A Quasistatic Crack Propagation Model Allowing for Cohesive Forces and Crack Reversibility
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Griffith Theory: Drawbacks


Physical / Mathematical Issues: Griffith theory predicts stress singularity at crack tip:

\[ \sigma_{\text{max}} = \frac{2a}{b} \sigma_0 \]  
(load acts entirely on undeformed state),

\[ \sigma_{\text{max}} = \frac{\epsilon}{\sigma_0} \ln \left( \cosh \frac{2\sigma_0}{\epsilon} + \frac{a}{b} \sinh \frac{2\sigma_0}{\epsilon} \right) \]  
(load applied incrementally),

\[ \lim_{b \to 0} \sigma_{\text{max}} = \infty \]  
in both cases.

For \( b \to 0 \), Griffith theory predicts its own failure: The assumptions used in its derivation (elasticity, small deformations) no longer apply. Its predictions become nonphysical.

Limited Scope: Griffith theory can not predict location of crack initiation and crack path.
Cohesive Forces According to Barenblatt and Sinclair


Qualitative properties of the cohesive stress-separation law ($s_e$: equilibrium separation):

Repulsion for $s < s_e$, nearly linear $\sigma_{co}(s) \approx k_e (s - s_e)$ for $s$ close to $s_e$, then $\sigma_{co}$ reaches maximum and gradually decays to 0 for large $s$.

Challenges:
(a) Compute realistic quantitative laws for $\sigma_{co}(s)$ for a given material from quantum mechanics.
(b) Accounting for the nonlinear law $\sigma_{co}(s)$ can render computation of the resulting stress field very difficult.
Predicting Crack Initiation and Path: Francfort-Marigo Theory


**Goal:** Formulate the problem such that the location and path of a crack, in contrast to Griffith theory, does not need to be described a priori, but is part of the problem’s solution.

Mathematical setting:
Let $\Omega \subseteq \mathbb{R}^N$, $N \in \{1, 2, 3\}$, be a body’s uncracked reference configuration.
For each time $t \in [0, T]$, a strained and cracked configuration of the body is described by a displacement function $u : \Omega \longrightarrow \mathbb{R}^N$ together with a crack $\Gamma \subseteq \Omega$.

Example with Dirichlet boundary conditions:
Prescribe $u = u_D$ on some part of the boundary $\partial \Omega$ of $\Omega$. For example $u_{D,1}(t, x) = (0, 0, 0)$, $u_{D,2}(t, x) = (0, -t, 0)$ (see figure).
Quasistatic Energy Minimization

The goal is to determine $u(t, x)$ by quasistatic energy minimization.

**Quasistatic:**

Assumption: There are two, decoupled time scales:

- Slow Time Scale: Variation of boundary conditions and loading.
- Fast Time Scale: The system instantaneously settles into an energy minimum for each time $t$ of the slow time scale.

**Quasistatic Evolution:**

Find $t \mapsto u(t)$ such that $u(t)$ satisfies an energy balance (energy spent in crack increase must equal the work of the external forces) and has minimal energy among all admissible displacement fields $v \in AD(t)$.

The choice for $AD(t)$ is not obvious. Tentative choice:

$$AD(t) := \{ u \in BV(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) : u_D(t) = u \text{ on } \partial_D u \}.$$
Allowing for Reversible Cracks -1-

For each time \( t \in [0, T] \), reversibility is described by a reversibility function

\[
    r : \Omega \longrightarrow \{0, 1\}, \quad r(x) = \begin{cases} 
        1 & \text{if there is an irreversible crack at } x, \\
        0 & \text{otherwise.}
    \end{cases}
\]

Irreversibility is triggered where a crack has opened more than a threshold value \( a_{th} \).

Cracks are now defined in terms of \( u \) and \( r \):

\[
    \Gamma(u, r) := r^{-1}\{1\} \cup \{ x \in J_u : ([u](x)) \cdot n_{J_u}(x) > 0 \}.
\]

Given \( u(t) \), \( r \) can be defined in terms of \( u \) as a memory function:

\[
    r_u(t, x) = \begin{cases} 
        0 & \text{if } ([u](t, x)) \cdot n_{J_u(t)}(x) < a_{th} \text{ for all } s \leq t, \\
        1 & \text{otherwise.}
    \end{cases}
\]
Due to the reversibility function, the formulation of the minimality condition at \( t \) makes use of the function \( u \) already defined for times smaller than \( t \):

Let \( v \in AD(t) \) be an admissible displacement field at time \( t \), and let \( u : [0, t] \rightarrow BV(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) \) be given. Then \( u \) can be extended to \( t \) by \( v \):

\[
\begin{align*}
u_v : [0, t] &\rightarrow BV(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N), \\
u_v(s) := \begin{cases} u(s) & \text{for } s < t, \\
v & \text{for } s = t. \end{cases}
\end{align*}
\]

Thereby, \( v \) also gives rise to a reversibility function \( r^v \):

\[
r^v : [0, t] \rightarrow \{0, 1\}, \quad r^v := r_{u_v}.
\]
Minimality condition at time $t$

$u(t)$ needs to satisfy

(1) $u(t) \in \text{AD}(t)$,

(2) $\mathcal{E}(t)(u_{u(t)}) \leq \mathcal{E}(t)(u_v)$ for each $v \in \text{AD}(t)$,

where the total energy is given by

$\mathcal{E}(t)(u) = \mathcal{E}_b(u) - \mathcal{F}(t)(u) + \mathcal{E}_{cr}(\Gamma(u, r_u))$:

$\mathcal{E}_b(u)$ is the strain energy of the bulk, $\mathcal{E}_b(u) := \int_{\Omega} W(x, \nabla u(x)) \, dx$

with a suitable material function $W : \Omega \times \mathbb{R}^{N^2} \to \mathbb{R}_0^+$;

$\mathcal{F}(t)(u)$ is the energy due to body and surface forces;

$\mathcal{E}_{cr}(\Gamma(u, r_u))$ is the crack energy:

$\mathcal{E}_{cr}(\Gamma) = \int_{\Gamma} \kappa(x, n_{\Gamma}(x), [u](x), r(x)) \, d\mathcal{H}^{N-1}(x)$,

where $\kappa : \Omega \times S^{N-1} \times \mathbb{R}^N \times \{0, 1\} \to \mathbb{R}_0^+ \cup \{\infty\}$ is a material function, describing the material's toughness, $n_{\Gamma}$ is the unit normal vector on the crack.

The dependence on $r(x)$ can account for crack reversibility:

Cohesive forces should play no role once the crack has become irreversible:

$\kappa$ depends nontrivially on the third variable if, and only if, the fourth variable is 0.
Example: Global minimization fails (const. body force, F. & M. 1998, Sec. 5.2):

\[ x = 2 \]
\[ x = 1 \]
\[ x = 0 \]

\[ \Omega := \{ x \in \mathbb{R} : 0 < x < 2 \}, \quad \partial_D \Omega := \{2\}, \]
\[ u_D : [0, T] \rightarrow L^\infty(\partial_D \Omega, \mathbb{R}), \quad u_D(t)(2) := 0, \]
\[ W : \Omega \times \mathbb{R} \rightarrow \mathbb{R}_0^+, \quad W(x, \xi) := \frac{\xi^2}{2}, \]
\[ F : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}, \quad F(t, x, z) := -tz, \]
\[ \kappa : \Omega \times \{-1, 1\} \times \mathbb{R}^N \times \{0, 1\} \rightarrow \mathbb{R}_0^+ \cup \{\infty\}, \]
\[ \kappa(x, n, z, 0) := \begin{cases} \infty & \text{for } z \cdot n \leq -a_{th}, \\ \text{continuous} & \text{for } -a_{th} < z \cdot n \leq a_{th}, \\ \kappa_{th} > 0 & \text{for } a_{th} \leq z \cdot n, \end{cases} \]
\[ \kappa(x, n, z, 1) := \kappa_{th}, \quad a_{th} := 1. \]

Let \( a > 1 \) and consider

\[ u_a : [0, T] \rightarrow BV^\infty(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R}), \quad u_a := \begin{cases} u_a(t, x) := 0 & \text{for } 1 < x < 2, \\ u_a(t, x) := -a & \text{for } 0 < x < 1. \end{cases} \]
Example where global minimization fails (constant body force):

\[ W(x, \xi) := \frac{\xi^2}{2}, \quad F(t, x, z) := -tz, \quad \kappa(x, n, z, 1) := \kappa_{th}, \]

\[ u_a := \begin{cases} 
  u_a(t, x) := 0 & \text{for } 1 < x < 2, \\
  u_a(t, x) := -a & \text{for } 0 < x < 1.
\end{cases} \]

\[ E(t)(u_a) = \int_{\Gamma(u_a, r_{u_a})} \kappa(x, n_{\Gamma(u_a, r_{u_a})}(x), [u_a](x), r_{u_a}(t, x)) \, d\mathcal{H}^0(x) \]
\[ + \int_{\Omega} W(x, \nabla u_a(x)) \, dx - \int_{\Omega} F(t, x, u_a(x)) \, dx \]
\[ = \int_{\{x=1\}} \kappa(x, 1, a, 1) \, d\mathcal{H}^0(x) \]
\[ + \int_{\Omega} (\nabla u_a(t, x))^2 / 2 \, dx - \int_0^1 ta \, dx \]
\[ = \kappa_{th} - ta. \]

Thus, for each \( t > 0 \), one has \( \lim_{a \to \infty} E(t)(u_a) = -\infty \Rightarrow \text{failure for arbitrarily small positive load.} \)
Global Versus Local Energy Minimization -3-

Constant body force with local energy minimization:

\[ W(x, \xi) := \frac{\xi^2}{2}, \quad F(t, x, z) := -t z, \quad a_{th} := 1, \]
\[ \kappa(x, n, z, 1) := \kappa_{th}, \]
\[ \kappa(x, n, z, 0) := \begin{cases} 
\infty & \text{for } z \cdot n \leq -a_{th}, \\
\text{cont. } \kappa_j(z \cdot n) & \text{for } -a_{th} < z \cdot n \leq a_{th}, \\
\kappa_{th} > 0 & \text{for } a_{th} \leq z \cdot n. 
\end{cases} \]

For \( t \in [0, T] \), let \( u_e(t) : \Omega \rightarrow \mathbb{R}, u_e(t) \leq 0 \), be the solution for the “perfectly elastic” limit of the material, i.e. \( u_e(t) \) is the (global) minimizer of

\[ E_e(t)(u) := \int_{\Omega} \frac{1}{2} \left( \nabla u(x) \right) \cdot \left( \nabla u(x) \right) \, dx + \int_{\Omega} t u(x) \, dx. \]

Consider a crack at \( y \in \Omega \): \( u_e(t) + \phi_{a,b,y} \), where \( \phi_{a,b,y}(x) := \begin{cases} 
b & \text{for } y < x < 2, \\
-a & \text{for } 0 < x < y. 
\end{cases} \)
Global Versus Local Energy Minimization -4-

Constant body force with local energy minimization:

\[ u_D(t)(2) := 0, \quad W(x, \xi) := \frac{\xi^2}{2}, \]
\[ F(t, x, z) := -t z, \quad \kappa(x, n, z, 0) := \kappa_j(n z), \]
\[ \phi_{a,b,y}(x) := \begin{cases} 
  b & \text{for } y < x < 2, \\
  -a & \text{for } 0 < x < y,
\end{cases} \quad \Rightarrow \quad b = 0. \]
\[ \phi_{a,y} := \phi_{a,0,y}, \quad \nabla \phi_{a,y} = 0, \quad \nabla (u_e(t) + \phi_{a,y}) = \nabla (u_e(t)). \]
\[ \mathcal{E}(t)(u_e(t) + \phi_{a,y}) = \kappa_j(a) + \int_\Omega \frac{1}{2} \left( \nabla u_e(t, x) \right) \cdot \left( \nabla u_e(t, x) \right) \, dx \]
\[ + \int_\Omega t \left( u_e(t, x) + \phi_{a,y}(x) \right) \, dx \]
\[ = \mathcal{E}_e(t)(u_e(t)) + \kappa_j(a) - ta y. \]

\[ \mathcal{E}(t)(u_e(t)) \] is a local min if, and only if, \( \kappa_j(a) - ta y \) has local min at 0.

Result: At critical \( t > 0 \), crack appears at \( x = 2 \) (the physically expected result).