} + R_m(\xi) = \sum_{k=0}^{m} \frac{D^k f(\xi)(h, \ldots, h)}{k!} + R_m(\xi)
\]

where
\[
f(\xi) + \frac{(h \nabla)(f)(\xi)}{1!} + \frac{(h \nabla)^2(f)(\xi)}{2!} + \cdots + \frac{(h \nabla)^m(f)(\xi)}{m!} + R_m(\xi), \quad (4.55)
\]

is the integral form of the remainder term. Also similar to the one-dimensional case, if \( K = \mathbb{R} \), then there is \( \theta \in [0, 1] \) such that

\[
R_m(\xi) = \frac{(h \nabla)^{m+1}(f)(\xi + \theta h)}{(m + 1)!}, \quad (4.57)
\]

called the Lagrange form of the remainder term.

**Proof.** Since \( S_{\xi, \xi + h} \subseteq G \) and \( G \) is open, there is \( \epsilon > 0 \) such that we can consider the auxiliary function

\[
\phi : ] - \epsilon, 1 + \epsilon [ \rightarrow K, \quad \phi(t) := f(\xi + th).
\]

This definition immediately implies \( \phi(0) = f(\xi) \) and \( \phi(1) = f(\xi + h) \). We can apply the chain rule to get

\[
\phi'(t) = \nabla f(\xi + th) \cdot h = (h \nabla)(f)(\xi + th),
\]

using the notation from (4.51). Since \( f \in C^{m+1}(G, K) \), we can use an induction to get, for each \( k \in \{0, \ldots, m + 1\} \),

\[
\phi^{(k)}(t) = (h \nabla)^k(f)(\xi + th). \quad (4.58)
\]

Applying the one-dimensional form of Taylor’s theorem [Phi16, Th. 10.27] with the remainder term in integral form to \( \phi \) with \( x = 1 \) and \( a = 0 \) together with (4.58) yields

\[
f(\xi + h) = \phi(1)
\]

\[
= \phi(0) + \phi'(0)(1 - 0) + \frac{\phi''(0)}{2!}(1 - 0)^2 + \cdots + \frac{\phi^{(m)}(0)}{m!}(1 - 0)^m
\]

\[
+ \int_{0}^{1} \frac{(1 - t)^m}{m!} \phi^{(m+1)}(t) \, dt
\]

\[
= f(\xi) + \frac{(h \nabla)(f)(\xi)}{1!} + \frac{(h \nabla)^2(f)(\xi)}{2!} + \cdots + \frac{(h \nabla)^m(f)(\xi)}{m!}
\]

\[
+ \int_{0}^{1} \frac{(1 - t)^m}{m!} (h \nabla)^{m+1}(f)(\xi + th) \, dt, \quad (4.59)
\]
which is precisely (4.55) with \( R_m(\xi) \) in the form (4.56). To prove the Lagrange form of the remainder term, we restate (4.59), this time applying \[\text{Phi16, Th. 10.27}\] to \( \phi \) with the remainder term in Lagrange form, yielding

\[
f(\xi + h) = \sum_{k=0}^{m} \frac{(h \nabla)^k(f)(\xi)}{k!} + \frac{\phi^{(m+1)}(\theta)}{(m+1)!}(1 - 0)^{m+1}
\]

for some suitable \( \theta \in [0, 1) \), thereby completing the proof.

\[\square\]

**Example 4.45.** Let us write Taylor’s formula (4.55) explicitly for the function

\[
f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) := \sin(xy)
\]

for \( m = 1 \) and for \( \xi = (0, 0) \). Here, we have for the gradient

\[
\nabla f(x, y) = \left( y \cos(xy), x \cos(xy) \right)
\]

and for the Hessian matrix of second order partials

\[
H_f(x, y) = \begin{pmatrix}
\partial_x^2 f(x, y) & \partial_y \partial_x f(x, y) \\
\partial_y \partial_x f(x, y) & \partial_y^2 f(x, y)
\end{pmatrix} = \begin{pmatrix}
-y^2 \sin(xy) & \cos(xy) - xy \sin(xy) \\
\cos(xy) - xy \sin(xy) & -x^2 \sin(xy)
\end{pmatrix}.
\]

For \( h = (h_1, h_2) \in \mathbb{R}^2 \), we obtain

\[
f(h) = -h_1^2 \theta^2 h_2^2 \sin(\theta^2 h_1 h_2) + 2h_1 h_2 \cos(\theta^2 h_1 h_2) - 2h_1^2 \theta^2 h_2^2 \sin(\theta^2 h_1 h_2) - h_2^2 \theta^2 h_1^2 \sin(\theta^2 h_1 h_2)
\]

\[
= -2h_1^2 \theta^2 h_2^2 \sin(\theta^2 h_1 h_2) + h_1 h_2 \cos(\theta^2 h_1 h_2)
\]

for some suitable \( 0 < \theta < 1 \).

**4.11 Implicit Function Theorem**

The implicit function Th. 4.49 below provides suitable hypotheses such that the equation \( f(x, y) = C \) can, locally, be solved for \( y \) (or \( x \)). To illustrate the situation, consider

\[
x^2 + y^2 = C, \quad x, y, C \in \mathbb{R},
\]

which, for \( C > 0 \), represents a circle with radius \( \sqrt{C} \) and center at \( (0, 0) \). This simple example already shows that one can not expect such an equation to have a solution for each \( C \), and, if it does have a solution, it need not be unique. However, if

\[
\xi^2 + \eta^2 - C = 0, \quad C > 0,
\]
and \( \eta \neq 0 \), then, in a neighborhood of \((\xi, \eta)\), (4.60) can, indeed, uniquely be solved for \( y \), namely

\[
\text{for } \eta > 0:\quad \forall_{(x,y) \in G_1} \quad \left( x^2 + y^2 = C \iff y = \sqrt{C - x^2} \right), \quad G_1 := \left] -\sqrt{C}, \sqrt{C} \right[ \times \mathbb{R}^+
\]

(note that \( G_1 \) is, indeed, an open neighborhood of \((\xi, \eta)\)), and

\[
\text{for } \eta < 0:\quad \forall_{(x,y) \in G_2} \quad \left( x^2 + y^2 = C \iff y = -\sqrt{C - x^2} \right), \quad G_2 := \left[ -\sqrt{C}, \sqrt{C} \right[ \times \mathbb{R}^-
\]

(note that \( G_2 \) is, indeed, an open neighborhood of \((\xi, \eta)\)). If \( \eta = 0 \), then, in each neighborhood of \((\xi, \eta)\), (4.60) has two distinct solutions for \( y \). However, in this case, there exists a neighborhood of \((\xi, \eta)\), where (4.60) can, indeed, uniquely be solved for \( x \) (one merely has to switch the roles of \( x \) and \( y \) in the above considerations).

In the example

\[
|x| - |y| = 0, \quad x, y \in \mathbb{R},
\]

the equation can not be solved uniquely for either \( x \) or \( y \) in any neighborhood of \((0, 0)\).

The implicit function Th. 4.49 will show that a sufficient condition for \( f(x, y) = 0 \) to be uniquely solvable for \( y \) in a neighborhood of \((\xi, \eta)\) is \( f \) to be continuously differentiable, with invertible derivative with respect to \( y \) in \((\xi, \eta)\).

In preparation for the proof of the implicit function theorem, we provide the following proposition:

**Proposition 4.46.** Let \( \| \cdot \| \) be some norm on \( \mathbb{R}^n \), \( n \in \mathbb{N} \). Moreover, let \( a \in \mathbb{R}^n \), \( r > 0 \), and let \( f : B_r(a) \to \mathbb{R}^n \) be defined on the open \( r \)-ball with center \( a \) with respect to \( \| \cdot \| \). If \( A \) is an invertible \( n \times n \) matrix over \( \mathbb{R} \) such that

\[
\| A^{-1}f(a) \| < \frac{r}{2}
\]

and such that the map

\[
F : B_r(a) \to \mathbb{R}^n, \quad F(x) := x - A^{-1}f(x),
\]

is Lipschitz continuous with Lipschitz constant \( L = 1/2 \), then \( f \) has a unique zero \( \xi \in B_r(a) \). Moreover, for each \( x_0 \in B_r(a) \), \( \xi \) is the limit of the sequence \((x_k)_{k \in \mathbb{N}_0} \), recursively defined by

\[
\forall_{k \in \mathbb{N}_0} \quad x_{k+1} := F(x_k).
\]

**Proof.** Let \( x_0 \in B_r(a) \) and set

\[
s_0 := \max \left\{ 2 \| A^{-1}f(a) \| , \| x_0 - a \| \right\}^{(4.64)} \in [0, r].
\]

The idea is to show that, for each \( s_0 < s < r \), the Banach fixed point Th. 2.29 applies to the contraction

\[
F_s : \overline{B}_s(a) \to \overline{B}_s(a), \quad F_s(x) := F(x).
\]
We verify that $F_s$, indeed, maps $B_s(a)$ into $B_s(a)$: If $x \in B_s(a)$, then
\[
\|F(x) - a\| \leq \|F(x) - F(a)\| + \|F(a) - a\| \leq \frac{1}{2}\|x - a\| + \|A^{-1}f(a)\|
\]
\[
\leq \frac{s}{2} + \frac{s_0}{2} < s,
\]
showing $F_s(x) \in B_s(a)$ (in particular, this shows the $x_k$ are well-defined by (4.66)). As $F$ is Lipschitz continuous with Lipschitz constant $L = 1/2$, so is $F_s$, i.e. $F_s$ is, indeed, a contraction. As $B_s(a)$ is closed, the Banach fixed point Th. 2.29 yields that $F_s$ has a unique fixed point $\xi$ and, moreover, $\xi = \lim_{k \rightarrow \infty} x_k$. Since this holds for each $s \in ]s_0, r[$, $\xi$ must also be the unique fixed point of $F$. The proof is concluded by noting
\[
\forall_{y \in B_s(a)} f(y) = 0 \iff F(y) = y - A^{-1}f(y) = y,
\]
that means $y$ is a zero of $f$ if, and only if, $y$ is a fixed point of $F$.  

**Remark 4.47.** If the map $f$ in Prop. 4.46 is differentiable with invertible derivatives $Df(x)$, and if, instead of using a constant matrix $A$ in the definition of (4.66), one uses $(Df(x_k))^{-1}$, then the iteration defined by (4.66) is known as **Newton’s method** (in $n$ dimensions, cf. [Phi17, Sec. 6.3]). In consequence, if $A \approx (Df(x_k))^{-1}$ in (4.66), then the defined iteration is sometimes referred to as a simplified Newton’s method.

**Notation 4.48.** Let $k, m, n \in \mathbb{N}$, let $G \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open, and consider a map $f : G \rightarrow \mathbb{R}^k$. If $(\xi, \eta) \in G$ and $f$ is $\mathbb{R}$-differentiable at $(\xi, \eta)$, then let $D_yf(\xi, \eta)$ and $D_xf(\xi, \eta)$ denote the $\mathbb{R}$-linear maps
\[
D_yf(\xi, \eta) : \mathbb{R}^m \rightarrow \mathbb{R}^k, \quad (D_yf(\xi, \eta))(h) := (Df(\xi, \eta))(0, h), \quad (4.67a)
\]
\[
D_xf(\xi, \eta) : \mathbb{R}^n \rightarrow \mathbb{R}^k, \quad (D_xf(\xi, \eta))(h) := (Df(\xi, \eta))(h, 0), \quad (4.67b)
\]
respectively.

**Theorem 4.49 (Implicit Function Theorem).** Let $m, n \in \mathbb{N}$, let $G \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open, and let $f : G \rightarrow \mathbb{R}^m$ be continuously differentiable, i.e. $f \in C^1(G, \mathbb{R}^m)$. If $(\xi, \eta) \in G$ is such that
\[
f(\xi, \eta) = 0 \quad \text{and} \quad A := D_yf(\xi, \eta) \text{ is invertible}, \quad (4.68)
\]
then there exist open neighborhoods $U_\xi \subseteq \mathbb{R}^n$ of $\xi$ and $V_\eta \subseteq \mathbb{R}^m$ of $\eta$, and a continuously differentiable map $g : U_\xi \rightarrow V_\eta$ such that the zeros of $f$ in $U_\xi \times V_\eta$ are given precisely by the graph of $g$, i.e.
\[
(U_\xi \times V_\eta) \cap f^{-1}\{0\} = \{(x, g(x)) : x \in U_\xi\}, \quad (4.69a)
\]
which can be restated as
\[
\forall_{(x,y) \in U_\xi \times V_\eta} \left( f(x, y) = 0 \iff y = g(x) \right). \quad (4.69b)
\]
Moreover,
\[
\forall_{x \in U_\xi} Dg(x) = -(D_yf(x, g(x)))^{-1} D_xf(x, g(x)) \quad (4.70)
\]
and, if $f \in C^\alpha(G, \mathbb{R}^m)$, $\alpha \in \mathbb{N} \cup \{\infty\}$, then $g \in C^\alpha(U_\xi, \mathbb{R}^m)$.
Proof. Fix some arbitrary norms on $\mathbb{R}^n$ and on the set $\mathcal{M}(m, \mathbb{R})$ of real $m \times m$ matrices (for readability’s sake, we will denote both norms by $\|\cdot\|$). On $\mathbb{R}^m$, we will use the 1-norm $\|\cdot\|_1$ to apply Th. 4.38. According to the hypothesis, $A$ is invertible. Thus, $\det(A) \neq 0$. Since the map $B \mapsto \det(B)$ is continuous (cf. Ex. 2.21(a)), and the map $D_y f : G \mapsto \mathcal{M}(m, \mathbb{R})$ is continuous due to the assumed continuous differentiability of $f$, the set

$$G_0 := \{(x, y) \in G : \det(D_y f(x, y)) \neq 0\} \subseteq G$$

is an open neighborhood of $(\xi, \eta)$. Next, we consider the map

$$F : G_0 \mapsto \mathbb{R}^m, \quad F(x, y) := y - A^{-1}f(x, y).$$

Then $F$ is continuously differentiable with

$$D_y F : G_0 \mapsto \mathcal{M}(m, \mathbb{R}), \quad D_y F(x, y) = \text{Id} - A^{-1}D_y f(x, y),$$

being continuous as well. Thus, since $D_y F(\xi, \eta) = \text{Id} - A^{-1}A = 0$, there exists $r > 0$ such that the open $r$-balls $B_r(\xi) \subseteq \mathbb{R}^n$ and $B_r(\eta) \subseteq \mathbb{R}^m$ satisfy

$$(\xi, \eta) \in B_r(\xi) \times B_r(\eta) \subseteq \left\{(x, y) \in G_0 : \forall_{k,l=1,\ldots,m} |\partial_{y_k} F_l(x, y)| \leq \frac{1}{2m}\right\} \subseteq G_0. \quad (4.71)$$

As we assume $f$ to be continuous with $f(\xi, \eta) = 0$, there exists $s \in [0, r]$ such that

$$B_s(\xi) \subseteq \left\{x \in B_r(\xi) : \|A^{-1}f(x, \eta)\|_1 < \frac{r}{2}\right\} \subseteq B_r(\xi). \quad (4.72)$$

To construct the map $g : B_s(\xi) \mapsto B_r(\eta)$, we fix $x \in B_s(\xi)$ and apply Prop. 4.46 to the function

$$f_x : B_r(\eta) \mapsto \mathbb{R}^m, \quad f_x(y) := f(x, y). \quad (4.73)$$

To verify that the hypotheses of Prop. 4.46 are satisfied, we observe $\|A^{-1}f_x(\eta)\|_1 < \frac{r}{2}$ holds due to $x \in B_s(\xi)$ and (4.72), the map $F_x : B_r(\eta) \mapsto \mathbb{R}^m, F_x(y) := y - A^{-1}f_x(y) = F(x, y)$, is Lipschitz continuous with Lipschitz constant $L = m\frac{1}{2m} = \frac{1}{2}$ due to (4.71) and Th. 4.38. Thus, according to Prop. 4.46, the function $f_x$ has a unique zero $g(x)$ in $B_r(\eta)$, which defines the function $g$.

Note that, in the above argument, for each $0 < \rho < r$, one can choose $s(\rho) < s$ such that (4.72) holds with $s$ replaced by $s(\rho)$ and $r$ replaced by $\rho$, then showing that $g$ maps $B_s(\xi)$ into $B_{\rho}(\eta)$. We now choose some arbitrary $\rho \in [0, r]$ and set

$$U_\xi := B_{s(\rho)}(\xi), \quad V_\eta := B_{\rho}(\eta)$$

for the desired neighborhoods of the theorem. We verify $g$ to be continuous on $U_\xi$: Let $x \in U_\xi$ and let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $U_\xi$ with $\lim_{k \to \infty} x_k = x$. We have to show $\lim_{k \to \infty} g(x_k) = g(x)$. If $\lim_{k \to \infty} g(x_k) = g(x)$ does not hold, then, without loss of generality, we may assume that there exists $\epsilon > 0$ such that $\|g(x_k) - g(x)\| > \epsilon$ for each $k \in \mathbb{N}$ (after having replaced $(x_k)_{k \in \mathbb{N}}$ with a suitable subsequence). Moreover, we may replace $(x_k)_{k \in \mathbb{N}}$ with another subsequence such that there exists $y \in \overline{B}_{\rho}(\eta) \subseteq B_{\rho}(\eta)$
satisfying \( y = \lim_{k \to \infty} g(x_k) \) (this is due to the Bolzano-Weierstrass Th. 1.31, as \( g(x_k) \in B_\rho(\eta) \) for each \( k \in \mathbb{N} \)). Then the continuity of \( f \) implies

\[
 f(x, y) = \lim_{k \to \infty} f(x_k, g(x_k)) = 0,
\]

showing \( g(x) = y = \lim_{k \to \infty} g(x_k) \) (due to (4.69b) – here we need \( y \in B_\rho(\eta) \), which was the reason for choosing \( \rho < r \)), which is in contradiction to the choice of the \( x_k \), and proves the continuity of \( g \).

Next, we show that \( g \) is differentiable at each \( x \in U_\xi \), where the derivative is given by (4.70): To this end, let \( x \in U_\xi \) and note the existence of \( (D_y f(x, g(x)))^{-1} \) due to \( (x, g(x)) \in U_\xi \times V_\eta \subseteq G_0 \). According to Def. 4.18, we have to show

\[
 \lim_{h \to 0} \frac{g(x + h) - g(x) + (D_y f(x, g(x)))^{-1} D_x f(x, g(x)) h}{\|h\|} = 0. \tag{4.74}
\]

Let \( 0 \neq h \in \mathbb{R}^n \) be sufficiently small such that \( x + h \in U_\xi \). Using the notation of the mean value Th. 4.35, for each \( l \in \{1, \ldots, m\} \), there exist \( x_{h,l} \in S_{x,x+h} \) and \( y_{h,l} \in S_{g(x),g(x+h)} \) such that

\[
0 = f_1(x + h, g(x + h)) - f_1(x, g(x))
= f_1(x + h, g(x + h)) - f_1(x, g(x + h)) + f_1(x, g(x + h)) - f_1(x, g(x))
= D_x f_1(x_{h,l}, g(x + h))(h) + D_y f_1(x, y_{h,l})(g(x + h) - g(x)). \tag{4.75}
\]

Note that the two derivatives occurring in (4.75) have the form of gradients, which, according to our usual convention, we can interpret as row vectors. Joining \( m \) row vectors into a matrix, we can write the \( m \) equations of (4.75) in matrix form as

\[
0 = X_h h + Y_h (g(x + h) - g(x)), \tag{4.76}
\]

where

\[
X_h := \begin{pmatrix}
D_x f_1(x_{h,1}, g(x + h)) \\
\vdots \\
D_x f_m(x_{h,m}, g(x + h))
\end{pmatrix}, \quad Y_h := \begin{pmatrix}
D_y f_1(x, y_{h,1}) \\
\vdots \\
D_y f_m(x, y_{h,m})
\end{pmatrix}.
\]

As we already know \( g \) to be continuous, \( h \to 0 \) implies \( g(x + h) \to g(x) \). Thus, since \( y_{h,l} \in S_{g(x),g(x+h)} \), \( h \to 0 \) implies \( y_{h,l} \to g(x) \) for each \( l \in \{1, \ldots, m\} \), and, as all partials of \( f \) are continuous as well, \( Y_h \to D_y f(x, g(x)) \). Since the maps \( B \mapsto \det(B) \) and \( B \mapsto \|B^{-1}\| \) are continuous (cf. Ex. 2.6(a) and Ex. 2.21(a),(b)), \( h \to 0 \) implies \( \det(Y_h) \to \det(D_y f(x, g(x))) \neq 0 \) and \( Y_h \) is invertible for sufficiently small \( h \) with \( (Y_h)^{-1} \to (D_y f(x, g(x)))^{-1} \). For such sufficiently small \( h \), we can rewrite (4.76) as

\[
g(x + h) - g(x) = -(Y_h)^{-1} X_h h.
\]

Also, since \( x_{h,l} \in S_{x,x+h} \), \( h \to 0 \) implies \( x_{h,l} \to x \) and, then, the continuity of \( g \) together with the continuity of the partials of \( f \) implies \( X_h \to D_x f(x, g(x)) \). Thus, we can finish
the proof of (4.70) by noting
\[
\lim_{h \to 0} \left\| \frac{g(x + h) - g(x) + (D_y f(x, g(x)))^{-1} D_x f(x, g(x)) h}{\|h\|} \right\|_1 = 0.
\]

It remains to prove that \( g \) is \( C^\alpha \) if \( f \) is \( C^\alpha \), \( \alpha \in \mathbb{N} \cup \{\infty\} \). To this end, for \( \alpha \in \mathbb{N} \), we will show by induction on \( \beta = 1, \ldots, \alpha \) that each partial derivative of \( g \) at \( x \in \xi \) of order \( \beta \) is a rational function of partials of \( f \) of order \( \leq \beta \), all taken at \( (x, g(x)) \), and of partials of \( g \) of order \( \leq \beta - 1 \), all taken at \( x \) (in particular, the denominator of this rational function does not have any zeros in \( \xi \)): For \( \beta = 1 \), the claim follows from (4.70): The entries of \( D_x f(x, g(x)) \) are polynomials of first partials of \( f \) taken at \( (x, g(x)) \); the entries of \( (D_y f(x, g(x)))^{-1} \) are rational functions, where both the numerator and the denominator polynomial are polynomials of first partials of \( f \) taken at \( (x, g(x)) \) (in particular, the entries of the right-hand side of (4.70) do not involve any first partials of \( g \)). For the induction step, let \( 1 < \beta \leq \alpha \). By induction, we know the partials of \( g \) of order \( \beta - 1 \) are rational functions of partials of \( f \) of order \( \leq \beta - 1 \), all taken at \( (x, g(x)) \), and of partials of \( g \) of order \( \leq \beta - 2 \), all taken at \( x \). Taking the derivative of partials of \( g \) of order \( \leq \beta - 2 \) evaluated at \( x \), yields partials of \( g \) of order \( \leq \beta - 1 \) still evaluated at \( x \); according to the chain rule of Th. 4.31, taking the derivative of partials of \( f \) of order \( \leq \beta - 1 \) evaluated at \( (x, g(x)) \), yields polynomials of partials of \( f \) of order \( \leq \beta - 1 \) evaluated at \( x \) and of first partials of \( g \) evaluated at \( x \). Thus, applying the product and the quotient rule establishes the case. \( \square \)

**Theorem 4.50 (Inverse Function Theorem).** Let \( n \in \mathbb{N} \), let \( G \subseteq \mathbb{R}^n \) be open, and let \( f : G \to \mathbb{R}^n \) be continuously differentiable, i.e. \( f \in C^1(G, \mathbb{R}^n) \). If \( \xi \in G \) is such that

\[
D f(\xi) \text{ is invertible,}
\]

then there exists an open neighborhood \( \xi \subseteq G \) of \( \xi \) such that \( V := f(\xi) \) is open and the restriction \( f : U \to V \) is bijective with continuously differentiable inverse function \( f^{-1} : V \to U \). Moreover,

\[
\forall y \in V \quad D(f^{-1})(y) = \left( D f(f^{-1}(y)) \right)^{-1}
\]

and, if \( f \in C^\alpha(U, \mathbb{R}^n), \alpha \in \mathbb{N} \cup \{\infty\} \), then \( f^{-1} \in C^\alpha(V, \mathbb{R}^n) \).

**Proof.** The idea is to apply the implicit function Th. 4.49 to the continuously differentiable map

\[
F : G \times \mathbb{R}^n \to \mathbb{R}^n, \quad F(x, y) := f(x) - y.
\]

Here, as compared to Th. 4.49, the roles of the variables \( x \) and \( y \) are switched. Letting \( \eta := f(\xi) \), we have

\[
F(\xi, \eta) = f(\xi) - \eta = 0, \quad \text{and} \quad D_x F(\xi, \eta) = D f(\xi) \text{ is invertible.}
\]
Thus, Th. 4.49 applies and yields an open neighborhood $\tilde{U} \subseteq G$ of $\xi$, an open neighborhood $V \subseteq \mathbb{R}^n$ of $\eta$, and a $C^1$ map $g : V \to \tilde{U}$ such that
\[
\forall_{(x,y)\in \tilde{U} \times V} \left( F(x,y) = f(x) - y = 0 \iff x = g(y) \right).
\]
(4.79)
If we let $U := g(V)$, then $U \subseteq \tilde{U}$ is a neighborhood of $\xi = g(f(\xi))$, and (4.79) implies that $f : U \to V$ and $g : V \to U$ are inverse to each other, in particular, they are both bijective with $f^{-1} = g$. To verify that $U$ is open, consider the (still continuous) map $f : \tilde{U} \to \mathbb{R}^n$ and observe $U = f^{-1}(V)$. As $V$ is open, Th. 2.7(iii) implies the existence of $O \subseteq \mathbb{R}^n$ open with $U = O \cap \tilde{U}$. Since both $O$ and $\tilde{U}$ are open, $U$ must be open as well.

Using (4.70), we obtain, for each $y \in V$,
\[
Dg(y) = -\left(D_x F(g(y), y)\right)^{-1} D_y F(g(y), y) = -\left(Df(g(y))\right)^{-1} (\text{Id}) = \left(Df(g(y))\right)^{-1},
\]
proving (4.78). Finally, if $f$ is $C^\alpha$ on $U$, then $F$ is $C^\alpha$ on $U \times \mathbb{R}^n$, such that Th. 4.49 implies $g = f^{-1}$ to be $C^\alpha$ as well.

**Corollary 4.51.** Let $n \in \mathbb{N}$, let $G \subseteq \mathbb{R}^n$ be open, and let $f : G \to \mathbb{R}^n$ be continuously differentiable, i.e. $f \in C^1(G, \mathbb{R}^n)$. If $Df(x)$ is invertible for each $x \in G$, then $f$ maps open sets to open sets, i.e. if $O \subseteq G$ is open, then $f(O)$ is open as well.

**Proof.** Let $O \subseteq G$ be open. We have to show that each point $\eta \in f(O)$ is an interior point of $f(O)$. To this end, let $\eta \in f(O)$ and let $\xi \in O$ be such that $f(\xi) = \eta$. Since $Df(\xi)$ is invertible by hypothesis, we can apply the inverse function Th. 4.50 to the restriction of $f$ to $O$, obtaining open neighborhoods $U \subseteq O$ of $\xi$ and $V \subseteq f(O)$ of $\eta$ such that $f : U \to V$ is bijective. In particular, $\eta$ is an interior point of $f(O)$, proving $f(O)$ to be open.

## 5 Extreme Values, Stationary Points, Optimization

### 5.1 Definitions of Extreme Values

The following Def. 5.1 is a generalization of [Phi16, Def. 7.50].

**Definition 5.1.** Let $(X,d)$ be a metric space, $M \subseteq X$, and $f : M \to \mathbb{R}$.

**(a)** Given $x \in M$, $f$ has a (strict) global min at $x$ if, and only if, $f(x) \leq f(y)$ ($f(x) < f(y)$) for each $y \in M \setminus \{x\}$. Analogously, $f$ has a (strict) global max at $x$ if, and only if, $f(x) \geq f(y)$ ($f(x) > f(y)$) for each $y \in M \setminus \{x\}$. Moreover, $f$ has a (strict) global extreme value at $x$ if, and only if, $f$ has a (strict) global min or a (strict) global max at $x$. 


(b) Given \( x \in M \), \( f \) has a (strict) local min at \( x \) if, and only if, there exists \( \epsilon > 0 \) such that \( f(x) \leq f(y) \) (\( f(x) < f(y) \)) for each \( y \in \{ y \in M : d(x, y) < \epsilon \} \setminus \{ x \} \). Analogously, \( f \) has a (strict) local max at \( x \) if, and only if, there exists \( \epsilon > 0 \) such that \( f(x) \geq f(y) \) (\( f(x) > f(y) \)) for each \( y \in \{ y \in M : d(x, y) < \epsilon \} \setminus \{ x \} \). Moreover, \( f \) has a (strict) local extreme value at \( x \) if, and only if, \( f \) has a (strict) local min or a (strict) local max at \( x \).

**Remark 5.2.** In the context of Def. 5.1, it is immediate from the respective definitions that \( f \) has a (strict) global min at \( x \in M \) if, and only if, \(-f\) has a (strict) global max at \( x \). Moreover, the same holds if "global" is replaced by "local". It is equally obvious that every (strict) global min/max is a (strict) local min/max.

In optimization problems, one often aims at finding global (or at least local) minima or maxima of real-valued functions. From Th. 3.19, we already know that continuous functions on compact topological spaces always have a global max and a global min. However, in general, one has no method to actually find such extrema. For differentiable functions defined on subsets of \( \mathbb{R}^n \), the situation is somewhat better, even though finding extrema of a complicated function can still be very difficult. To prove sufficient conditions for extrema of differentiable functions of several variables, we will make use of Taylor’s Th. 4.44 and so-called quadratic forms.

## 5.2 Quadratic Forms

Before we get to the quadratic forms, we briefly need to consider the Euclidean norm of matrices.

**Notation 5.3.** Let \( A = (a_{kl})_{(k,l) \in \{1,\ldots,m\} \times \{1,\ldots,n\}} \) a real \( m \times n \) matrix, \( m, n \in \mathbb{N} \). We introduce the quantity

\[
\|A\|_{\text{HS}} := \sqrt{\sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^2},
\]

(5.1)
called the Hilbert-Schmidt norm or the Frobenius norm of \( A \). Thus, \( \|A\|_{\text{HS}} \) is the Euclidean norm of \( A \) if we consider \( A \) as an element of \( \mathbb{R}^{mn} \). **Caveat:** For \( m, n > 1 \), the Hilbert-Schmidt norm is not! the operator norm of \( A \) with respect to the Euclidean norms on \( \mathbb{R}^m \) and \( \mathbb{R}^n \) – it is actually not an operator norm at all (see [Phi17, Ex. B.9]). We could actually use the mentioned operator norm in the following and everything would work just the same (since (5.2) also holds for the operator norm) – the reason we prefer the Hilbert-Schmidt norm here, is that it is much easier to compute and, thus, less abstract.

**Lemma 5.4.** Let \( A = (a_{kl})_{(k,l) \in \{1,\ldots,m\} \times \{1,\ldots,n\}} \) a real \( m \times n \) matrix, \( m, n \in \mathbb{N} \). Then, for each \( x \in \mathbb{R}^n \), it holds that

\[
\|Ax\|_2 \leq \|A\|_{\text{HS}} \|x\|_2.
\]

(5.2)
Proof. This follows easily from the Cauchy-Schwarz inequality. For each \( k \in \{1, \ldots, m\} \), let \( a_k := (a_{k1}, \ldots, a_{kn}) \) denote the \( k \)th row vector of the matrix \( A \). Then one computes

\[
\|Ax\|_2 = \sqrt{\sum_{k=1}^{m} \left( \sum_{l=1}^{n} a_{kl}x_l \right)^2} \leq \sqrt{\sum_{k=1}^{m} \|a_k\|_2^2 \|x\|_2^2} = \|A\|_{\text{HS}} \|x\|_2,
\]

thereby establishing the case. \[\blacksquare\]

Definition 5.5. Let \( n \in \mathbb{N} \). A quadratic form is a map

\[
Q_A : \mathbb{R}^n \rightarrow \mathbb{R}, \quad Q_A(x) := x^tAx = \sum_{k,l=1}^{n} a_{kl}x_kx_l,
\]

(5.3)

where \( x^t \) denotes the transpose of \( x \), and \( A = (a_{kl})_{k,l=1}^{n} \) is a symmetric real \( n \times n \)-matrix, i.e. a quadratic real matrix with \( a_{kl} = a_{lk} \).

Remark 5.6. Each quadratic form is a polynomial and, thus, continuous by Th. 2.20. Moreover, if \( \lambda \in \mathbb{R} \) and \( A \) and \( B \) are symmetric real \( n \times n \)-matrices, then \( \lambda A \) and \( A + B \) are also symmetric real \( n \times n \)-matrices, and \( Q_{\lambda A} = \lambda Q_A \) as well as \( Q_{A+B} = Q_A + Q_B \), showing, in particular, that the symmetric real \( n \times n \)-matrices form a real vector space and that the quadratic forms also form a real vector space.

Example 5.7. If \( G \subseteq \mathbb{R}^n \) is open and \( f : G \rightarrow \mathbb{R} \) is \( C^2 \), then, for each \( \xi \in G \), the Hessian matrix

\[
H_f(\xi) = \left( \partial_k \partial_l f(\xi) \right)_{k,l=1}^{n}
\]

is symmetric, i.e. \( Q_{H_f(\xi)} : \mathbb{R}^n \rightarrow \mathbb{R} \) is a quadratic form.

Lemma 5.8. Let \( A = (a_{kl})_{k,l=1}^{n} \) is a symmetric real \( n \times n \)-matrix, \( n \in \mathbb{N} \), and let \( Q_A \) be the corresponding quadratic form.

(a) \( Q_A \) is homogeneous of degree 2, i.e.

\[
Q_A(\lambda x) = \lambda^2 Q_A(x) \quad \text{for each } x \in \mathbb{R}^n \text{ and each } \lambda \in \mathbb{R}.
\]

(b) For each \( \alpha \in \mathbb{R} \), the following statements are equivalent:

(i) \( Q_A(x) \geq \alpha \|x\|_2^2 \) for all \( x \in \mathbb{R}^n \).

(ii) \( Q_A(x) \geq \alpha \) for all \( x \in \mathbb{R}^n \) with \( \|x\|_2 = 1 \).

(c) For each \( x \in \mathbb{R}^n \):

\[
|Q_A(x)| \leq \|A\|_{\text{HS}} \|x\|_2.
\]

Proof. (a) is an immediate consequence of (5.3).
(b): That (i) implies (ii) is trivial, since (ii) is a special case of (i). It remains to show that (ii) implies (i). For \( x = 0 \), one has \( 0 = Q_A(x) = \alpha \|x\|^2 \), so let \( x \neq 0 \) and assume (ii). Then one obtains

\[
Q_A(x) = Q_A \left( \|x\| \frac{x}{\|x\|} \right) = \alpha \|x\|^2 Q_A \left( \frac{x}{\|x\|} \right) \geq \alpha \|x\|^2,
\]

proving (i).

(c): Let \( x \in \mathbb{R}^n \). Since \( Q_A(x) = x \cdot (Ax) \), the Cauchy-Schwarz inequality yields \( |Q_A(x)| \leq \|Ax\| \|x\| \), and (5.2) then implies (c).

**Definition 5.9.** Let \( A = (a_{kl})_{k,l=1}^n \) is a symmetric real \( n \times n \)-matrix, \( n \in \mathbb{N} \), and let \( Q_A \) be the corresponding quadratic form.

(a) \( A \) and \( Q_A \) are called **positive definite** if, and only if, \( Q_A(x) > 0 \) for every \( 0 \neq x \in \mathbb{R}^n \).

(b) \( A \) and \( Q_A \) are called **positive semidefinite** if, and only if, \( Q_A(x) \geq 0 \) for every \( x \in \mathbb{R}^n \).

(c) \( A \) and \( Q_A \) are called **negative definite** if, and only if, \( Q_A(x) < 0 \) for every \( 0 \neq x \in \mathbb{R}^n \).

(d) \( A \) and \( Q_A \) are called **negative semidefinite** if, and only if, \( Q_A(x) \leq 0 \) for every \( x \in \mathbb{R}^n \).

(e) \( A \) and \( Q_A \) are called **indefinite** if, and only if, they are neither positive semidefinite nor negative semidefinite, i.e. if, and only if, there exist \( a, b \in \mathbb{R}^n \) with \( Q_A(a) > 0 \) and \( Q_A(b) < 0 \).

**Example 5.10.** Let \( n = 2 \) and consider the real symmetric matrix \( A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \). One then obtains

\[
Q_A : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad Q_A(x,y) = ax^2 + 2bxy + cy^2.
\]

One can now use the value of \( \det A = ac - b^2 \), which is also called the **discriminant** of \( Q_A \), to determine the definiteness of \( A \). This is due to the following identity, that holds for each \( (x, y) \in \mathbb{R}^2 \):

\[
aQ_A(x, y) = a(ax^2 + 2bxy + cy^2) = (ax + by)^2 + (\det A)y^2.
\]

One obtains the following cases:

\( \det A > 0 \): This implies \( a \neq 0 \). Then (5.6) provides:

\[
a > 0 \quad \iff \quad Q_A \text{ positive definite},
\]

\[
a < 0 \quad \iff \quad Q_A \text{ negative definite}.
\]

\( \det A < 0 \): In this case, we claim:

\( Q_A \) is indefinite.
To verify this claim, first consider $a > 0$. Then $Q_A(1,0) = a > 0$ and, according to (5.6), $Q_A(-b/a, 1) = (\det A)/a < 0$, showing that $Q_A$ is indefinite. Now let $a < 0$. Then $Q_A(1,0) = a < 0$ and, according to (5.6), $Q_A(-b/a, 1) = (\det A)/a > 0$, again showing that $Q_A$ is indefinite. Finally, let $a = 0$. Then $\det A < 0$ implies $b \neq 0$. If $c > 0$, then $Q_A(0,1) = c > 0$ and $Q_A(1/(2b), -1/(2c)) = -1/(2c) + 1/(4c) = -1/(2c) < 0$, i.e. $Q_A$ is indefinite. If $c < 0$, then $Q_A(0,1) = c < 0$ and $Q_A(1/(2b), -1/(2c)) = -1/(2c) + 1/(4c) = -1/(4c) > 0$, i.e. $Q_A$ is again indefinite. If $c = 0$, then $Q_A(1/(2b), 1) = 1$ and $Q_A(1/(2b), -1) = -1$ and $Q_A$ is indefinite also in this last case.

$\det A = 0$: Here, we claim:

$$a > 0 \text{ or } (a = 0 \text{ and } c \geq 0) \iff Q_A \text{ positive semidefinite},$$

$$a < 0 \text{ or } (a = 0 \text{ and } c \leq 0) \iff Q_A \text{ negative semidefinite}.$$  

Once again, for the proof, we need to distinguish the different possible cases. If $a > 0$, then $Q_A(x,y) = (ax + by)^2/a \geq 0$, i.e. $Q_A$ is positive semidefinite. If $a < 0$, then $Q_A(x,y) = (ax + by)^2/a \leq 0$, i.e. $Q_A$ is negative semidefinite. Now let $a = 0$. Then $\det A = 0$ implies $b = 0$. Thus, $Q_A(x,y) = cy^2$, i.e. $Q_A$ is positive semidefinite for $c \geq 0$ and negative semidefinite for $c \leq 0$.

**Proposition 5.11.** Let $A = (a_{kl})_{k,l=1}^n$ is a symmetric real $n \times n$-matrix, $n \in \mathbb{N}$, and let $Q_A$ be the corresponding quadratic form.

(a) $A$ and $Q_A$ are positive definite if, and only if, there exists $\alpha > 0$ such that

$$Q_A(x) \geq \alpha > 0 \text{ for each } x \in \mathbb{R}^n \text{ with } \|x\|_2 = 1. \quad (5.7a)$$

Analogously, $A$ and $Q_A$ are negative definite if, and only if, there exists $\alpha < 0$ such that

$$Q_A(x) \leq \alpha < 0 \text{ for each } x \in \mathbb{R}^n \text{ with } \|x\|_2 = 1. \quad (5.7b)$$

(b) If $A$ and $Q_A$ are positive definite (respectively negative definite, or indefinite), then there exists $\epsilon > 0$ such that each symmetric real $n \times n$ matrix $B$ with $\|A - B\|_{\text{HS}} < \epsilon$ is also positive definite (respectively negative definite, or indefinite).

(c) If $A$ and $Q_A$ are indefinite, then there exists $\epsilon > 0$ and $a, b \in \mathbb{R}^n$ with $\|a\|_2 = \|b\|_2 = 1$ such that, for each symmetric real $n \times n$ matrix $B$ with $\|A - B\|_{\text{HS}} < \epsilon$ and each $0 \neq \lambda \in \mathbb{R}$, it holds that $Q_B(\lambda a) > 0$ and $Q_B(\lambda b) < 0$.

**Proof.** (a): We consider the positive definite case; the negative definite case is proved completely analogously. First note that (5.7a) implies that $A$ and $Q_A$ are positive definite according to Lem. 5.8(b). Conversely, assume that $A$ and $Q_A$ are positive definite. The 1-sphere $S_1(0) = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ is a closed and bounded subset of $\mathbb{R}^n$ and, hence, compact. Since $Q_A$ is continuous, it must assume its min on $S_1(0)$ according to Th. 3.19, i.e. there is $\alpha \in \mathbb{R}$ and $x_0 \in S_1(0)$ such that $Q_A(x_0) = \alpha$ and $Q_A(x) \geq \alpha$ for each $x \in \mathbb{R}$ with $\|x\|_2 = 1$. Since $Q_A$ is positive definite, $\alpha > 0$, proving (5.7a).

(b) and (c): We begin by employing (5.7a) to show (b) for $A$ and $Q_A$ being positive definite (employing (5.7b), the case of $A$ and $Q_A$ being negative definite can be treated
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completely analogously). If $A$ and $Q_A$ are positive definite, then there is $\alpha > 0$ such that (5.7a) holds. Choose $\epsilon := \alpha/2$. If $B$ is a symmetric real $n \times n$ matrix with $\|A - B\|_{\text{HS}} < \epsilon$, then, using Lem. 5.8(c), for each $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$:

$$|Q_A(x) - Q_B(x)| = |Q_{A-B}(x)| \leq \|A - B\|_{\text{HS}} < \epsilon = \frac{\alpha}{2}.$$  

(5.8)

Since $Q_A(x) \geq \alpha > 0$, this implies $Q_B(x) \geq \alpha/2 > 0$ for each $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$. Due to (a), this proves that $B$ is positive definite. Now consider the case that $A$ and $Q_A$ are indefinite. Then there are $0 \neq a, b \in \mathbb{R}^n$ such that $Q_A(a) > 0$ and $Q_A(b) < 0$. By normalizing and using Lem. 5.8(a), one can even additionally assume $\|a\|_2 = \|b\|_2 = 1$. Set $\alpha := \min\{Q_A(a), |Q_A(b)|\}$. Then $\alpha > 0$. If $\epsilon := \alpha/2$ and $B$ is a symmetric real $n \times n$ matrix with $\|A - B\|_{\text{HS}} < \epsilon$, then, as above, (5.8) holds for each $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$. In particular, $Q_B(a) \geq \alpha/2 > 0$ and $Q_B(b) \leq -\alpha/2 < 0$, showing that $Q_B$ is indefinite, concluding the proof of (b). To complete the proof of (c) as well, it merely remains to remark that, for each $0 \neq \lambda \in \mathbb{R}$, one has $Q_B(\lambda a) \geq \lambda^2 \alpha/2 > 0$ and $Q_B(\lambda b) \leq -\lambda^2 \alpha/2 < 0$. □

5.3 Extreme Values and Stationary Points of Differentiable Functions

**Definition 5.12.** Let $G \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $f : G \rightarrow \mathbb{K}$, and let $\xi$ be an interior point of $G$. If all first partials of $f$ exist in $\xi$, then $\xi$ is called a stationary or critical point of $f$ if, and only if,

$$\nabla f(\xi) = 0.$$  

(5.9)

The following Th. 5.13 generalizes [Phi16, Th. 9.15] to functions defined on subsets of $\mathbb{R}^n$:

**Theorem 5.13.** Let $G \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $f : G \rightarrow \mathbb{R}$, and let $\xi$ be an interior point of $G$. If all first partials of $f$ exist in $\xi$ and $f$ has a local min or max at $\xi$, then $\xi$ is a stationary point of $f$, i.e. $\nabla f(\xi) = 0$.

**Proof.** Since $\xi$ is an interior point of $G$ and since $f$ has a local min or max at $\xi$, there is $\epsilon > 0$ such that $B_\epsilon(\xi) \subseteq G$ and such that $f(\xi) \leq f(x)$ for each $x \in B_\epsilon(\xi)$ or such that $f(\xi) \geq f(x)$ for each $x \in B_\epsilon(\xi)$. Let $j \in \{1, \ldots, n\}$. Then there is $\delta > 0$ such that $(\xi_1, \ldots, \xi_{j-1}, t, \xi_{j+1}, \ldots, \xi_n) \in B_\delta(\xi)$ for each $t \in [\xi_j - \delta, \xi_j + \delta]$. Thus, the one-dimensional function $g : [\xi_j - \delta, \xi_j + \delta] \rightarrow \mathbb{R}$, $g(t) := f(\xi_1, \ldots, \xi_{j-1}, t, \xi_{j+1}, \ldots, \xi_n)$, has a local min or max at $\xi_j$, and, since $\partial_j f(\xi)$ exists, $g$ is differentiable in $\xi_j$, implying $0 = g'(\xi_j) = \partial_j f(\xi)$ according to [Phi16, Th. 9.15]. Since $j \in \{1, \ldots, n\}$ was arbitrary, $\nabla f(\xi) = 0$. □

One already knows from simple one-dimensional examples such as $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) := x^3$ and $\xi = 0$ that $\nabla f(\xi) = 0$ is not a sufficient condition for $f$ to have a local extreme value at $\xi$. However, the following Th. 5.14 does provide such sufficient conditions.
Example 5.15. Consider the case being an open subset of \( \mathbb{R} \) \( \mathbb{H} \) \( \theta \) \( Q \) form (which, by definition, is the same as the discriminant of the corresponding quadratic function). For each \( 0 \) sign of \( Q \) \( \parallel \) definite, an analogous argument shows that \( f \) \( \delta > 0 \) is indefinite, then, by Prop. 5.11(c), there is \( h \) such that \( \parallel h \parallel > \delta \) also positive definite. Moreover, the continuity of \( H \) \( H \) \( H \) \( H \) implies the continuity of \( H \). Thus, if \( f \) \( \parallel h \parallel < \delta \) in the following cases, one can use the Hessian matrix \( H_f(\xi) \) to determine if \( f \) has a local extreme value at \( \xi \):

\[
H_f(\xi) \text{ positive definite } \Rightarrow f \text{ has a strict local min at } \xi, \quad (5.10a)
\]
\[
H_f(\xi) \text{ negative definite } \Rightarrow f \text{ has a strict local max at } \xi, \quad (5.10b)
\]
\[
H_f(\xi) \text{ indefinite } \Rightarrow f \text{ does not have a local extreme value at } \xi. \quad (5.10c)
\]

Proof. Since \( G \) is open, there is \( \epsilon > 0 \) such that \( \xi + h \in G \) for each \( h \in \mathbb{R}^n \) with \( \parallel h \parallel < \epsilon \). For each such \( h \), by an application of Taylor’s Th. 4.44 with \( m = 1 \), we obtain the existence of \( \theta \in ]0,1[ \) satisfying

\[
f(\xi + h) = f(\xi) + h \cdot \nabla f(\xi) + \frac{1}{2} \sum_{k,l=1}^n \partial_k \partial_l f(\xi + \theta h) h_k h_l \]
\[
= f(\xi) + \frac{h^T H_f(\xi + \theta h) h}{2} = f(\xi) + \frac{Q_{H_f(\xi + \theta h)}(h)}{2}. \quad (5.11)
\]

Rewriting (5.11), one gets

\[
f(\xi + h) - f(\xi) = \frac{Q_{H_f(\xi + \theta h)}(h)}{2}. \quad (5.12)
\]

Note that the assumed continuity of the functions \( \partial_k \partial_l f : G \rightarrow \mathbb{R} \) \( (k,l \in \{1, \ldots, n\}) \) implies the continuity of \( H_f : G \rightarrow \mathbb{R}^n^2 \), \( x \mapsto H_f(x) \) (the \( \partial_k \partial_l f \) are the coordinate functions of \( H_f \)). Thus, if \( H_f(\xi) \) is positive definite, then, by Prop. 5.11(b), there is \( \delta > 0 \) such that \( \parallel h \parallel < \epsilon \) and \( \parallel H_f(\xi) - H_f(\xi + \theta h)\parallel_{\text{HS}} < \delta \) imply that \( H_f(\xi + \theta h) \) is also positive definite. Moreover, the continuity of \( H_f \) means that there exists \( 0 < \alpha < \epsilon \) such that \( \parallel h \parallel < \alpha \) implies \( \parallel H_f(\xi) - H_f(\xi + \theta h)\parallel_{\text{HS}} < \delta \) for each \( \theta \in ]0,1[ \). For such \( h \neq 0 \), the right-hand side of (5.12) must be positive, showing that \( f \) has a strict local min at \( \xi \) \( (f(\xi) < f(x) \text{ for each } x \in B_{\alpha}(\xi) \setminus \{\xi\}) \). For \( H_f(\xi) \) being negative definite, an analogous argument shows that \( f \) has a strict max at \( \xi \). Similarly, if \( H_f(\xi) \) is indefinite, then, by Prop. 5.11(c), there is \( \delta > 0 \) and \( a, b \in \mathbb{R}^n \) with \( \parallel a \parallel = \parallel b \parallel = 1 \) such that, \( \parallel h \parallel < \epsilon \) and \( \parallel H_f(\xi) - H_f(\xi + \theta h)\parallel_{\text{HS}} < \delta \) imply that \( Q_{H_f(\xi + \theta h)}(\lambda a) > 0 \) and \( Q_{H_f(\xi + \theta h)}(\lambda b) < 0 \) for each \( 0 \neq \lambda \in \mathbb{R} \). The continuity of \( H_f \) provides some \( 0 < \alpha < \epsilon \) such that \( \parallel h \parallel < \alpha \) implies \( \parallel H_f(\xi) - H_f(\xi + \theta h)\parallel_{\text{HS}} < \delta \) for each \( \theta \in ]0,1[ \). For each \( 0 < \lambda < \alpha \), we get \( \parallel \lambda a \parallel < \alpha \) and \( \parallel \lambda b \parallel < \alpha \), such that (5.12) implies \( f(\xi + \lambda b) < f(\xi) < f(\xi + \lambda a) \), i.e. \( f \) has neither a local min nor a local max at \( \xi \). 

Example 5.15. Consider the case \( n = 2 \), i.e. the case of a \( C^2 \) function \( f : G \rightarrow \mathbb{R} \), \( G \) being an open subset of \( \mathbb{R}^2 \). Let \( (x_0, y_0) \in G \) be a stationary point of \( f \). Then, according to Example 5.10, the definiteness of the Hessian matrix \( H_f(x_0, y_0) \) is determined by the sign of

\[
\det H_f(x_0, y_0) = \partial_x \partial_y f(x_0, y_0) \partial_y \partial_x f(x_0, y_0) - (\partial_x \partial_y f(x_0, y_0))^2 \quad (5.13)
\]

(which, by definition, is the same as the discriminant of the corresponding quadratic form \( Q_{H_f(x_0,y_0)} \)). If \( \det H_f(x_0, y_0) > 0 \), then Th. 5.14 tells us that \( f \) has a strict local
Consider \( f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) := x^2 + y^2 \). Then \( \nabla f(x, y) = (2x, 2y) \) and \( H_f(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \). Thus, \((0, 0)\) is the only stationary point of \( f \). Since \( \det H_f(0, 0) = 4 > 0 \), \( f \) has a strict local min at \((0, 0)\) and this is the only point, where \( f \) has a local extreme value. Moreover, since \( f(x, y) > 0 \) for \((x, y) \neq (0, 0)\), \( f \) also has a strict global min at \((0, 0)\).

(b) Consider \( f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) := x^2 - y^2 \). Then \( \nabla f(x, y) = (2x, -2y) \) and \( H_f(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \). Thus, \((0, 0)\) is the only stationary point of \( f \). Here, one has \( \det H_f(0, 0) = -4 < 0 \), i.e. \( f \) does not have a local min or max at \((0, 0)\) (or anywhere else). Thus, \((0, 0)\) is an example of a saddle point.

Let us summarize the general strategy for determining extreme values of differentiable functions \( f \) defined on a set \( G \): One starts by seeking all stationary points of \( f \), that means the points \( \xi \), where \( \nabla f(\xi) = 0 \). Every min or max of \( f \) that lies in the interior of \( G \) must be included in the set of stationary points. To investigate if a stationary point is, indeed, a max or a min, one will compute the Hessian matrix \( H_f \) at this point, and one will determine the definiteness properties of \( H_f \). Then one can use Th. 5.14 to decide if the stationary point is a max, a min, or neither, except for cases, where \( H_f \) is only (positive or negative) semidefinite, in which case Th. 5.14 does not help and one has to resort to other means (which can be difficult). As is already know from functions defined on \( G \subseteq \mathbb{R} \), one also has to investigate the behavior of \( f \) at the boundary of \( G \) if one wants to find out if one of the local extrema is actually a global extremum. Moreover, if \( f \) is defined on \( \partial G \), then \( \partial G \) might contain further local extrema of \( f \).

### 5.4 Constrained Optimization, Lagrange Multipliers

Constrained optimization can be seen as restricting the function \( f : A \rightarrow \mathbb{R} \) to be minimized or maximized to some subset of \( A \), determined by the constraints. Constraints
can be given explicitly or implicitly. If $A = \mathbb{R}$, then an explicit constraint might be to seek nonnegative solutions. Implicit constraints can be given in the form of equation constraints – for example, if $A = \mathbb{R}^n$, then one might want to minimize $f$ on the set of all solutions to $Ma = b$, where $M$ is some real $m \times n$ matrix.

If one is seeking extrema of a differentiable function $f : G \to \mathbb{R}$, $G \subseteq \mathbb{R}^{n+m}$, under the constraint $g(x) = 0$, where $g : G \to \mathbb{R}^m$, then, under suitable hypotheses, one can obtain necessary conditions using the trick of introducing additional (auxiliary) variables. These additional variables are called Lagrange multipliers. The proof is an application of the implicit function Th. 4.49.

**Theorem 5.16.** Let $m, n \in \mathbb{N}$, let $G \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open, $f \in C^1(G, \mathbb{R})$, and $g \in C^1(G, \mathbb{R}^m)$. Suppose $(\xi, \eta) \in G$ and, using the notation of (4.67), suppose $Dy g(\xi, \eta)$ is invertible. If $f$ has a local min or local max at $(\xi, \eta)$ under the constraint $g = 0$ (that means, more precisely, $f \big|_{\{(x,y) \in G : g(x,y) = 0\}}$ has a local min or local max at $(\xi, \eta)$), then there exists $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$ such that

$$H : G \times \mathbb{R}^m \to \mathbb{R}, \quad H(x, y, \lambda) := f(x, y) + \lambda \cdot g(x, y) = f(x, y) + \sum_{k=1}^m \lambda_k g_k(x, y), \quad (5.14)$$

has a stationary point at $(\xi, \eta, \mu)$, i.e. $\nabla H(\xi, \eta, \mu) = 0$, i.e.

$$\forall \ j \in \{1, \ldots, n\} \quad \partial_{x_j} H(\xi, \eta, \mu) = \partial_{x_j} f(\xi, \eta) + \sum_{k=1}^m \mu_k \partial_{x_j} g_k(\xi, \eta) = 0, \quad (5.15a)$$

$$\forall \ j \in \{1, \ldots, m\} \quad \partial_{y_j} H(\xi, \eta, \mu) = \partial_{y_j} f(\xi, \eta) + \sum_{k=1}^m \mu_k \partial_{y_j} g_k(\xi, \eta) = 0, \quad (5.15b)$$

$$\forall \ j \in \{1, \ldots, m\} \quad \partial_{\lambda_j} H(\xi, \eta, \mu) = g_j(\xi, \eta) = 0. \quad (5.15c)$$

The additional variables $\lambda_1, \ldots, \lambda_m$ are called Lagrange multipliers.

**Proof.** According to the hypotheses, the implicit function Th. 4.49 applies to $g$ at $(\xi, \eta)$. Thus, there exist $U \subseteq \mathbb{R}^n$ open and $V \subseteq \mathbb{R}^m$ open such that $\xi \in U$, $\eta \in V$, and such that there exists a continuously differentiable $h : U \to V$, satisfying

$$\forall \ (x,y) \in U \times V \quad \big( g(x,y) = 0 \iff y = h(x) \big)$$

as well as

$$\forall \ x \in U \quad Dh(x) = -(Dy g(x, h(x)))^{-1} Dx g(x, h(x)).$$

If $f$ has a local min or local max at $(\xi, \eta)$ under the constraint $g = 0$, then $\varphi : U \to \mathbb{R}$, $\varphi(x) := f(x, h(x))$ has a local min or local max at $\xi$ (with no constraint). Thus, according to Th. 5.13, $\nabla \varphi(\xi) = 0$. On the other hand, the chain rule of Th. 4.31 yields

$$\nabla \varphi(\xi) = Df(\xi, h(\xi)) \left( \begin{array}{c} \text{Id}_n \\ Dh(\xi) \end{array} \right) = \left( \begin{array}{cc} D_x f(\xi, \eta) & D_y f(\xi, \eta) \end{array} \right) \left( \begin{array}{c} \text{Id}_n \\ Dh(\xi) \end{array} \right) - D_x f(\xi, \eta) - D_y f(\xi, \eta) (D_y g(\xi, \eta))^{-1} D_x g(\xi, \eta) = 0. \quad (5.16)$$
Letting \( \mu := D_y f(\xi, \eta)(D_y g(\xi, \eta))^{-1} \in \mathbb{R}^m \), (5.16) reads
\[
D_x f(\xi, \eta) - \mu \cdot D_x g(\xi, \eta) = 0,
\]
i.e. (5.15a) holds. On the other hand,
\[
D_y f(\xi, \eta) - \mu \cdot D_y g(\xi, \eta) = D_y f(\xi, \eta) - D_y f(\xi, \eta)(D_y g(\xi, \eta))^{-1} D_y g(\xi, \eta) = 0,
\]
showing that (5.15b) holds as well. As (5.15c) holds simply due to \( g(\xi, \eta) = 0 \), the proof is complete. \( \square \)

In the formulation of Th. 5.16, there is an a priori distinction between the variables \( y \) such that \( D_y g(\xi, \eta) \) is invertible and the remaining variables \( x \). In practice, however, there is often no such a priori distinction. Thus, we provide the following reformulation of Th. 5.16:

**Corollary 5.17.** Let \( m, n \in \mathbb{N}, m < n \), let \( G \subseteq \mathbb{R}^n \) be open, \( f \in C^1(G, \mathbb{R}) \), and \( g \in C^1(G, \mathbb{R}^m) \). Suppose \( \xi \in G \) and \( D g(\xi) \) has rank \( m \). If \( f \) has a local min or local max at \( \xi \) under the constraint \( g = 0 \) (that means, more precisely, \( f |_{\{x \in G : g(x) = 0\}} \) has a local min or local max at \( \xi \)), then there exists \( \mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m \) such that
\[
H : G \times \mathbb{R}^m \rightarrow \mathbb{R}, \quad H(x, \lambda) := f(x) + \lambda \cdot g(x) = f(x) + \sum_{k=1}^m \lambda_k g_k(x), \quad (5.17)
\]
has a stationary point at \( (\xi, \mu) \), i.e. \( \nabla H(\xi, \mu) = 0 \), i.e.
\[
\forall j \in \{1, \ldots, n\} \quad \partial_{x_j} H(\xi, \mu) = \partial_{x_j} f(\xi) + \sum_{k=1}^m \mu_k \partial_{x_j} g_k(\xi) = 0, \quad (5.18a)
\]
\[
\forall j \in \{1, \ldots, m\} \quad \partial_{\lambda_j} H(\xi, \mu) = g_j(\xi) = 0. \quad (5.18b)
\]

**Proof.** Since \( D g(\xi) \) has rank \( m < n \), there exist \( j_1, \ldots, j_m \in \{1, \ldots, n\} \) such that one can let \( y := (x_{j_1}, \ldots, x_{j_m}) \) and apply Th. 5.16. \( \square \)

**Example 5.18.** Let us maximize
\[
f : [0, \pi]^n \rightarrow \mathbb{R}, \quad f(x) := \sum_{j=1}^n \sin x_j,
\]
for \( n \in \mathbb{N}, n \geq 3 \), under the constraint that \( g(x) = 0 \), where\(^8\)
\[
g : [0, \pi]^n \rightarrow \mathbb{R}, \quad g(x) := -2\pi + \sum_{j=1}^n x_j.
\]

\(^8\)Geometrically, \( f(x) \) is twice the area of an \( n \)-gon with vertices on the unit circle, where the \( j \)th vertex \( P_j \in \mathbb{R}^2 \) has coordinates \( (\cos s_j, \sin s_j) \), \( s_j = \sum_{k=1}^n x_k \): Here, \( \sin x_j \) is twice the area of the triangle \( P_j 0 P_{j+1} \), since, if one uses \( 0 P_j \) as the base, then \( \sin x_j \) is the height of the triangle \( (x_j \) is the size of the angle between \( 0 P_j \) and \( 0 P_{j+1} \)). Thus, we are aiming at maximizing the area among all \( n \)-gons with vertices on the unit circle.
Then $f$ and $g$ are defined on $\overline{G}$ with $G := (0, \pi]^n$. Clearly, $f$ and $g$ are continuously differentiable on $G$ and
\[ \forall_{j \in \{1, \ldots, n\}} \forall_{x \in G} \partial_j g(x) = 1, \]
i.e. each $D_xj g(x)$ is invertible. Thus, the hypotheses of Cor. 5.17 are satisfied and, to apply Cor. 5.17, we define
\[ H : G \times \mathbb{R} \rightarrow \mathbb{R}, \quad H(x, \lambda) := f(x) + \lambda g(x) = -2\pi\lambda + \sum_{j=1}^{n} (\sin x_j + \lambda x_j). \]
According to (5.18a), for $f$ to have a local min or max at $\xi \in G$, under the constraint $g = 0$, it is a necessary condition that
\[ \partial_x H(\xi, \mu) = \cos \xi_j + \mu = 0, \]
for some suitable $\mu \in \mathbb{R}$. As each $\xi_j \in I := ]0, \pi[$ and the range of $\cos$ on $I$ is $]-1, 1[$, $\mu$ must be in $]-1, 1[$. Moreover, since $\cos$ is injective on $I$, there exists $\alpha \in I$ such that $\xi_j = \alpha$ for each $j \in \{1, \ldots, n\}$. Thus,
\[ 0 = g(\xi) = -2\pi + n\alpha \]
implies $\alpha = \frac{2\pi}{n}$. We can now show that, under the constraint $g = 0$, $f : \overline{G} \rightarrow \mathbb{R}$ actually has its global max\(^9\) at $\xi$: Since $C := \{x \in \overline{G} : g(x) = 0\}$ is compact and $f$ is continuous, $f$ must have a global max on $C$. As every global max is a local max, if it is not at $\xi$, then it must occur at some $x^0 \in C \cap \partial G$. The value of $f$ at $\xi$ is
\[ A_n := f(\xi) = n \sin \frac{2\pi}{n} = 2\pi \phi \left( \frac{2\pi}{n} \right), \]
where
\[ \phi : \mathbb{R}^+ \rightarrow \mathbb{R}, \quad \phi(t) := \frac{\sin t}{t}. \quad (5.19) \]
If we can show $\phi$ to be strictly decreasing on $I$, then
\[ \forall_{n \geq 3} A_n < A_{n+1}. \quad (5.20) \]
Using (5.20), we are in a position to show that the global max of $f$ on $C$ is at $\xi$ via induction on $n \geq 3$: Note
\[ x^0 \in \partial G \Rightarrow \exists_{j \in \{1, \ldots, n\}} x^0_j \in \{0, \pi\}. \quad (5.21) \]
For $n = 3$, if some $x^0_j = 0$, then the remaining coordinates of $x^0$ must equal $\pi$ (due to $g(x^0) = 0$). Thus $f(x^0) = 0 < f(\xi)$. If some $x^0_j = \pi$, then, as $\sin(x^0_j) = 0$,
\[ f(x^0) \leq 2 < \frac{3\sqrt{3}}{2} = 3 \sin \frac{2\pi}{3} = f(\xi). \]
\(^9\)Geometrically, we have then shown that among all $n$-gons with vertices on the unit circle, the regular $n$-gon maximizes the area.
Now let \( n > 3 \). If some \( x_j^0 = 0 \), then the sum of the remaining coordinates of \( x^0 \) must equal \( 2\pi \) and, by induction

\[
F(x^0) \overset{\text{ind.hyp.}}{\leq} A_{n-1} < A_n,
\]

showing \( f \) does not have a global max at \( x^0 \). If some \( x_j^0 = \pi \), then the sum of the remaining coordinates of \( x^0 \) must equal \( \pi \), implying

\[
f(x^0) = \sum_{k=1}^{n} \sin(x_k^0) \leq \sum_{k=1, k \neq j}^{n} x_k^0 = \pi < 4 = A_4 \overset{(5.20)}{\leq} f(\xi),
\]

again showing \( f \) does not have a global max at \( x^0 \). So it only remains to verify that the function \( \phi \) of (5.19) is strictly decreasing on \( I \). To this end, note

\[
\phi'(t) = \frac{t \cos t - \sin t}{t^2},
\]

i.e. \( \phi'(t) < 0 \) for \( t \in I: t < \tan t \) for \( 0 < t < \frac{\pi}{2} \), \( \phi'(\frac{\pi}{2}) = -1 < 0 \), and both \( \cos t \) and \( -\sin t \) are negative for \( \frac{\pi}{2} < t < \pi \).

### A Set-Theoretic Rules for Cartesian Products

**Proposition A.1.** Let \( I \) be a nonempty index set and \( (X_i)_{i \in I} \) a family of sets. We consider the Cartesian product \( X := \prod_{i \in I} X_i \) (cf. [Phi16, Def. 2.15(c)]). Let \( J \) also be a nonempty index set and, for each \( i \in I \), let \( (A_{ij})_{j \in J} \) be a family of subsets of \( X_i \). We then have the following rules:

\[
\bigcap_{j \in J} \left( \prod_{i \in I} A_{ij} \right) = \prod_{i \in I} \left( \bigcap_{j \in J} A_{ij} \right), \tag{A.1}
\]

\[
\bigcup_{j \in J} \left( \prod_{i \in I} A_{ij} \right) \subseteq \prod_{i \in I} \left( \bigcup_{j \in J} A_{ij} \right). \tag{A.2}
\]

**Proof.** Define

\[
\forall_{j \in J} A_j := \prod_{i \in I} A_{ij} \subseteq X. \tag{A.3}
\]

Consider \( x := (x_i)_{i \in I} \in X \), i.e. \( x_i \in X_i \) for each \( i \in I \). Then

\[
x \in \bigcap_{j \in J} A_j \iff \forall_{j \in J} x \in A_j \iff \forall_{j \in J} \forall_{i \in I} x_i \in A_{ij} \iff \forall_{i \in I} x_i \in \bigcap_{j \in J} A_{ij} \iff x \in \prod_{i \in I} \left( \bigcap_{j \in J} A_{ij} \right), \tag{A.4}
\]
proving (A.1). Moreover,
\[
x \in \bigcup_{j \in J} A_j \iff \exists j \in J \quad x \in A_j \iff \exists j \in J \quad \forall i \in I \quad x_i \in A_{ij} \Rightarrow \forall i \in I \quad \exists j \in J \quad x_i \in A_{ij}
\]
proving (A.2).

**Example A.2.** We give an example that show that, in general, equality does not hold in (A.2): Let \(X_1 := X_2 := \{1, 2\}\), \(A_1 := \{1\}\), \(A_2 := \{2\}\), \(Y_1 := A_1 \times A_2 = \{(1, 2)\}\), \(Y_2 := A_2 \times A_1 = \{(2, 1)\}\). Then
\[
Y_1 \cup Y_2 = \{(1, 2), (2, 1)\} \neq \{1, 2\} \times \{1, 2\} = (A_1 \cup A_2) \times (A_2 \cup A_1).
\]

## B Box Topology

**Example B.1.** Let \((X_i, T_i)\) be arbitrary topological spaces. Consider the set
\[
B_b := \left\{ \prod_{i \in I} O_i : \left( \forall i \in I \quad O_i \in T_i \right) \right\} = \left\{ \bigcap_{i \in I} \pi_i^{-1}(O_i) : \forall i \in I \quad O_i \in T_i \right\},
\]
where
\[
\pi_j : X \to X_j, \quad \pi_j((x_i)_{i \in I}) := x_j,
\]
denote the projections. Analogous to the base \(B_p\) of the product topology of Ex. 1.53(a), \(B_b\) also satisfies conditions (i) and (ii) of Prop. 1.48 (since \(X \in B_b\) and since \(B_b\) is closed under finite intersections by (A.1) of Appendix A and Def. 1.1(iii)) and, thus, also forms a base for a topology \(T_b\) on \(X\), called the **box topology** on \(X\). Analogous to Ex. 1.53(a), another base for \(T_b\) is
\[
B_b^\circ := \left\{ \prod_{i \in I} B_i : \left( \forall i \in I \quad B_i \in B_i \right) \right\} = \left\{ \bigcap_{i \in I} \pi_i^{-1}(B_i) : \forall i \in I \quad B_i \in B_i \right\}.
\]

Let us compare the box topology on \(X\) with the product topology of Ex. 1.53(a): Since \(B_p \subseteq B_b\), we always have \(T_p \subseteq T_b\). If \(I\) is finite, then \(B_p = B_b\) and \(T_p = T_b\). The same is true if one \(X_i = \emptyset\) (then \(X = \emptyset\)) or if all, but finitely many, of the \(T_i\) are indiscrete. In all other cases, \(T_b\) is strictly finer than \(T_p\): Each set \(A := \bigcap_{i \in I} \pi_i^{-1}(O_i)\), each \(O_i \in T_i\) is in \(T_b\), but not in \(T_p\) if infinitely many \(O_i \neq X_i\) (since, in that case, \(A\) does not contain a set from \(B_p\) as a subset). Moreover, in this case, the sets \(S_p\) and \(S_p^\circ\) of Ex. 1.53(a) are not subbases of \(T_b\), since \(A\) is not a finite intersection of sets of the form \(\pi_i^{-1}(O_i)\).

\(T_b\) is not separable if \(I\) is not finite and each \(T_i\) contains two disjoint open sets \(O_{i,1}\) and \(O_{i,2}\) (in particular, \(\mathbb{K}^\infty\) is not separable in the box topology): If \(f : I \to \{0, 1\}\), then \(O_f := \prod_{i \in I} O_{i,f(i)} \in T_b\). Then the elements of \(O := \{O_f : (f : I \to \{0, 1\})\}\) are
pairwise disjoint. Since $f \mapsto O_f$ is an injective map from $2^I$ to $\mathcal{O}$ and $2^I$ is not countable if $I$ is not finite, $\mathcal{O}$ is not countable, i.e. no countable subset of $X$ can be dense.

Next, we prove that the box topology on $X = \mathbb{K}^\mathbb{R}$ is not first countable (in particular, not metrizable) – a property shared with the (coarser) product topology on $X = \mathbb{K}^\mathbb{R}$ (note that there is no easy way to infer one from the other – both the indiscrete (the coarsest) and the discrete (the finest) topology on $X$ are first countable): Seeking a contradiction, let $f \in X$ and let $\mathcal{N}$ be a countable local base at $f$, where $(N_k)_{k \in \mathbb{N}}$ is an enumeration of the elements of $\mathcal{N}$. For each $k \in \mathbb{N}$, consider $\pi_k(N_k)$, which, as it is open in $\mathbb{K}$, must contain an open interval $I_k$ with $f(k) \in I_k$, i.e. we can choose $a_k, b_k \in I_k$ with $a_k < f(k) < b_k$. Letting $O_k := ]a_k, b_k[$, we have $O_k \varsubsetneq I_k$ and $O := \bigcap_{k \in \mathbb{N}} \pi_k^{-1}(O_k)$ is a box topology neighborhood of $f$. However, $O$ does not contain any of the $N_k$ (since $\pi_k(O) = O_k$ is strictly contained in $\pi_k(N_k)$), a contradiction to $\mathcal{N}$ being a local base at $f$ (note that the argument even shows that $\mathcal{T}_b$ on $X = \mathbb{K}^\mathbb{N}$ is not first countable).

To finish this example, let us check explicitly that the box topology on $X = \mathbb{K}^\mathbb{R}$ is not the topology of pointwise convergence. Since $\mathcal{T}_p \varsubsetneq \mathcal{T}_b$, every sequence in $X$ converging with respect to $\mathcal{T}_b$ must also converge with respect to $\mathcal{T}_p$. However, we claim that the sequence $(f^k)_{k \in \mathbb{N}}$ in $X$, where $f^k \equiv \frac{1}{k}$, which, clearly, converges pointwise (even uniformly) to $f \equiv 0$, does not converge to $f \equiv 0$ in the box topology: For each $0 \neq s \in \mathbb{R}$, let $O_s := B_s(0) \subseteq \mathbb{K}$, $O_0 := \mathbb{K}$, $O := \prod_{s \in \mathbb{R}} O_s$. Then $O \in \mathcal{B}_b$, in particular, open with respect to the box topology. However, $O$ contains none of the $f^k$: $f^k(s) \notin O_s$ for each $0 < s < \frac{1}{k}$.

\section{Uniform Continuity and Lipschitz Continuity}

This section provides some additional important results regarding uniformly continuous functions and Lipschitz continuous functions (see Def. 2.3(b)). We start with an auxiliary result:

\textbf{Lemma C.1.} If $f, g$ are real-valued functions on a set $X$, i.e. if $f, g : X \rightarrow \mathbb{R}$, then, for each $x, y \in X$,

\begin{align*}
\|\max(f, g)(x) - \max(f, g)(y)\| &\leq \max\{\|f(x) - f(y)\|, \|g(x) - g(y)\|\}, \\
\|\min(f, g)(x) - \min(f, g)(y)\| &\leq \max\{\|f(x) - f(y)\|, \|g(x) - g(y)\|\}.
\end{align*}

\textbf{Proof.} By possibly switching the names of $f$ and $g$, one can assume, without loss of generality, that $\max(f, g)(x) = f(x)$, i.e. $g(x) \leq f(x)$. If $g(y) \leq f(y)$ as well, then $\|\max(f, g)(x) - \max(f, g)(y)\| = \|f(x) - f(y)\|$ and $\|\min(f, g)(x) - \min(f, g)(y)\| = \|g(x) -
uniformly continuous. For \( \lambda f \) showing that together with Lem. C.1 implies showing the uniform continuity of \( \max(f,g) \), i.e. \( C \) holds in all cases.

\[ | \max(f,g)(x) - \max(f,g)(y) | = |f(x) - g(y)| \leq \begin{cases} |g(x) - g(y)| & \text{for } f(x) \leq g(y), \\ f(x) - f(y) & \text{for } f(x) > g(y), \end{cases} \quad \text{(C.2a)} \]

\[ | \min(f,g)(x) - \min(f,g)(y) | = |g(x) - f(y)| \leq \begin{cases} |g(x) - g(y)| & \text{for } g(x) \leq f(y), \\ f(x) - f(y) & \text{for } g(x) > f(y), \end{cases} \quad \text{(C.2b)} \]

\[ \text{Theorem C.2. Let } (X,d) \text{ be a metric space (e.g. a normed space), } (Y,\| \cdot \|) \text{ a normed vector space over } \mathbb{K}, \text{ and assume that } f, g : X \rightarrow Y \text{ are uniformly continuous. Then } f + g \text{ and } \lambda f \text{ are uniformly continuous for each } \lambda \in \mathbb{K}, \text{ i.e. the set of all uniformly continuous functions from } X \text{ into } Y \text{ constitutes a subspace of the vector space } F(X,Y) \text{ over } \mathbb{K}. \text{ Moreover, if } Y = \mathbb{K} = \mathbb{R}, \text{ then } \max(f,g), \min(f,g), f^+, f^-, |f| \text{ are all uniformly continuous.} \]

\[ \text{Proof. As } f \text{ and } g \text{ are uniformly continuous, given } \epsilon > 0, \text{ there exist } \delta_f > 0 \text{ and } \delta_g > 0 \text{ such that, for each } x, y \in X, \]

\[ d(x,y) < \delta_f \Rightarrow \| f(x) - f(y) \| < \epsilon/2, \quad \text{(C.3a)} \]

\[ d(x,y) < \delta_g \Rightarrow \| g(x) - g(y) \| < \epsilon/2. \quad \text{(C.3b)} \]

Thus, if \( d(x,y) < \min\{\delta_f, \delta_g\} \), then

\[ \| (f + g)(x) - (f + g)(y) \| \leq \| f(x) - f(y) \| + \| g(x) - g(y) \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \text{(C.3c)} \]

showing that \( f + g \) is uniformly continuous. Next, if \( \lambda = 0 \), then \( \lambda f \equiv 0 \), and obviously uniformly continuous. For \( \lambda \neq 0 \), choose \( \delta > 0 \) such that \( d(x,y) < \delta \) implies \( \| f(x) - f(y) \| < \epsilon/|\lambda| \). Then

\[ \| (\lambda f)(x) - (\lambda f)(y) \| = |\lambda| \| f(x) - f(y) \| < |\lambda| \frac{\epsilon}{|\lambda|} = \epsilon, \quad \text{(C.3d)} \]

showing that \( \lambda f \) is uniformly continuous. If \( Y = \mathbb{K} = \mathbb{R} \), then \( d(x,y) < \min\{\delta_f, \delta_g\} \) together with Lem. C.1 implies

\[ | \max(f,g)(x) - \max(f,g)(y) | < \epsilon/2 < \epsilon, \quad \text{(C.3e)} \]

\[ | \min(f,g)(x) - \min(f,g)(y) | < \epsilon/2 < \epsilon, \quad \text{(C.3f)} \]

showing the uniform continuity of \( \max(f,g) \) and \( \min(f,g) \) and, in turn, also of \( f^+, f^- \), and \( |f| \).

\[ \text{Theorem C.3. Let } (X,d) \text{ be a metric space (e.g. a normed space), } (Y,\| \cdot \|) \text{ a normed vector space over } \mathbb{K}, \text{ and assume that } f, g : X \rightarrow Y \text{ are Lipschitz continuous. Then } f + g \text{ and } \lambda f \text{ are Lipschitz continuous for each } \lambda \in \mathbb{K}, \text{ i.e. the set } \text{Lip}(X,Y) \text{ constitutes a subspace of the vector space } F(X,Y) \text{ over } \mathbb{K}. \text{ Moreover, if } Y = \mathbb{K} = \mathbb{R}, \text{ then } \max(f,g), \min(f,g), f^+, f^-, |f| \text{ are all Lipschitz continuous.} \]

\[ \]
Proof. As \( f \) and \( g \) are Lipschitz continuous, there exist \( L_f \geq 0 \) and \( L_g \geq 0 \) such that, for each \( x, y \in X \),
\[
\|f(x) - f(y)\| \leq L_f d(x, y), \tag{C.4a}
\]
\[
\|g(x) - g(y)\| \leq L_g d(x, y). \tag{C.4b}
\]
Thus,
\[
\|(f + g)(x) - (f + g)(y)\| \leq \|f(x) - f(y)\| + \|g(x) - g(y)\|
\leq L_f d(x, y) + L_g d(x, y) = (L_f + L_g) d(x, y), \tag{C.4c}
\]
showing that \( f + g \) is Lipschitz continuous with Lipschitz constant \( L_f + L_g \). Next, for \( \lambda \in \mathbb{K} \),
\[
\|(\lambda f)(x) - (\lambda f)(y)\| = |\lambda| \|f(x) - f(y)\| \leq |\lambda| L_f d(x, y), \tag{C.4d}
\]
showing that \( \lambda f \) is Lipschitz continuous with Lipschitz constant \(|\lambda| L_f \). For \( Y = \mathbb{K} = \mathbb{R} \), Lem. C.1 shows \( \max(f, g) \) and \( \min(f, g) \) are Lipschitz continuous with Lipschitz constant \( \max\{L_f, L_g\} \), \( f^+ \) and \( f^- \) are Lipschitz continuous with Lipschitz constant \( L_f \), and \( |f| \) is Lipschitz continuous with Lipschitz constant \( 2L_f \).

Caveat C.4. Products and quotients of uniformly continuous functions are not necessarily uniformly continuous; products and quotients of Lipschitz continuous functions are not necessarily Lipschitz continuous: Even though \( f \equiv 1 \) and \( g(x) = x \) are Lipschitz continuous, it was shown in Examples 2.5(a),(b), respectively, that \( f/g \) and \( g^2 \) are not even uniformly continuous on \( \mathbb{R}^+ \).

D  Viewing \( \mathbb{C}^n \) as \( \mathbb{R}^{2n} \)

Remark D.1. Recall that the set of complex numbers \( \mathbb{C} \) is \textit{defined} to be \( \mathbb{R}^2 \), where the imaginary unit is \( i := (0, 1) \in \mathbb{R}^2 \), which allows to write each \( z = (x, y) \in \mathbb{C} = \mathbb{R}^2 \) as \( z = x + iy \), where \( x = \text{Re} \, z \) and \( y = \text{Im} \, z \). This, for each \( n \in \mathbb{N} \), gives rise to the \( \mathbb{R} \)-linear bijective map
\[
I : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}, \quad I((x_1, y_1), \ldots, (x_n, y_n)) := (x_1, y_1, \ldots, x_n, y_n), \tag{D.1}
\]
allowing to canonically identify \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \).

The identification (D.1) allows the identification of metric structures on \( \mathbb{C}^n \) and \( \mathbb{R}^{2n} \) due to the following general result:

Proposition D.2. Let \( X, Y \) be sets, let \( d : X \times X \longrightarrow \mathbb{R}^+_0 \) be a metric on \( X \), and let \( I : X \longrightarrow Y \) be bijective. Then
\[
d_Y : Y \times Y \longrightarrow \mathbb{R}^+_0, \quad d_Y(x, y) := d(I^{-1}(x), I^{-1}(y)), \tag{D.2}
\]
defines a metric on \( Y \) such that \((X, d)\) and \((Y, d_Y)\) are isometric (with the map \( I \) providing the isometry).
Proof. Let \( x, y, z \in Y \). Then
\[
d_Y(x, y) = 0 \iff d(I^{-1}(x), I^{-1}(y)) = 0 \iff I^{-1}(x) = I^{-1}(y) \iff x = y,
\]
showing that \( d_Y \) is positive definite. Moreover,
\[
d_Y(x, y) = d(I^{-1}(x), I^{-1}(y)) = d(I^{-1}(y), I^{-1}(x)) = d_Y(y, x),
\]
showing \( d_Y \) is symmetric. Finally,
\[
d_Y(x, z) = d(I^{-1}(x), I^{-1}(z)) \leq d(I^{-1}(x), I^{-1}(y)) + d(I^{-1}(y), I^{-1}(z)) = d_Y(x, y) + d_Y(y, z),
\]
proving the triangle inequality for \( d_Y \) and completing the proof that \( d_Y \) constitutes a metric. That \( I \) provides an isometry between \((X, d)\) and \((Y, d_Y)\) is immediate from (D.2).

**Corollary D.3.** Let \( n \in \mathbb{N} \), let \( d : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}^+_0 \) be a metric, and let \( I \) be the map from (D.1). Then
\[
d_t : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^+_0, \quad d_t(x, y) := d(I^{-1}(x), I^{-1}(y)),
\]
defines a metric on \( \mathbb{R}^{2n} \) such that \((\mathbb{C}^n, d)\) and \((\mathbb{R}^{2n}, d_t)\) are isometric (with the map \( I \) providing the isometry). Moreover, the map \( d \mapsto d_t \) is bijective between the set of metrics on \( \mathbb{C}^n \) and the set of metrics on \( \mathbb{R}^{2n} \).

**Proposition D.4.** Let \( n \in \mathbb{N} \). If \( \| \cdot \| \) constitutes a norm on the vector space \( \mathbb{C}^n \) over \( \mathbb{C} \), then
\[
\| \cdot \|_I : \mathbb{R}^{2n} \rightarrow \mathbb{R}^+_0, \quad \|(x_1, y_1, \ldots, x_n, y_n)\|_I := \|((x_1, y_1), \ldots, (x_n, y_n))\|
\]
defines a norm on the vector space \( \mathbb{R}^{2n} \) over \( \mathbb{R} \) such that \((\mathbb{C}^n, \| \cdot \|)\) and \((\mathbb{R}^{2n}, \| \cdot \|_I)\) are isometric (with the map \( I \) from (D.1) providing the isometry – even more precisely, if \( d \) and \( d_t \) denote the respective induced metrics, then the relation between \( d \) and \( d_t \) is given by (D.6)).

**Proof.** Exercise.

**Example D.5.** Let \( n \in \mathbb{N} \), \( p \in [1, \infty] \), and let \( \| \cdot \| \) denote the \( p \)-norm on the vector space \( \mathbb{R}^n \) over \( \mathbb{R} \), i.e. \( \| x \| := \left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \) for \( p < \infty \) and \( \| x \| = \max\{|x_j| : j = 1, \ldots, n\} \) for \( p = \infty \). Then it is an exercise to show
\[
\| \cdot \|_C : \mathbb{C}^n \rightarrow \mathbb{R}^+_0, \quad \|(z_1, \ldots, z_n)\|_C := \|(|z_1|, \ldots, |z_n|)\|
\]
defines a norm on the vector space \( \mathbb{C}^n \) over \( \mathbb{C} \).
Remark D.6. As a consequence of Th. 1.72, every norm on the normed vector space \( \mathbb{C}^n \) over \( \mathbb{C} \) generates precisely the same open subsets of \( \mathbb{C}^n \) – in other words, there is only one norm topology on \( \mathbb{C}^n \). Analogously, there is only one norm topology on \( \mathbb{R}^n \) as every norm on the normed vector space \( \mathbb{R}^n \) over \( \mathbb{R} \) generates precisely the same open subsets of \( \mathbb{R}^n \). Moreover, Prop. D.4 shows that the open sets of the norm topology on \( \mathbb{C}^n \) are actually precisely the same as the open sets of the norm topology on \( \mathbb{R}^{2n} \).

Theorem D.7. Let \( n \in \mathbb{N}, A \subseteq \mathbb{C}^n \). Then \( A \) is bounded in the normed vector space \( \mathbb{C}^n \) over \( \mathbb{C} \) if, and only if, \( A \) is bounded in the normed vector space \( \mathbb{R}^{2n} \) over \( \mathbb{R} \).

Proof. Exercise. ■

E Pseudometrics and Seminorms

If one omits the requirement of definiteness in the definitions of metric and norm, then one obtains what is called a pseudometric and a seminorm, respectively. In spite of the possible loss of definiteness, one can still carry out many parts of the theory analogously. In particular, a pseudometric still induces a topology (however, one can no longer expect this topology to be Hausdorff (not even \( T_1 \), actually), as it can happen that there are points \( x, y \) such that every open set that contains \( x \) also contains \( y \) (i.e., \( y \in \{x\} \)).

Definition E.1. Let \( X \) be a set. A function \( d : X \times X \to \mathbb{R}_0^+ \) is called a pseudometric or semimetric on \( X \) if, and only if, the following three conditions are satisfied:

(i) For each \( x \in X \), one has \( d(x, x) = 0 \).

(ii) \( d \) is symmetric, i.e., for each \( (x, y) \in X \times X \), \( d(y, x) = d(x, y) \).

(iii) \( d \) satisfies the triangle inequality, i.e., for each \( (x, y, z) \in X^3 \), \( d(x, z) \leq d(x, y) + d(y, z) \).

If \( d \) constitutes a pseudometric on \( X \), then the pair \((X, d)\) is called a pseudometric space.

Definition E.2. Let \( X \) be a vector space over the field \( \mathbb{K} \). Then a function \( \|\cdot\| : X \to \mathbb{R}_0^+ \) is called a seminorm on \( X \) if, and only if, the following three conditions are satisfied:

(i) \( \|0\| = 0 \).

(ii) \( \|\cdot\| \) is homogeneous of degree 1, i.e.

\[ \|\lambda x\| = |\lambda|\|x\| \quad \text{for each } \lambda \in \mathbb{K}, x \in X. \]

(iii) \( \|\cdot\| \) satisfies the triangle inequality, i.e.

\[ \|x + y\| \leq \|x\| + \|y\| \quad \text{for each } x, y \in X. \]
If \( \| \cdot \| \) constitutes a seminorm on \( X \), then the pair \( (X, \| \cdot \|) \) is called a \textit{seminormed vector space} or just \textit{seminormed space}.

**Remark E.3.** The proof of Lem. 1.8 shows that, if \( (X, \| \cdot \|) \) is a seminormed space, then
\[
d : X \times X \longrightarrow \mathbb{R}^+_0, \quad d(x, y) := \|x - y\|,
\]
constitutes a pseudometric on \( X \). One calls \( d \) the pseudometric induced by the seminorm \( \| \cdot \| \).

**Remark E.4.** Let \( (X, d) \) be a pseudometric space. Given \( x \in X \) and \( r \in \mathbb{R}^+ \), the open ball \( B_r(x) \), the closed ball \( \overline{B}_r(x) \), and the sphere \( S_r(x) \) are still defined precisely as in Def. 1.10. And analogous to Def. 1.10, we define a set \( U \) as an \( \mathcal{T} \)-open subset of \( X \) if, and only if, there is \( \epsilon \in \mathbb{R}^+ \) such that \( B_\epsilon(x) \subseteq U \). We call \( O \subseteq X \) \textit{open} if, and only if,
\[
\forall x \in O \quad \exists \epsilon \in \mathbb{R}^+ \quad B_\epsilon(x) \subseteq O.
\]

**Theorem E.5.** Let \( (X, d) \) be a pseudometric space. Then \( \mathcal{T} := \{O \subseteq X : O \text{ open}\} \) constitutes a topology on \( X \): One also calls \( \mathcal{T} \) the topology induced by the pseudometric \( d \), making each pseudometric space into a topological space.

**Proof.** Clearly, \( \emptyset \in \mathcal{T} \) and \( X \in \mathcal{T} \). Now consider finitely many open sets \( O_1, \ldots, O_N \in \mathcal{T}, \ N \in \mathbb{N} \), and let \( O := \bigcap_{j=1}^N O_j \). We have to prove that \( O \) is open. Hence, let \( x \in O \). Then \( x \in O_j \) for each \( j \in \{1, \ldots, N\} \). Since each \( O_j \) is open, for each \( j \in \{1, \ldots, N\} \), there is \( \epsilon_j > 0 \) such that \( B_{\epsilon_j}(x) \subseteq O_j \). If we let \( \epsilon := \min\{\epsilon_j : j \in \{1, \ldots, N\}\} \), then \( \epsilon > 0 \) and \( B_\epsilon(x) \subseteq B_{\epsilon_j}(x) \subseteq O_j \) for each \( j \in \{1, \ldots, N\} \), i.e. \( B_\epsilon(x) \subseteq O \), showing \( O \) is open. Now let \( I \) be an arbitrary index set. For each \( j \in I \), let \( O_j \in \mathcal{T} \). We have to verify that \( O := \bigcup_{j \in I} O_j \) is open. Let \( x \in O \). Then there is \( j \in I \) such that \( x \in O_j \). Since \( O_j \) is open, there is \( \epsilon > 0 \) such that \( B_\epsilon(x) \subseteq O_j \subseteq O \), showing \( O \) to be open. \( \blacksquare \)

**Definition E.6.** A topological space \( (X, \mathcal{T}) \) is called \textit{pseudometrizable} if, and only if, there exists a pseudometric \( d \) on \( X \) such that \( \mathcal{T} \) is induced by \( d \).

**Remark E.7.** Let \( (X, \mathcal{T}) \) be a topological space, where \( \mathcal{T} \) is induced by the pseudometric \( d \) on \( X \).

(a) As for metric spaces, one can still characterize the convergence in pseudometric spaces via the convergence of distances: Let \( (x_i)_{i \in I} \) be a net in \( X \), and \( x \in X \). Since every ball \( B_\epsilon(x), \ \epsilon > 0 \), is a neighborhood of \( x \) and, conversely, every \( U \in \mathcal{U}(x) \) contains some ball \( B_\epsilon(x) \subseteq U, \ \epsilon > 0 \), we have the equivalence
\[
\lim_{i \in I} x_i = x \Leftrightarrow \forall \epsilon \in \mathbb{R}^+ \exists i \in I \forall j \geq i \quad d(x_i, x) < \epsilon.
\]

(b) Example 1.33 still works exactly the same for pseudometric spaces: Given \( x \in X \) and \( r \in \mathbb{R}^+ \), the open ball \( B_r(x) \) is an open set and the closed ball \( \overline{B}_r(x) \) is a closed set.
(c) As for metric spaces, one still has that pseudometric spaces are first countable: For each \( x \in X \),
\[
\mathcal{B}(x) := \{ B_\varepsilon(x) : \varepsilon \in \mathbb{Q}^+ \}
\]
constitutes a countable local base at \( x \).

(d) Proposition 1.55(d) still works exactly the same for pseudometric spaces, showing that, for \( M \subseteq X \), the subspace topology \( T_M \) on \( M \) is pseudometrizable by \( d|_{M \times M} \).

(e) Lemma 1.57 still works exactly the same for pseudometric spaces, i.e.
\[
|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y') \quad \text{for each } x, x', y, y' \in X,
\]
and, if \( (X, \| \cdot \|) \) is a seminormed space, then
\[
\| \|x\| - \|y\| \| \leq \|x - y\| \quad \text{for each } x, y \in X.
\]
In particular, seminorms are still continuous (and even Lipschitz continuous, if one extends this notion to pseudometric spaces).

We will now see how to obtain metric spaces from pseudometric spaces (and normed spaces from seminormed spaces) by identifying points with pseudometrics.

**Theorem E.8.** Let \((X, d)\) be a pseudometric space. Define an equivalence relation on \( X \) by letting
\[
x \sim y :\iff d(x, y) = 0.
\]
Let \( Y := \{ [x] : x \in X \} \) be the set of the corresponding equivalence classes and define
\[
\rho : Y \times Y \longrightarrow \mathbb{R}_+^+, \quad \rho([x], [y]) := d(x, y).
\]
Then \( \rho \) is a metric on \( Y \); \( f : X \longrightarrow Y, \ f(x) := [x], \) is surjective and continuous.

**Proof.** We start by verifying that \( \sim \) is, indeed, an equivalence relation on \( X \): \( x \sim x \), since \( d(x, x) = 0 \); \( x \sim y \) implies \( y \sim x \), since \( d(x, y) = 0 \) implies \( d(y, x) = 0 \); \( x \sim y \) and \( y \sim z \) implies \( x \sim z \), since \( d(x, y) = 0 = d(y, z) \) implies \( 0 \leq d(x, z) \leq d(x, y) + d(y, z) = 0 \), i.e. \( d(x, z) = 0 \). Next, we show \( \rho \) to be well-defined: If \([x] = [\tilde{x}]\) and \([y] = [\tilde{y}]\), then \( x \sim \tilde{x} \) and \( y \sim \tilde{y} \), i.e. \( d(x, \tilde{x}) = d(y, \tilde{y}) = 0 \). Thus,
\[
|\rho([x], [y]) - \rho([\tilde{x}], [\tilde{y}])| = |d(x, y) - d(\tilde{x}, \tilde{y})| \leq d(x, \tilde{x}) + d(y, \tilde{y}) = 0,
\]
showing \( \rho([x], [y]) = \rho([\tilde{x}], [\tilde{y}]) \) as desired. In the next step, we verify \( \rho \) to be a metric: \( \rho([x], [x]) = d(x, x) = 0 \). If \( \rho([x], [y]) = 0 \), then \( d(x, y) = 0 \), i.e. \( x \sim y \) and \([x] = [y]\), showing \( \rho \) to be positive definite. As one also has \( \rho([x], [y]) = d(x, y) = d(y, x) = \rho([y], [x]) \) and \( \rho([x], [y]) = d(x, y) \leq d(x, z) + d(z, y) = \rho([x], [z]) + \rho([z], [y]) \), \( \rho \) is a
metric. That $f$ is surjective is immediate. For the continuity, let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $X$ such that $\lim_{k \to \infty} x_k = x \in X$. Then

$$\lim_{k \to \infty} \rho(f(x_k), f(x)) = \lim_{k \to \infty} \rho([x_k], [x]) = \lim_{k \to \infty} d(x_k, x) = 0,$$

proving $\lim_{k \to \infty} f(x_k) = f(x) \in Y$. Since $(X, d)$ is first countable, this shows $f$ to be continuous, completing the proof. □

The corresponding result for seminorms and norms is analogous:

**Theorem E.9.** Let $(X, N)$ be a seminormed vector space over $\mathbb{K}$. Then

$$V := N^{-1}\{0\}$$

is a subspace (over $\mathbb{K}$) of $X$. Let $Y := X/V$ be the corresponding factor space and define

$$\| \cdot \| : Y \to \mathbb{R}_{+}^{*}, \quad \|V + x\| := N(x).$$

Then $\| \cdot \|$ is a norm on $Y$; $f : X \to Y$, $f(x) := V + x$, is linear, surjective, and continuous.

**Proof.** We start by verifying that $V$ is, indeed, a subspace: $0 \in V$, since $N(0) = 0$; if $x, y \in V$, then $N(x) = N(y) = 0$, implying $0 \leq N(x + y) \leq N(x) + N(y) = 0$, i.e. $N(x + y) = 0$ and $x + y \in V$; if $x \in V$ and $\lambda \in \mathbb{K}$, then $N(\lambda x) = |\lambda| N(x) = 0$, i.e. $\lambda x \in V$. Next, we show $\| \cdot \|$ to be well-defined: If $V + x = V + \tilde{x}$, then $x - \tilde{x} \in V$ i.e. $N(x - \tilde{x}) = 0$. Thus,

$$\left| N(x) - N(\tilde{x}) \right| \leq N(x - \tilde{x}) = 0,$$

showing $\|V + x\| = \|V + \tilde{x}\|$ as desired. In the next step, we verify $\| \cdot \|$ to be a norm: $\|V + 0\| = N(0) = 0$. If $\|V + x\| = 0$, then $N(x) = 0$, i.e. $x \in V$ and $V + x = V + 0$, showing $\| \cdot \|$ to be positive definite. As one also has, for each $\lambda \in \mathbb{K}$, $\|V + \lambda x\| = N(\lambda x) = |\lambda| N(x) = |\lambda| \|V + x\|$ and $\|V + x + y\| = N(x + y) \leq N(x) + N(y) = \|V + x\| + \|V + y\|$, $\| \cdot \|$ is a norm. That $f$ is linear and surjective, since $Y = X/V$ and $f$ is the corresponding cannonical epimorphism. For the continuity, let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $X$ such that $\lim_{k \to \infty} x_k = x \in X$. Then

$$\lim_{k \to \infty} \|f(x_k) - f(x)\| = \lim_{k \to \infty} \|V + x_k - x\| = \lim_{k \to \infty} N(x_k - x) = 0,$$

proving $\lim_{k \to \infty} f(x_k) = f(x) \in Y$. Since $(X, N)$ is first countable, this shows $f$ to be continuous, completing the proof. □

**F Initial and Final Topologies, Quotient Spaces**

In this section, we will briefly study two very general construction principles that are useful in topology as well as in some other branches of mathematics (one of the construction principles we have actually already used when constructing subspace and product topologies, cf. Ex. F.4(a),(b) below).
Definition F.1. Let $X$ be a set and let $(X_i, T_i)_{i \in I}$ be a family of topological spaces, $I \neq \emptyset$.

(a) Given a family of functions $(f_i)_{i \in I}$, $f_i : X \rightarrow X_i$, the initial or weak topology on $X$ with respect to the family $(f_i)_{i \in I}$ is the coarsest topology $T$ on $X$ that makes all $f_i$ continuous (i.e. $T$ is the intersection of all topologies that make all $f_i$ continuous – this intersection is well-defined, since the discrete topology on $X$ always makes all $f_i$ continuous). The name initial topology stems from the $f_i$ being initially in $X$.

(b) Given a family of functions $(f_i)_{i \in I}$, $f_i : X_i \rightarrow X$, the final topology on $X$ with respect to the family $(f_i)_{i \in I}$ is

$$T := \left\{ O \subseteq X : \forall i \in I \ f_i^{-1}(O) \in T_i \right\}. \quad \text{(F.1)}$$

We will see in Lem. F.2(b) below that $T$ is, indeed, a topology, and that it is the finest topology on $X$ that makes all $f_i$ continuous. The name final topology stems from the $f_i$ being finally in $X$.$^{10}$

Lemma F.2. Let $X$ be a set and let $(X_i, T_i)_{i \in I}$ be a family of topological spaces, $I \neq \emptyset$.

(a) Given a family of functions $(f_i)_{i \in I}$, $f_i : X \rightarrow X_i$, the set

$$S := \left\{ f_i^{-1}(O_i) : O_i \in T_i, \ i \in I \right\} \quad \text{(F.2)}$$

is a subbase of the initial topology $T$ on $X$ with respect to the family $(f_i)_{i \in I}$.

(b) Given a family of functions $(f_i)_{i \in I}$, $f_i : X_i \rightarrow X$, the final topology $T$ on $X$ with respect to the family $(f_i)_{i \in I}$ as defined in (F.1) is, indeed, a topology on $X$, and it is the finest topology on $X$ that makes all $f_i$ continuous.

Proof. (a): Let $\tau(S)$ be the topology on $X$ generated by $S$, and let $T'$ be an arbitrary topology on $X$ that makes all $f_i$ continuous. Then, clearly, $S \subseteq T'$, also implying $\tau(S) \subseteq T'$. Thus, $\tau(S) \subseteq T$. On the other hand, by the definition of $S$, $\tau(S)$ also has the property of making every $f_i$ continuous, proving $\tau(S) = T$.

(b): We verify that $T$ is a topology. Fix $i \in I$. Then $f_i^{-1}(\emptyset) = \emptyset \in T_i$ and $f_i^{-1}(X) = X_i \in T_i$, showing $\emptyset, X \in T$. If $O_1, O_2 \in T$, then $f_i^{-1}(O_1 \cap O_2) = f_i^{-1}(O_1) \cap f_i^{-1}(O_2) \in T_i$, showing $O_1 \cap O_2 \in T$. If $O_j \in T$, $j \in J$, then $f_i^{-1}\left( \bigcup_{j \in J} O_j \right) = \bigcup_{j \in J} f_i^{-1}(O_j) \in T_i$, showing $\bigcup_{j \in J} O_j \in T$. Thus, $T$ is a topology. It is immediate from (F.1) that every $f_i$, $i \in I$, is continuous with respect to $T$. To see that $T$ is the finest topology on $X$ with this property, we still need to show that every topology $\mathcal{A}$ on $X$ making all $f_i$ continuous is contained in $T$. To this end, let $\mathcal{A}$ be such a topology on $X$. If $O \in \mathcal{A}$, then, for each $i \in I$, $f_i^{-1}(O) \in T_i$, i.e. $O \in T$, showing $\mathcal{A} \subseteq T$. $\blacksquare$

$^{10}$In the language of so-called Category Theory, we can say that the category of topological spaces has initial and final objects – in Analysis III, we will see that the category of measurable spaces has that same property.
Proposition F.3. Let \( X \) be a set and let \( (X_i, T_i)_{i \in I} \) be a family of topological spaces, \( I \neq \emptyset \).

(a) Given a family of functions \( (f_i)_{i \in I}, f_i : X \to X_i \), let \( T \) denote the initial topology on \( X \) with respect to the family \( (f_i)_{i \in I} \). Then \( T \) has the property that each map \( g : Z \to X \) from a topological space \( (Z, T_Z) \) into \( X \) is continuous if, and only if, each map \( (f_i \circ g) : Z \to X_i \) is continuous. Moreover, \( T \) is the only topology on \( X \) with this property.

(b) Given a family of functions \( (f_i)_{i \in I}, f_i : X_i \to X \), let \( T \) denote the final topology on \( X \) with respect to the family \( (f_i)_{i \in I} \). Then \( T \) has the property that each map \( g : X \to Z \) from \( X \) into a topological space \( (Z, T_Z) \) is continuous if, and only if, each map \( (g \circ f_i) : X_i \to Z \) is continuous. Moreover, \( T \) is the only topology on \( X \) with this property.

Proof. (a): If \( g \) is continuous, then each composition \( f_i \circ g, i \in I \), is also continuous. For the converse, let \( z \in Z \) and assume each \( f_i \circ g, i \in I \), to be continuous in \( z \). Let \( (z_j)_{j \in J} \) be a net in \( Z \) such that \( \lim_{j \in J} z_j = z \). Let \( O \in T_i, i \in I \), such that \( g(z) \in f_i^{-1}(O) \), i.e. such that \( (f_i \circ g)(z_j) \in O \). Then the continuity of \( f_i \circ g \) in \( z \) implies \( \lim_{j \in J} (f_i \circ g)(z_j) = (f_i \circ g)(z) \). Thus,

\[
\exists j_0 \in J \quad \forall j \geq j_0 \quad ((f_i \circ g)(z_j) \in O \quad \text{i.e.} \quad g(z_j) \in f_i^{-1}(O)),
\]

implying \( \lim_{j \in J} g(z_j) = z \) by Lem. F.2(a) and Cor. 1.50(a). Thus, we obtain the continuity of \( g \) in \( z \). Now let \( \mathcal{A} \) be an arbitrary topology on \( X \) with the property stated in the hypothesis. Letting \( (Z, T_Z) := (X, \mathcal{A}) \) and \( g := \text{Id}_X \), we see that each \( f_i \) is continuous with respect to \( \mathcal{A} \), implying \( T \subseteq \mathcal{A} \). Now let \( T' \) be an arbitrary topology on \( X \) that makes all \( f_i \) continuous. Letting \( (Z, T_Z') := (X, T') \), we see that \( g := \text{Id}_X \) is \( T'-\mathcal{A} \) continuous (since each \( f_i = \text{Id}_X \circ f_i \) is \( T'-T_i \) continuous) i.e., for each \( O \in \mathcal{A} \), we have \( g^{-1}(O) = O \in T' \), showing \( \mathcal{A} \subseteq T' \) and \( \mathcal{A} \subseteq T \), also completing the proof of \( \mathcal{A} = T \).

(b): If \( g \) is continuous, then each composition \( g \circ f_i, i \in I \), is also continuous. For the converse, assume each \( g \circ f_i, i \in I \), to be continuous. If \( O \in T_Z \), then, for each \( i \in I, f_i^{-1}(g^{-1}(O)) \in T_i \), showing \( g^{-1}(O) \in T \) according to (F.1). Thus, \( g \) is continuous.

Now let \( \mathcal{A} \) be an arbitrary topology on \( X \) with the property stated in the hypothesis. Letting \( (Z, T_Z) := (X, \mathcal{A}) \) and \( g := \text{Id}_X \), we see that each \( f_i \) is continuous with respect to \( \mathcal{A} \), implying \( \mathcal{A} \subseteq T \). Now let \( T' \) be an arbitrary topology on \( X \) that makes all \( f_i \) continuous. Letting \( (Z, T_Z') := (X, T') \), we see that \( g := \text{Id}_X \) is \( \mathcal{A}-T' \) continuous (since each \( f_i = f_i \circ \text{Id}_X \) is \( T'-T_i \) continuous) i.e., for each \( O \in T' \), we have \( g^{-1}(O) = O \in \mathcal{A} \), showing \( T' \subseteq \mathcal{A} \) and \( \mathcal{A} \subseteq T \), also completing the proof of \( \mathcal{A} = T \).  

Example F.4. (a) The product topology on \( X = \prod_{i \in I} X_i \) (cf. Ex. 1.53) is the initial topology with respect to the projections \( (\pi_i)_{i \in I}, \pi_i : X \to X_i \) (as is clear from Lem. F.2(a)).

(b) The subspace topology on \( M \subseteq X \), where \( (X, T) \) is a topological space (cf. Prop. 1.54), is the initial topology with respect to the identity inclusion map \( \iota : M \to X \).
\( \iota(x) := x \): This is also clear from Lem. F.2(a), since
\[
T_M = \{ O \cap M : O \in \mathcal{T} \} = \{ \iota^{-1}(O) : O \in \mathcal{T} \}.
\]

(c) An important example of a final topology is given by the quotient topology. Let \((X, \mathcal{T})\) be a topological space and let \(\sim\) be an equivalence relation on \(X\). Moreover, let \(Y := X/\sim = \{ [x] : x \in X \}\) be the corresponding quotient set (i.e. the set of corresponding equivalence classes). Then the quotient topology on \(Y\) with respect to \(\sim\), denoted \(\mathcal{T}/\sim\), is defined as the final topology with respect to the canonical projection
\[
\pi : X \longrightarrow Y, \quad \pi(x) := [x].
\]
Thus, by (F.1),
\[
\mathcal{T}/\sim = \{ O \subseteq Y : \pi^{-1}(O) \in \mathcal{T} \}.
\]

It is an exercise to show that \(S_1(0) \subseteq \mathbb{R}^2\), i.e. the unit sphere in \(\mathbb{R}^2\), endowed with the subspace topology, is homeomorphic to \(Y := (\mathbb{R} \cup \{ \infty, -\infty \})/\sim\), where \(\sim\) identifies \(\infty\) and \(-\infty\), and where \(Y\) is endowed with the corresponding quotient topology.

G Separation: More Counterexamples

Example G.1. (a) \(\mathbb{R}^2\) with a double origin is an example of a topological space that is \(T_2\), but not \(T_3\) (see [SS95, Sec. 74]).

(b) For examples of spaces that are regular, but not \(T_4\), cf. Ex. G.1(d) and Ex. G.1(e) below.

(c) Let \(X := \{0, 1\}, \mathcal{T} := \{ \emptyset, \{1\}, X \}\). Clearly, \((X, \mathcal{T})\) is a topological space (it is known as the Sierpinski space). The space is \(T_4\), since \(\{0\}\) and \(X\) are the only nonempty closed sets and these are not disjoint. However, the space is not \(T_1, T_2, T_3\): Due to Lem. 3.2(c) it suffices to see it is not \(T_1\). It is not \(T_1\), since \(\{1\}, \{0\}\) are disjoint; \(\{0\}\) is closed, but \(\{1\}, \{0\}\) can not be separated by open sets.

(d) The following simple example shows that a subspace of a \(T_4\) space does not need to be a \(T_4\) space: Let \(X := \{0, 1, 2, 3\}, \mathcal{T} := \{ \emptyset, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X \}\).

Clearly, \(\mathcal{T}\) is a topology on \(X\). The closed subsets of \(X\) are precisely \(\emptyset, \{0, 1, 2\}, \{0, 2\}, \{0, 1\}, \{0\}, X\). We see that, if \(A, B\) are closed disjoint subsets of \(X\), then \(A = \emptyset\) or \(B = \emptyset\), showing \((X, \mathcal{T})\) to be \(T_4\). However, if we consider \(M := \{1, 2, 3\}\), then \(T_M = \mathcal{T} \setminus \{X\}\), and the closed sets in \(M\) are precisely \(\emptyset, \{1, 2\}, \{2\}, \{1\}, X\). Now \(\{2\}, \{1\}\) are closed subsets of \(M\) that can not be separated, showing that \((M, T_M)\) is not \(T_4\).

Finding spaces that are normal, but have subspaces that are not \(T_4\) is not so easy, but they do exist: For example \([0, 1]^{[0,1]}\) with the product topology (see [SS95, Sec.
105]) or the so-called Tychonoff plank (see [SS95, Sec. 86,87]). Each subspace of a
normal space that is not $T_4$ provides an example of a space that is regular (as a
subspace of a regular space), but not $T_4$.

(e) The following example shows that the product of normal spaces does not need to
be $T_4$: If $(S, T_S)$ is $\mathbb{R}$ with the Sorgenfrey topology of Ex. 1.52(d), then $(S, T_S)$
is normal (see [SS95, Sec. 51]). If $X := S \times S$ with the corresponding product
topology $T_X$, then $(X, T_X)$ is called the Sorgenfrey plane. The Sorgenfrey plane is
not $T_4$ (see [SS95, Sec. 84]). On the other hand, $(X, T_X)$ must be regular, since it
is a product of regular spaces.

H Compactness

H.1 Intersections of Compact Sets

We provide an example that shows that in spaces that are not $T_2$, it can happen that
the intersection of two compact sets is not compact:

Example H.1. Let $a, b \notin \mathbb{N}$, $a \neq b$. Consider $X := \mathbb{N} \cup \{a, b\}$. Define

\[ T := \mathcal{P}(\mathbb{N}) \cup \{X\} \cup \{\mathbb{N} \cup \{a\}\} \cup \{\mathbb{N} \cup \{b\}\}. \]  

(H.1)

Clearly, $T$ is a topology on $X$. Moreover $C_1 := \mathbb{N} \cup \{a\}$ and $C_2 := \mathbb{N} \cup \{b\}$ are compact:
Each open cover of $C_1$ must have $X$ or $C_1$ as a member (they are the only open sets
containing $a$), providing a finite subcover (the analogous argument shows $C_2$ to be
compact). However, $C_1 \cap C_2 = \mathbb{N}$ and, as the subspace topology on $\mathbb{N}$ is discrete, $\mathbb{N}$ is
not compact.

H.2 Unit Balls in Normed Vector Spaces

The goal of this section is to prove Th. 3.18, i.e. that the closed unit ball in a normed
vector space $X$ is compact if, and only if, $X$ is finite-dimensional. In preparation, we
show that finite-dimensional subspaces of normed vector spaces are always closed:

Theorem H.2. Let $(X, \| \cdot \|)$ be a normed vector space over $\mathbb{K}$. If $U \subseteq X$ is a subspace
such that $\dim U = n \in \mathbb{N}$, then $U$ is closed.

Proof. Let $(b_1, \ldots, b_n)$ be a basis of $U$. Then, clearly,

\[ A : U \rightarrow \mathbb{K}^n, \quad A \left( \sum_{k=1}^n \alpha_k b_k \right) := (\alpha_1, \ldots, \alpha_n), \]  

(H.2)

defines a linear isomorphism. We define a norm on $\mathbb{K}^n$ by letting

\[ \| \cdot \| : \mathbb{K}^n \rightarrow \mathbb{R}_0^+, \quad \| z \| := \| A^{-1}(z) \|. \]  

(H.3)
Moreover, there exists a convergent subsequence. Therefore, 

\[ \| \lambda z \| = \| A^{-1}(\lambda z) \| = |\lambda| \| A^{-1}(z) \| = |\lambda| \| z \|, \]

showing \( \| \cdot \| \) to be homogeneous of degree 1. Finally,

\[ \forall z, w \in \mathbb{K} \quad \| z + w \| = \| A^{-1}(z + w) \| \leq \| A^{-1}(z) \| + \| A^{-1}(w) \| = \| z \| + \| w \|, \]

showing the triangle inequality to hold for \( \| \cdot \| \).

Let \((u^k)_{k \in \mathbb{N}}\) be a sequence in \(U\) such that \(\lim_{k \to \infty} u^k = x \in X\). Then \((u^k)_{k \in \mathbb{N}}\) is a Cauchy sequence and, as \(A\) is norm-preserving in consequence of (H.3), \((Au^k)_{k \in \mathbb{N}}\) is a Cauchy sequence in \(\mathbb{K}^n\). Since \(\mathbb{K}^n\) is complete, there is \(z \in \mathbb{K}^n\) such that \(\lim_{k \to \infty} Au^k = z\) and \(\lim_{k \to \infty} u^k = A^{-1}z\), showing \(x = A^{-1}z \in U\), i.e. \(U\) is closed.

**Proof of Th. 3.18.** Let \(X\) be finite-dimensional. If \((b_1, \ldots, b_n)\) denotes a basis of \(X\), then (H.2) defines a linear isomorphism \(A : X \to \mathbb{K}^n\). If we define a norm on \(\mathbb{K}^n\) via (H.3), then \(A^{-1}\) becomes norm-preserving and, in particular, continuous. Then \(\overline{B}_1(0)\) in \(X\) must be compact as the continuous image (under \(A^{-1}\)) of \(\overline{B}_1(0)\) in \(\mathbb{K}^n\).

Conversely, let \(X\) be infinite-dimensional. To show that \(\overline{B}_1(0)\) is not compact, we construct, via recursion, a sequence \((x^k)_{k \in \mathbb{N}}\) in \(\overline{B}_1(0)\) (actually in the sphere \(S_1(0)\)) that does not have a convergent subsequence: Fix \(n \in \mathbb{N}\) and assume \((x^1, \ldots, x^n)\) to be already constructed such that

\begin{align*}
\forall k \in \{1, \ldots, n\} & \quad \|x^k\| = 1, \quad (H.4a) \\
\forall k, l \in \{1, \ldots, n\}, k \neq l & \quad \|x^k - x^l\| \geq \frac{1}{2}. \quad (H.4b)
\end{align*}

Let \(U := \text{span}\{x^1, \ldots, x^n\}\). Since \(X\) is infinite-dimensional, we have \(U \neq X\). Let \(x \in X \setminus U\). Since \(U\) is closed by Th. H.2, it is

\[ d := \inf \{ \|x - u\| : u \in U \} > 0. \]

Moreover, there exists \(u_0 \in U\) such that \(\|x - u_0\| \leq 2d\). Set

\[ x^{n+1} := \frac{x - u_0}{\|x - u_0\|} \]

Then \(\|x^{n+1}\| = 1\) and, for each \(u \in U\) is \(\|x - u_0\| u + u_0 \in U\), implying

\[ \|u - x^{n+1}\| = \|\|x - u_0\| u + u_0\| \|x - u_0\| \geq \|x - u_0\| \geq \frac{d}{\|x - u_0\|} \geq \frac{1}{2}. \]

Thus, (H.4) holds with \(n\) replaced by \(n + 1\), where (H.4b) means that \((x^k)_{k \in \mathbb{N}}\) can not have a convergent subsequence. \(\blacksquare\)
H.3 Proof of Tychonoff’s Theorem

Proof of Th. 3.25. When using net convergence, the proof of the theorem can be carried out rather elegantly. A standard method in the literature is to first show that every net has a so-called universal subnet. Once this is established, Tychonoff’s theorem is a simple corollary. The following proof is essentially the one given in [Che92], which avoids the use of universal nets. Let \( \nu := (x^j)_j \) be a net in \( X \). We have to show that \( \nu \) has a convergent subnet. By Prop. 1.27, it suffices to show this \( \nu \) has a cluster point. The idea is to show that by an application of Zorn’s lemma. By definition, each \( \nu \) has a convergent subnet. By Prop. 1.27, it suffices to show this. Thus, \( x \in X \) since \( \emptyset \in P \). To apply Zorn’s lemma, we have to show that each totally ordered subset of \( P \) is a cluster point of \( \nu \), and only if, \( \nu \) is a cluster point of \( \nu \). Let \( P \) be the set of all partial cluster points of \( \nu \). Then \( P \neq \emptyset \), since \( \emptyset \in P \). The empty function \( \emptyset \) is a cluster point (in fact the limit) of the constant net \( \nu \mid = (\emptyset)_j \). Even if you are not fond of the empty set, you need not be concerned, as we will now show

\[
\forall y \in P \cap X_K, K \subseteq I \quad \Rightarrow \quad \exists \ i_0 \in I \setminus K, \exists \ z \in P \cap X_K \cup \{i_0\} \quad z \mid K = y : \quad (H.5)
\]

Let \( y \in P \cap X_K, K \subseteq I \). Then \( y \) is a cluster point of \( \nu \mid K \). Thus, \( \nu \mid K \) has a subnet \( (x^a \mid K)_{a \in A} \) such that \( \lim_{a \in A} x^a \mid K = y \). Let \( i_0 \in I \setminus K, L := K \cup \{i_0\} \). Since \( X_{i_0} \) is compact, the net \( (x_{i_0}^a)_{a \in A} \) has a subnet \( (x_{i_0}^b)_{b \in B} \) that converges to some \( z_{i_0} \in X_{i_0} \). Define

\[
z \in \prod_{i \in L} X_i, \quad z(i) := \begin{cases} y(i) & \text{for } i \in K, \\ z_{i_0} & \text{for } i = i_0. \end{cases}
\]

Then \( z \mid K = y \) and it remains to show \( z \in P \), i.e. \( z \) is a cluster point of \( \nu \mid L \). Indeed, the subnet \( (x_{i_0}^b)_{b \in B} \) of \( \nu \mid L \) converges to \( z \): Let \( O \in U(z) \) and suppose \( O = \pi_i^{-1}(O_i) \) with \( i \in L \) and \( O_i \in \mathcal{T} \). If \( i \in K \), then \( \lim_{b \in B} x_{i_0}^b \mid K = y \) implies there is \( b_0 \in B \) such that, for each \( b \geq b_0 \), one has \( x_{i_0}^b \mid L \subseteq O_i \). If \( i = i_0 \), then \( \lim_{b \in B} x_{i_0}^b = z_{i_0} \) implies there is \( b_0 \in B \) such that, for each \( b \geq b_0 \), one has \( x_{i_0}^b \mid L \subseteq O_i \). According to Cor. 1.50(a), this shows \( \lim_{b \in B} x_{i_0}^b \mid L = z \) and establishes (H.5). We now define a partial order on \( P \) by setting

\[
\forall y,z \in P, y \leq z \quad \Leftrightarrow \quad \left( y \in X_{K_y}, z \in X_{K_z}, K_y \subseteq K_z \subseteq I, z \mid K_y = y \right).
\]

To apply Zorn’s lemma, we have to show that each totally ordered subset of \( P \) has an upper bound. Let \( Q = \{y \in \prod_{i \in K_y} X_i\} \) be a totally ordered subset of \( P \). Let \( K := \bigcup_{y \in Q} K_y \) and define

\[
z \in \prod_{i \in K} X_i, \quad z(i) := y(i) \quad \text{for } i \in K_y
\]

(note that \( z \) is well-defined since \( Q \) is totally ordered). To see that \( z \in P \), let \( K_0 \subseteq K \) be finite and \( O := \bigcap_{i \in K_0} \pi_i^{-1}(O_i), O_i \in \mathcal{T} \). If \( K_0 = \{i_1, \ldots, i_N\}, N \in \mathbb{N} \), then there are \( y_1, \ldots, y_N \in Q \) such that \( i_l \in K_{y_l} \) for each \( l \in \{1, \ldots, N\} \). If \( y := \max\{y_1, \ldots, y_N\} \),
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then \( y \) is a cluster point of \( \nu \mid_{K_{q}} \) and \( \nu \mid_{K} \) is frequently in \( O \), showing \( z \) is a cluster point of \( \nu \mid_{K} \). Clearly, \( z \) is an upper bound for \( \mathcal{P} \). In consequence, Zorn’s lemma applies and \( \mathcal{P} \) must contain a maximal element \( c \). Due to (H.5), \( c \) must be defined on all of \( I \), i.e. \( c \) is a cluster point of \( \nu \) as desired.

I Topological Invariants

The following Prop. I.1 lists topological invariants (i.e. properties preserved under homeomorphisms) that are relevant to this class:

Proposition I.1. Let \((X, \mathcal{T}_{X})\) and \((Y, \mathcal{T}_{Y})\) be topological spaces, let \( f : X \longrightarrow Y \) be a homeomorphism, \( A \subseteq X, x \in X \).

(a) \( A \) is open if, and only if, \( f(A) \) is open; \( A \) is closed if, and only if, \( f(A) \) is closed; \( A \in \mathcal{U}(x) \) if, and only if, \( f(A) \in \mathcal{U}(f(x)) \).

(b) The net \((x_{i})_{i \in I}\) in \( X \) converges to \( x \) (has \( x \) as a cluster point) if, and only if, the net \((f(x_{i}))_{i \in I}\) in \( Y \) converges to \( f(x) \) (has \( f(x) \) as a cluster point).

(c) \((X, \mathcal{T}_{X})\) is (pseudo)metrizable if, and only if, \((Y, \mathcal{T}_{Y})\) is (pseudo)metrizable.

(d) \( x \) is an interior point (a boundary point, a point in the closure, a cluster point, an isolated point) of \( A \) if, and only if, \( f(x) \) is an interior point (a boundary point, a point in the closure, a cluster point, an isolated point) of \( f(A) \).

(e) \( A \) is dense in \( X \) if, and only if, \( f(A) \) is dense in \( Y \); \((X, \mathcal{T}_{X})\) is separable if, and only if, \((Y, \mathcal{T}_{Y})\) is separable.

(f) \( A \subseteq \mathcal{T}_{X} \) is a local base at \( x \) (a base of \( \mathcal{T}_{X} \), a subbase of \( \mathcal{T}_{X} \)) if, and only if, \( f(A) := \{ f(A) : A \in A \} \) is a local base at \( f(x) \) (a base of \( \mathcal{T}_{Y} \), a subbase of \( \mathcal{T}_{Y} \)); \((X, \mathcal{T}_{X})\) is first (second) countable if, and only if, \((Y, \mathcal{T}_{Y})\) is first (second) countable.

(g) If \( M \subseteq X \), then \( A \subseteq M \) is \( M \)-open (\( M \)-closed) if, and only if, \( f(A) \) is \( f(M) \)-open (\( f(M) \)-closed).

(h) For each \( n \in \{1, 2, 3, 4\} \), \((X, \mathcal{T}_{X})\) is \( T_{n} \) if, and only if, \((Y, \mathcal{T}_{Y})\) is \( T_{n} \).

(i) \( A \) is compact if, and only if, \( f(A) \) is compact.

(j) \( A \) is connected (resp. path-connected) if, and only if, \( f(A) \) is connected (resp. path-connected). Moreover, \( A \) is a connected component (resp. a path-component) of \( X \) if, and only if, \( f(A) \) is a connected component (resp. a path-component) of \( Y \).

Proof. Since \( f \) is a homeomorphism if, and only if, \( f^{-1} \) is a homeomorphism, it always suffices to prove one direction of the claimed equivalences.

(a): If \( A \) is open, then \( f(A) \) is open by Th. 2.7(ii), since \( f^{-1} \) is continuous. If \( A \) is closed, then \( f(A) \) is closed by Th. 2.7(iv), since \( f^{-1} \) is continuous. If \( A \in \mathcal{U}(x) \), then there
exists $O \in T_X$ such that $x \in O \subseteq A$. Then $f(x) \in f(O) \subseteq f(A)$. Since $f(O) \in T_Y$, this shows $f(A) \in U(f(x))$.

(b): Let $(x_i)_{i \in I}$ be a net in $X$, converging to $x$. Let $U \in U(f(x))$. Then, by (a), $f^{-1}(U) \in U(x)$ and $(x_i)_{i \in I}$ is eventually in $f^{-1}(U)$. Then $(f(x_i))_{i \in I}$ is eventually in $U = f(f^{-1}(U))$, showing $(f(x_i))_{i \in I}$ to converge to $f(x)$. If $x$ is a cluster point of $(x_i)_{i \in I}$, then there is a subnet $(x_j)_{j \in J}$ of $(x_i)_{i \in I}$ such that $\lim_{j \in J} x_j = x$. Then $(f(x_j))_{j \in J}$ is a subnet of $(f(x_i))_{i \in I}$ such that $\lim_{j \in J} f(x_j) = f(x)$, showing $f(x)$ to be a cluster point of $(f(x_i))_{i \in I}$.

(c): Let $T_X$ be induced by the (pseudo)metric $d_X$ on $X$. Then, clearly,

$$d_Y : Y \times Y \to \mathbb{R}^+_0, \quad d_Y(y_1, y_2) := d_X(f^{-1}(y_1), f^{-1}(y_2)),$$

defines a (pseudo)metric on $Y$. We show that $d_Y$ induces $T_Y$: We have the equivalences

$$O \in T_Y \iff f^{-1}(O) \in T_X \iff \forall x \in f^{-1}(O) \exists \varepsilon \in \mathbb{R}^+ B_{\varepsilon}(x) \subseteq f^{-1}(O) \iff \forall y \in O \exists \varepsilon \in \mathbb{R}^+ B_{\varepsilon}(y) \subseteq O,$$

establishing the case.

(d): If $x$ is an interior point of $A$, then there is $O \in T_X$ such that $x \in O \subseteq A$. Then $f(O) \in T_Y$ and $f(x) \in f(O) \subseteq f(A)$, showing $f(x)$ to be an interior point of $f(A)$. If $x \in \partial A$ and $U \in U(f(x))$, then $O := f^{-1}(U) \in U(x)$, $O \cap A \neq \emptyset$ and $O \cap A^c \neq \emptyset$. Thus, $f(O) \cap f(A) = U \cap f(A) \neq \emptyset$ and $f(O) \cap f(A^c) = U \cap f(A^c) \neq \emptyset$, showing $f(x) \in \partial f(A)$. If $x$ is a cluster point of $A$, then there is a net $(a_i)_{i \in I}$ in $A \setminus \{x\}$ such that $\lim_{i \in I} a_i = x$. Then $(f(a_i))_{i \in I}$ is a net in $f(A) \setminus \{f(x)\}$ such that $\lim_{i \in I} f(a_i) = f(x)$ (by (b)), showing $f(x)$ to be a cluster point of $f(A)$. Finally, the set $P$ of isolated points of $A$ is $P = A \setminus H(A)$ (where $H(A)$ is set of cluster points). Then $f(P) = f(A) \setminus f(H(A)) = f(A) \setminus H(f(A))$, i.e. $f(P)$ is the set of isolated points of $f(A)$.

(e): According to (e) $\overline{A} = X$ if and only if $f(\overline{A}) = Y$. The claim regarding separability then also follows, as $A$ is countable if, and only if, $f(A)$ is countable.

(f): If $A$ is a local base at $x$ and $U \in U(f(x))$, then $f^{-1}(U) \in U(x)$. Thus, there is $B \in A$ such that $x \in B \subseteq f^{-1}(U)$. Then $f(x) \in f(B) \subseteq U$, showing that $f(A)$ is a local base at $f(x)$. Now let $A$ be a base of $T_X$ and let $O \in T_Y$. Then $f^{-1}(O) \in T_X$, i.e. $f^{-1}(O) = \bigcup_{i \in I} B_i$ with suitable $I$ and $B_i \in A$. Then $O = \bigcup_{i \in I} f(B_i)$, proving $f(A)$ to be a base of $T_Y$. Since $f$ is bijective, $A$ is countable if, and only if, $f(A)$ is countable. Finally, let $A$ be a subbase of $T_X$ and then set $\beta(A)$, then set of finite intersections of sets in $A$ is a base of $T_X$. Then $f(\beta(A))$ is a base of $T_Y$. If $B \in \beta(A)$, then

$$B = \bigcap_{i=1}^n A_i, \quad \forall i \in \{1, \ldots, n\}, \quad A_i \in A, \quad n \in \mathbb{N},$$

and

$$f(B) = \bigcap_{i=1}^n f(A_i)$$
showing \( f(A) \) to be a subbase of \( \mathcal{T}_Y \).

(g): If \( A \subseteq M \subseteq X \) and \( A \) is \( M \)-open (\( M \)-closed), then there is \( B \subseteq X \) such that \( B \) is \( X \)-open (\( X \)-closed) and \( A = M \cap B \). Then \( f(A) = f(M) \cap f(B) \), and, since \( f(B) \) is \( Y \)-open (\( Y \)-closed) this shows \( f(A) \) to be \( (f(M)) \)-open \( (f(M)) \)-closed).

(h): Suppose \( (X, \mathcal{T}_X) \) is \( T_1 \) (resp. \( T_2 \)) and let \( y_1, y_2 \in Y \) such that \( y_1 \neq y_2 \). Then \( x_1 := f^{-1}(y_1) \neq x_2 := f^{-1}(y_2) \) and there are open \( O_1 \in \mathcal{U}(x_1) \) and open \( O_2 \in \mathcal{U}(x_2) \) such that \( x_2 \notin O_1 \) and \( x_1 \notin O_2 \) (resp. \( O_1 \cap O_2 = \emptyset \)). Then \( U_1 := f(O_1) \) is open, \( U_2 := f(O_2) \) is open, \( y_1 \in U_1 \), \( y_2 \in U_2 \), and \( y_2 \notin U_1 \) as well as \( y_1 \notin U_2 \) (resp. \( U_1 \cap U_2 = \emptyset \)), showing \( (Y, \mathcal{T}_Y) \) to be \( T_1 \) (resp. \( T_2 \)). Now suppose \( (X, \mathcal{T}_X) \) is \( T_3 \), let \( y \in Y \), and let \( B \subseteq Y \) be closed such that \( y \notin B \). Then \( x := f^{-1}(y) \notin A := f^{-1}(B) \), \( A \) is closed, and there are open \( O_1 \in \mathcal{U}(x) \) and open \( O_2 \subseteq X \) such that \( A \subseteq O_2 \) and \( O_1 \cap O_2 = \emptyset \). Then \( U_1 := f(O_1) \) is open, \( U_2 := f(O_2) \) is open, \( y \in U_1 \), \( B \subseteq U_2 \), and \( U_1 \cap U_2 = \emptyset \), showing \( (Y, \mathcal{T}_Y) \) to be \( T_3 \). Finally, suppose \( (X, \mathcal{T}_X) \) is \( T_4 \) and let \( B_1, B_2 \subseteq Y \) be closed such that \( B_1 \cap B_2 = \emptyset \). Then \( A_1 := f^{-1}(B_1) \cap A_2 := f^{-1}(B_2) = \emptyset \), \( A_1 \) and \( A_2 \) are closed, and there are open \( O_1, O_2 \subseteq X \) such that \( A_1 \subseteq O_1 \), \( A_2 \subseteq O_2 \) and \( O_1 \cap O_2 = \emptyset \). Then \( U_1 := f(O_1) \) is open, \( U_2 := f(O_2) \) is open, \( B_1 \subseteq U_1 \), \( B_2 \subseteq U_2 \), and \( U_1 \cap U_2 = \emptyset \), showing \( (Y, \mathcal{T}_Y) \) to be \( T_4 \).

(i): Suppose \( A \) is compact and let \((O_i)_{i \in I}\) be an open cover of \( f(A) \). Letting \( U_i := f^{-1}(O_i) \) for each \( i \in I \), we see that \((U_i)_{i \in I}\) is an open cover of \( A \). Since \( A \) is compact, there exists a finite \( J \subseteq I \) such that \((U_i)_{i \in J}\) is still a cover of \( A \). But then \((O_i)_{i \in J}\) is a finite subcover of \((O_i)_{i \in I}\) that still covers \( f(A) \), proving \( f(A) \) to be compact.

(j): For the first part, without loss of generality, we may assume \( A = X \). Assume \( Y \) is not connected and let \( O_1, O_2 \in \mathcal{T}_Y \) such that \( O_1 \cap O_2 = \emptyset \), \( Y = O_1 \cup O_2 \), \( O_1, O_2 \neq \emptyset \). If \( U_1 := f^{-1}(O_1) \), \( U_2 := f^{-1}(O_2) \), then \( U_1, U_2 \in \mathcal{T}_X \), \( U_1 \cap U_2 = \emptyset \), \( X = U_1 \cup U_2 \), \( U_1, U_2 \neq \emptyset \), showing \( X \) is not connected. If \( x, y \in A \) and \( \phi : [0, 1] \rightarrow A \) is a path in \( A \) connecting \( x \) and \( y \), then \( f \circ \phi \) is a path in \( f(A) \) connecting \( f(x) \) and \( f(y) \). Thus, if \( A \) is connected, so is \( f(A) \). If \( A \) is a connected component (resp. a path-component) of \( X \) and \( x \in A \), then \( A \) is the union of all connected sets containing \( x \). Then \( f(A) \) is the union of all connected (resp. path-connected) sets containing \( f(x) \), i.e. \( f(A) \) is a connected component (resp. a path-component) of \( Y \).

\[ \blacksquare \]

### J Multilinear Maps

We are mostly interested in vector spaces over the fields \( F = \mathbb{R} \) and \( F = \mathbb{C} \). However, the following considerations hold for an arbitrary field \( F \).

**Definition J.1.** Let \( X \) and \( Y \) be vector spaces over the field \( F \), \( \alpha \in \mathbb{N} \). We call a map

\[ L : X^\alpha \rightarrow Y \]  

(J.1)

**multilinear** (more precisely, \( \alpha \) times linear) if, and only if, it is linear in each component,
i.e., for each \(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{\alpha}, v, w \in X\), each \(\lambda, \mu \in F\):
\[
L(x_1, \ldots, x_{i-1}, \lambda v + \mu w, x_{i+1}, \ldots, x_{\alpha}) = \lambda L(x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_{\alpha}) + \mu L(x_1, \ldots, x_{i-1}, w, x_{i+1}, \ldots, x_{\alpha}). \tag{J.2}
\]

We denote the set of all \(\alpha\) times linear maps from \(X\) into \(Y\) by \(\mathcal{L}^\alpha(X, Y)\). We also set \(\mathcal{L}^0(X, Y) := Y\).

**Remark J.2.** In the situation of Def. J.1, each \(\mathcal{L}^\alpha(X, Y), \alpha \in \mathbb{N}_0\), constitutes a vector space over \(F\): It is a subspace of the vector space over \(F\) of all functions from \(X\) into \(Y\), since, clearly, if \(K, L : X^\alpha \rightarrow Y\) are both \(\alpha\) times linear and \(\lambda, \mu \in F\), then \(\lambda K + \mu L\) is also \(\alpha\) times linear.

**Theorem J.3.** Let \(X\) and \(Y\) be vector spaces over the field \(F\), \(\alpha \in \mathbb{N}\). Then, as vector spaces over \(F\), \(\mathcal{L}(X, \mathcal{L}^{\alpha-1}(X, Y))\) and \(\mathcal{L}^\alpha(X, Y)\) are isomorphic via the isomorphism
\[
\Phi : \mathcal{L}(X, \mathcal{L}^{\alpha-1}(X, Y)) \rightarrow \mathcal{L}^\alpha(X, Y),
\Phi(L)(x^1, \ldots, x^\alpha) := L(x^1)(x^2, \ldots, x^\alpha). \tag{J.3}
\]

**Proof.** Since \(L\) is linear and \(L(x^1)\) is \((\alpha - 1)\) times linear, \(\Phi(L)\) is, indeed, an element of \(\mathcal{L}^\alpha(X, Y)\), showing that \(\Phi\) is well-defined by (J.3). Next, we verify \(\Phi\) to be linear: If \(\lambda \in F\) and \(K, L \in \mathcal{L}(X, \mathcal{L}^{\alpha-1}(X, Y))\), then
\[
\Phi(\lambda L)(x^1, \ldots, x^\alpha) = (\lambda L)(x^1)(x^2, \ldots, x^\alpha) = \lambda(L(x^1)(x^2, \ldots, x^\alpha)) = \lambda \Phi(L)(x^1, \ldots, x^\alpha)
\]
and
\[
\Phi(K + L)(x^1, \ldots, x^\alpha) = (K + L)(x^1)(x^2, \ldots, x^\alpha) = (K(x^1) + L(x^1))(x^2, \ldots, x^\alpha)
\]
\[
= K(x^1)(x^2, \ldots, x^\alpha) + L(x^1)(x^2, \ldots, x^\alpha)
\]
\[
= \Phi(K)(x^1, \ldots, x^\alpha) + \Phi(L)(x^1, \ldots, x^\alpha)
\]
\[
= (\Phi(K) + \Phi(L))(x^1, \ldots, x^\alpha),
\]
proving \(\Phi\) to be linear. Now we show \(\Phi\) to be injective. To this end, we show that, if \(L \neq 0\), then \(\Phi(L) \neq 0\). If \(L \neq 0\), then there exist \(x^1, \ldots, x^\alpha \in X\) such that \(L(x^1)(x^2, \ldots, x^\alpha) \neq 0\), showing that \(\Phi(L) \neq 0\) as needed. To verify \(\Phi\) is also surjective, let \(K \in \mathcal{L}^\alpha(X, Y)\). Define \(L : X \rightarrow \mathcal{L}^{\alpha-1}(X, Y)\) by letting
\[
L(x^1)(x^2, \ldots, x^\alpha) := K(x^1, \ldots, x^\alpha). \tag{J.4}
\]
Then, clearly, for each \(x^1 \in X\), \(L(x^1) \in \mathcal{L}^{\alpha-1}(X, Y)\). Moreover, \(L\) is linear, i.e. \(L \in \mathcal{L}(X, \mathcal{L}^{\alpha-1}(X, Y))\). Comparing (J.4) with (J.3) shows \(\Phi(L) = K\), i.e. \(\Phi\) is surjective, completing the proof.

**Remark J.4.** For simplicity, we will now restrict ourselves to finite-dimensional \(X\). Suppose \(\dim X = n, n \in \mathbb{N}\). Moreover, let \(\{b_1, \ldots, b_n\}\) be a basis of \(X\) over \(F\). If
$x_1^i, \ldots, x_\alpha^i \in X$, then there are $x_j^i \in F$, $j \in \{1, \ldots, \alpha\}$, $i \in \{1, \ldots, n\}$, such that

$$x^i = \sum_{i=1}^{n} x_i^j b_j. $$

Thus, if $L \in \mathcal{L}^\alpha(X,Y)$, then

$$L(x^i, \ldots, x^\alpha) = \sum_{i_1, \ldots, i_n} x_{i_1}^1 \cdots x_{i_n}^\alpha L(b_{i_1}, \ldots, b_{i_n}),$$

showing $L$ is uniquely determined by its values $L(b_{i_1}, \ldots, b_{i_n})$, $(i_1, \ldots, i_n) \in \{1, \ldots, n\}^\alpha$. Conversely, if, for each $(i_1, \ldots, i_n) \in \{1, \ldots, n\}^\alpha$, one is given a vector $y_{i_1, \ldots, i_n} \in Y$, then

$$L(x^i, \ldots, x^\alpha) = \sum_{i_1, \ldots, i_n=1}^{n} x_{i_1}^1 \cdots x_{i_n}^\alpha y_{i_1, \ldots, i_n},$$

clearly, defines an element $L \in \mathcal{L}^\alpha(X,Y)$.

**Theorem J.5.** Let $X$ and $Y$ be vector spaces over the field $F$, $\alpha \in \mathbb{N}$. Moreover, let $\dim X = n$, $n \in \mathbb{N}$, let $\{b_1, \ldots, b_n\}$ be a basis of $X$ over $F$, and let $B$ be a basis of $Y$ over $F$. For each $(i_1, \ldots, i_n) \in I := \{1, \ldots, n\}^\alpha$ and each $b \in B$, define

$$L_{i_1, \ldots, i_n,b}(b_{j_1}, \ldots, b_{j_n}) := \begin{cases} b & \text{for } (j_1, \ldots, j_n) = (i_1, \ldots, i_n), \\ 0 & \text{otherwise.} \end{cases} \quad (J.7)$$

According to Rem. J.4, (J.7) uniquely defines an element of $\mathcal{L}^\alpha(X,Y)$. Then $\mathcal{B} := \{L_{i_1, \ldots, i_n,b} : (i_1, \ldots, i_n) \in I, b \in B\}$ constitutes a basis of $\mathcal{L}^\alpha(X,Y)$ over $F$ and, in particular, if $\dim Y = m$, then $\dim \mathcal{L}^\alpha(X,Y) = n^\alpha m$.

**Proof.** We verify that the elements of $\mathcal{B}$ are linearly independent: Let $M, N \in \mathbb{N}$. Let $(i^1_1, \ldots, i^1_\alpha), \ldots, (i^M_1, \ldots, i^M_\alpha) \in I$ be distinct and let $b^1, \ldots, b^M \in B$ be distinct as well. Assume $\lambda_{i^l k} \in F$ to be such that

$$L := \sum_{l=1}^{M} \sum_{k=1}^{N} \lambda_{i^l k} L_{i^l_1, \ldots, i^l_\alpha, b^l} = 0.$$

Let $k \in \{1, \ldots, N\}$. Then

$$0 = L(b_{i^k_1}, \ldots, b_{i^k_\alpha}) = \sum_{l=1}^{M} \sum_{k=1}^{N} \lambda_{i^l k} L_{i^l_1, \ldots, i^l_\alpha, b^l}(b_{i^k_1}, \ldots, b_{i^k_\alpha}) = \sum_{l=1}^{M} \lambda_{i^l k} b^l$$

implies $\lambda_{i^1 k} = \cdots = \lambda_{i^M k} = 0$ due to the linear independence of the $b^l$. As this holds for each $k \in \{1, \ldots, N\}$, we have established the linear independence of $\mathcal{B}$. It remains to verify that $\mathcal{B}$ spans $\mathcal{L}^\alpha(X,Y)$. According to Rem. J.4, if $L \in \mathcal{L}^\alpha(X,Y)$, then $L$ has the form (J.6), where $y_{i_1, \ldots, i_n} = L(b_{i_1}, \ldots, b_{i_n}) \in Y$ for each $(i_1, \ldots, i_n) \in I$. Thus, if, for each $(i_1, \ldots, i_n) \in I$,

$$L_{i_1, \ldots, i_n}(b_{j_1}, \ldots, b_{j_n}) := \begin{cases} y_{i_1, \ldots, i_n} & \text{for } (j_1, \ldots, j_n) = (i_1, \ldots, i_n), \\ 0 & \text{otherwise,} \end{cases} \quad (J.6)$$

then $L = \sum_{i_1, \ldots, i_n} L_{i_1, \ldots, i_n}(b_{i_1}, \ldots, b_{i_n})$.
then

\[ L = \sum_{(i_1, \ldots, i_\alpha) \in I} L_{i_1, \ldots, i_\alpha}. \]

It merely remains to write each \( L_{i_1, \ldots, i_\alpha} \) as a linear combination of elements of \( \mathcal{B} \). To this end, write

\[ y_{i_1, \ldots, i_\alpha} = \sum_{k=1}^{N} \lambda^k b^k, \]

where \( N \in \mathbb{N}, \lambda^k \in F, b^k \in B \). Then

\[ L_{i_1, \ldots, i_\alpha} = \sum_{k=1}^{N} \lambda^k L_{i_1, \ldots, i_\alpha, b^k}. \]

Let \( K := \sum_{k=1}^{N} \lambda^k L_{i_1, \ldots, i_\alpha, b^k} \). Then, for each \((j_1, \ldots, j_\alpha) \in I\),

\[ K(b_{j_1}, \ldots, b_{j_\alpha}) = \begin{cases} y_{i_1, \ldots, i_\alpha} & \text{for } (j_1, \ldots, j_\alpha) = (i_1, \ldots, i_\alpha), \\ 0 & \text{otherwise}, \end{cases} \]

thereby completing the proof. \( \blacksquare \)

In Th. J.5, if \( X \) is infinite-dimensional, then the set corresponding to \( \mathcal{B} \) is still linearly independent, but, if \( Y \neq \{0\} \), then \( \mathcal{B} \) does no longer generate \( \mathcal{L}^\alpha(X,Y) \) and, in particular, it is no longer a basis of \( \mathcal{L}^\alpha(X,Y) \).

## K Differential Calculus

### K.1 Bounded Derivatives Imply Lipschitz Continuity

It is sometimes useful if the bound on the derivatives is the same as the resulting Lipschitz constant (which, for \( m > 1 \), is not the case in Th. 4.38). The following Th. K.2 provides a variant, where the constants are the same, formulated for functions \( f : I \rightarrow \mathbb{R}^n \), defined on open intervals \( I \subseteq \mathbb{R} \), and making use of the Euclidean norm \( \| \cdot \|_2 \) on \( \mathbb{R}^n \). We will start with some auxiliary results regarding the Euclidean norm and the Euclidean inner product:

**Proposition K.1.** Let \( I \subseteq \mathbb{R} \) be an open interval, and let \( g, h : I \rightarrow \mathbb{R}^n \) be differentiable, \( n \in \mathbb{N} \).

(a) The function

\[ f : I \rightarrow \mathbb{R}, \quad f(x) := g(x) \bullet h(x) = \sum_{j=1}^{n} g_j(x)h_j(x), \quad (K.1) \]

is differentiable and

\[ f' : I \rightarrow \mathbb{R}, \quad f'(x) = g'(x) \bullet h(x) + h(x) \bullet h'(x). \quad (K.2) \]
(b) The function \( \alpha : I \to \mathbb{R} \), \( \alpha(x) := \|g(x)\|_2 = \sqrt{g(x) \cdot g(x)} \), is differentiable at each \( x \in I \) such that \( g(x) \neq 0 \). Moreover,

\[
\alpha'(x) = \frac{g(x) \cdot g'(x)}{\alpha(x)} = \frac{g(x) \cdot g'(x)}{\|g(x)\|_2}.
\]

Proof. (a) is immediate from the product rule.

(b) is an easy consequence of (a), as (a) implies \( \alpha \) to be differentiable at each \( x \in I \) such that \( g(x) \neq 0 \), and

\[
\forall x \in I, \quad \|g(x)\|_2 = g(x) \cdot g'(x) \alpha'(x) = \frac{2 g(x) \cdot g'(x)}{\sqrt{g(x) \cdot g(x)}},
\]

completing the proof. \( \blacksquare \)

Theorem K.2. Let \( a, b \in \mathbb{R} \) with \( a < b \) and let \( f : ]a, b[ \to \mathbb{R}^n \) be differentiable with uniformly bounded derivative, i.e. with

\[
\exists M \in \mathbb{R}^+_0, \quad \forall x \in ]a, b[ \quad \|f'(x)\|_2 = \sqrt{\sum_{j=1}^n |f_j(x)|^2} \leq M.
\]

Then \( f \) is \( M \)-Lipschitz, i.e.

\[
\forall x_1, x_2 \in ]a, b[ \quad \|f(x_1) - f(x_2)\|_2 \leq M \|x_1 - x_2\|.
\]

Proof. For \( x_1 = x_2 \), there is nothing to prove. Thus, assume \( x_1 \neq x_2 \) and define the auxiliary function

\[
g : [0, 1] \to \mathbb{R}^n, \quad g(t) := f(x_1 + t(x_2 - x_1)) - f(x_1).
\]

According to the chain rule of Th. 4.31, \( g \) is differentiable on \( ]0, 1[ \) and

\[
\forall t \in ]0, 1[ \quad g'(t) = (x_2 - x_1)f'(x_1 + t(x_2 - x_1)),
\]

implying

\[
\forall t \in ]0, 1[ \quad \|g'(t)\|_2 \leq M \|x_1 - x_2\|.
\]

We now introduce another auxiliary function, namely

\[
\alpha : [0, 1] \to \mathbb{R}, \quad \alpha(t) := \|g(t)\|_2.
\]
The mean value theorem \[\Phi_{16}, \text{Th. 9.18}\] implies the existence of \(\rho\). We have to define \(\alpha\) as follows:

\[
\alpha(t) = \begin{cases}
\rho & \text{if } n \text{ is odd}, \\
\epsilon & \text{if } n \text{ is even};
\end{cases}
\]

which establishes the case.

### K.2 Surjectivity of Directional Derivatives

We finish the proof of Th. 4.41 by showing that, for \(n \geq 2\), the map

\[
D : S_1(0) \rightarrow [-\alpha, \alpha], \quad D(e) := \nabla f(\xi) \cdot e = \sum_{j=1}^{n} \epsilon_j \partial_j f(\xi), \quad \alpha = \| \nabla f(\xi) \|_2, \quad (K.14)
\]

is surjective (we already know from (4.50) that \(D(e) \in [-\alpha, \alpha] \) for each \(e \in S_1(0)\)). We also recall \(e_{\max} = \nabla f(\xi) / \alpha, \; e_{\min} = -e_{\max}, D(e_{\max}) = \alpha, D(e_{\min}) = -\alpha.\)

The idea is to rotate \(e_{\max}\) into \(e_{\min}\). This can be achieved using a suitable function

\[
\rho : [0, \pi] \rightarrow S_1(0) \subseteq \mathbb{R}^n, \quad \rho = (\rho_1, \ldots, \rho_n).
\]

We have to define \(\rho\) differently, depending on \(n \geq 2\) being even or odd. To this end, let \((\epsilon_1, \ldots, \epsilon_n) := e_{\max}\). If \(n\) is even, then define

\[
\forall j \in \{1, \ldots, n\}, \quad \rho_j : [0, \pi] \rightarrow [-1, 1], \quad \rho_j(\theta) := \begin{cases}
\epsilon_j \cos \theta + \epsilon_{j+1} \sin \theta & \text{if } j \text{ is odd}, \\
-\epsilon_{j-1} \sin \theta + \epsilon_j \cos \theta & \text{if } j \text{ is even};
\end{cases}
\]

if \(n\) is odd (note \(n \geq 3\) in this case), then define

\[
\forall j \in \{1, \ldots, n\}, \quad \rho_j(\theta) := \begin{cases}
\epsilon_j \cos \theta + \epsilon_{j+1} \sin \theta & \text{if } j < n - 2 \text{ is odd}, \\
-\epsilon_{j-1} \sin \theta + \epsilon_j \cos \theta & \text{if } j < n - 2 \text{ is even}, \\
\epsilon_{n-2} \cos \theta + \frac{\epsilon_{n-1} + \epsilon_n}{\sqrt{\epsilon_{n-1}^2 + \epsilon_n^2}} \sin \theta & \text{if } j = n - 2, \\
\epsilon_{n-1} \cos \theta - \frac{\epsilon_{n-2} \epsilon_{n-1}}{\sqrt{\epsilon_{n-1}^2 + \epsilon_n^2}} \sin \theta & \text{if } j = n - 1, \\
\epsilon_n \cos \theta - \frac{\epsilon_{n-2} \epsilon_n}{\sqrt{\epsilon_{n-1}^2 + \epsilon_n^2}} \sin \theta & \text{if } j = n.
\end{cases}
\]
For the sake of readability, we assumed $\epsilon_{n-1} \neq 0$ or $\epsilon_n \neq 0$ in (K.15b). There is always at least one $j_0 \in \{1, \ldots, n\}$ such that $\epsilon_{j_0} \neq 0$. If $j_0 \notin \{n - 1, n\}$, then one merely needs to interchange the roles of $j_0$ and $n$ in (K.15b).

Clearly, for every $n \geq 2$, each $\rho_j$ is continuous, i.e. $\rho$ is continuous.

Next, we verify that $\rho$, indeed, maps into $S_1(0)$ (which, in particular, implies each $\rho_j$ maps into $[-1, 1]$): If $n \geq 2$ is even, then, for each odd $j \leq n - 1$, one has

$$
(\rho_j(\theta))^2 + (\rho_{j+1}(\theta))^2
= (\epsilon_j \cos \theta + \epsilon_{j+1} \sin \theta)^2 + (-\epsilon_j \sin \theta + \epsilon_{j+1} \cos \theta)^2
= \epsilon_j^2 \cos^2 \theta + 2\epsilon_j \epsilon_{j+1} \cos \theta \sin \theta + \epsilon_{j+1}^2 \sin^2 \theta
+ \epsilon_j^2 \sin^2 \theta - 2\epsilon_j \epsilon_{j+1} \cos \theta \sin \theta + \epsilon_{j+1}^2 \cos^2 \theta
= \epsilon_j^2 (\cos^2 \theta + \sin^2 \theta) + \epsilon_{j+1}^2 (\cos^2 \theta + \sin^2 \theta) = \epsilon_j^2 + \epsilon_{j+1}^2,
$$

implying

$$
\forall \theta \in [0, \pi] \quad \|\rho(\theta)\|^2 = \sum_{j=1}^{n} (\rho_j(\theta))^2 = \sum_{j=1}^{n} \epsilon_j^2 = 1. \tag{K.17}
$$

If $n \geq 3$ is odd, then (K.16) still holds for each odd $j \leq n - 4$. Additionally,

$$
(\rho_{n-2}(\theta))^2 + (\rho_{n-1}(\theta))^2 + (\rho_n(\theta))^2
= \epsilon_1^2 \cos^2 \theta + 2\epsilon_1 \sqrt{\epsilon_2^2 + \epsilon_3^2} \sin \theta \cos \theta + (\epsilon_2^2 + \epsilon_3^2) \sin^2 \theta
+ \epsilon_2^2 \cos^2 \theta - 2\frac{\epsilon_1 \epsilon_2}{\sqrt{\epsilon_2^2 + \epsilon_3^2}} \sin \theta \cos \theta + \frac{\epsilon_1^2 \epsilon_2^2}{\epsilon_2^2 + \epsilon_3^2} \sin^2 \theta
+ \epsilon_3^2 \cos^2 \theta - 2\frac{\epsilon_1 \epsilon_3}{\sqrt{\epsilon_2^2 + \epsilon_3^2}} \sin \theta \cos \theta + \frac{\epsilon_1^2 \epsilon_3^2}{\epsilon_2^2 + \epsilon_3^2} \sin^2 \theta
= (\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) \cos^2 \theta
+ \frac{2\epsilon_1 (\epsilon_2^2 + \epsilon_3^2 - \epsilon_2^2 - \epsilon_3^2)}{\sqrt{\epsilon_2^2 + \epsilon_3^2}} \sin \theta \cos \theta
+ \left(\epsilon_2^2 + \epsilon_3^2 + \frac{\epsilon_1^2 (\epsilon_2^2 + \epsilon_3^2)}{\epsilon_2^2 + \epsilon_3^2}\right) \sin^2 \theta
= (\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) (\cos^2 \theta + \sin^2 \theta) = \epsilon_{n-2}^2 + \epsilon_{n-1}^2 + \epsilon_n^2,
$$

i.e. (K.17) is true also for each $n \geq 3$ odd.

Clearly, $D$ is also continuous and, thus, so is $D \circ \rho : [0, \pi] \to [-\alpha, \alpha]$. Moreover, as $\sin(0) = \sin(\pi) = 0$, $\cos(0) = 1$, $\cos(\pi) = -1$, we obtain

$$
\forall n \geq 2 \quad \forall j \in \{1, \ldots, n\} \quad (\rho_j(0) = \epsilon_j \quad \land \quad \rho_j(\pi) = -\epsilon_j), \tag{K.19}
$$

implying

$$
\forall n \geq 2 \quad (\rho(0) = e_{\max} \quad \land \quad (D \circ \rho)(0) = \alpha \quad \land \quad \rho(\pi) = e_{\min} \quad \land \quad (D \circ \rho)(\pi) = -\alpha). \tag{K.20}
$$

The continuity of $D \circ \rho$ and the intermediate value theorem [Phi16, Th. 7.57] imply $D \circ \rho$ to be surjective, i.e. $D$ must be surjective as well.
References


