Mathematical Logic

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Contents

CHAPTER 1

Constructive Mathematics and Classical Mathematics

1.1. The fundamental thesis of constructivism

One of the most fundamental concepts of mathematics is that of a natural number. Maybe the best way to introduce it is though the following *inductive definition* of the set of natural numbers $\mathbb{N}$:

$$\frac{}{0 \in \mathbb{N}}, \quad \frac{n \in \mathbb{N}}{S(n) \in \mathbb{N}},$$

where $S(n)$ is called the *successor* of n. The natural numbers that are generated from these two inductive rules are called the *canonical* elements of $\mathbb{N}$ i.e, a natural number is canonical if and only if it is 0, or the successor of some canonical natural number. E.g.,

$$S(S(S(S(0))))$$

is a canonical natural number. Since there are many sets that satisfy the above rules, an essential complement to this inductive definition is its corresponding *induction principle*:

$$\frac{A(0), \quad \forall_{n \in \mathbb{N}}(A(n) \Rightarrow A(S(n)))}{\forall_{n \in \mathbb{N}}(A(n))},$$

where $A(n)$ is any property on natural numbers. According to this induction principle, if A is a set that satisfies the defining rules of $\mathbb{N}$, i.e., it is a *competitor* set to $\mathbb{N}$, then it has to be "larger" than $\mathbb{N}$. In other words, $\mathbb{N}$ is the *least* set that satisfies its defining rules. We can define the equality $n =_{\mathbb{N}} m$ on natural numbers also inductively, so that the other standard Peano axioms will hold.

In mathematical practice though, we usually work with *representations* of natural numbers rather than with canonical natural numbers. E.g., we write

$$4 \equiv S(S(S(S(0)))),$$

where the symbol

$$\sigma \equiv \tau$$

means that the mathematical expression σ is *by definition* the mathematical expression τ, and that we can *substitute* σ, in any other expression that contains σ, by τ. E.g., the canonical natural number $S(S(S(S(0))))$ is represented by the term 4 and $S(0)$ by the term 1. The mathematical terms 10^2 and 10^{100} are examples of representations of natural numbers, which facilitate the writing of mathematics dramatically. Note that, in principle, it is possible for someone to write down the canonical natural numbers that correspond to 10^2 and 10^{100}.

The question that arises naturally is if all representations of natural numbers are meaningful, and, if not, which of them are going to be accepted as such.

If the property Goldbach(n) on natural numbers is defined by

$$\text{Goldbach}(n) \equiv n \text{ is the sum of two primes},$$

we consider the following representations of natural numbers:

$$n_1 \equiv \begin{cases} 0 & , \forall_{n\in\mathbb{N}}\Big((4 \leq n \leq 10^2 \ \&\ \text{Even}(n)) \Rightarrow \text{Goldbach}(n)\Big) \\ 1 & , \text{otherwise} \end{cases}$$

$$n_2 \equiv \begin{cases} 0 & , \forall_{n\in\mathbb{N}}\Big((4 \leq n \leq 10^{100} \ \&\ \text{Even}(n)) \Rightarrow \text{Goldbach}(n)\Big) \\ 1 & , \text{otherwise} \end{cases}$$

$$n_3 \equiv \begin{cases} 0 & , \forall_{n\in\mathbb{N}}\Big((4 \leq n \ \&\ \text{Even}(n)) \Rightarrow \text{Goldbach}(n)\Big) \\ 1 & , \text{otherwise} \end{cases}$$

With some patience a mathematician can compute n_1, and, possibly with the help of some computing machine, he can, in principle, compute n_2. There is no known finite, purely routine, process to convert n_3 to canonical form. The provability of the formula

$$\forall_{n\in\mathbb{N}}\Big((4 \leq n \ \&\ \text{Even}(n)) \Rightarrow \text{Goldbach}(n)\Big)$$

is known as the *Goldbach conjecture*, which is one of the oldest and best-known unsolved problems in number theory and all of mathematics. The current computing machines have verified the Goldbach conjecture up to 4×10^{18}.

DEFINITION 1.1.1. A representation m of a natural number is called *real*, if it can be converted, *in principle*, to a canonical natural number m^* by a finite, purely routine, process. If m is a real representation of a natural number, we say that m is *constructively defined*. Two constructively defined natural numbers l, m are *equal* if their canonical forms l^*, m^* are equal i.e.,

$$l =_{\mathbb{N}} m \equiv l^* =_{\mathbb{N}} m^*.$$

The, necessarily unique, canonical form of a real representation of a natural number is called the *normal form* of the representation. A representation of a natural number that cannot be accepted as real is called *ideal*.

Fundamental thesis of constructivism for the natural numbers (FTC-$\mathbb{N}$): Only real representations of natural numbers are accepted constructively.

Since the above representation of n_3 is ideal, according to FTC-$\mathbb{N}$, it cannot be accepted constructively. It might be the case though, to find a finite, purely routine, process to convert n_3 to decimal form in the future. As we explain later, there are situations for which such processes are not expected to be found even in the future!

DEFINITION 1.1.2. An *operation from* $\mathbb{N}$ *to* $\mathbb{N}$ is a *rule* R that associates to each canonical number n a canonical number $R(n)$. A representation R of an operation from $\mathbb{N}$ to $\mathbb{N}$ is a rule that associates to each representation of a natural number m a representation of a natural number $R(m)$. A representation R of an operation from $\mathbb{N}$ to $\mathbb{N}$ is called *real*, if it associates to each constructively defined natural number m a constructively defined natural number $R(m)$. A representation of an operation that cannot be accepted as real is called *ideal*.

E.g., the rule $f(n) \equiv SS(n)$, for every $n \in \mathbb{N}$, is an operation from $\mathbb{N}$ to $\mathbb{N}$, the rule

$$g(n) \equiv n^{10}, \quad n \in \mathbb{N},$$

is a real operation from $\mathbb{N}$ to $\mathbb{N}$, and the rule i.e., rules that define such functions using representations of natural numbers. E.g., we define

$$h(n) \equiv n_3, \quad n \in \mathbb{N}$$

is an ideal operation from $\mathbb{N}$ to $\mathbb{N}$.

Fundamental thesis of constructivism for operations from $\mathbb{N}$ to $\mathbb{N}$: Only real representations of operations from $\mathbb{N}$ to $\mathbb{N}$ are accepted constructively.

Definition 1.1.3. A *function* $f : \mathbb{N} \to \mathbb{N}$ from $\mathbb{N}$ to $\mathbb{N}$ is a real operation from $\mathbb{N}$ to $\mathbb{N}$, which is *extensional* i.e., it satisfies the condition

$$n =_{\mathbb{N}} m \Rightarrow f(n) =_{\mathbb{N}} f(m),$$

for every constructively defined natural numbers n, m. We denote by $\mathbb{F}(\mathbb{N}, \mathbb{N})$ the set of functions from $\mathbb{N}$ to $\mathbb{N}$. If $f, g \in \mathbb{F}(\mathbb{N}, \mathbb{N})$ we define

$$f =_{\mathbb{F}(\mathbb{N},\mathbb{N})} g \equiv \forall_{n\in\mathbb{N}}(f(n) =_{\mathbb{N}} g(n)).$$

For simplicity we just write $f = g$.

Note that the extensional property of function f from $\mathbb{N}$ to $\mathbb{N}$ relies on the fact that f is a real operation from $\mathbb{N}$ to $\mathbb{N}$.

We can define the set of integers $\mathbb{Z}$ with its equality $=_{\mathbb{Z}}$, and the set of rational numbers $\mathbb{Q}$ with its equality $=_{\mathbb{Q}}$ in the standard manner. The notion of a real representation of an integer is similar to that of a real representation of a natural number and the fundamental thesis of constructivism for integers (FTC-$\mathbb{Z}$) is formulated similarly to FTC-$\mathbb{N}$.

Definition 1.1.4. A representation of a rational number is *real*, if it can be converted, *in principle*, to the normal form $\frac{k}{l}$, where k, l are canonical integers and $l \neq 0$, by a finite, purely routine, process. If p is a rational number with a real representation, we say that p is *constructively defined*. Two constructively defined rational numbers p, q are *equal* ($p =_{\mathbb{Q}} q$), if their normal forms are equal. A representation of a rational number that cannot be accepted as real is called *ideal*.

The fundamental thesis of constructivism for rationals (FTC-$\mathbb{Q}$) is formulated in a way similar to FTC-$\mathbb{N}$. If $p, q \in \mathbb{Q}$, the absolute value $|q|$, the operations $p + q, p - q$, and the relations $p < q$ and $p \leq q$ are defined as usual.

The operations from $\mathbb{N}$ to $\mathbb{Z}$, or to $\mathbb{Q}$, the operations from $\mathbb{Z}$ to $\mathbb{Z}$ or $\mathbb{Q}$, and the operations from $\mathbb{Q}$ to $\mathbb{Q}$ are defined as in Definition 1.1.2. The fundamental thesis of constructivism for operations from $\mathbb{N}$ to $\mathbb{N}$ is extended to all these operations. The sets of functions $\mathbb{F}(\mathbb{N}, \mathbb{Z})$, $\mathbb{F}(\mathbb{N}, \mathbb{Q})$, $\mathbb{F}(\mathbb{Z}, \mathbb{Z})$, $\mathbb{F}(\mathbb{Z}, \mathbb{Q})$, $\mathbb{F}(\mathbb{Q}, \mathbb{Q})$ with their corresponding equalities are defined as in Definition 1.1.3. The functions in $\mathbb{F}(\mathbb{N}, \mathbb{Q})$ from $\mathbb{N}$ to $\mathbb{Q}$ are called *sequences* of rationals, while if

$$\mathbb{N}^+ \equiv \{n \in \mathbb{N} \mid n \geq 1\}$$

and its equality is "inherited" from the equality of $\mathbb{N}$, we call the functions in $\mathbb{F}(\mathbb{N}^+, \mathbb{Q})$ from $\mathbb{N}^+$ to $\mathbb{Q}$ *strict sequences* of rationals.

Next we start fixing some fundamental *logical notions*, although we have used many of them already.

Definition 1.1.5. If A is a mathematical formula and X is one of $\mathbb{N}, \mathbb{N}^+, \mathbb{Z}, \mathbb{Q}$, the *universal* formula "for all x in X, A" is denoted by $\forall_{x \in X} A$. A proof of $\forall_{x \in X} A$ is a method that generates a proof of A, for every x in X. If A, B are formulas, their *implication* "if A, then B" is denoted by $A \Rightarrow B$. A proof of $A \Rightarrow B$ is a method that generates a proof of B, given a proof of A.

Although $\mathbb{Q}$ satisfies all expected properties e.g., with respect to its orderings $\leq, <$, things change when we treat the real numbers constructively. The reason behind this different behavior is that the rationals are simple, or *finite*, objects, while the reals are *infinite* objects.

1.2. The reals under the fundamental thesis of constructivism

A real number is defined constructively as a special Cauchy sequence of rational numbers.

Definition 1.2.1. A *real number* x is a strict sequence $x : \mathbb{N}^+ \to \mathbb{Q}$ of rationals, which is *regular* i.e.,

$$\forall_{n,m \in \mathbb{N}^+} \left(|x_m - x_n| \leq \frac{1}{m} + \frac{1}{n} \right).$$

We also denote a real number x by $(x_n)_{n \in \mathbb{N}^+}$, and we write $x \equiv (x_n)_{n \in \mathbb{N}^+}$. If $x \equiv (x_n)_{n \in \mathbb{N}^+}$ is a real number, we call x_n the *n-th rational approximation* to x, for every $n \in \mathbb{N}^+$. The set of real numbers is denoted by $\mathbb{R}$. If $x \equiv (x_n)_{n \in \mathbb{N}^+}$ and $y \equiv (y_n)_{n \in \mathbb{N}^+}$ are reals, we define their equality by

$$x =_{\mathbb{R}} y \equiv \forall_{n \in \mathbb{N}^+} \left(|x_n - y_n| \leq \frac{2}{n} \right).$$

Let X be one of the sets $\mathbb{N}, \mathbb{N}^+, \mathbb{Z}, \mathbb{Q}$. A function $f : X \to \mathbb{R}$ is a real operation from X to $\mathbb{R}$ i.e., it associates to every constructively defined element a of X a real number $f(a)$, which is also extensional i.e., for every constructively defined elements a, b of X we have that

$$a =_X b \Rightarrow f(a) =_{\mathbb{R}} f(b).$$

A function $g : \mathbb{R} \to X$, is a real operation from $\mathbb{R}$ to X i.e., it associates to each real number x a constructively defined element $g(x)$ of X, which is also extensional i.e., for every $x, y \in \mathbb{R}$

$$x =_{\mathbb{R}} y \Rightarrow g(x) =_X g(y).$$

The definition of equality of the elements of the sets $\mathbb{F}(X,\mathbb{R}), \mathbb{F}(\mathbb{R},X)$ of functions from X to $\mathbb{R}$ and of functions from $\mathbb{R}$ to X, respectively, is similar to Definition 1.1.3.

Corollary 1.2.2. *If $p \in \mathbb{Q}$, let $p^* : \mathbb{N}^+ \to \mathbb{Q}$ be defined by $p^*(n) \equiv p$, for every $n \in \mathbb{N}$. The following hold:*
(i) $p^* \in \mathbb{R}$.
(ii) The rule * that sends a rational p to p^* is a function from $\mathbb{Q}$ to $\mathbb{R}$.
(iii) The function $^* : \mathbb{Q} \to \mathbb{R}$ is an injection i.e., $p^* =_{\mathbb{R}} q^* \Rightarrow p =_{\mathbb{Q}} q$, for every $p, q \in \mathbb{Q}$.

Proof. Exercise. □

Because of Corollary 1.2.2, we identify p with p^*.

Definition 1.2.3. If A is a mathematical formula and X is one of the sets $\mathbb{N}, \mathbb{N}^+, \mathbb{Z}, \mathbb{Q}$, the *existential* formula "there exists x in X, such that A" is denoted by $\exists_{x\in X} A$. A proof of $\exists_{x\in X} A$ is a method that generates an element x of X and a proof of A for that x.

The *conjunction* of two mathematical formulas A, B is denoted by $A \wedge B$ i.e., we read $A \wedge B$ as "A and B". A proof of $A \wedge B$ is a proof of A and a proof of B. The *equivalence* of two mathematical formulas A, B is denoted by $A \Leftrightarrow B$ i.e., we read $A \Leftrightarrow B$ as "A if and only if B". The equivalence $A \Leftrightarrow B$ is defined as the conjunction $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

According to the next lemma, the equality of two reals means that their rational approximations are *eventually* arbitrarily close.

Lemma 1.2.4. *If $x \equiv (x_n)_{n\in\mathbb{N}^+}$ and $y \equiv (y_n)_{n\in\mathbb{N}^+}$, then*

$$x =_{\mathbb{R}} y \Leftrightarrow \forall_{k\in\mathbb{N}^+}\exists_{E_k\in\mathbb{N}^+}\forall_{n\geq E_k}\left(|x_n - y_n| \leq \frac{1}{k}\right).$$

Proof. Suppose first that $x =_{\mathbb{R}} y$ and fix $k \in \mathbb{N}^+$. We need to find $E_k \in \mathbb{N}^+$ such that for every $n \geq E_k$ we have that $|x_n - y_n|(\leq \frac{2}{n}) \leq \frac{1}{k}$. We take $E_k \equiv 2k$. For the converse we fix $n \in \mathbb{N}^+$ and $l \in \mathbb{N}^+$. Let $m \in \mathbb{N}^+$ such that $m > \max\{l, E_l\}$. Hence

$$\begin{aligned}
|x_n - y_n| &\leq |x_n - x_m| + |x_m - y_m| + |y_m - y_n| \\
&\leq \left(\frac{1}{n} + \frac{1}{m}\right) + \frac{1}{l} + \left(\frac{1}{m} + \frac{1}{n}\right) \\
&< \frac{1}{n} + \frac{1}{l} + \frac{1}{l} + \frac{1}{l} + \frac{1}{n}
\end{aligned}$$

$$= \frac{2}{n} + \frac{3}{l}.$$

Since l is arbitrary, we get that $|x_n - y_n| \leq \frac{2}{n}$. □

PROPOSITION 1.2.5. *The relation $=_{\mathbb{R}}$ is an equivalence relation on $\mathbb{R}$.*

PROOF. Exercise. □

DEFINITION 1.2.6. The *Royden number* is the sequence $\varrho : \mathbb{N}^+ \to \mathbb{Q}$, where, for every $n \in \mathbb{N}^+$,

$$\varrho_n \equiv \sum_{k=1}^{n} \frac{a_k}{2^k},$$

$$a_k = \begin{cases} 0 & , \forall_{n \in \mathbb{N}} \Big((4 \leq n \leq k \ \& \ \mathrm{Even}(n)) \Rightarrow \mathrm{Goldbach}(n) \Big) \\ 1 & , \text{otherwise} \end{cases}$$

Note that each a_k is a constructively defined natural number, and each n-th rational approximation ϱ_n to ϱ is a constructively defined rational.

PROPOSITION 1.2.7. *The Royden number ϱ is a real number.*

PROOF. Exercise. □

In the previous proof we do not need to calculate the value of some a_k, we only use that a_k is in $\{0, 1\}$. We can also write the Royden number as

$$\varrho \equiv \sum_{k=1}^{\infty} \frac{a_k}{2^k}.$$

DEFINITION 1.2.8. If $x \equiv (x_n)_{n \in \mathbb{N}^+}$ is a real number, we define

$$x \text{ is strictly positive } \equiv \exists_{n \in \mathbb{N}^+} \left(x_n > \frac{1}{n} \right),$$

$$x \text{ is positive } \equiv \forall_{n \in \mathbb{N}^+} \left(x_n \geq -\frac{1}{n} \right),$$

Let $\mathbb{R}^+$ and $\mathbb{R}^{\pm}$ be the sets of strictly positive and positive reals, respectively.

The next proposition expresses the fact that "x is strictly positive" means that its rational approximations are eventually above *some* $\frac{1}{N}$, and the fact that "x is positive" means that its rational approximations are eventually above *every* $-\frac{1}{k}$.

PROPOSITION 1.2.9. *If $x \equiv (x_n)_{n \in \mathbb{N}^+}$ is a real number, then*

$$x \in \mathbb{R}^+ \Leftrightarrow \exists_{N \in \mathbb{N}^+} \forall_{m \geq N} \left(x_m \geq \frac{1}{N} \right),$$

$$x \in \mathbb{R}^{\pm} \Leftrightarrow \forall_{k \in \mathbb{N}^+} \exists_{P_k \in \mathbb{N}^+} \forall_{m \geq P_k} \left(x_m \geq -\frac{1}{k} \right).$$

PROOF. Exercise. $\square$

PROPOSITION 1.2.10. *Let $x, y \in \mathbb{R}$ such that $x =_{\mathbb{R}} y$.*
(i) If $x \in \mathbb{R}^+$, then $y \in \mathbb{R}^+$.
(ii) If $x \in \mathbb{R}^{\pm}$, then $y \in \mathbb{R}^{\pm}$.

PROOF. (i) Let $N \in \mathbb{N}^+$ such that $\forall_{m \geq N}(x_m \geq \frac{1}{N})$. If $N' = 4N$ and $m \geq N'$, then $|x_m - y_m| \leq \frac{2}{m} \leq \frac{2}{4N} = \frac{1}{2N}$. Hence

$$\begin{aligned} y_m &\geq x_m - \frac{1}{2N} \\ &= (x_m - \frac{1}{N}) + \frac{1}{2N} \\ &\geq 0 + \frac{1}{2N} \\ &\geq \frac{1}{4N} \\ &\equiv \frac{1}{N'}. \end{aligned}$$

(ii) Since $x \in \mathbb{R}^{\pm}$, by Proposition 1.2.9 we get $\forall_{k \in \mathbb{N}^+} \exists_{P_k \in \mathbb{N}^+} \forall_{m \geq P_k}(x_m \geq -\frac{1}{k})$. Since $x =_{\mathbb{R}} y$, by Lemma 1.2.4 we get $\forall_{k \in \mathbb{N}^+} \exists_{E_k \in \mathbb{N}^+} \forall_{n \geq E_k}(|x_n - y_n| \leq \frac{1}{k})$. If $k \in \mathbb{N}^+$ and ${P_k}' \equiv \max\{P_{2k}, E_{2k}\}$, then for every $m \geq {P_k}'$ we get

$$\begin{aligned} y_m &= (y_m - x_m) + x_m \\ &\geq (-\frac{1}{2k}) + (-\frac{1}{2k}) \\ &= -\frac{1}{k}. \end{aligned}$$

$\square$

DEFINITION 1.2.11. If $x \equiv (x_n)_{n \in \mathbb{N}^+}$ and $y \equiv (y_n)_{n \in \mathbb{N}^+}$ are in $\mathbb{R}$, we define

$$x + y \equiv (x_{2n} + y_{2n})_{n \in \mathbb{N}^+},$$

$$x \vee y \equiv \max\{x, y\} \equiv (\max\{x_n, y_n\})_{n \in \mathbb{N}^+},$$
$$-x \equiv (-x_n)_{n \in \mathbb{N}^+},$$
$$x - y \equiv x + (-y)$$
$$x \wedge y \equiv \min\{x, y\} \equiv -\max\{-x, -y\},$$
$$|x| \equiv x \vee (-x).$$

It is immediate to see that the above sequences $x + y, x \vee y, -x$ are real numbers, and, consequently, $x - y, x \wedge y$ and $|x|$ are also real numbers. It is easy to show that these operations on reals preserve the equality of $\mathbb{R}$, therefore these operations are functions, and that the embedding $^* : \mathbb{Q} \to \mathbb{R}$ preserves the corresponding algebraic structure of $\mathbb{Q}$.

DEFINITION 1.2.12. If $x, y \in \mathbb{R}$, we define

$$x < y \ (y > x) \equiv y - x \in \mathbb{R}^+,$$
$$x \leq y \ (y \geq x) \equiv y - x \in \mathbb{R}^{\pm},$$

It is immediate to see that the embedding $^* : \mathbb{Q} \to \mathbb{R}$ preserves the order structure of $\mathbb{Q}$.

PROPOSITION 1.2.13. *If* $x \equiv (x_n)_{n \in \mathbb{N}^+} \in \mathbb{R}$, *then*

$$x \in \mathbb{R}^+ \Leftrightarrow x > 0,$$
$$x \in \mathbb{R}^{\pm} \Leftrightarrow x \geq 0.$$

PROOF. It is easy to see that $x' \equiv (x_{2n})_{n \in \mathbb{N}^+}$ is also in $\mathbb{R}$ and $x =_{\mathbb{R}} x'$. By Proposition 1.2.10 and Definition 1.2.11 we have that

$$\begin{aligned} x > 0 &\equiv x - 0 \in \mathbb{R}^+ \\ &\equiv x + (-0) \in \mathbb{R}^+ \\ &\equiv x' \in \mathbb{R}^+ \\ &\Leftrightarrow x \in \mathbb{R}^+. \end{aligned}$$

□

Next we gather without proof some properties of $(\mathbb{R}, +, <, \leq)$.

PROPOSITION 1.2.14. *Let* $x, y, z, w \in \mathbb{R}$.

(i) If $x, y > 0$, *then* $x + y > 0$.
(ii) If $x > 0$ *and* $y \geq 0$, *then* $x + y > 0$.
(iii) $|x| \geq 0$.
(iv) If $x > 0$, *then* $x \vee y > 0$.
(v) If $x > 0$ *and* $y > 0$, *then* $x \wedge y > 0$.

(vi) If $x < y$ and $y < z$, then $x < z$.
(vii) If $x \leq z$ and $y \leq w$, then $x + y \leq z + w$.
(viii) If $x < y$, then $-x > -y$.
(ix) $x \leq x \vee y$.
(x) $x \wedge y \leq x$.
(xi) $|x + y| \leq |x| + |y|$.

Proof. Left to the reader. □

Definition 1.2.15. The *disjunction* of two mathematical formulas A, B is denoted by $A \vee B$ i.e., we read $A \vee B$ as "A or B". A proof of $A \vee B$ is a proof of A, or a proof of B. The *negation* of a A is denoted by $\neg A$ i.e., we read $\neg A$ as "not A". A proof of $\neg A$ is a proof of a contradiction, like $0 =_{\mathbb{N}} 1$, given a proof of A.

Proposition 1.2.16. *If $x, y \in \mathbb{R}$ such that $x+y > 0$, then $x > 0 \vee y > 0$.*

Proof. Since $x + y \equiv (x_{2n} + y_{2n})_{n \in \mathbb{N}^+} > 0 \Leftrightarrow x + y \in \mathbb{R}^+$, there is some $n \in \mathbb{N}^+$ such that $x_{2n} + y_{2n} > \frac{1}{n}$. Since $x_{2n}, y_{2n}, \frac{1}{n} \in \mathbb{Q}$, if $x_{2n} \leq \frac{1}{2n}$ and $y_{2n} \leq \frac{1}{2n}$, we would have $x_{2n} + y_{2n} \leq \frac{1}{n}$, which contradicts our hypothesis. Hence $x_{2n} > \frac{1}{2n}$ or $y_{2n} > \frac{1}{2n}$. In the first case we get $x \in \mathbb{R}^+ \Leftrightarrow x > 0$, and similarly in the second we get $y > 0$. □

Note that in the previous proof we use the fact that if $p, q, r \in \mathbb{Q}$, then $\neg(p \leq r \wedge q \leq r) \Rightarrow p > r \vee q > r$. As we explain later, this property cannot be accepted for the order of reals, but since the rationals are finite objects, their ordering is *decidable*. In connection to the proposition that follows, we see that the "logic" of the mathematical objects under study depends on the objects themselves, and especially on their finite or infinite character.

Proposition 1.2.17. *For the Royden number ϱ the following hold.*
(i) $\varrho \geq 0$.
(ii) If there is a proof of the disjunction

$$\varrho > 0 \ \vee \ \varrho = 0,$$

then the Goldbach conjecture is decided i.e., there is a proof of the Goldbach conjecture or a proof of the negation of the Goldbach conjecture.

Proof. Exercise. □

The above result explains why we gave a separate definition of $x \geq 0$ and didn't define it as the disjunction $x > 0 \vee x = 0$.

Proposition 1.2.18. *If $x \in \mathbb{R}$, such that $x \geq 0$ and $x \leq 0$, then $x = 0$.*

Proof. Since $x \leq 0 \equiv (0 - x) \in \mathbb{R}^{\pm} \equiv (-x_{2n})_{n \in \mathbb{N}^+} \in \mathbb{R}^{\pm}$, we get $\forall_{n \in \mathbb{N}^+}(-x_{2n} \geq -\frac{1}{n})$. Since $x \geq 0 \equiv (x - 0) \in \mathbb{R}^{\pm} \equiv (x_{2n})_{n \in \mathbb{N}^+} \in \mathbb{R}^{\pm}$, we get $\forall_{n \in \mathbb{N}^+}(x_{2n} \geq -\frac{1}{n})$. Hence $\forall_{n \in \mathbb{N}^+}(\frac{1}{n} \leq x_{2n} \leq \frac{1}{n})$ i.e., $\forall_{n \in \mathbb{N}^+}(|x_{2n}| \leq \frac{1}{n} \leq \frac{2}{n})$, and $x' \equiv (x_{2n})_{n \in \mathbb{N}^+} =_{\mathbb{R}} 0$. Since $x =_{\mathbb{R}} x'$, we also get $x =_{\mathbb{R}} 0$. $\square$

Consequently, $(x \leq y \wedge y \leq x) \Rightarrow x =_{\mathbb{R}} y$, for every $x, y \in \mathbb{R}$.

Corollary 1.2.19. *If there is a proof of $\varrho \leq 0$, where ϱ is the Royden number, then there is a proof of the Goldbach conjecture.*

Proof. Suppose that there is a proof that $\varrho \leq 0$. Since by Proposition 1.2.17(i) we have that $\rho \geq 0$, by Proposition 1.2.18 we get $\rho = 0$, and we use Proposition 1.2.17(ii). $\square$

Note that, although we don't have a proof of $\varrho \leq 0$, we cannot prove that $\varrho > 0$, a fact which indicates that we cannot accept the classical property

$$\neg(x \leq 0) \Rightarrow x > 0.$$

Next we use the notation $A \vee B \vee C \equiv A \vee (B \vee C)$.

Corollary 1.2.20. *If there is a proof of the disjunction*

$$\varrho > 0 \ \vee \ \varrho = 0 \ \vee \varrho < 0,$$

then the Goldbach conjecture is decided.

Proof. If $\varrho < 0$, then $\varrho < 0 \leq \varrho$, which is absurd. The remaining disjunction is Proposition 1.2.17(ii). $\square$

Although constructively the classical trichotomy

$$x < y \ \vee \ x = y \ \vee x > y$$

cannot be accepted, the following property is its constructive alternative.

Proposition 1.2.21. *If $x, y, z \in \mathbb{R}$ such that $x < y$, then*

$$x < z \ \vee \ z < y.$$

Proof. Since $0 < y - x = (y - z) + (z - x)$, by Proposition 1.2.16 we have that $y > z$ or $z > x$. $\square$

Proposition 1.2.22. *If $x \in \mathbb{R}$, then*

$$\neg(x < 0) \Leftrightarrow x \geq 0.$$

Proof. ($\Rightarrow$) We show that $\forall_{n\in\mathbb{N}^+}(x_n \geq -\frac{1}{n})$. If $n \in \mathbb{N}^+$ such that $x_n < -\frac{1}{n}$, then $-x_n > \frac{1}{n}$, hence $-x > 0$, and consequently $x < 0$. By our hypothesis $\neg(x < 0)$ we get a contradiction. Hence, necessarily $x_n \geq -\frac{1}{n}$.
($\Leftarrow$) Suppose that $x < 0$, hence by Proposition 1.2.14(viii) $-x > 0$, which by definition means that there is some $n \in \mathbb{N}^+$ such that $-x_n > \frac{1}{n} \Leftrightarrow x_n < -\frac{1}{n}$. Since $x \geq 0 \Leftrightarrow \forall_{n\in\mathbb{N}^+}(x_n \geq -\frac{1}{n})$, for this n we get $-\frac{1}{n} \leq x_n < -\frac{1}{n}$, which is a contradiction. □

Consequently, $\neg(x < y) \Leftrightarrow x \geq y$, for every $x, y \in \mathbb{R}$.

Proposition 1.2.23. *Let $x, y \in \mathbb{R}$.*
(i) If $x < x \vee y$, then $x \leq y$.
(ii) If $x < x \vee y$, then $x \vee y = y$.
(iii) $|x| > 0 \Rightarrow x > 0 \vee x < 0$.

Proof. (i) Suppose that $x > y$. It is easy to show that in this case $x \vee y = x$, which contradicts the hypothesis $x < x \vee y$. Hence $x \leq y$.
(ii) Clearly, $y \leq x \vee y$. Suppose that $y < x \vee y$. By case (i) we get $y \leq x$. By case (i) we also have that $x \leq y$, hence $x = y = x \vee y$, which contradicts the hypothesis $x < x \vee y$. Hence, $y \geq x \vee y$, which together with $y \leq x \vee y$ imply that $y = x \vee y$.
(iii) By Proposition 1.2.21 we have that $0 < x$, or $x < |x|$. In the first case we get automatically the required disjunction. In the second case we have that $x < |x| \equiv x \vee (-x)$, hence by case (ii) we get $|x| = -x$, therefore $-x > 0 \Leftrightarrow x < 0$. □

The converse implication to Proposition 1.2.23(iii) holds trivially. The next concept is a notion of inequality between real numbers, which is defined positively i.e., without using negation.

Definition 1.2.24. If $x, y \in \mathbb{R}$ we define

$$x \bowtie y \equiv |x - y| > 0,$$

and we read $x \bowtie y$ as "*x is apart from y*".

By Proposition 1.2.23(iii) we have that

$$x \bowtie y \Leftrightarrow x > y \ \vee \ x < y.$$

Proposition 1.2.25. *Let $x, y, z \in \mathbb{R}$.*
(i) $x \bowtie y \Rightarrow \neg(x =_{\mathbb{R}} y)$.
(ii) $\neg(x \bowtie x)$.
(iii) $x \bowtie y \Rightarrow y \bowtie x$.

(iv) $x \bowtie y \Rightarrow x \bowtie z \vee z \bowtie y$.
(v) $\neg(x \bowtie y) \Rightarrow x =_{\mathbb{R}} y$.

PROOF. Left to the reader. □

Regarding the converse to Proposition 1.2.25(i), note that although we cannot prove $\varrho = 0$, we cannot also prove that $\varrho \bowtie 0$.

DEFINITION 1.2.26. If $x \equiv (x_n)_{n\in\mathbb{N}^+}, y \equiv (y_n)_{n\in\mathbb{N}^+} \in \mathbb{R}$, their *multiplication* $x \cdot y$, or simpler xy, is defined by

$$x \cdot y \equiv (x_{2kn} \cdot y_{2kn})_{n\in\mathbb{N}^+},$$
$$k \equiv \max\{k_x, k_y\},$$

where, for every $x \in \mathbb{R}$, k_x is the least natural number, which is larger than $|x_1| + 2$, and it is called the *canonical bound* of x.

Note that since

$$|x_n| \leq |x_n - x_1| + |x_1| \leq \frac{1}{n} + 1 + |x_1| \leq 2 + |x_1|,$$

we conclude that k_x is a bound of the approximations of x i.e.,

$$\forall_{n\in\mathbb{N}^+}(|x_n| < k_x).$$

It is easy to see that $x \cdot y \in \mathbb{R}$ and that the operation of multiplication is a function. Next follows a positive definition of an irrational real number.

DEFINITION 1.2.27. A real number x is called *irrational*, if

$$\forall_{p\in\mathbb{Q}}(x \bowtie p).$$

We denote the set of irrational numbers by $\mathbb{I}r$.

PROPOSITION 1.2.28. *Let $x : \mathbb{N}^+ \to \mathbb{Q}$ be defined recursively by*

$$x_1 \equiv 1,$$
$$x_{n+1} \equiv \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

If $n \in \mathbb{N}^+$, the following hold:
(i) $x_n > 0$.
(ii) $x_{n+1}^2 \geq 2$.
(iii) $x_{n+2} \leq x_{n+1}$.
(iv) $x \in \mathbb{R}$.
(v) $x^2 =_{\mathbb{R}} 2$.

PROOF. Left to the reader. □

We denote the real number of the previous proposition by $\sqrt{2}$.

LEMMA 1.2.29. *If $k, l \in \mathbb{Z}$, such that $l \neq 0$, then*

$$\neg\left(\frac{k^2}{l^2} =_{\mathbb{Q}} 2\right).$$

PROOF. Without loss of generality k, l are natural numbers, which are not both of them even (why?). If $k^2 = 2l^2$, then k^2 is even, therefore k is even. Let $k = 2m$, for some $m \in \mathbb{N}^+$. Since then $k^2 = 4m^2 = 2l^2$, we get $l^2 = 2m^2$, hence l^2, and therefore l are even, which contradicts our hypothesis. $\square$

PROPOSITION 1.2.30. $\sqrt{2} \in \mathbb{I}r$.

PROOF. We show that if $\frac{k}{l} \in \mathbb{Q}$, then $\sqrt{2} \bowtie \frac{k}{l}$. Since $x_2 \equiv \frac{1}{2}(1 + \frac{2}{1}) = \frac{3}{2} > \frac{1}{2}$, we get that $\sqrt{2} > 0$. Hence, if $\frac{k}{l} < 0$, we have that $\frac{k}{l} < \sqrt{2}$. Clearly, $\sqrt{2} \leq 2$, since if $\sqrt{2} > 2$, then $2 > 4$, which is absurd. Hence, if $\frac{k}{l} > 2$, we get $\frac{k}{l} > 2 \geq \sqrt{2}$, hence $\frac{k}{l} > \sqrt{2}$. Suppose next that $0 \leq \frac{k}{l} \leq 2$. By Lemma 1.2.29 we have that $|k^2 - 2l^2| \geq 1$, hence

$$\begin{aligned}
\left|\frac{k^2}{l^2} - 2\right| &= \left|\frac{k}{l} - \sqrt{2}\right|\left|\frac{k}{l} + \sqrt{2}\right| \\
&= \left|\frac{k - \sqrt{2}l}{l}\right|\left|\frac{k + \sqrt{2}l}{l}\right| \\
&= \left|k^2 - 2l^2\right|\frac{1}{l^2} \\
&\geq \frac{1}{l^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\frac{k}{l} - \sqrt{2}\right| &= \frac{|\frac{k}{l} + \sqrt{2}|}{|\frac{k}{l} + \sqrt{2}|}\left|\frac{k}{l} - \sqrt{2}\right| \\
&\geq \frac{1}{|\frac{k}{l} + \sqrt{2}|}\frac{1}{l^2} \\
&\geq \left(\frac{1}{2+2}\right)\frac{1}{l^2} \\
&= \frac{1}{4l^2} \\
&> 0.
\end{aligned}$$

$\square$

PROPOSITION 1.2.31. *Let ϱ be the Royden number. If there is a proof of the disjunction*

$$\varrho \in \mathbb{Q} \ \vee \ \varrho \in \mathbb{I}r,$$

then the Goldbach conjecture is decided.

PROOF. If $\varrho \in \mathbb{Q}$, then, since $\varrho \geq 0$, either $\varrho > 0$, or $\varrho = 0$. By Proposition 1.2.17(ii) the Goldbach conjecture is in this case decided. If $\varrho \in \mathbb{I}r$, then $\varrho \bowtie 0$, which implies that $\varrho > 0$, and we use again Proposition 1.2.17(ii). $\square$

Hence, although $\mathbb{Q} \cup \mathbb{I}r \subseteq \mathbb{R}$, we cannot show that $\mathbb{R} \subseteq \mathbb{Q} \cup \mathbb{I}r$. Next we show that larger approximations of x are closer to x.

LEMMA 1.2.32. *If $x \equiv (x_n)_{n\in\mathbb{N}^+} \in \mathbb{R}$, then*

$$\forall_{n\in\mathbb{N}^+}\left(|x - x_n| \leq \frac{1}{n}\right).$$

PROOF. If $m \in \mathbb{N}^+$, then unfolding the identifications made in the formulation of the formula we want to show we get

$$\begin{aligned}
|x - x_m| \leq \frac{1}{m} &\equiv |x - (x_m)^*| \leq \left(\frac{1}{m}\right)^* \\
&\equiv [x - (x_m)^*] \vee [-(x - (x_m)^*)] \leq \left(\frac{1}{m}\right)^* \\
&\equiv (x_{2n} - x_m)_{n\in\mathbb{N}^+} \vee (x_m - x_{2n})_{n\in\mathbb{N}^+} \leq \left(\frac{1}{m}\right)^* \\
&\equiv \big((x_{2n} - x_m) \vee (x_m - x_{2n})\big)_{n\in\mathbb{N}^+} \leq \left(\frac{1}{m}\right)^* \\
&\equiv (|x_{2n} - x_m|)_{n\in\mathbb{N}^+} \leq \left(\frac{1}{m}\right)^*.
\end{aligned}$$

Since $|x_{2n} - x_m| \leq \frac{1}{2n} + \frac{1}{m}$, we have that

$$\begin{aligned}
\frac{1}{m} - |x_{2n} - x_m| &\geq \frac{1}{m} - \left(\frac{1}{2n} + \frac{1}{m}\right) \\
&= -\frac{1}{2n} \\
&\geq -\frac{1}{n},
\end{aligned}$$

and we use Definition 1.2.8 to conclude that $\frac{1}{m} \geq |x - x_m|$.

$\square$

The next proposition expresses the density of $\mathbb{Q}$ in $\mathbb{R}$.

PROPOSITION 1.2.33. *If* $x \equiv (x_n)_{n\in\mathbb{N}^+}, y \equiv (y_n)_{n\in\mathbb{N}^+} \in \mathbb{R}$ *such that* $x < y$, *there exists* $q \in \mathbb{Q}$ *such that*

$$x < q < y \equiv x < q \ \wedge \ q < y.$$

PROOF. Since $y > x$, there is some $n \in \mathbb{N}^+$ such that $(y - x)_n > \frac{1}{n} \equiv y_{2n} - x_{2n} > \frac{1}{n}$, hence $y_{2n} - x_{2n} = \frac{1}{n} + \sigma$, for some $\sigma \in \mathbb{Q}$ such that $\sigma > 0$. The midpoint of x_{2n}, y_{2n} is going to be $<$-between x and y. Let

$$q \equiv \frac{x_{2n} + y_{2n}}{2}.$$

By Lemma 1.2.32 we have that $x_{2n} - x \geq -\frac{1}{2n}$, hence $x_{2n} - x = -\frac{1}{2n} + \tau$, for some $\tau \in \mathbb{Q}$ such that $\tau \geq 0$. Using the obvious identifications and Proposition 1.2.14(ii) we have that

$$\begin{aligned} q - x &= \frac{x_{2n} + y_{2n}}{2} - x \\ &= x_{2n} - \frac{x_{2n}}{2} + \frac{y_{2n}}{2} - x \\ &= (x_{2n} - x) + \frac{1}{2}(y_{2n} - x_{2n}) \\ &= -\frac{1}{2n} + \tau + \frac{1}{2}\left(\frac{1}{n} + \sigma\right) \\ &= \tau + \frac{\sigma}{2} \\ &> 0. \end{aligned}$$

Similarly we show that $y - q > 0$. $\square$

PROPOSITION 1.2.34. *Let* $x, y, z \in \mathbb{R}$.
(i) If $x < y$ *and* $z < 0$, *then* $zx > zy$.
(ii) If $x, y > 0$, *then* $xy > 0$.
(iii) $|xy| = |x||y|$.

PROOF. Left to the reader. $\square$

PROPOSITION 1.2.35. *Let* $x \in \mathbb{R}$, *such that* $x \bowtie 0$. *Since there exists* $N \in \mathbb{N}^+$, *such that* $|x_m| \geq \frac{1}{N}$, *for every* $m \geq N$, *we define*

$$y_n \equiv \begin{cases} \frac{1}{x_{N^3}} & , n < N \\ \frac{1}{x_{nN^2}} & , n \geq N \end{cases}$$

(i) The sequence

$$x^{-1} \equiv (y_n)_{n\in\mathbb{N}^+}$$

is in $\mathbb{R}$.
(ii) $xx^{-1} = 1$.
(iii) If $x > 0$*, then* $x^{-1} > 0$.
(iv) If $s \in \mathbb{R}$ *such that* $xs =_{\mathbb{R}} 1$*, then* $s =_{\mathbb{R}} x^{-1}$.
(v) The rule $x \mapsto x^{-1}$ *is a function from* $\mathbb{R}_0 \equiv \{x \in \mathbb{R} \mid x \bowtie 0\}$ *to* $\mathbb{R}$.
(vi) If $x, y \in \mathbb{R}_0$*, then* $(xy)^{-1} =_{\mathbb{R}} x^{-1}y^{-1}$.
(vii) If $q \in \mathbb{Q}$ *and* $q \neq 0$*, then* $(q^*)^{-1} =_{\mathbb{R}} (q^{-1})^*$.
(viii) If $x \in \mathbb{R}_0$*, then* $(x^{-1})^{-1} =_{\mathbb{R}} x$.

PROOF. Left to the reader. □

PROPOSITION 1.2.36. *Let* $x, y \in \mathbb{R}$.
(i) If $x \geq 0$ *and* $x^2 > 0$*, then* $x > 0$.
(ii) If $x, y \geq 0$ *and* $xy > 0$*, then* $x > 0$ *and* $y > 0$.
(iii) $xy > 0 \Rightarrow x \bowtie 0 \ \wedge \ y \bowtie 0$.
(iv) $xy < 0 \Rightarrow x < 0 \ \vee \ y < 0$.

PROOF. By constructive trichotomy $x > 0$, or $x < x^2 \Leftrightarrow x(x-1) > 0$. It suffices to treat the second case. Since $0 < 1$, we have that $x > 0$, or $x < 1 \Leftrightarrow x - 1 < 0$. Suppose the latter case. By Proposition 1.2.35 and Propositin 1.2.34(i) we get that $\frac{1}{x-1}x(x-1) < 0 \Leftrightarrow x < 0$, which contradicts our hypothesis $x \geq 0$. Hence $x > 0$ is the case for x.
(ii) If $xy > 0$, then $(x+y)^2 = x^2+y^2+2xy > 0$, hence, since, if $x, y \geq 0$, then $x + y \geq 0$, by case (i) we get that $x + y > 0$, therefore by Proposition 1.2.16 we have that $x > 0$, or $y > 0$. If $x > 0$, then by Proposition 1.2.34(ii) $y = \frac{1}{x}(xy) > 0$, Similarly, if $y > 0$, we get $x > 0$.
(iii) Since $0 < xy = |xy| = |x||y|$, by case (ii) we get $|x| > 0$ and $|y| > 0$, i.e., $x \bowtie 0$ and $y \bowtie 0$.
(iv) Since $xy \bowtie 0$, by case (iii) we have that $x \bowtie 0$ and $y \bowtie 0$. If $x > 0$ and $y > 0$, then by Proposition 1.2.34(ii) we get $xy > 0$, which contradicts our hypothesis $xy < 0$. Hence $x < 0$, or $y < 0$. □

PROPOSITION 1.2.37. *The modified Royden number* ϱ^* *is defined by*

$$\varrho^* \equiv \sum_{k=1}^{\infty} \frac{a_{2k}}{(-2)^k},$$

$$a_{2k} = \begin{cases} 0 & , \forall_{n\in\mathbb{N}}\Big((4 \leq n \leq 2k \ \& \ \mathrm{Even}(n)) \Rightarrow \mathrm{Goldbach}(n)\Big) \\ 1 & , \textit{otherwise} \end{cases}$$

(i) $\varrho^* \in \mathbb{R}$.
(ii) If there is a proof of the disjunction

$$\varrho^* \geq 0 \ \vee \ \varrho^* \leq 0,$$

then the Goldbach conjecture is decided.

Proof. (i) We work as in the proof of Proposition 1.2.7.
(ii) We use the fact that, if $l \in \mathbb{N}^+$ is the first index such that $a_{2l} = 1$, then

$$\varrho^* = \sum_{k=l}^{\infty} \frac{1}{(-2)^k} = \frac{1}{(-2)^l}\left(1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \dots\right)$$

and the sign of ϱ^* is determined by l i.e., if l is odd, or even. □

Corollary 1.2.38. *Consider the following equation* (E)*:*

$$x(x - \varrho^*) = 0.$$

(i) The real number $0 \wedge \varrho^*$ *is a solution of* (E)*.*
(ii) If there is a proof of the disjunction

$$(0 \wedge \varrho^*) = 0 \ \vee \ (0 \wedge \varrho^*) = \varrho^*,$$

then the Goldbach conjecture is decided.
(iii) If there is a proof of the implication

$$x(x - \varrho^*) = 0 \Rightarrow (x = 0 \ \vee \ x = \varrho^*),$$

then the Goldbach conjecture is decided.

Proof. (i) Let $(0 \wedge \varrho^*)(0 \wedge \varrho^* - \varrho^*) > 0$. By Proposition 1.2.36(iii) we have that $(0 \wedge \varrho^*) \bowtie 0$, or $(0 \wedge \varrho^* - \varrho^*) \bowtie 0$. In the first case we get $0 \geq (0 \wedge \varrho^*) > 0$, which is a contradiction, or $0 \wedge \varrho^* < 0$. Since $0 \wedge \varrho^* \equiv -(-0 \vee -\varrho^*) = -(0 \vee -\varrho^*)$, we have that $0 \wedge \varrho^* < 0 \Leftrightarrow -(0 \vee -\varrho^*) < 0 \Leftrightarrow (0 \vee -\varrho^*) > 0$, therefore by Proposition 1.2.23(ii) we get $(0 \vee -\varrho^*) = -\varrho^*$. Hence, $0 \wedge \varrho^* = -(-\varrho^*) = \varrho^*$, and the hypothesis $\varrho^* < 0$ decides, according to Proposition 1.2.37(ii), the Goldbach conjecture. Hence, $(0 \wedge \varrho^*)(0 \wedge \varrho^* - \varrho^*) \leq 0$.

If $(0 \wedge \varrho^*)(0 \wedge \varrho^* - \varrho^*) < 0$, then by Proposition 1.2.36(iv) either $0 \wedge \varrho^* < 0$, or $(0 \wedge \varrho^* - \varrho^*) < 0$. In the first case we work exactly as in the previous case. In the second case we get $0 \wedge \varrho^* < \varrho^*$, and similarly we conclude that $0 = 0 \wedge \varrho^* < \varrho^*$, hence by Proposition 1.2.37(ii) the Goldbach conjecture is decided. Hence, $(0 \wedge \varrho^*)(0 \wedge \varrho^* - \varrho^*) \geq 0$.
(ii) If there is a proof of $(0 \wedge \varrho^*) = 0$, then $\varrho^* \geq 0$. If there is a proof of $(0 \wedge \varrho^*) = \varrho^*$, then $\varrho^* \leq 0$.
(iii) In this case $(0 \wedge \varrho^*) = 0 \vee (0 \wedge \varrho^*) = \varrho^*$, and we use (ii). □

Consequently, the classical property of an integral domain

$$xy = 0 \Rightarrow (x = 0 \ \vee \ y = 0)$$

cannot be accepted constructively for the real numbers. It is easy to show though, that the following implication holds:

$$x^2 = 0 \Rightarrow x = 0.$$

1.3. The trichotomy of algebraic numbers

Definition 1.3.1. A function $f : \mathbb{R} \to \mathbb{R}$ is called *continuous*, if for every $n \in \mathbb{N}^+$ there is a function $\omega_{f,n} : \mathbb{R}^+ \to \mathbb{R}^+$

$$0 < \epsilon \mapsto \omega_{f,n}(\epsilon) > 0,$$

which is called the *modulus of continuity* of f on the closed interval $[-n, n]$, and satisfies

$$|x - y| < \omega_{f,n}(\epsilon) \Rightarrow |f(x) - f(y)| \leq \epsilon,$$

for every $\epsilon > 0$, and $x, y \in [-n, n]$. We denote by $\mathbb{B}(\mathbb{R})$ the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Lemma 1.3.2. *If $x, y \in \mathbb{R}$, there is $n \in \mathbb{N}^+$ such that $x, y \in [-n, n]$.*

Proof. By Lemma 1.2.32 we have that $x - x_m \leq \frac{1}{m}$, hence $x \leq x_m + \frac{1}{m} \leq N$, for some $N \in \mathbb{N}^+$. Similarly $-x \leq M \Leftrightarrow x \geq -M$, for some $M \in \mathbb{N}^+$. If $k \equiv \max\{N, M\}$, then $x \in [-k, k]$. Similarly, there is $l \in \mathbb{N}^+$ such that $y \in [-l, l]$. Take $n \equiv \max\{k, l\}$. $\square$

Proposition 1.3.3. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $x, y \in \mathbb{R}$, then*

$$f(x) \bowtie f(y) \Rightarrow x \bowtie y.$$

Proof. By Lemma 1.3.2 there is $n \in \mathbb{N}^+$ such that $x, y \in [-n, n]$. Let $0 < \epsilon \equiv |f(x) - f(y)|$. Suppose that $|x - y| < \omega_{f,n}(\frac{\epsilon}{2})$. Hence, $|f(x) - f(y)| \leq \frac{\epsilon}{2}$, which contradicts our hypothesis. So, $|x - y| \geq \omega_{f,n}(\frac{\epsilon}{2}) > 0$. $\square$

Proposition 1.3.4. *Let $f, g \in \mathbb{B}(\mathbb{R})$ and $\lambda \in \mathbb{R}$.*
(i) $f + g \in \mathbb{B}(\mathbb{R})$.
(ii) $\lambda f \in \mathbb{B}(\mathbb{R})$.
(iii) $f^2 \in \mathbb{B}(\mathbb{R})$.
(iv) $f \cdot g \in \mathbb{B}(\mathbb{R})$.

Proof. Exercise. For the proof of (iii) use without proof the fact that if $f \in \mathbb{B}(\mathbb{R})$, then its restriction $f_{|[-n,n]}$ to $[-n, n]$ is bounded i.e.,

$$\exists_{M_n \in \mathbb{N}^+} \forall_{x \in [-n,n]} (|f(x)| \leq M_n).$$

$\square$

Definition 1.3.5. If $p \in \mathbb{Q}$, then $\bar{p} : \mathbb{R} \to \mathbb{R}$ denotes the constant function on $\mathbb{R}$ with value p. The set of polynomials with rational coefficients $\mathbb{Q}[x]$ is defined by

$$\mathbb{Q}[x] = \bigcup_{n \in \mathbb{N}} \mathbb{Q}^n[x],$$

$$\mathbb{Q}^0[x] \equiv \{\bar{p} \mid p \in \mathbb{Q}\},$$

$$\mathbb{Q}^n[x] \equiv \left\{ f \in \mathbb{F}(\mathbb{R}, \mathbb{R}) \mid \exists_{p_0,\dots,p_{n-1} \in \mathbb{Q}} \exists_{p_n \in \mathbb{Q} \setminus \{0\}} \forall_{x \in \mathbb{R}} \left(f(x) \equiv \sum_{i=0}^{n} p_i x^i \right) \right\},$$

if $n \in \mathbb{N}^+$. If $f \in \mathbb{Q}^n[x]$, where $n > 0$, then $n = \deg(f)$ is the *degree* of f. If $f \equiv \sum_{i=0}^{n} p_i x^i \in \mathbb{Q}[x]$, we say that f is *non-constant*, if $f \in \mathbb{Q}^n[x]$, for some $n \in \mathbb{N}^+$, and the *derivative* f' of f is the polynomial

$$f' \equiv \sum_{i=1}^{n} i p_i x^{i-1}.$$

A non-constant polynomial $f \equiv \sum_{i=0}^{n} p_i x^i$ is called *monic*, if $p_n = 1$, Let $\mathbb{Q}^*[x] \equiv \mathbb{Q}[x] \setminus \{\bar{0}\}$. The set of *algebraic* real numbers $\mathbb{A}$ is defined by

$$\mathbb{A} \equiv \{x \in \mathbb{R} \mid \exists_{f \in \mathbb{Q}^*[x]} (f(x) = 0)\}.$$

Clearly, $\mathbb{Q} \subset \mathbb{A}$, and if $f(x) = 0$, for some $f \in \mathbb{Q}^*[x]$, then f is non-constant. With the use of the Euclidean division algorithm one can show that if $f, g \in \mathbb{Q}[x]$, such that their greatest common divisor (f, g) is 1, or, in other words, if f, g are relatively prime, there exist $s, t \in \mathbb{Q}[x]$ with $sf + tg = 1$.

Corollary 1.3.6. *A polynomial $f(x)$ in $\mathbb{Q}[x]$ is in $\mathbb{B}(\mathbb{R})$.*

Proof. Exercise. Note that one can prove this without using Proposition 1.3.4(iii). □

Lemma 1.3.7. *Let $x_0 \in \mathbb{R}$, and $f, g \in \mathbb{Q}[x]$ such that $(f, g) = 1$. Then*

$$f(x_0)g(x_0) = 0 \Rightarrow (f(x_0) = 0 \ \vee \ g(x_0) = 0).$$

Proof. Let $s, t \in \mathbb{Q}[x]$ such that $sf + tg = 1$. Hence, $s(x_0)f(x_0) + t(x_0)g(x_0) = 1$, and

$$s(x_0)f(x_0) + t(x_0)g(x_0) > 0,$$

which by Proposition 1.2.16 implies

$$s(x_0)f(x_0) > 0 \ \vee \ t(x_0)g(x_0) > 0.$$

Suppose that $s(x_0)f(x_0) > 0$. Since

$$0 = s(x_0)\big(f(x_0)g(x_0)\big) = \big(s(x_0)f(x_0)\big)g(x_0),$$

we conclude that $g(x_0) = 0$. Similarly, if we suppose that $t(x_0)g(x_0) > 0$, we conclude that $f(x_0) = 0$. □

LEMMA 1.3.8. *If $S = \{f_1, \ldots, f_n\}$ is a finite set of monic polynomials in $\mathbb{Q}[x]$, there is a finite set $T = \{g_1, \ldots, g_m\}$ of monic polynomials in $\mathbb{Q}[x]$ such that:*
(i) If $g_i, g_j \in T$, then

$$g_i = g_j \ \vee \ (g_i, g_j) = 1.$$

(ii) If $f_k \in S$, there are $g_{i_1} \ldots, g_{i_l} \in T$ such that for every $x \in \mathbb{R}$:

$$f_k(x) \equiv \prod_{j=i_1}^{i_l} g_j(x).$$

PROOF. Left to the reader. □

E.g., if $S_1 = \{x^2, x^3\}$, then $T_1 = \{x\}$, and if $S_2 = \{x^2 - 4, (x-2)^3\}$, then $T_2 = \{x - 2, x + 2\}$. Note that if $S = \{f\}$ and f has a proper factor, then the degree of the elements of T is smaller than $\deg(f)$.

LEMMA 1.3.9. *Let $g \in \mathbb{Q}[x]$ a non-constant polynomial. There are $f_1, \ldots, f_r \in \mathbb{Q}[x]$, for some $r \in \mathbb{N}^+$, and $m_1, \ldots, m_r \in \mathbb{N}^+$ such that:*
(i) $(f_i, f'_i) = 1$, for every $1 \leq i \leq r$.
(ii) $(f_i^{m_i}, f_j^{m_j}) = 1$, for every $1 \leq i \neq j \leq r$.
(iii) For every $x \in \mathbb{R}$ we have that

$$g(x) = \prod_{i=1}^{r} f_i^{m_i}(x).$$

PROOF. Without loss of generality let g be monic. If $\deg(g) = 1$, then $g(x) = x + p_0$, for some $p_0 \in \mathbb{Q}$. Hence, $(g, g') = 1$, and we take $r = 1$, $f_1 = g$ and $m_1 = 1$. If $\deg(g) > 1$, we compute (g, g'). If $(g, g') = 1$, we work as in the previous case. If not, then (g, g') is a proper factor of g, hence by Lemma 1.3.8 g is the product of monic polynomials $h_1, \ldots, h_m$ with degree smaller than $\deg(g)$, which are pairwise relatively prime. By inductive hypothesis we write them such that (i)-(iii) are satisfied, and then the required writing of g follows, since if $(h_i, h_j) = 1$ and

$$h_i = \prod_{\rho=1}^{s} u_\rho^{k_\rho}, \quad h_j = \prod_{\sigma=1}^{t} w_\sigma^{l_\sigma},$$

then $(u_\rho^{k_\rho}, w_\sigma^{l_\sigma}) = 1$, for every ρ, σ. $\square$

THEOREM 1.3.10 (Julian, Mines, Richman). *If $a, b \in \mathbb{A}$, then*

$$a < b \ \vee \ a = b \ \vee a > b.$$

PROOF. Without loss of generality we can take monic polynomials g, h in $\mathbb{Q}[x]$ such that $g(a) = 0 = h(b)$. If $f = gh$, then f is a monic polynomial in $\mathbb{Q}[x]$ such that

$$f(a) = 0 = f(b).$$

Since g, h are non-constant, f is also non-constant, hence by Lemma 1.3.9 there are $f_1, \ldots, f_r \in \mathbb{Q}[x]$ and $m_1, \ldots, m_r \in \mathbb{N}^+$ such that:
(i) $(f_i, {f'}_i) = 1$, for every $1 \leq i \leq r$.
(ii) $(f_i^{m_i}, f_j^{m_j}) = 1$, for every $1 \leq i \neq j \leq r$.
(iii) For every $x \in \mathbb{R}$ we have that

$$f(x) = \prod_{i=1}^{r} f_i^{m_i}(x).$$

Since

$$f(a) = \prod_{i=1}^{r} f_i^{m_i}(a) = 0,$$

and because of condition (ii), by Lemma 1.3.7 we get some index $i \in \{1, \ldots, r\}$ such that

$$f_i^{m_i}(a) = 0.$$

Similarly, we get some index $j \in \{1, \ldots, r\}$ such that

$$f_j^{m_j}(b) = 0.$$

If $i \neq j$, and because of condition (ii), there are polynomials $s(x), t(x)$ in $\mathbb{Q}[x]$ such that for every $x \in \mathbb{R}$

$$s(x) f_i^{m_i}(x) + t(x) f_j^{m_j}(x) = 1.$$

Hence, the equality

$$s(a) f_i^{m_i}(a) + t(a) f_j^{m_j}(a) = 1$$

implies the equality $t(a) f_j^{m_j}(a) = 1$, hence by Proposition 1.2.36(iii) $t(a) \bowtie 0$ and $f_j^{m_j}(a) \bowtie 0$, therefore

$$f_j^{m_j}(a) \bowtie f_j^{m_j}(b).$$

By Proposition 1.3.3 and the continuity of $f_j^{m_j}$ we get $a \bowtie b$.

If $i = j$, then,
$$f_i(a)^{m_i} = 0 = f_i(b)^{m_i},$$
hence
$$f_i(a) = 0 = f_i(b).$$
Using the elementary theory of Taylor series for the infinitely differentiable polynomial functions we have that
$$f_i(y) = (y-b){f_i}'(b) + (y-b)^2 K(y) = (y-b)[{f_i}'(b) + (y-b)K(y)].$$
Hence
$$(*) \qquad 0 = f_i(a) = (a-b)[{f_i}'(b) + (a-b)K(a)].$$
Since $(f_i, {f_i}') = 1$, there are $k(x), l(x)$ in $\mathbb{Q}[x]$ such that
$$k(x)f_i(x) + l(x){f_i}'(x) = 1,$$
for every $x \in \mathbb{R}$. Since $f_i(b) = 0$, we get ${f_i}'(b) \bowtie 0$, and
$$\begin{aligned} 0 \bowtie {f_i}'(b) &= {f_i}'(b) + (a-b)K(a) - (a-b)K(a) \\ &= [{f_i}'(b) + (a-b)K(a)] + [-(a-b)K(a)]. \end{aligned}$$
By the obvious generalisation of Proposition 1.2.16 we get
$$[{f_i}'(b) + (a-b)K(a)] \bowtie 0 \ \vee \ [-(a-b)K(a)] \bowtie 0,$$
and consequently
$$[{f_i}'(b) + (a-b)K(a)] \bowtie 0 \ \vee \ (a-b)K(a) \bowtie 0.$$
If $[{f_i}'(b)+(a-b)K(a)] \bowtie 0$, then the equation $(*)$ implies $a = b$. If otherwise $(a-b)K(a) \bowtie 0$, then by Proposition 1.2.36(iii) we get $a \bowtie b$. $\square$

The classical behavior of the algebraic numbers $\mathbb{A}$ is anticipated from the "finite" information included in the definition of $\mathbb{A}$, which makes $\mathbb{A}$ behave like $\mathbb{Q}$.

1.4. Notes

The fundamental thesis of constructivism is formulated in [**5**], an unpublished lecture of Bishop on which the first chapter is based. We also include some notions and results from [**6**], a book of Bishop with Bridges, which has a lot in common with Bishop's original book [**4**], but it is a "different" book in many respects.

With his remarkable book *Foundations of constructive analysis* Errett Bishop (1928-1983), an important analyst, wanted to revolutionize mathematics. Bishop's achievement was to develop large parts of mathematics using intuitionistic logic without contradicting classical mathematics i.e., mathematics based roughly on the principle of the excluded middle. Before Bishop, it was the great topologist Luitzen Egbertus Jan Brouwer (1881-1966) who used intuitionistic logic in his intuitionistic mathematics, without avoiding though contradicting with classical mathematics. Although Bishop's work didn't influence the every day mathematician, it had an enormous impact on mathematical logic and formal studies on the foundations of mathematics (see e.g., [**2**], for the influence of Bishop's book in the logical studies of the 70's and the 80's). Today, the influence of Bishop's paradigm is evident in theoretical computer science and especially in the current use of type theory in Voevodsky's univalent foundations of mathematics (see [**23**]).

One of the formal systems of the 70's that was motivated a lot from Bishop's book was Martin-Löf's type theory (see [**19**] and [**20**]). A formal version, or an implementation, of FTC-$\mathbb{N}$ in Martin-Löf's type theory is the canonicity property of the type of natural numbers, according to which every closed term of type $\mathbb{N}$ is reduced (simplified) to a numeral.

The definition of the equality $=_X$ between the elements of a set X is specific to each set X and an essential part of the definition of X itself. This is a fundamental idea of Bishop's set theory, which is in contrast to the standard, "global" set-theoretic equality, and it is implemented in Martin-Löf's type theory through the identity type $x =_A y$.

The set-theoretical definition of a function $f : X \to Y$ is that $f \subseteq X \times Y$ such that $(x, y) \in f$ and $(x, z) \in f$ implies $y = z$, for every $x \in X$ and $y, z \in Y$. There are many reasons not to consider this definition of the concept of function, since it does not reveal the dynamic character of the concept (see [**12**], section 2.1) In Bishop's approach to constructive mathematics, and in many formalizations of Bishop's constructive mathematics the notion of function, or a rule, is taken as primitive, not reduced to some other concept, such the concept of set.

The definition of a real number, Definition 1.2.1, differs from the classical one, as classically a real number is the equivalence class of the reals, as defined here, with respect to the equivalence relation of their equality. The avoidance of equivalence classes is a central feature of Bishop-style constructive mathematics.

The definition of $x \leq 0$ is not given through the negation of $x > 0$, but it is defined *positively* in Definition 1.2.12. Since negation does not behave constructively as in the classical setting, *negatively* defined concepts are avoided when a positive formulation of them can be given.

In classical mathematics one finds important theorems in disjunctive form for which no method is known (yet) that decides which disjunct is the case. E.g., Jensen proved in the early 70's that the universe of sets V is either "very close" to Gödel's constructible universe L, which is an inner model of Zermelo-Fraenkel axiomatic set theory ZF in which the axiom of choice and the generalised continuum hypothesis are true in it, or "very far" from it.

THEOREM 1.4.1. *Exactly one of the following hold:*
(i) Every singular cardinal γ *is singular in* L, *and* $(\gamma^+)^L = \gamma^+$.
(ii) Every uncountable cardinal is inaccessible in L.

Note that the proof of this theorem cannot specify which one of the two cases holds. The existence of large cardinals implies (ii), but this existence is unprovable in ZFC, which is ZF with the axiom of choice. A similar dichotomy for the inner model HOD was proved by Woodin a few years ago. Assuming the existence of an extendible cardinal, the first alternative of Woodin's dichotomy implies that HOD is close to V, and the second that HOD is far from V. At the moment there is no evidence which one of the two alternatives is the right one, a fact with important consequences for the future of set theory (see [**1**]).

Hence, a proof of the impossibility of (not A) and (not B) is not generally a constructive proof of $A \vee B$ (Definition 1.2.15), since such a proof does not always imply a finite process that determines which one of the two disjuncts is the case.

Definitions 1.1.5, 1.2.3, and 1.2.15 constitute the so-called *Brouwer-Heyting-Kolmogorov interpretation* of logical connectives and quantifiers.

The definition of a continuous function (Definition 1.3.1) is one of the major keys in Bishop's development of constructive analysis. By "replacing" pointwise continuity of a real-valued function on $\mathbb{R}$ with uniform continuity on the compact intervals $[-n, n]$ of $\mathbb{R}$ he managed to avoid clashing with classical analysis.

Section 1.3 draws from the paper [**17**] of Julian, Mines and Richman. For a development of Bishop-style constructive algebra see [**21**].

CHAPTER 2

Constructive Logic and Classical Logic

2.1. First-order languages

Definition 2.1.1. Let $\mathtt{Var} = \{ v_i \mid i \in \mathbb{N} \}$ be a fixed countably infinite set of *variables*. We also denote the elements of $\mathtt{Var}$ by x, y, z, etc. . Let $L = \{\to, \wedge, \vee, \forall, \exists, (,), , \}$, where each element of L is called a *logical symbol*. A *first-order language* over $\mathtt{Var}$ and L is a pair $\mathcal{L} = (\mathtt{Rel}, \mathtt{Fun})$, where $\mathtt{Var}, L, \mathtt{Rel}, \mathtt{Fun}$ are pairwise disjoint sets such that

$$\mathtt{Rel} = \bigcup_{n \in \mathbb{N}} \mathtt{Rel}^{(n)},$$

where for every $n \in \mathbb{N}$, $\mathtt{Rel}^{(n)}$ is a (possible empty) set of n-ary *relation symbols* (or *predicate symbols*). Moreover, $\mathtt{Rel}^{(n)} \cap \mathtt{Rel}^{(m)} = \emptyset$, for every $n \neq m$. A 0-ary relation symbol is called a *propositional symbol*. The symbol $\perp$ (read "falsum") is required as a fixed propositional symbol (i.e., $\mathtt{Rel}^{(0)}$ is inhabited by $\perp$). The language will *not*, unless stated otherwise, contain the equality symbol $=$, which is a 2-ary relation symbol. Moreover,

$$\mathtt{Fun} = \bigcup_{n \in \mathbb{N}} \mathtt{Fun}^{(n)},$$

where for every $n \in \mathbb{N}$, $\mathtt{Fun}^{(n)}$ is a (possible empty) set of n-ary *function symbols*. Moreover, $\mathtt{Fun}^{(n)} \cap \mathtt{Fun}^{(m)} = \emptyset$, for every $n \neq m$. A 0-ary function symbol is called *constant*, and we define

$$\mathtt{Const} \equiv \mathtt{Fun}^{(0)}.$$

The pair $(\mathtt{Rel}, \mathtt{Fun})$ is called the *signature* of $\mathcal{L}$.

Note that this definition uses the notion of set, therefore it is given within some theory of sets, which is the *meta-theory* of our theory of first-order languages. Here we choose as meta-theory the classical theory of sets. All proofs of properties of first-order languages are given within set-theory. One could use as meta-theory the constructive theory of sets that was roughly introduced in the previous chapter, and also use constructive arguments in the related proofs.

If our formal language includes one more fixed countably infinite set of variables $\mathtt{VAR} = \{V_i \mid i \in \mathbb{N}\}$, where V_i is a variable of another sort, e.g., a set-variable, then one could define the notion of a *second-order* language in a similar fashion.

The first-order language of *arithmetic* has as signature the pair $(\{\bot, =\}, \{0, S, +, \cdot\})$, which is written for simplicity as $(\bot, =, 0, S, +, \cdot)$, such that $0 \in \mathtt{Const}, S \in \mathtt{Fun}^{(1)}$, and $+, \cdot \in \mathtt{Fun}^{(2)}$. The first-order language of *set theory* has signature the pair $(\{\bot, =, \in\}, \emptyset)\})$, which is written for simplicity as $(\bot, =, \in)$, such that $\in$ is in $\mathtt{Rel}^{(2)}$.

Next, the terms $\mathtt{Term}_{\mathcal{L}}$ of a first-order language $\mathcal{L}$ are inductively defined. For simplicity we omit the subscript $\mathcal{L}$.

Definition 2.1.2. The terms $\mathtt{Term}$ of a first-order language $\mathcal{L}$ are defined by the following inductive rules:

$$\frac{x \in \mathtt{Var}}{x \in \mathtt{Term}}, \qquad \frac{c \in \mathtt{Const}}{c \in \mathtt{Term}},$$

$$\frac{n \in \mathbb{N}^+, \quad t_1, \ldots, t_n \in \mathtt{Term}, \quad f \in \mathtt{Fun}^{(n)}}{f(t_1, \ldots, t_n) \in \mathtt{Term}}.$$

In words, every variable is a term, every constant is a term, and if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol with $n \geq 1$, then $f(t_1, \ldots, t_n)$ is a term. If r, s are terms and $\circ$ is a binary function symbol, we usually write $(r \circ s)$ instead of $\circ(r, s)$. E.g.,

$$0, \; S(0), \; S(S(0)), \; (S(0) + S(S(0)))$$

are terms of the language of arithmetic.

As in the case of the inductive definition of $\mathbb{N}$, we associate to Definition 2.1.2 the following induction principle:

$$\frac{\begin{array}{c}\forall_{x \in \mathtt{Var}}(P(x)),\\ \forall_{c \in \mathtt{Const}}(P(c)),\\ \forall_{n \in \mathbb{N}^+} \forall_{t_1, \ldots, t_n \in \mathtt{Term}} \forall_{f \in \mathtt{Fun}^{(n)}}((P(t_1) \wedge \ldots \wedge P(t_n)) \Rightarrow P(f(t_1, \ldots, t_n))\end{array}}{\forall_{t \in \mathtt{Term}}(P(t))},$$

where $P(t)$ is any property of our meta-language that concerns the set of terms. E.g., $P(t)$ could be "the number of left parentheses, (, occurring in t is equal to the number of right parentheses,), occurring in t". We need of course, to express this property in mathematical terms. As in the case of the induction principle for natural numbers, the induction principle for `Term` expresses that `Term` is the *least* set satisfying its defining rules.

As one can show for $\mathbb{N}$ that if X is a set, $x_0 \in X$ and $g : X \to X$, there is a unique function $f : \mathbb{N} \to X$ such that

$$f(0) \equiv x_0,$$
$$f(S(n)) \equiv g(f(n)),$$

for every $n \in \mathbb{N}$, the following recursion theorem holds for `Term`.

PROPOSITION 2.1.3 (Recursion theorem for `Term`). *Let X be a set. If there are functions*

$$F_{\texttt{Var}} : \texttt{Var} \to X,$$
$$F_{\texttt{Const}} : \texttt{Const} \to X,$$
$$F_{f,n} : X^n \to X,$$

for every $f \in \texttt{Fun}^{(n)}$ and $n \in \mathbb{N}^+$, then there is a unique function

$$F : \texttt{Term} \to X$$

such that, for every $n \in \mathbb{N}^+, t_1, \ldots, t_n \in$ `Term`, and $f \in \texttt{Fun}^{(n)}$,

$$F(x) \equiv F_{\texttt{Var}}(x), \quad x \in \texttt{Var},$$
$$F(c) \equiv F_{\texttt{Const}}(c), \quad c \in \texttt{Const},$$
$$F(f(t_1, \ldots, t_n)) \equiv F_{f,n}(F(t_1), \ldots, F(t_n)).$$

PROOF. Let $F \subseteq \texttt{Term} \times X$ defined as follows:

$$F \equiv \{(u_i, F_{\texttt{Var}}(u_i)) \mid u_i \in \texttt{Var}\} \cup \{(c, F_{\texttt{Const}}(c)) \mid c \in \texttt{Const}\}$$
$$\cup \big\{(f(t_1, \ldots, t_n), F_{f,n}(x_1, \ldots, x_n) \mid t_1, \ldots, t_n \in \texttt{Term},$$
$$(t_1, x_1), \ldots, (t_n, x_n) \in F, f \in \texttt{Fun}^{(n)}, n \in \mathbb{N}^+\big\}.$$

Using the induction principle for `Term` we show that F is a function i.e.,

$$\forall_{t \in \texttt{Term}} \big(\forall_{x,y \in X}((t, x) \in F \wedge (t, y) \in F \Rightarrow x = y)\big).$$

If $t \equiv u_i$, for some $i \in \mathbb{N}$, then $(u_i, x) \in F \Leftrightarrow x = F_{\texttt{Var}}(u_i)$ and $(u_i, y) \in F \Leftrightarrow y = F_{\texttt{Var}}(u_i)$. Since $F_{\texttt{Var}}$ is a function, we get $x = y$. If $t \equiv c$, for some $c \in$ `Const`, then $(c, x) \in F \Leftrightarrow c = F_{\texttt{Const}}(c)$ and $(c, y) \in F \Leftrightarrow y = F_{\texttt{Const}}(c)$. Since $F_{\texttt{Const}}$ is a function, we get $x = y$. If $t \equiv f(t_1, \ldots, t_n)$, for some $t_1, \ldots, t_n \in$ `Term` and $f \in \texttt{Fun}^{(n)}$, then

$$(f(t_1, \ldots, t_n), x) \in F \Leftrightarrow x = F_{f,n}(x_1, \ldots, x_n) \wedge (t_1, x_1) \in F \wedge \ldots (t_n, x_n) \in F,$$

for some $x_1, \ldots, x_n \in X$. Similarly,

$$(f(t_1, \ldots, t_n), y) \in F \Leftrightarrow y = F_{f,n}(y_1, \ldots, y_n) \wedge (t_1, y_1) \in F \wedge \ldots (t_n, y_n) \in F,$$

for some $y_1, \ldots, y_n \in X$. By the inductive hypothesis on $t_1, \ldots, t_n$ we get

$$x_1 = y_1 \wedge \ldots \wedge x_n = y_n,$$

and since $F_{f,n}$ is a function, from

$$((x_1, \ldots, x_n), x) \in F_{f,n} \ \wedge \ ((x_1, \ldots, x_n), y) \in F_{f,n}$$

we get $x = y$. Using the induction principle for $\mathtt{Term}$ we get $\mathtt{Term} \subseteq \mathrm{dom}(F)$ i.e.,

$$\forall_{t \in \mathtt{Term}} \big(t \in \mathrm{dom}(F)\big).$$

If $t \equiv u_i$, for some $i \in \mathbb{N}$, then $(u_i, F_{\mathtt{Var}}(x)) \in F$, therefore $u_i \in \mathrm{dom}(F)$. If $t \equiv c$, for some $c \in \mathtt{Const}$, then $(c, F_{\mathtt{Const}}(c)) \in F$, therefore $c \in \mathrm{dom}(F)$. If $t \equiv f(t_1, \ldots, t_n)$, for some $t_1, \ldots, t_n \in \mathtt{Term}$ and $f \in \mathtt{Fun}^{(n)}$, such that $t_1, \ldots, t_n \in \mathrm{dom}(F)$. Hence, there are $x_1, \ldots, x_n \in$ such that $(t_1, x_1) \in F \wedge \ldots (t_n, x_n) \in F$. By the definition of F we get

$$(f(t_1, \ldots, t_n), F_{f,n}(x_1, \ldots, x_n) \in F,$$

hence $f(t_1, \ldots, t_n) \in \mathrm{dom}(F)$. The uniqueness of F is also shown again with the use of the induction principle for $\mathtt{Term}$. If $G : \mathtt{Term} \to X$ satisfies the defining properties of F, it is easy to show now that

$$\forall_{t \in \mathtt{Term}} \big(F(t) = G(t)\big).$$

□

Using the recursion theorem for $\mathtt{Term}$ one can define e.g., the function $P_{\mathtt{left}} : \mathtt{Term} \to \mathbb{N}$ such that $P_{\mathtt{left}}(t)$ is the number of left parentheses occurring in $t \in \mathtt{Term}$. It suffice to define it on the variables, the constants, and the complex terms $f(t_1, \ldots, t_n)$ supposing that $P_{\mathtt{left}}$ is defined on the terms $t_1, \ldots, t_n$. Namely, we define

$$P_{\mathtt{left}}(u_i) \equiv 0,$$
$$P_{\mathtt{left}}(c) \equiv 0,$$
$$P_{\mathtt{left}}(f(t_1, \ldots, t_n)) \equiv 1 + \sum_{i=1}^{n} P_{\mathtt{left}}(t_i).$$

Here we used the recursion theorem for $\mathtt{Term}$ with respect the functions

$$F_{\mathtt{Var}}(x) \equiv 0 \equiv F_{\mathtt{Const}}(c),$$
$$F_{f,n}(x_1, \ldots, x_n) \equiv 1 + \sum_{i=1}^{n} x_i.$$

Similarly, one defines the function $P_{\mathtt{right}} : \mathtt{Term} \to \mathbb{N}$ such that $P_{\mathtt{right}}(t)$ is the number of right parentheses occurring in $t \in \mathtt{Term}$.

PROPOSITION 2.1.4. $\forall_{t\in\texttt{Term}}\big(P_{\texttt{left}}(t) = P_{\texttt{right}}(t)\big)$.

PROOF. Exercise. $\square$

DEFINITION 2.1.5. The formulas Form of a first-order language $\mathcal{L}$ are defined by the following inductive rules:

$$\frac{n \in \mathbb{N}, \quad t_1, \ldots, t_n \in \texttt{Term}, \quad R \in \texttt{Rel}^{(n)}}{R(t_1, \ldots, t_n) \in \texttt{Form}},$$

$$\frac{A, B \in \texttt{Form}}{(A \to B),\ (A \wedge B),\ (A \vee B) \in \texttt{Form}},$$

$$\frac{A \in \texttt{Form}, \quad x \in \texttt{Var}}{\forall_x A,\ \exists_x A \in \texttt{Form}}.$$

The formulas of the form $R(t_1, \ldots, t_n)$ are called *prime formulas*, or *atomic formulas*, or just *atoms*. If r, s are terms and $\sim$ is a binary relation symbol, we also write $(r \sim s)$ for the prime formula $\sim (r, s)$. Since $\bot \in \texttt{Rel}^{(0)}$, we get $\bot \in \texttt{Form}$. The *negation* $\neg A$ of a formula A is defined as the formula

$$\neg A \equiv A \to \bot.$$

The formulas generated by the prime formulas are called *complex*, or *non-atomic* formulas. Usually, we denote $(A\square B)$ by $A\square B$, where $\square \in \{\to, \wedge, \vee\}$. We also define

$$A \to B \to C \equiv A \to (B \to C).$$

These are some examples of formulas:

$$(\bot \to \bot), \quad \forall_x(\bot \to \bot), \quad \exists_x(R(x) \vee S(x)).$$

To the Definition 2.1.5 we associate the following induction principle:

$$\frac{\begin{array}{c}\forall_{n\in\mathbb{N}}\forall_{t_1,\ldots,t_n\in\texttt{Term}}\forall_{R\in\texttt{Rel}^{(n)}}(P(R(t_1,\ldots,t_n))),\\ \forall_{A,B\in\texttt{Form}}\big(P(A) \wedge P(B) \Rightarrow \big(P(A \to B) \wedge P(A \wedge B) \wedge P(A \vee B)\big)\big),\\ \forall_{A\in\texttt{Form}}\forall_{x\in\texttt{Var}}\big(P(A) \Rightarrow P(\forall_x A) \wedge P(\exists_x A)\big)\end{array}}{\forall_{A\in\texttt{Form}}(P(A))},$$

where $P(A)$ is any property of our meta-language that concerns the set of formulas. E.g., $P(A)$ could be "the number of left parentheses occurring in A is equal to the number of right parentheses occurring in A". The induction principle for Form expresses that Form is the *least* set satisfying its defining rules.

Note that the induction principle for Form consists of formulas of our meta-theory, where the same quantifiers and logical symbols, except from the meta-theoretic implication symbol $\Rightarrow$, are used. Since the variables occurring in these meta-theoretic formulas are different from $\mathtt{Var}$, it is easy to understand from the context the difference between the formulas in $\mathtt{Form}$ and the formulas in our meta-theory. As in the case of terms, we have a recursion theorem for $\mathtt{Form}$.

PROPOSITION 2.1.6 (Recursion theorem for $\mathtt{Form}$). *Let X be a set. If there are functions*

$$F_{\mathtt{Rel}} : \{R(t_1, \ldots, t_n) \mid R \in \mathtt{Rel}^{(n)}, t_1, \ldots, t_n \in \mathtt{Term}, n \in \mathbb{N}\} \to X$$

$$F_{\to}, F_{\wedge}, F_{\vee} : X \times X \to X,$$

$$F_{\forall,x}, F_{\exists,x} : X \to X,$$

for every $x \in \mathtt{Var}$, then there is a unique function

$$F : \mathtt{Form} \to X$$

such that

$$F(R(t_1, \ldots, t_n)) \equiv F_{\mathtt{Rel}}(R(t_1, \ldots, t_n)),$$

$$F(A \to B) \equiv F_{\to}(F(A), F(B)),$$

$$F(A \wedge B) \equiv F_{\wedge}(F(A), F(B)),$$

$$F(A \vee B) \equiv F_{\vee}(F(A), F(B)),$$

$$F(\forall_x A) \equiv F_{\forall,x}(F(A)),$$

$$F(\exists_x A) \equiv F_{\exists,x}(F(A)).$$

PROOF. We work similarly to the proof of Proposition 2.1.3. $\square$

DEFINITION 2.1.7. The function $|.| : \mathtt{Form} \to \mathbb{N}$ determines the *height* $|A|$ of a formula A and it is defined by the clauses

$$|P| \equiv 0, \quad P \text{ is atomic},$$

$$|A \Box B| \equiv \max\{|A|, |B|\} + 1, \quad \Box \in \{\to, \wedge, \vee\},$$

$$|\triangle_x A| \equiv |A| + 1, \quad \triangle \in \{\forall, \exists\}.$$

In the previous definition we used the recursion theorem for $\mathtt{Form}$ with respect the following functions:

$$F_{\mathtt{Rel}}(P) \equiv 0,$$

$$F_{\Box}(a, b) \equiv \max\{a, b\} + 1,$$

$$F_{\triangle,x}(a) \equiv a + 1.$$

DEFINITION 2.1.8. The function $||.|| : \texttt{Form} \to \mathbb{N}$ determines the *length* $||A||$ of a formula A and it is defined by the clauses $||.|| : \texttt{Form} \to \mathbb{N}$ by the following conditions:

$$||P|| \equiv 1, \quad P \text{ is atomic},$$
$$||A \,\square\, B|| \equiv ||A|| + ||B||, \quad \{\to, \wedge, \vee\},$$
$$||\triangle_x A|| = 1 + ||A||, \quad \triangle \in \{\forall, \exists\}.$$

PROPOSITION 2.1.9. $\forall_{A \in \texttt{Form}}(||A|| + 1 \leq 2^{|A|+1})$.

PROOF. Exercise. $\square$

DEFINITION 2.1.10. The function $\mathrm{FV}_{\texttt{Term}} : \texttt{Term} \to \mathcal{P}^{\mathrm{fin}}(\texttt{Var})$, where $\mathcal{P}^{\mathrm{fin}}(X)$ denotes the finite subsets of some set X, expresses the set of (free) variables occurring in a term and it is defined by the clauses

$$\mathrm{FV}_{\texttt{Term}}(x) \equiv \{x\},$$
$$\mathrm{FV}_{\texttt{Term}}(c) \equiv \emptyset,$$
$$\mathrm{FV}_{\texttt{Term}}(f(t_1, \ldots, t_n)) \equiv \bigcup_{i=1}^{n} \mathrm{FV}_{\texttt{Term}}(t_i).$$

The function $\mathrm{FV}_{\texttt{Form}} : \texttt{Form} \to \mathcal{P}^{\mathrm{fin}}(\texttt{Var})$ expresses the set of *free* variables occurring in a formula and it is defined by the clauses

$$\mathrm{FV}_{\texttt{Form}}(R) = \emptyset, \quad R \in \texttt{Rel}^{(0)},$$
$$\mathrm{FV}_{\texttt{Form}}(R(t_1, \ldots, t_n)) \equiv \bigcup_{i=1}^{n} \mathrm{FV}_{\texttt{Term}}(t_i), \quad R \in \texttt{Rel}^{(n)}, n \in \mathbb{N}^+,$$
$$\mathrm{FV}_{\texttt{Form}}(A \square B) \equiv \mathrm{FV}_{\texttt{Form}}(A) \cup \mathrm{FV}_{\texttt{Form}}(B),$$
$$\mathrm{FV}_{\texttt{Form}}(\triangle_x A) \equiv \mathrm{FV}_{\texttt{Form}}(A) \setminus \{x\}.$$

If $\mathrm{FV}(A) = \emptyset$, we call A a *sentence*, or a *closed* formula.

According to Definition 2.1.10, a variable y is free in a prime formula A, if just occurs in A, it is free in $A \square B$, if it is free in A or free in B, and it is free in $\triangle_x A$, if it is free in A and $y \neq x$. E.g.,

$$\forall_y(R(y) \to S(y)), \quad \forall_y(R(y) \to \forall_z S(z))$$

are sentences, and y is free in

$$(\forall_y(R(y)) \to S(y).$$

DEFINITION 2.1.11. $\mathbb{W}(\mathcal{L})$ is the set of finite lists of symbols from the set $\texttt{Var} \cup L \cup \texttt{Rel} \cup \texttt{Fun}$. The set $\mathbb{W}(\mathcal{L})$ can be defined inductively, and its elements are called *words* of $\mathcal{L}$.

Note that $\texttt{Term}, \texttt{Form} \subset \mathbb{W}(\mathcal{L})$, and $fR \wedge g(\bot, u_8$ is a word which is neither in $\texttt{Term}$ nor in $\texttt{Form}$.

DEFINITION 2.1.12. If $s \in \texttt{Term}$ and $x \in \texttt{Var}$ are fixed, the function

$$\mathrm{Sub}_{s/x} : \texttt{Term} \to \mathbb{W}(\mathcal{L})$$
$$t \mapsto t[x := s] \equiv \mathrm{Sub}_{s/x}(t),$$

determines the word generated by substituting x from s in t and it is defined by the clauses

$$v_i[x := s] \equiv \begin{cases} s & , x \equiv v_i \\ v_i & , \text{otherwise}, \end{cases}$$
$$c[x := s] \equiv c,$$
$$f(t_1, \ldots, t_n)[x := s] \equiv f(t_1[x := s], \ldots, t_n[x := s]).$$

PROPOSITION 2.1.13. $\forall_{t \in \texttt{Term}}(t[x := s] \in \texttt{Term})$.

PROOF. Exercise. $\square$

PROPOSITION 2.1.14. $\forall_{t \in \texttt{Term}}(x \notin \mathrm{FV}(t) \Rightarrow t[x := s] \equiv t)$.

PROOF. We use induction on $\texttt{Term}$. If $t \equiv v_i$, for some $v_i \in \mathrm{Var}$, then $x \notin \mathrm{FV}(v_i) \leftrightarrow x \notin \{v_i\} \Leftrightarrow x \neq v_i$, hence $v_i[x := s] \equiv v_i$. If $t \equiv c$, for some $c \in \texttt{Const}$, then $x \notin \mathrm{FV}(c) \leftrightarrow x \notin \emptyset$, which is always the case. By definition we get $c[x := s] \equiv c$. If $t \equiv f(t_1, \ldots, t_n)$, for some $f \in \texttt{Fun}^{(n)}$ and $t_1, \ldots, t_n \in \texttt{Term}$, then $x \notin \mathrm{FV}(f(t_1, \ldots, t_n)) \leftrightarrow x \notin \mathrm{FV}(t_i)$, for every $i \in \{1, \ldots, n\}$. By the inductive hypothesis on $t_1, \ldots, t_n$ we get $t_i[x := s] \equiv t_i$, for every $i \in \{1, \ldots, n\}$. Hence,

$$f(t_1, \ldots, t_n)[x := s] \equiv f(t_1[x := s], \ldots, t_n[x := s]) \equiv f(t_1, \ldots, t_n).$$

$\square$

If we consider the formula

$$A \equiv \exists_y(\neg(y = x)),$$

then the possible substitution of x from y would generate the formula

$$\exists_y(\neg(y = y)),$$

which cannot be true in any "interpretation" of these symbols i.e., when y ranges over some collection of objects and $=$ is the equality of the objects in this collection. Hence, we need to be careful with substitution on semantical (see chapter 4), rather than syntactical, grounds. Note also that x is free in A, and if it is substituted by y, then y is bound in A. This is often called a "*capture*", and we want to avoid them.

DEFINITION 2.1.15. Let $s \in \texttt{Term}$, such that $\mathrm{FV}(s) = \{y_1, \ldots, y_m\}$, and $x \in \texttt{Var}$. If $\mathbf{2} \equiv \{0, 1\}$, the function

$$\mathrm{Free}_{s,x} : \texttt{Form} \to \mathbf{2}$$

expresses when "the variable x is substitutable (free to be substituted) from s in some formula" i.e., if $\mathrm{Free}_{s,x}(A) = 1$, then x is substitutable from s in A, and if $\mathrm{Free}_{s,x}(A) = 0$, then x is not substitutable from s in A. The function $\mathrm{Free}_{s,x}$ is defined by the clauses

$$\mathrm{Free}_{s,x}(P) \equiv 1, \quad P \text{ is atomic},$$

$$\mathrm{Free}_{s,x}(A \square B) \equiv \mathrm{Free}_{s,x}(A) \cdot \mathrm{Free}_{s,x}(B),$$

$$\mathrm{Free}_{s,x}(\triangle_y A) \equiv \begin{cases} 0 & , x = y \ \vee \ [x \neq y \ \wedge \ y \in \{y_1, \ldots, y_m\}] \\ 1, & , x \neq y \wedge x \notin \mathrm{FV}(A) \setminus \{y\} \\ \mathrm{Free}_{s,x}(A) & , x \neq y \ \wedge \ y \notin \{y_1, \ldots, y_m\} \wedge x \in \mathrm{FV}(A). \end{cases}$$

According to Definition 2.1.15, x is substitutable from s in a prime formula, since there are no quantifiers in it that can generate a capture. It is substitutable in $A \square B$, if it is substitutable both in A and B. In the case of an $\exists$, or $\forall$-formula $\triangle_y A$, if x is not free in A (which is equivalent to $x \neq y \wedge x \notin \mathrm{FV}(A) \setminus \{y\}$), then we set $\mathrm{Free}_{s,x}(\triangle_y A) \equiv 1$, since no capture is possible to be generated.

If $A \equiv \exists_y(\neg(y = x))$, then, according to Definition 2.1.15, we get

$$\mathrm{Free}_{y,x}\big(\exists_y(\neg(y = x))\big) = 0.$$

If x, y, z are distinct variables, it is easy to see that

$$\mathrm{Free}_{z,x}(R(x)) = 1,$$

$$\mathrm{Free}_{z,x}\big(\forall_z R(x)\big) = 0,$$

$$\mathrm{Free}_{f(x,z),x}\big(\forall_y S(x, y)\big) = 1,$$

$$\mathrm{Free}_{f(x,z),x}\big(\exists_z \forall_y (S(x, y) \Rightarrow R(x))\big) = 0.$$

From now on, when we define a function on `Form` that is based on a function on `Term`, as in the case of $\mathrm{FV}_{\texttt{Form}}$ and $\mathrm{FV}_{\texttt{Term}}$, we omit the subscripts and we understand from the context their domain of definition.

DEFINITION 2.1.16. If $s \in \texttt{Term}$ and $x \in \texttt{Var}$ are fixed, the function

$$\mathrm{Sub}_{s/x} : \texttt{Form} \to \mathbb{W}(\mathcal{L})$$

$$A \mapsto A[x := s] \equiv \mathrm{Sub}_{s/x}(A),$$

determines the word generated by substituting x from s in A, and it is defined as follows:

$$\text{if } \mathrm{Free}_{s,x}(A) = 0, \text{ then } A[x := s] \equiv A,$$

while if $\mathrm{Free}_{s,x}(A) = 1$, we use the following clauses:

$$R[x := s] \equiv R, \quad R \in \mathtt{Rel}^{(0)},$$

$$R(t_1, \ldots, t_n)[x := s] \equiv R(t_1[x := s], \ldots, t_n[x := s]), \quad R \in \mathtt{Rel}^{(n)}, n \in \mathbb{N}^+,$$

$$(A \Box B)[x := s] \equiv (A[x := s] \Box B[x := s]),$$

$$(\triangle_y A)[x := s] \equiv \triangle_y (A[x := s]).$$

Often, we write for simplicity $A(s)$ instead of $A[x := s]$.

Note that if $\mathrm{Free}_{s,x}(A \Box B) = 1$, then $\mathrm{Free}_{s,x}(A) = \mathrm{Free}_{s,x}(B) = 1$, and if $\mathrm{Free}_{s,x}(\triangle_y A) = 1$, then $\mathrm{Free}_{s,x}(A) = 1$.

Proposition 2.1.17. *$\forall_{A \in \mathtt{Form}}(A[x := s] \in \mathtt{Form})$.*

Proof. Exercise. □

Proposition 2.1.18. *$\forall_{A \in \mathtt{Form}}(x \notin \mathrm{FV}(A) \Rightarrow A[x := s] \equiv A)$.*

Proof. We use induction on $\mathtt{Form}$. If $A \equiv R$, for some $R \in \mathtt{Rel}^{(0)}$, then $x \notin \mathrm{FV}(R) \leftrightarrow x \notin \emptyset$, which is always the case. Since $\mathrm{Free}_{s,x}(R) = 1$, by definition of substitution we get $R[x := s] \equiv R$. If $A \equiv R(t_1, \ldots, t_n)$, for some $R \in \mathtt{Rel}^{(n)}$, $n \in \mathbb{N}^+$, and $t_1, \ldots, t_n \in \mathtt{Term}$, then $x \notin \mathrm{FV}(R(t_1, \ldots, t_n)) \leftrightarrow x \notin \bigcup_{i=1}^n \mathrm{FV}(t_i)$, for every $i \in \{1, \ldots, n\}$. By Proposition 2.1.14 we get $t_i[x := s] \equiv t_i$, for every $i \in \{1, \ldots, n\}$, hence, since $\mathrm{Free}_{s,x}(R(t_1, \ldots, t_n)) = 1$, we have that

$$R(t_1, \ldots, t_n)[x := s] \equiv R(t_1[x := s], \ldots, t_n[x := s]) \equiv R(t_1, \ldots, t_n).$$

If our formula is of the the form $A \Box B$, then $x \notin \mathrm{FV}(A \Box B) \Leftrightarrow x \notin \mathrm{FV}(A) \cup \mathrm{FV}(B) \Leftrightarrow x \notin \mathrm{FV}(A)$ and $x \notin \mathrm{FV}(B)$. If $\mathrm{Free}_{s,x}(A \Box B) = 0$, then we get immediately what we want. If $\mathrm{Free}_{s,x}(A \Box B) = 1$, then by the inductive hypothesis on A, B we get $A[x := s] \equiv A$ and $B[x := s] \equiv B$, hence by Definition 2.1.16 we have that

$$(A \Box B)[x := s] \equiv (A[x := s] \Box B[x := s]) \equiv (A \Box B).$$

If our formula is of the form $\triangle_y A$, then $x \notin \mathrm{FV}(\triangle_y A) \Leftrightarrow x \notin \mathrm{FV}(A) \setminus \{y\} \Leftrightarrow x \notin \mathrm{FV}(A)$ or $x = y$. If $x = y$, then $\mathrm{Free}_{s,x}(\triangle_y A) = 0$, hence $(\triangle_y A)[x := s] \equiv \triangle_y A$. If $x \notin \mathrm{FV}(A) \setminus \{y\}$ and $x \neq y$, then $x \notin \mathrm{FV}(A)$, and by inductive hypothesis on A we get

$$(\triangle_y A)[x := s] \equiv \triangle_y (A[x := s]) \equiv \triangle_y A.$$

If $x \in \mathrm{FV}(A)$, and $x \neq y \;\wedge\; y \notin \{y_1, \dots, y_m\}$, the required implication follows trivially. $\square$

If $\vec{x} \equiv (x_1, \dots, x_n)$ is a given n-tuple of distinct variables in `Var` and $\vec{s} \equiv (s_1, \dots, s_n)$ is a given n-tuple of terms in `Term`, for some $n \in \mathbb{N}^+$, we can define similarly for every formula A the formula $A[\vec{x} := \vec{s}]$ generated by the substitution of x_i from s_i in A, for every $i \in \{1, \dots, n\}$.

2.2. Derivations in minimal logic

To motivate the rules for natural deduction, let us start with informal proofs of some simple logical facts. We consider the following formula

$$D \equiv (A \to B \to C) \to (A \to B) \to A \to C,$$

which, according to our notational convention, is the formula

$$(A \to B \to C) \to ((A \to B) \to (A \to C)).$$

First we give an informal proof of D using the Brouwer-Heyting-Kolmogorov interpretation of $\to$ (Definition 1.1.5). According to it, a proof

$$p : (A \to B \to C) \to ((A \to B) \to (A \to C))$$

is a rule that sends a supposed proof $q : A \to B \to C$ to a proof

$$p(q) : (A \to B) \to (A \to C),$$

which is a rule that sends a proof $r : A \to B$ to a proof

$$p(q, r) \equiv [p(q)](r) : A \to C,$$

which is a rule that sends a proof $s : A$ to a proof

$$p(q, r, s) \equiv \{[p(q)](r)\}(s) : C.$$

We define this proof by

$$p(q, r, s) \equiv q(s, r(s)).$$

Another informal proof of D goes as follows: Assume $A \to B \to C$. To show $(A \to B) \to A \to C$, we assume $A \to B$. To show $A \to C$ we assume A. We show C by using the third assumption twice and we have $B \to C$ by the first assumption, and B by the second assumption. From $B \to C$ and B we obtain C. Then we obtain $A \to C$ by cancelling the assumption on A, and $(A \to B) \to A \to C$ by cancelling the second assumption; and the result follows by cancelling the first assumption.

We consider next the formula

$$E \equiv \forall_x (A \to B) \to A \to \forall_x B, \quad \text{if } x \notin \mathrm{FV}(A).$$

First we give an informal proof of E using the Brouwer-Heyting-Kolmogorov interpretation of $\to, \forall_x A$ (Definition 1.1.5), without specifying though some set X in which the variable x ranges over. According to it, a proof

$$p : \forall_x (A \to B) \to A \to \forall_x B$$

is a rule that sends a supposed proof $q : \forall_x (A \to B)$ to a proof

$$p(q) : A \to \forall_x B,$$

which is a rule that sends a proof $r : A$ to some proof

$$p(q, r) : \forall_x B.$$

The proof $q : \forall_x (A \to B)$ is understood as a family of proofs

$$q \equiv \big(q_x : A \to B\big)_x,$$

and, similarly, the required proof $p(q, r) : \forall_x B$ is a family of proofs

$$p(q, r) \equiv \big([p(q, r)]_x : B\big)_x.$$

We define this family of proofs by

$$[p(q, r)]_x \equiv q_x(r).$$

Another informal proof of E goes as follows: Assume $\forall_x (A \to B)$. To show $A \to \forall_x B$ we assume A. To show $\forall_x B$ let x be arbitrary; note that we have not made any assumptions on x. To show B we have $A \to B$ by the first assumption, and hence also B by the second assumption. Hence $\forall_x B$. Hence $A \to \forall_x B$, cancelling the second assumption. Hence E, cancelling the first assumption.

A characteristic feature of the second kind of informal proofs is that assumptions are introduced and eliminated again. At any point in time during the proof the free or "open" assumptions are known, but as the proof progresses, free assumptions may become cancelled or "closed" because of the implies-introduction rule.

We reserve the word *proof* for the informal level; a formal representation of a proof will be called a *derivation*.

An intuitive way to communicate derivations is to view them as labelled trees each node of which denotes a rule application. The labels of the inner nodes are the formulas derived as conclusions at those points, and the labels of the leaves are formulas or terms. The labels of the nodes immediately above a node k are the *premises* of the rule application. At the root of the tree we have the conclusion (or end formula) of the whole derivation. In natural deduction systems one works with *assumptions* at leaves of the tree; they can be either *open* or *closed* (cancelled). Any of these assumptions carries a *marker*. As markers we use *assumption variables* denoted $u, v, w, u_0, u_1, \dots$. The variables in `Var` will now often be called *object variables*, to distinguish them from assumption variables. If at a node below an assumption the dependency on this assumption is removed (it becomes closed), we record this by writing down the assumption variable. Since the same assumption may be used more than once (this was the case in the first example above), the assumption marked with u (written $u\colon A$) may appear many times. Of course we insist that distinct assumption formulas must have distinct markers.

An inner node of the tree is understood as the result of passing from premises to the conclusion of a given rule. The label of the node then contains, in addition to the conclusion, also the name of the rule. In some cases the rule binds or closes or cancels an assumption variable u (and hence removes the dependency of all assumptions $u\colon A$ thus marked). An application of the $\forall$-introduction rule similarly binds an object variable x (and hence removes the dependency on x). In both cases the bound assumption or object variable is added to the label of the node.

First we have an assumption rule, allowing to write down an arbitrary formula A together with a marker u:

$$u\colon A \qquad \text{assumption.}$$

The other rules of natural deduction split into introduction rules (I-rules for short) and elimination rules (E-rules) for the logical connectives which, for the time being, are just $\to$ and $\forall$. For implication $\to$ there is an introduction rule $\to^+$ and an elimination rule $\to^-$ also called *modus ponens*. The left premise $A \to B$ in $\to^-$ is called the *major* (or *main*) premise, and the right premise A the *minor* (or *side*) premise. Note that with an application of the $\to^+$-rule *all* assumptions above it marked with $u\colon A$ are cancelled (which is denoted by putting square brackets around these assumptions), and the u then gets written alongside. There may of course be other uncancelled assumptions $v\colon A$ of the same formula A, which may get cancelled at a later stage.

Definition 2.2.1. The introduction and elimination rules for implication are:

$$\dfrac{\begin{matrix}[u\colon A] \\ \mid M \\ B\end{matrix}}{A \to B}\,\to^+ u \qquad\qquad \dfrac{\begin{matrix}\mid M \\ A \to B\end{matrix} \qquad \begin{matrix}\mid N \\ A\end{matrix}}{B}\,\to^-$$

For the universal quantifier $\forall$ there is an introduction rule $\forall^+$ (again marked, but now with the bound variable x) and an elimination rule $\forall^-$ whose right premise is the term r to be substituted. The rule $\forall^+ x$ with conclusion $\forall_x A$ is subject to the following *(eigen-)variable condition* to avoid capture: the derivation M of the premise A must not contain any open assumption having x as a free variable.

$$\frac{\begin{matrix}\mid M\\ A\end{matrix}}{\forall_x A}\,\forall^+ x \qquad \frac{\begin{matrix}\mid M\\ \forall_x A\end{matrix} \qquad r \in \mathtt{Term}}{A(r)}\,\forall^-$$

For disjunction the introduction and elimination rules are

$$\frac{\begin{matrix}\mid M\\ A\end{matrix}}{A \vee B}\,\vee_0^+ \qquad \frac{\begin{matrix}\mid N\\ B\end{matrix}}{A \vee B}\,\vee_1^+ \qquad \frac{\begin{matrix}\\ \mid M\\ A \vee B\end{matrix} \qquad \begin{matrix}[u\colon A]\\ \mid N\\ C\end{matrix} \qquad \begin{matrix}[v\colon B]\\ \mid K\\ C\end{matrix}}{C}\,\vee^- u, v$$

For conjunction we have the rules

$$\frac{\begin{matrix}\mid M\\ A\end{matrix} \qquad \begin{matrix}\mid N\\ B\end{matrix}}{A \wedge B}\,\wedge^+ \qquad \frac{\begin{matrix}\mid M\\ A \wedge B\end{matrix} \qquad \begin{matrix}[u\colon A] \quad [v\colon B]\\ \mid N\\ C\end{matrix}}{C}\,\wedge^- u, v$$

and for the existential quantifier we have the rules

$$\frac{r \in \mathtt{Term} \qquad \begin{matrix}\mid M\\ A(r)\end{matrix}}{\exists_x A}\,\exists^+ \qquad \frac{\begin{matrix}\mid M\\ \exists_x A\end{matrix} \qquad \begin{matrix}[u\colon A]\\ \mid N\\ B\end{matrix}}{B}\,\exists^- x, u \text{ (var.cond.)}$$

Similar to $\forall^+ x$ the rule $\exists^- x, u$ is subject to an *(eigen-)variable condition*: in the derivation N the variable x (i) should not occur free in the formula of any open assumption other than $u\colon A$, and (ii) should not occur free in B. Again, in each of the elimination rules $\vee^-$, $\wedge^-$ and $\exists^-$ the left premise is called *major* (or *main*) premise, and the right premise is called the *minor* (or *side*) premise.

The rule $\vee^- u, v$

$$\frac{\begin{matrix}\\ \mid M\\ A \vee B\end{matrix} \qquad \begin{matrix}[u\colon A]\\ \mid N\\ C\end{matrix} \qquad \begin{matrix}[v\colon B]\\ \mid K\\ C\end{matrix}}{C}\,\vee^- u, v$$

is understood as follows: given a derivation tree for $A \vee B$ and derivation trees for C with assumption variables $u : A$ and $v : B$, respectively, a derivation tree for C is formed, such that $u : A$ and $v : B$ are cancelled. Similarly we understand the rules $\to^+ u$, $\wedge^- u, v$ and $\exists^- x, u$.

Definition 2.2.2. A formula A is called *derivable* (in *minimal logic*), written

$$\vdash A,$$

if there is a derivation of A (without free assumptions) using the natural deduction rules of Definition 2.2.1. A formula A is called *derivable from assumptions* $A_1, \dots, A_n$, written

$$\{A_1, \dots, A_n\} \vdash A, \quad \text{or simpler} \quad A_1, \dots, A_n \vdash A,$$

if there is a derivation of A with free assumptions among $A_1, \dots, A_n$. The following tree

$$\frac{u\colon A}{A}\ \texttt{ax}$$

is a derivation tree of a formula A from assumption A i.e., we always have

$$A \vdash A.$$

If $\Gamma \subseteq \texttt{Form}$, a formula A is called *derivable from* Γ, written

$$\Gamma \vdash A,$$

if A is derivable from finitely many assumptions $A_1, \dots, A_n \in \Gamma$.

Note that the rules of Definition 2.2.1 are used in the presence of free assumptions in the same way. E.g., next follows a derivation tree for C with assumption formula G:

$$\dfrac{\begin{array}{c} w\colon G \\ \mid M \\ A \vee B \end{array} \qquad \begin{array}{c} [u\colon A] \\ \mid N \\ C \end{array} \qquad \begin{array}{c} [v\colon B] \\ \mid K \\ C \end{array}}{C}\ \vee^- u, v$$

We now give derivations of the two example formulas D, E, treated informally above. Since in many cases the rule used is determined by the conclusion, we suppress in such cases the name of the rule. First we give the derivation of D:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[u\colon A\to B\to C] \qquad [w\colon A]}{B\to C} \qquad \cfrac{[v\colon A\to B] \qquad [w\colon A]}{B}}{C}}{A\to C}\,\to^+ w}{(A\to B)\to A\to C}\,\to^+ v}{(A\to B\to C)\to(A\to B)\to A\to C}\,\to^+ u$$

Next we give the derivation of E:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[u\colon \forall_x(A\to B)] \qquad x\in\texttt{Var}}{A\to B} \qquad [v\colon A]}{B}}{\forall_x B}\,\forall^+ x}{A\to\forall_x B}\,\to^+ v}{\forall_x(A\to B)\to A\to\forall_x B}\,\to^+ u$$

Note that the variable condition is satisfied: In the derivation of B the still open assumption formulas are A and $\forall_x(A\to B)$; by hypothesis x is not free in A, and by Definition 2.1.10 it is also not free in $\forall_x(A\to B)$.

Proposition 2.2.3. *The following formulas are derivable:*
(i) $A\to A$.
(ii) $A\to\neg\neg A$.
(iii) (*Brouwer*) $\neg\neg\neg A\to\neg A$.

Proof. The derivation for (i) is

$$\cfrac{\cfrac{[u\colon A]}{A}\,\texttt{ax}}{A\to A}\,\to^+ u$$

The derivation for (ii) is

$$\cfrac{\cfrac{\cfrac{[v\colon A\to\bot] \qquad [u\colon A]}{\bot}}{(A\to\bot)\to\bot}\,\to^+ v}{A\to(A\to\bot)\to\bot}\,\to^+ u$$

The derivation for (iii) is an exercise. □

Note that the formula DNS $\equiv\neg\neg A\to A$, which is known as the *double negation shift*, is in general *not* derivable in minimal logic.

PROPOSITION 2.2.4. *The following are derivable.*

(i) $(A \to B) \to \neg B \to \neg A$,
(ii) $\neg(A \to B) \to \neg B$,
(iii) $\neg\neg(A \to B) \to \neg\neg A \to \neg\neg B$,
(iv) $(\bot \to B) \to (\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)$,
(v) $\neg\neg\forall_x A \to \forall_x \neg\neg A$.

PROOF. Exercise. □

PROPOSITION 2.2.5. *We consider the following formulas:*

$$\begin{aligned}
&\mathtt{ax}\vee_0^+ \equiv A \to A \vee B,\\
&\mathtt{ax}\vee_1^+ \equiv B \to A \vee B,\\
&\mathtt{ax}\vee^- \equiv A \vee B \to (A \to C) \to (B \to C) \to C,\\
&\mathtt{ax}\wedge^+ \equiv A \to B \to A \wedge B,\\
&\mathtt{ax}\wedge^- \equiv A \wedge B \to (A \to B \to C) \to C,\\
&\mathtt{ax}\exists^+ \equiv A \to \exists_x A,\\
&\mathtt{ax}\exists^- \equiv \exists_x A \to \forall_x (A \to B) \to B \qquad (x \notin \mathrm{FV}(B)).
\end{aligned}$$

(i) *The formulas* $\mathtt{ax}\vee_0^+, \mathtt{ax}\vee_1^+$ *and* $\mathtt{ax}\vee^-$ *are equivalent, as axioms, to the rules* $\vee_0^+$, $\vee_1^+$ *and* $\vee^- u, v$ *over minimal logic.*
(ii) *The formulas* $\mathtt{ax}\wedge^+$ *and* $\mathtt{ax}\wedge^-$ *as axioms are equivalent, as axioms, to the rules* $\wedge^+$ *and* $\wedge^-$ *over minimal logic.*
(iii) *The formulas* $\mathtt{ax}\exists^+$ *and* $\mathtt{ax}\exists^-$ *are equivalent, as axioms, to the rules* $\exists^+$ *and* $\exists^- x, u$ *over minimal logic.*

PROOF. (i) First we show that from the axiom $\mathtt{ax}\vee_0^+$, a derivation of which is considered the formula itself, and a supposed derivation M of A we get the following derivation of $A \vee B$

$$\dfrac{A \to A \vee B \qquad \begin{array}{c} \mid M \\ A \end{array}}{A \vee B} \to^-$$

Similarly we show that from the formula $\mathtt{ax}\vee_1^+$ and a supposed derivation N of A we get a derivation of $A \vee B$. Next we show that from the formula $\mathtt{ax}\vee^-$ and supposed derivations M of $A \vee B$, N of C with assumption A, and K of C with assumption B we get the following derivation of C

$$\dfrac{\dfrac{\dfrac{\mathtt{ax}\vee^- \qquad \begin{array}{c}\mid M\\ A\vee B\end{array}}{(A\to C)\to(B\to C)\to C}\to^- \qquad \dfrac{\begin{array}{c}[u\colon A]\\ \mid N\\ C\end{array}}{A\to C}\to^+u}{(B\to C)\to C}\to^- \qquad \dfrac{\begin{array}{c}[v\colon B]\\ \mid K\\ C\end{array}}{B\to C}\to^+u}{C}\to^-$$

Conversely, from the rule $\vee_0^+$ we get the following derivation of $\mathtt{ax}\vee_0^+$

$$\dfrac{\dfrac{\dfrac{[u\colon A]}{A}\,\mathtt{ax}}{A\vee B}\,\vee_0^+}{A\to A\vee B}\to^+u$$

Similarly, from the rule $\vee_1^+$ we get a derivation of $\mathtt{ax}\vee_1^+$. From the elimination rule for disjunction we get the following derivation of $\mathtt{ax}\vee^-$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[u\colon A\vee B]}{A\vee B}\,\mathtt{ax} \qquad \dfrac{[v\colon A\to C] \qquad [v'\colon A]}{C} \qquad \dfrac{[w\colon B\to C] \qquad [w'\colon B]}{C}}{C}\,\vee^-v',w'}{(B\to C)\to C}\to^+w}{(A\to C)\to(B\to C)\to C}\to^+v}{A\vee B\to(A\to C)\to(B\to C)\to C}\to^+u$$

(ii) and (iii) Exercises. $\square$

Note that in the above derivation of C

$$\dfrac{\dfrac{u\colon A\vee B}{A\vee B}\,\mathtt{ax} \qquad \dfrac{[v\colon A\to C] \qquad [v'\colon A]}{C} \qquad \dfrac{[w\colon B\to C] \qquad [w'\colon B]}{C}}{C}\,\vee^-v',w'$$

we used the rule $\vee^-v', w'$ in the "extended" way described previously, where the assumption variable $u\colon A\vee B$ is still open. Of course, it will be cancelled later in the derivation of $\mathtt{ax}\vee^-$.

The notation $B\leftarrow A$ means $A\to B$.

PROPOSITION 2.2.6. *The following formulas are derivable*

(i) $(A\wedge B\to C)\leftrightarrow(A\to B\to C)$,

(ii) $(A\to B\wedge C)\leftrightarrow(A\to B)\wedge(A\to C)$,

(*iii*) $(A \vee B \to C) \leftrightarrow (A \to C) \wedge (B \to C)$,
(*iv*) $(A \to B \vee C) \leftarrow (A \to B) \vee (A \to C)$,

(*v*) $(\forall_x A \to B) \leftarrow \exists_x(A \to B)$ *if* $x \notin \mathrm{FV}(B)$,
(*vi*) $(A \to \forall_x B) \leftrightarrow \forall_x(A \to B)$ *if* $x \notin \mathrm{FV}(A)$,

(*vii*) $(\exists_x A \to B) \leftrightarrow \forall_x(A \to B)$ *if* $x \notin \mathrm{FV}(B)$,
(*viii*) $(A \to \exists_x B) \leftarrow \exists_x(A \to B)$ *if* $x \notin \mathrm{FV}(A)$.

PROOF. (i) - (vii) Exercise. A derivation of the final formula is

$$\dfrac{\dfrac{\dfrac{[u\colon \exists_x(A \to B)] \qquad \dfrac{x \in \texttt{Var} \qquad \dfrac{[w\colon A \to B] \qquad [v\colon A]}{B}}{\exists_x B}}{\exists_x B}\ \exists^- x, w}{A \to \exists_x B}\ \to^+ v}{\exists_x(A \to B) \to A \to \exists_x B}\ \to^+ u$$

The variable condition for $\exists^-$ is satisfied since the variable x (i) is not free in the formula A of the open assumption $v\colon A$, and (ii) is not free in $\exists_x B$. Of course, it is not a problem that it occurs free in $A \to B$. □

PROPOSITION 2.2.7. *If* $\Gamma, \Delta \subseteq \texttt{Form}$ *and* $A, B \in \texttt{Form}$, *the following rules hold:*

$$\dfrac{\Gamma \vdash A, \quad \Gamma \subseteq \Delta}{\Delta \vdash A}\ \texttt{ext}$$

$$\dfrac{\Gamma \vdash A, \quad \Delta \cup \{A\} \vdash B}{\Gamma \cup \Delta \vdash B}\ \texttt{cut}$$

PROOF. The $\texttt{ext}$-rule is an immediate consequence of the definition of $\Gamma \vdash A$. Suppose next that there are $C_1, \ldots, C_n \in \Gamma$ and $D_1, \ldots, D_m \in \Delta$ such that $C_1, \ldots, C_n \vdash A$ and $D_1, \ldots, D_m, A \vdash B$. The following is a derivation of B from assumptions in $\Gamma \cup \Delta$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix} u_1\colon D_1 \ldots u_m\colon D_m\ [u\colon A] \\ \mid M \end{matrix}}{B}}{A \to B}\ \to^+ u \qquad \begin{matrix} w_1\colon C_1 \ldots w_n\colon C_n \\ \mid N \\ A \end{matrix}}{B}\ \to^-$$

□

The following rules are special cases of the cut-rule for $\Gamma = \Delta$ and $\Gamma = \Delta = \emptyset$, respectively.

$$\frac{\Gamma \vdash A, \quad \Gamma \cup \{A\} \vdash B}{\Gamma \vdash B}$$

$$\frac{\vdash A, \quad A \vdash B}{\vdash B}$$

From now on, we also denote $\Gamma \vdash A$ with the tree

$$\begin{array}{c} \Gamma \\ \mid M \\ A \end{array}$$

Proposition 2.2.8. *Let* $\Gamma \subseteq \mathtt{Form}$ *and* $A, B \in \mathtt{Form}$.

(i) $\Gamma \vdash (A \to B) \Rightarrow (\Gamma \vdash A \Rightarrow \Gamma \vdash B)$.
(ii) $(\Gamma \vdash A$ *or* $\Gamma \vdash B) \Rightarrow \Gamma \vdash A \vee B$.
(iii) $\Gamma \vdash (A \wedge B) \Leftrightarrow (\Gamma \vdash A$ *and* $\Gamma \vdash B)$.
(iv) $\Gamma \vdash \forall_y A \Rightarrow \Gamma \vdash A(s)$*, for every* $s \in \mathtt{Term}$.
(v) *If* $s \in \mathtt{Term}$ *such that* $\Gamma \vdash A(s)$*, then* $\Gamma \vdash \exists_y A$.

Proof. (i) If $\Gamma \vdash (A \to B)$ and $\Gamma \vdash A$, the following is a derivation of B from Γ:

$$\dfrac{\begin{array}{c} \Gamma \\ \mid M \\ A \to B \end{array} \qquad \begin{array}{c} \Gamma \\ \mid N \\ A \end{array}}{B} \to^-$$

(ii) If $\Gamma \vdash A$, the following is a derivation of $A \vee B$ from Γ:

$$\dfrac{\begin{array}{c} \Gamma \\ \mid M \\ A \end{array}}{A \vee B} \vee_0^+$$

(iii) If $\Gamma \vdash A \wedge B$, the following is a derivation of A from Γ:

$$\dfrac{\begin{array}{c} \Gamma \\ \mid M \\ A \wedge B \end{array} \qquad \dfrac{[u\colon A][v\colon B]}{A}\,\mathtt{ax}}{A} \wedge^- u, v$$

If $\Gamma \vdash A$ and $\Gamma \vdash B$, the following is a derivation of $A \wedge B$ from Γ:

$$\dfrac{\begin{array}{c} \Gamma \\ \mid M \\ A \end{array} \qquad \begin{array}{c} \Gamma \\ \mid N \\ B \end{array}}{A \wedge B} \wedge^+$$

(iv) and (v) If $\Gamma \vdash \forall_y A$, the left derivation is a derivation of $A(s)$ from Γ, and if $\Gamma \vdash A(s)$, the right derivation is a derivation of $\exists_y A$ from Γ:

$$\dfrac{\begin{array}{c}\Gamma \\ \mid M \\ \forall_y A\end{array} \qquad s \in \mathtt{Term}}{A(s)}\ \forall^- \qquad\qquad \dfrac{s \in \mathtt{Term} \qquad \begin{array}{c}\Gamma \\ \mid M \\ A(s)\end{array}}{\exists_y A}\ \exists^+$$

$\square$

Proposition 2.2.9. *Let* $\Gamma \subseteq \mathtt{Form}$ *and* $A, B \in \mathtt{Form}$.

(i) $\Gamma \cup \{A\} \vdash B \Leftrightarrow \Gamma \vdash A \to B$.

(ii) If we define, for every $A_1, \ldots, A_n, A_{n+1} \in \mathtt{Form}$,

$$\bigwedge_{i=1}^{1} A_i \equiv A_1,$$

$$\bigwedge_{i=1}^{n+1} A_i \equiv \left(\bigwedge_{i=1}^{n} A_i\right) \wedge A_{n+1},$$

then

$$\forall_{n \in \mathbb{N}^+}\left(\forall_{A_1,\ldots,A_n,A \in \mathtt{Form}}\big(\{A_1, \ldots, A_n\} \vdash A \Leftrightarrow \vdash \left(\bigwedge_{i=1}^{n} A_i\right) \to A\big)\right).$$

Proof. (i) If $C_1, \ldots, C_n \in \Gamma$ such that $C_1, \ldots, C_n, A \vdash B$, then

$$\dfrac{\begin{array}{c}u_1\colon C_1 \ldots u_n\colon C_n\ [u\colon A] \\ \mid M \\ B\end{array}}{A \to B}\ \to^+ u$$

is a derivation of $A \to B$ from Γ. Conversely, if $C_1, \ldots, C_n \in \Gamma$ such that $C_1, \ldots, C_n, \vdash A \to B$, the following is a derivation of B from $\Gamma \cup \{A\}$:

$$\dfrac{\begin{array}{c}u_1\colon C_1 \ldots u_n\colon C_n \\ \mid M \\ A \to B\end{array} \qquad \dfrac{u\colon A}{A}\ \mathtt{ax}}{B}\ \to^-$$

(ii) We use induction on $\mathbb{N}^+$. If $n = 1$, our goal-formula becomes

$$\forall_{A,B \in \mathtt{Form}}\big(\{A\} \vdash B \Leftrightarrow \vdash A \to B\big),$$

which follows from (i) for $\Gamma = \emptyset$. Our inductive hypothesis is

$$\forall_{A_1,\ldots,A_n,A\in\texttt{Form}}\Bigg(\{A_1,\ldots,A_n\}\vdash A \Leftrightarrow \vdash\left(\bigwedge_{i=1}^{n} A_i\right)\to A\Bigg),$$

and we show

$$\forall_{A_1,\ldots,A_n,A_{n+1},A\in\texttt{Form}}\Bigg(\{A_1,\ldots,A_n,A_{n+1}\}\vdash A \Leftrightarrow \vdash\left(\bigwedge_{i=1}^{n+1} A_i\right)\to A\Bigg).$$

If we fix $A_1,\ldots,A_n,A_{n+1},A$, we have that

$$\begin{aligned}
\{A_1,\ldots,A_n,A_{n+1}\}\vdash A &\Leftrightarrow \{A_1,\ldots,A_n\}\cup\{A_{n+1}\}\vdash A\\
&\overset{(i)}{\Leftrightarrow} \{A_1,\ldots,A_n\}\vdash A_{n+1}\to A\\
&\overset{(*)}{\Leftrightarrow} \vdash\left(\bigwedge_{i=1}^{n} A_i\right)\to(A_{n+1}\to A)\\
&\overset{(**)}{\Leftrightarrow} \vdash\left(\bigwedge_{i=1}^{n} A_i\right)\wedge A_{n+1}\to A\\
&\equiv \vdash\left(\bigwedge_{i=1}^{n+1} A_i\right)\to A,
\end{aligned}$$

where $(*)$ follows by the inductive hypothesis on $A_1,\ldots,A_n$ and the formula $A_{n+1}\to A$, and $(**)$ follows by the derivation

$$\vdash (A\to B\to C)\leftrightarrow(A\wedge B\to C)$$

and the corollary of Proposition 2.2.8(i)

$$\vdash A\leftrightarrow B\Rightarrow(\vdash A\Leftrightarrow\vdash B).$$

□

2.3. Derivations in intuitionistic logic

DEFINITION 2.3.1. Let Efq be the following set of formulas:

$$\begin{aligned}
\mathrm{Efq}\equiv\{&\forall_{x_1,\ldots,x_n}(\bot\to R(x_1,\ldots,x_n))\mid n\in\mathbb{N}^+, R\in\texttt{Rel}^{(n)}, x_1,\ldots,x_n\in\texttt{Var}\}\\
&\cup\{\bot\to R\mid R\in\texttt{Rel}^{(0)}\setminus\{\bot\}\}.
\end{aligned}$$

We define when a formula A is *intuitionistically derivable* (from assumptions $\Gamma\subseteq\texttt{Form}$) , written $\vdash_i A$ $(\Gamma\vdash_i A)$, by

$$\vdash_i A\equiv \mathrm{Efq}\vdash A,$$

$$\Gamma \vdash_i A \equiv \Gamma \cup \mathrm{Efq} \vdash A.$$

The case $\bot \to \bot$ is not considered, as it is derived in minimal logic. Clearly, we have that

$$\Gamma \vdash A \Rightarrow \Gamma \vdash_i A.$$

Next we show that we can derive intuitionistically $\bot \to A$, for an *arbitrary* formula A, using the introduction rules for the logical connectives.

THEOREM 2.3.2 (Ex-falso-quodlibet). $\forall_{A \in \mathtt{Form}}\big(\vdash_i (\bot \to A)\big)$.

PROOF. If $A \equiv R(t_1, \dots, t_n)$, where $n \in \mathbb{N}^+$, $R \in \mathtt{Rel}^{(n)}$ and $t_1, \dots, t_n \in \mathtt{Term}$, the following is an intuitionistic derivation of $\bot \to R(t_1, \dots, t_n)$:

$$\dfrac{\dfrac{\forall_{x_1,\dots,x_n}(\bot \to R(x_1, \dots, x_n)) \qquad t_1 \in \mathtt{Term}}{\forall_{x_2,\dots,x_n}(\bot \to R(t_1, x_2 \dots, x_n))}\,\forall^- \qquad t_2 \in \mathtt{Term}}{\forall_{x_3,\dots,x_n}(\bot \to R(t_1, t_2, x_3 \dots, x_n))}\,\forall^-$$

$$\dots\dots\dots \qquad\qquad \dots\dots\dots$$

$$\dfrac{\forall_{x_n}(\bot \to R(t_1, \dots, t_{n-1}, x_n)) \qquad t_n \in \mathtt{Term}}{\bot \to R(t_1, t_2 \dots, t_n)}\,\forall^-$$

If we suppose that $\vdash_i (\bot \to A)$ and $\vdash_i (\bot \to B)$ i.e., that there are intuitionistic derivations M_i, N_i of $\bot \to A$ and $\bot \to B$, respectively, the following are intuitionistic derivations of $\bot \to A \to B, \bot \to A \vee B, \bot \to A \wedge B, \bot \to \forall_x A$ and $\bot \to \exists_x A$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix} \mid N_i \\ \bot \to B \end{matrix} \qquad [v\colon \bot]}{B}\,\to^-}{A \to B}\,\to^+ u\colon A}{\bot \to (A \to B)}\,\to^+ v$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix} \mid M_i \\ \bot \to A \end{matrix} \qquad [u\colon \bot]}{A}\,\to^- \qquad \dfrac{\begin{matrix} \mid N_i \\ \bot \to B \end{matrix} \qquad [u\colon \bot]}{B}\,\to^-}{A \wedge B}\,\wedge^+}{\bot \to (A \wedge B)}\,\to^+ u$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix} \mid M_i \\ \bot \to A \end{matrix} \qquad [u\colon \bot]}{A}\,\to^-}{A \vee B}\,\vee_0^+}{\bot \to (A \vee B)}\,\to^+ u \qquad\qquad \dfrac{\dfrac{\dfrac{\begin{matrix} \mid M_i \\ \bot \to A \end{matrix} \qquad [u\colon \bot]}{A}\,\to^-}{\forall_x A}\,\forall^+ x}{\bot \to \forall_x A}\,\to^+ u$$

$$\dfrac{\dfrac{x \in \mathtt{Var} \qquad \dfrac{\begin{matrix}\mid M_i \\ \bot \to A\end{matrix} \qquad [u\colon \bot]}{A}\,\to^-}{\exists_x A}\,\exists^+}{\bot \to \exists_x A}\,\to^+ u$$

Note that in the above use of the $\forall^+ x$-rule the variable condition is satisfied, as $x \notin \mathrm{FV}(\bot) \equiv \emptyset$. $\square$

2.4. Derivations in classical logic

DEFINITION 2.4.1. Let Stab be the following set of formulas:

$$\mathrm{Stab} \equiv \{\forall_{x_1,\dots,x_n}(\neg\neg R(x_1,\dots,x_n) \to R(x_1,\dots,x_n)) \mid n \in \mathbb{N}^+, R \in \mathtt{Rel}^{(n)},$$
$$x_1,\dots,x_n \in \mathtt{Var}\} \;\cup\; \{\neg\neg R \to R \mid R \in \mathtt{Rel}^{(0)} \setminus \{\bot\}\}.$$

We define when a formula A is *classically derivable* (from assumptions $\Gamma \subseteq \mathtt{Form}$) , written $\vdash_c A$ ($\Gamma \vdash_c A$), by

$$\vdash_c A \equiv \mathrm{Stab} \vdash A,$$
$$\Gamma \vdash_c A \equiv \Gamma \cup \mathrm{Stab} \vdash A.$$

The case $\neg\neg\bot \to \bot$ is not considered, as it is derived in minimal logic:

$$\dfrac{\dfrac{[v\colon (\bot \to \bot) \to \bot] \qquad \dfrac{\dfrac{[u\colon \bot]}{\bot}\,\mathtt{ax}}{\bot \to \bot}\,\to^+ u}{\bot}\,\to^-}{((\bot \to \bot) \to \bot) \to \bot}\,\to^+ v$$

Clearly, we have that

$$\Gamma \vdash A \Rightarrow \Gamma \vdash_c A.$$

PROPOSITION 2.4.2. *Let* $A \in \mathtt{Form}$ *and* $\Gamma \subseteq \mathtt{Form}$.
(*i*) $\vdash (\neg\neg A \to A) \to (\bot \to A)$.
(*ii*) $\Gamma \vdash_i A \Rightarrow \Gamma \vdash_c A$.

PROOF. (i) The required derivation is

$$\dfrac{\dfrac{\dfrac{[u\colon \neg\neg A \to A] \qquad \dfrac{\dfrac{[v\colon \bot]}{\bot}\,\mathtt{ax}}{\neg\neg A}\,\to^+ w\colon \neg A}{A}\,\to^-}{\bot \to A}\,\to^+ v}{(\neg\neg A \to A) \to \bot \to A}\,\to^+ u$$

(ii) If M is a derivation $\Gamma \vdash_i A$, then we add in all, finitely many, places in M the derivation of the element of Efq by the corresponding element of Stab, using case (i). I.e., there is some derivation $M(i)$ from case (i) such that

$$\dfrac{\begin{array}{c}\mid M(i)\\ (\neg\neg R(\vec{x}) \to R(\vec{x})) \to (\bot \to R(\vec{x}))\end{array} \qquad \dfrac{\forall_{\vec{x}}(\neg\neg R(\vec{x}) \to R(\vec{x}))\ \vec{x}}{\neg\neg R(\vec{x}) \to R(\vec{x})}\,\forall^-}{\dfrac{\bot \to R(\vec{x})}{\forall_{\vec{x}}(\bot \to R(\vec{x}))}\,\forall^+\vec{x}}\,\to^-$$

Note that the variable condition above is satisfied, and the resulted tree is a classical derivation of A. $\square$

Hence, we have that

$$\Gamma \vdash A \Rightarrow \Gamma \vdash_i A \Rightarrow \Gamma \vdash_c A.$$

A result similar to Theorem 2.3.2 can be shown only for formulas that do not involve $\vee, \exists$. These can be defined inductively as follows.

Definition 2.4.3. The formulas $\texttt{Form}^*$ without $\vee, \exists$ are defined by the following inductive rules:

$$\frac{P \text{ prime}}{P \in \texttt{Form}^*}, \qquad \frac{A, B \in \texttt{Form}^*}{(A \to B),\ (A \wedge B) \in \texttt{Form}^*}, \qquad \frac{A \in \texttt{Form}^*, \quad x \in \texttt{Var}}{\forall_x A \in \texttt{Form}^*}.$$

To the definition of $\texttt{Form}^*$ corresponds the obvious induction principle.

Lemma 2.4.4. *Let* $A, B \in \texttt{Form}$.
(i) $\vdash (\neg\neg A \to A) \to (\neg\neg B \to B) \to \neg\neg(A \wedge B) \to A \wedge B$.
(ii) $\vdash (\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B$.
(iii) $\vdash (\neg\neg A \to A) \to \neg\neg\forall_x A \to A$.

Proof. For simplicity, in the derivation to be constructed we leave out applications of $\to^+$ at the end.
(i) Left to the reader.
(ii)

$$\dfrac{u\colon \neg\neg B \to B \qquad \dfrac{\dfrac{v\colon \neg\neg(A \to B) \qquad \dfrac{\dfrac{u_1\colon \neg B \qquad \dfrac{u_2\colon A \to B \qquad w\colon A}{B}}{\bot}}{\neg(A \to B)}\,\to^+ u_2}{\bot}}{\neg\neg B}\,\to^+ u_1}{B}$$

(iii)

$$\dfrac{u\colon \neg\neg A \to A \qquad \dfrac{\dfrac{v\colon \neg\neg\forall_x A \qquad \dfrac{\dfrac{u_1\colon \neg A \qquad \dfrac{u_2\colon \forall_x A \qquad x}{A}}{\bot}}{\neg\forall_x A}\to^+ u_2}{\bot}}{\neg\neg A}\to^+ u_1}{A}$$

$\square$

THEOREM 2.4.5 (Stability). $\forall_{A\in\texttt{Form}^*}(\vdash_c \neg\neg A \to A)$.

PROOF. We use induction on $\texttt{Form}^*$. If A is atomic we work exactly as in the corresponding case of the proof of Theorem 2.3.2. Next we suppose that there are classical derivations of $\vdash_c \neg\neg A \to A, \vdash_c \neg\neg B \to B$ and we find classical derivations of $\vdash_c \neg\neg(A \to B) \to A \to B, \vdash_c \neg\neg(A \wedge B) \to A \wedge B$ and $\vdash_c \neg\neg\forall_x A \to \forall_x A$. By Lemma 2.4.4(ii) there is a derivation M of $(\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B$, and the required classical derivation is

$$\dfrac{\begin{array}{c}\mid M\\ (\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B\end{array} \qquad \begin{array}{c}\mid M_c\\ \neg\neg B \to B\end{array}}{\neg\neg(A \to B) \to A \to B}\to^-$$

By Lemma 2.4.4(i) there is a derivation N of

$$C \equiv (\neg\neg A \to A) \to (\neg\neg B \to B) \to \neg\neg(A \wedge B) \to A \wedge B,$$

and the required classical derivation is

$$\dfrac{\dfrac{\begin{array}{c}\mid N\\ C\end{array} \qquad \begin{array}{c}\mid N_c\\ \neg\neg A \to A\end{array}}{(\neg\neg B \to B) \to \neg\neg(A \wedge B) \to A \wedge B}\to^- \qquad \begin{array}{c}\mid M_c\\ \neg\neg B \to B\end{array}}{\neg\neg(A \wedge B) \to A \wedge B}\to^-$$

By Lemma 2.4.4(iii) there is a derivation K of

$$D \equiv (\neg\neg A \to A) \to \neg\neg\forall_x A \to A,$$

and the required classical derivation, where the variable condition is easy to see that it is satisfied, is

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mid K \\ D\end{matrix} \qquad \begin{matrix}\mid N_c \\ \neg\neg A \to A\end{matrix}}{\neg\neg\forall_x A \to A} \to^- \qquad u\colon \neg\neg\forall_x A}{A} \to^-}{\forall_x A} \forall^+ x}{\neg\neg\forall_x A \to \forall_x A} \to^+ u
$$

□

We distinguish between two kinds of "exists" and two kinds of "or": the "weak" or classical ones and the "strong" or non-classical ones, with constructive content. In the present context both kinds occur together and hence we must mark the distinction; we do so by writing a tilde above the weak disjunction and existence symbols thus

Definition 2.4.6. If $A, B \in \mathtt{Form}$, we define

$$A \mathrel{\tilde{\vee}} B \equiv \neg A \to \neg B \to \bot,$$

$$\tilde{\exists}_x A \equiv \neg\forall_x \neg A.$$

These are weak variants of $\vee$ and $\exists$, since

$$A \vee B \to A \mathrel{\tilde{\vee}} B, \qquad \exists_x A \to \tilde{\exists}_x A.$$

are derivable by putting $C \equiv \bot$ in $\vee^-$ and $B \equiv \bot$ in $\exists^-$. Note that Theorem 2.4.5 implies the classical derivability of the double negation shift of $A \mathrel{\tilde{\vee}} B$ and $\tilde{\exists}_x A$, if $A, B \in \mathtt{Form}^*$. By Brouwer's double negation shift of a negated formula, Proposition 2.2.3(iii), though we get the derivability of these double negation shifts in minimal logic, for every $A, B \in \mathtt{Form}$. The case $\tilde{\exists}_x A$ is immediate. For the case of $A \mathrel{\tilde{\vee}} B$ we use Brouwer's double negation shift of the negated formula $\neg(\neg A \wedge \neg B)$, the derivation

$$\vdash A \mathrel{\tilde{\vee}} B \leftrightarrow \neg(\neg A \wedge \neg B),$$

and the following fact.

Proposition 2.4.7. *Let* $C, D \in \mathtt{Form}$ *such that* $\vdash C \leftrightarrow D$. *Then*

$$\vdash (\neg\neg C \to C) \Rightarrow \vdash (\neg\neg D \to D).$$

Proof. If M is a derivation of $\neg\neg C \to C$, N a derivation of $C \to D$, and K a derivation of $D \to C$, then the following is a derivation of $\neg\neg D \to D$:

$$
\begin{prooftree}
\AxiomC{$\mid N$}
\noLine
\UnaryInfC{$C \to D$}
\AxiomC{$\mid M$}
\noLine
\UnaryInfC{$\neg\neg C \to C$}
\AxiomC{$[u\colon \neg\neg D]$}
\AxiomC{$[v\colon \neg C]$}
\AxiomC{$\mid K$}
\noLine
\UnaryInfC{$D \to C$}
\AxiomC{$[w\colon D]$}
\BinaryInfC{C}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ w$}
\UnaryInfC{$\neg D$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ v$}
\UnaryInfC{$\neg\neg C$}
\RightLabel{$\to^-$}
\BinaryInfC{C}
\BinaryInfC{D}
\RightLabel{$\to^+ u$}
\UnaryInfC{$\neg\neg D \to D$}
\end{prooftree}
$$

□

Proposition 2.4.8. *The following formulas are derivable.*

(*i*) $(\tilde{\exists}_x A \to B) \to \forall_x (A \to B)$, *if* $x \notin \mathrm{FV}(B)$.

(*ii*) $(\neg\neg B \to B) \to \forall_x (A \to B) \to \tilde{\exists}_x A \to B$, *if* $x \notin \mathrm{FV}(B)$.

(*iii*) $(\bot \to B(c)) \to (A \to \tilde{\exists}_x B) \to \tilde{\exists}_x (A \to B)$, *if* $x \notin \mathrm{FV}(A)$.

(*iv*) $\tilde{\exists}_x (A \to B) \to A \to \tilde{\exists}_x B$, *if* $x \notin \mathrm{FV}(A)$.

Proof. The following is a derivation of (*i*):

$$
\begin{prooftree}
\AxiomC{$\tilde{\exists}_x A \to B$}
\AxiomC{$[u_1\colon \forall_x \neg A]$}
\AxiomC{x}
\BinaryInfC{$\neg A$}
\AxiomC{$[w\colon A]$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_1$}
\UnaryInfC{$\neg \forall_x \neg A$}
\BinaryInfC{B}
\RightLabel{$\to^+ w$}
\UnaryInfC{$A \to B$}
\RightLabel{$\forall^+ x$}
\UnaryInfC{$\forall_x (A \to B)$}
\end{prooftree}
$$

The following is a derivation of (*ii*) without the last $\to^+$-rules:

$$
\begin{prooftree}
\AxiomC{$\neg\neg B \to B$}
\AxiomC{$\neg \forall_x \neg A$}
\AxiomC{$[u_2\colon \neg B]$}
\AxiomC{$\forall_x (A \to B)$}
\AxiomC{x}
\BinaryInfC{$A \to B$}
\AxiomC{$[u_1\colon A]$}
\BinaryInfC{B}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_1$}
\UnaryInfC{$\neg A$}
\UnaryInfC{$\forall_x \neg A$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_2$}
\UnaryInfC{$\neg\neg B$}
\BinaryInfC{B}
\end{prooftree}
$$

The following is a derivation of (iii) without the last $\to^+$-rules:

$$
\begin{prooftree}
\AxiomC{$\forall_x\neg(A\to B)$}\AxiomC{c}\BinaryInfC{$\neg(A\to B(c))$}
\AxiomC{$\bot\to B(c)$}
\AxiomC{$A\to\tilde{\exists}_x B$}\AxiomC{$u_2\colon A$}\BinaryInfC{$\tilde{\exists}_x B$}
\AxiomC{$\forall_x\neg(A\to B)$}\AxiomC{x}\BinaryInfC{$\neg(A\to B)$}
\AxiomC{$u_1\colon B$}\UnaryInfC{$A\to B$}
\BinaryInfC{$\bot$}\RightLabel{$\to^+ u_1$}\UnaryInfC{$\neg B$}\UnaryInfC{$\forall_x\neg B$}
\BinaryInfC{$\bot$}
\BinaryInfC{$B(c)$}\RightLabel{$\to^+ u_2$}\UnaryInfC{$A\to B(c)$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

Note that above we used the fact that if $x \notin \mathrm{FV}(A)$, then $A(c) \equiv A$ (Proposition 2.1.18). The following is a derivation of (iv) without the last $\to^+$-rules:

$$
\begin{prooftree}
\AxiomC{$\tilde{\exists}_x(A\to B)$}
\AxiomC{$\forall_x\neg B$}\AxiomC{x}\BinaryInfC{$\neg B$}
\AxiomC{$u_1\colon A\to B$}\AxiomC{A}\BinaryInfC{B}
\BinaryInfC{$\bot$}\RightLabel{$\to^+ u_1$}\UnaryInfC{$\neg(A\to B)$}\UnaryInfC{$\forall_x\neg(A\to B)$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

$\square$

PROPOSITION 2.4.9. *The following formulas are derivable.*

(i)
$$\forall_x(\bot\to A)\to(\forall_x A\to B)\to\forall_x\neg(A\to B)\to\neg\neg A.$$

(ii)
$$\forall_x(\neg\neg A\to A)\to(\forall_x A\to B)\to\tilde{\exists}_x(A\to B)\quad \textit{if } x\notin \mathrm{FV}(B).$$

PROOF. Writing Ax, Ay for $A(x), A(y)$ we get the following derivation M of (i) without the last $\to^+$-rules:

$$
\begin{prooftree}
\AxiomC{$\forall_x\neg(Ax\to B)$}\AxiomC{x}\BinaryInfC{$\neg(Ax\to B)$}
\AxiomC{$\forall_x Ax\to B$}
\AxiomC{$\forall_y(\bot\to Ay)$}\AxiomC{y}\BinaryInfC{$\bot\to Ay$}
\AxiomC{$u_1\colon\neg Ax$}\AxiomC{$u_2\colon Ax$}\BinaryInfC{$\bot$}
\BinaryInfC{Ay}\UnaryInfC{$\forall_y Ay$}
\BinaryInfC{B}\RightLabel{$\to^+ u_2$}\UnaryInfC{$Ax\to B$}
\BinaryInfC{$\bot$}\RightLabel{$\to^+ u_1$}\UnaryInfC{$\neg\neg Ax$}
\end{prooftree}
$$

Using this derivation M we obtain

$$\dfrac{\dfrac{\forall_x\neg(Ax\to B)\quad x}{\neg(Ax\to B)} \qquad \dfrac{\dfrac{\forall_x Ax\to B \qquad \dfrac{\dfrac{\dfrac{\forall_x(\neg\neg Ax\to Ax)\quad x}{\neg\neg Ax\to Ax} \qquad \begin{matrix}\mid M\\ \neg\neg Ax\end{matrix}}{Ax}}{\forall_x Ax}}{B}}{Ax\to B}}{\bot}$$

Note that the assumption $\forall_x(\neg\neg A \to A)$ in (ii) is used to derive the assumption $\forall_x(\bot \to A)$ in (i), since $\vdash (\neg\neg A \to A) \to \bot \to A$ (Proposition 2.4.2(i)). $\square$

Corollary 2.4.10. *If $R \in \mathtt{Rel}^{(1)}$, then* $\vdash_c \tilde{\exists}_x(R(x) \to \forall_x R(x))$.

Proof. let $A \equiv R(x)$ and $B \equiv \forall_x R(x)$ in Proposition 2.4.9(ii). $\square$

The formula $\tilde{\exists}_x(R(x) \to \forall_x R(x))$ is known as the *drinker formula*, and can be read as "in every non-empty bar there is a person such that, if this person drinks, then everybody drinks". The next proposition on weak disjunction is similar to Proposition 2.4.8.

Proposition 2.4.11. *The following are derivable.*

$$\begin{aligned}
&(A \tilde{\vee} B \to C) \to (A \to C) \wedge (B \to C),\\
(\neg\neg C \to C) \to\ &(A \to C) \to (B \to C) \to A \tilde{\vee} B \to C,\\
(\bot \to B) \to\ &(A \to B \tilde{\vee} C) \to (A \to B) \tilde{\vee} (A \to C),\\
&(A \to B) \tilde{\vee} (A \to C) \to A \to B \tilde{\vee} C,\\
(\neg\neg C \to C) \to\ &(A \to C) \tilde{\vee} (B \to C) \to A \to B \to C,\\
(\bot \to C) \to\ &(A \to B \to C) \to (A \to C) \tilde{\vee} (B \to C).
\end{aligned}$$

Proof. Exercise. $\square$

The weak disjunction and the weak existential quantifier satisfy the same axioms as the strong variants, if one restricts the conclusion of the elimination axioms to formulas without $\vee, \exists$.

Proposition 2.4.12. *The following are derivable.*

$$\begin{aligned}
&\vdash A \to A \tilde{\vee} B,\\
&\vdash B \to A \tilde{\vee} B,\\
&\vdash_c A \tilde{\vee} B \to (A \to C) \to (B \to C) \to C \quad (C \in \mathtt{Form}^*),
\end{aligned}$$

$$\vdash A \to \tilde{\exists}_x A,$$

$$\vdash_c \tilde{\exists}_x A \to \forall_x (A \to B) \to B \quad (x \notin \mathrm{FV}(B),\ B \in \mathtt{Form}^*).$$

PROOF. Left to the reader. □

2.5. The Gödel-Gentzen translation

DEFINITION 2.5.1. The *Gödel-Gentzen translation* is the mapping

$$^g : \mathtt{Form} \to \mathtt{Form}$$

$$A \mapsto A^g$$

defined by the following clauses

$$\begin{aligned}
\bot^g &\equiv \bot, \\
R^g &\equiv \neg\neg R, \quad R \in \mathtt{Rel}^{(0)} \setminus \{\bot\}, \\
(R\vec{t}\,)^g &\equiv \neg\neg R\vec{t}, \quad R \in \mathtt{Rel}^{(n)}, n \in \mathbb{N}^+, \vec{t} \in \mathtt{Term}^n, \\
(A \circ B)^g &\equiv A^g \circ B^g, \quad \circ \in \{\to, \wedge\}, \\
(\forall_x A)^g &\equiv \forall_x A^g, \\
(A \vee B)^g &\equiv A^g \mathbin{\tilde{\vee}} B^g, \\
(\exists_x A)^g &\equiv \tilde{\exists}_x A^g.
\end{aligned}$$

If $\Gamma \subseteq \mathtt{Form}$, we define the set

$$\Gamma^g \equiv \{C^g \mid C \in \Gamma\}.$$

It is immediate to see that

$$(\neg A)^g \equiv \neg A^g,$$

$$(A \to B \to C)^g \equiv A^g \to B^g \to C^g.$$

PROPOSITION 2.5.2. *Let $x \in \mathtt{Var}$ and $s \in \mathtt{Term}$.*

(i) $\forall_{A \in \mathtt{Form}} \big(A^g \in \mathtt{Form}^*\big)$.

(ii) $\forall_{A \in \mathtt{Form}} \big(\mathrm{FV}(A) = \mathrm{FV}(A^g)\big)$.

(iii) $\forall_{A \in \mathtt{Form}} \big(\mathrm{Free}_{s,x}(A) = \mathrm{Free}_{s,x}(A^g)\big)$.

(iv) $\forall_{A \in \mathtt{Form}} \big((A[x := s])^g = A^g[x := s]\big)$.

PROOF. Exercise. □

Since $R\vec{t}$ is not in the range $\mathtt{Form}^g$ of the mapping ^{g}, we get that $\mathtt{Form}^g$ is a proper subset of $\mathtt{Form}^*$.

DEFINITION 2.5.3. The *negative* formulas $\texttt{Form}^-$ of $\texttt{Form}$, or the *negative fragment* of $\texttt{Form}$, is defined by the following inductive rules:

$$\frac{}{\bot \in \texttt{Form}^-}, \quad \frac{P \text{ prime}}{P \to \bot \in \texttt{Form}^-}, \quad \frac{A, B \in \texttt{Form}^-}{(A \circ B) \in \texttt{Form}^-}, \quad \frac{A \in \texttt{Form}^-, \quad x \in \texttt{Var}}{\forall_x A \in \texttt{Form}^-},$$

where $\circ \in \{\to, \wedge\}$. To the definition of $\texttt{Form}^-$ corresponds the obvious induction principle.

PROPOSITION 2.5.4. *The following hold:*
(i) $\forall_{A \in \texttt{Form}^-}\big(A \in \texttt{Form}^*\big)$.
(ii) $\forall_{A \in \texttt{Form}}\big(A^g \in \texttt{Form}^-\big)$.

PROOF. Exercise. $\square$

Because of Proposition 2.5.4(ii) the Gödel-Gentzen translation is also called the *negative* translation.

Since $R\vec{t} \to \bot$ is not in the range $\texttt{Form}^g$, we get that

$$\texttt{Form}^g \subsetneq \texttt{Form}^- \subsetneq \texttt{Form}^*.$$

Since $A^g \in \texttt{Form}^*$, by Theorem 2.4.5 we get that $\vdash_c \neg\neg A^g \to A^g$. For the formulas in $\texttt{Form}^*$ of the form A^g one can prove though the minimal derivability of their stability. First we show this for the formulas in $\texttt{Form}^-$.

PROPOSITION 2.5.5. $\forall_{A \in \texttt{Form}^-}\big(\vdash \neg\neg A \to A\big)$.

PROOF. By induction on $\texttt{Form}^-$. If $A \equiv \bot$, we use $\vdash \neg\neg\bot \to \bot$. If $A \equiv \neg R\vec{t}$ with R distinct from $\bot$, we must show $\neg\neg\neg R\vec{t} \to \neg R\vec{t}$, which is a special case of $\vdash \neg\neg\neg B \to \neg B$, Proposition 2.2.3(iii). Next we suppose that $\vdash \neg\neg A \to A$, $\vdash \neg\neg B \to B$ and we show $\vdash \neg\neg(A \to B) \to (A \to B)$. If

$$C \equiv (\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B,$$

we use Lemma 2.4.4(ii) as follows:

$$\frac{\begin{matrix} \mid M \\ C \end{matrix} \qquad \begin{matrix} \mid N \\ \neg\neg B \to B \end{matrix}}{\neg\neg(A \to B) \to A \to B}$$

For the derivation of $\vdash \neg\neg(A \wedge B) \to (A \wedge B)$ we use Lemma 2.4.4(i) in a similar manner. If

$$D \equiv (\neg\neg A \to A) \to \neg\neg\forall_x A \to A,$$

for the derivation of $\vdash \neg\neg\forall_x A \to \forall_x A$ we use Lemma 2.4.4(iii) as follows:

$$\cfrac{\cfrac{\cfrac{D \qquad \begin{matrix} \mid M \quad\quad \mid K \\ \neg\neg A \to A \end{matrix}}{\neg\neg\forall_x A \to A} \qquad u\colon \neg\neg\forall_x A}{\cfrac{A}{\forall_x A}\,\forall^+ x}}{\neg\neg\forall_x A \to \forall_x A}\,\to^+ u$$

It is immediate to check that the variable condition is satisfied in the previous use of the rule $\forall^+ x$. □

Corollary 2.5.6. $\forall_{A\in\mathtt{Form}}\big(\vdash \neg\neg A^g \to A^g\big)$.

Proof. Immediately from Propositions 2.5.4(i) and 2.5.5. □

The Gödel-Gentzen translation is important because of the next theorem (Theorem 2.5.7). According to it, a classical derivation $\Gamma \vdash_c A$ is *translated* to a minimal derivation $\Gamma^g \vdash A^g$. In the proof we will use induction on the set of derivations $\mathcal{D}$. Recall that Definition 2.2.1, extended, for every $A \in \mathtt{Form}$, with the derivation M_A

$$\frac{u\colon A}{A}\,\mathtt{ax}$$

is a definition of $M \in \mathcal{D}$ *and* of the set $\mathtt{Assumptions}(M)$ of all formulas that are used as assumptions[1] in M, *and* of the formula $\mathtt{root}(M)$ that is actually derived in M, and is the *root* of the tree M. Of course, $\mathtt{Assumptions}(M_A) = \{A\}$ and $\mathtt{root}(M_A) = A$. We also denote by $\mathtt{Assumptions}^*(M)$ the set of all assumptions of M that do not belong in Stab. To the inductive definition of $\mathcal{D}$ corresponds a cumbersome to write in full induction principle. If P is a property of our metatheory on $\mathcal{D}$, its first two premises are:

$$\forall_{A\in\mathtt{Form}}(P(M_A),$$

and

$$\forall_{M\in\mathcal{D}}\forall_{n\in\mathbb{N}^+}\forall_{C_1,\ldots,C_n\in\mathtt{Form}}\forall_{N\in\mathcal{D}}\forall_{m\in\mathbb{N}^+}\forall_{D_1,\ldots,D_m\in\mathtt{Form}}$$
$$\Big(\exists_{A,B\in\mathtt{Form}}(\mathtt{root}(M) = A \to B) \wedge \mathtt{Assumptions}(M) \subseteq \{C_1,\ldots,C_n\}$$
$$\wedge\ \mathtt{root}(N) = A \wedge \mathtt{Assumptions}(N) \subseteq \{D_1,\ldots,D_m\}$$
$$\wedge\ P(M) \wedge P(N) \Rightarrow P(\to^-(M,N)\Big),$$

[1]Since a derivation M is a "finite" and "completed" object, the finite set $\mathtt{Assumptions}(M)$ is uniquely determined.

where $\to^-(M, N)$ denotes the derivation resulting from M and N with the use of $\to^-$. Of course, the conclusion of the induction principle on $\mathcal{D}$ is

$$\forall_{M \in \mathcal{D}}(P(M).$$

Note that the relation $\Gamma \vdash A$, defined by

$$\Gamma \vdash A \equiv \exists_{M \in \mathcal{D}}(\mathtt{Assumptions}(M) \subseteq \Gamma \ \wedge \ \mathtt{root}(M) = A)$$

is not itself inductively defined, but it is defined through the inductively defined set $\mathcal{D}$.

Theorem 2.5.7. *If* $\Gamma \subseteq \mathtt{Form}$ *and* $A \in \mathtt{Form}$*, then*

$$\Gamma \vdash_c A \Rightarrow \Gamma^g \vdash A^g.$$

Proof. It suffices to show, that if we have a derivation M

$$\begin{array}{c} u_1 \colon C_1 \dots u_n \colon C_n \ s_1 \colon S_1 \dots s_m \colon S_m \\ \mid M \\ A \end{array}$$

where $S_1, \dots, S_m \in \mathrm{Stab}$ and $C_1, \dots, C_n \notin \mathrm{Stab}$, there is a derivation M^g

$$\begin{array}{c} {u_1}^g \colon {C_1}^g \dots {u_n}^g \colon {C_n}^g \\ \mid M^g \\ A^g \end{array}$$

This derivation M^g with $\mathtt{Assumptions}(M^g) = \{{C_1}^g, \dots, {C_n}^g\}$ is minimal, since it is easy to see that

$$\mathtt{Form}^g \cap \mathrm{Stab} = \emptyset.$$

Hence, we prove by induction on $\mathcal{D}$ the following formula

$$\forall_{M \in \mathcal{D}} \exists_{M^g \in \mathcal{D}} \Big(\mathtt{Assumptions}(M^g) = [\mathtt{Assumptions}^*(M)]^g$$

$$\wedge \ \mathtt{root}(M^g) = \mathtt{root}(M)^g \Big).$$

($\mathtt{ax}$) If $S \in \mathrm{Stab}$, then for the derivation M_S

$$\frac{u \colon S}{S}\,\mathtt{ax}$$

we have that $\mathtt{Assumptions}^*(M_S) = \emptyset$. Using Proposition 2.2.3(iii), and since

$$[\forall_{\vec{x}}(\neg\neg R\vec{x} \to R\vec{x}]^g \equiv \forall_{\vec{x}}(\neg\neg\neg\neg R\vec{x} \to \neg\neg R\vec{x}\,)$$

the required minimal derivation M_{S^g} is the following:

$$\dfrac{\begin{array}{c}\mid N\\ \neg\neg\neg\neg R\vec{x} \to \neg\neg R\vec{x}\end{array}}{\forall_{\vec{x}}(\neg\neg\neg\neg R\vec{x} \to \neg\neg R\vec{x})}\;\forall^+\vec{x}$$

If $A \notin \mathrm{Stab}$, then for the derivation M_A

$$\dfrac{u\colon A}{A}\;\mathtt{ax}$$

the required minimal derivation M_{S^g} is the following:

$$\dfrac{u^g\colon A^g}{A^g}\;\mathtt{ax}$$

From now on we omit for simplicity the assumptions $s_1\colon S_1 \dots s_m\colon S_m$, where $S_1, \dots, S_m \in \mathrm{Stab}$, from the given classical derivation, and we write only the assumptions $u_1\colon C_1 \dots u_n\colon C_n$, where $C_1, \dots, C_m \notin \mathrm{Stab}$.

($\to^+$) If we consider the following derivation and the inductive hypothesis

$$\dfrac{\begin{array}{c}[u\colon A]\; u_1\colon C_1 \dots u_n\colon C_n\\ \mid M\\ B\end{array}}{A \to B}\;\to^+ u \qquad \text{and} \qquad \begin{array}{c}u^g\colon A^g\; {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g\\ \mid M^g\\ B^g\end{array}$$

we get the required derivation

$$\dfrac{\begin{array}{c}[u^g\colon A^g]\; {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g\\ \mid M^g\\ B^g\end{array}}{A^g \to B^g}\;\to^+ u^g$$

($\to^-$) If we consider the derivation

$$\dfrac{\begin{array}{c}u_1\colon C_1 \dots u_n\colon C_n\\ \mid M\\ A \to B\end{array} \qquad \begin{array}{c}v_1\colon D_1 \dots v_m\colon D_m\\ \mid N\\ A\end{array}}{B}\;\to^-$$

with the inductive hypotheses

$$\begin{array}{c}{u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g\\ \mid M^g\\ (A \to B)^g\end{array} \qquad \text{and} \qquad \begin{array}{c}{v_1}^g\colon {D_1}^g \dots {v_m}^g\colon {D_m}^g\\ \mid N^g\\ A^g\end{array}$$

we get the required derivation

$$\dfrac{\begin{array}{c} u_1{}^g\colon C_1{}^g \dots u_n{}^g\colon C_n{}^g \\ \mid M^g \\ A^g \to B^g \end{array} \qquad \begin{array}{c} v_1{}^g\colon D_1{}^g \dots v_m{}^g\colon D_m{}^g \\ \mid N^g \\ A^g \end{array}}{B^g} \to^-$$

($\forall^+$) If we consider the derivation and the inductive hypothesis

$$\begin{array}{c} u_1\colon C_1 \dots u_n\colon C_n \\ \mid M \\ \dfrac{A}{\forall_x A}\,\forall^+ x \end{array} \qquad \text{and} \qquad \begin{array}{c} u_1{}^g\colon C_1{}^g \dots u_n{}^g\colon C_n{}^g \\ \mid M^g \\ A^g \end{array}$$

with the variable condition $x \notin \mathrm{FV}(C_1) \wedge \dots \wedge x \notin \mathrm{FV}(C_n)$, we get the required derivation

$$\begin{array}{c} u_1{}^g\colon C_1{}^g \dots u_n{}^g\colon C_n{}^g \\ \mid M^g \\ \dfrac{A^g}{\forall_x A^g}\,\forall^+ x \end{array}$$

where the variable condition $x \notin \mathrm{FV}(C_1^g) \wedge \dots \wedge x \notin \mathrm{FV}(C_n^g)$, is satisfied, since by Proposition 2.5.2(ii) $\mathrm{FV}(C_i) = \mathrm{FV}(C_i{}^g)$, for every $i \in \{1, \dots, n\}$.
($\forall^-$) If we consider the derivation and the inductive hypothesis

$$\dfrac{\begin{array}{c} u_1\colon C_1 \dots u_n\colon C_n \\ \mid M \\ \forall_x A \end{array} \qquad r \in \texttt{Term}}{A(r)}\,\forall^- \qquad \text{and} \qquad \begin{array}{c} u_1{}^g\colon C_1{}^g \dots u_n{}^g\colon C_n{}^g \\ \mid M^g \\ (\forall_x A)^g \end{array}$$

by Proposition 2.5.2(iv) we get the required derivation

$$\dfrac{\begin{array}{c} u_1{}^g\colon C_1{}^g \dots u_n{}^g\colon C_n{}^g \\ \mid M^g \\ \forall_x A^g \end{array} \qquad r \in \texttt{Term}}{A^g(r) = A(r)^g}\,\forall^-$$

($\wedge^+$) If we consider the derivation

$$\dfrac{\begin{array}{c} u_1\colon C_1 \dots u_n\colon C_n \\ \mid M \\ A \end{array} \qquad \begin{array}{c} v_1\colon D_1 \dots v_m\colon D_m \\ \mid N \\ B \end{array}}{B}\,\wedge^+$$

with the inductive hypotheses

$$\begin{array}{c} {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g \\ \mid M^g \\ A^g \end{array} \quad \text{and} \quad \begin{array}{c} {v_1}^g\colon {D_1}^g \dots {v_m}^g\colon {D_m}^g \\ \mid N^g \\ B^g \end{array}$$

we get the required derivation

$$\dfrac{\begin{array}{c} {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g \\ \mid M^g \\ A^g \end{array} \qquad \begin{array}{c} {v_1}^g\colon {D_1}^g \dots {v_m}^g\colon {D_m}^g \\ \mid N^g \\ B^g \end{array}}{B^g}\,\wedge^+$$

$(\wedge^-)$ If we consider the derivation

$$\dfrac{\begin{array}{c} u_1\colon C_1 \dots u_n\colon C_n \\ \mid M \\ A \wedge B \end{array} \qquad \begin{array}{c} [u\colon A]\ [v\colon B] \\ \mid N \\ C \end{array}}{C}\,\wedge^- u, v$$

with the inductive hypotheses

$$\begin{array}{c} {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g \\ \mid M^g \\ (A \wedge B)^g \end{array} \quad \text{and} \quad \begin{array}{c} u^g\colon A^g\ v^g\colon B^g \\ \mid N^g \\ C^g \end{array}$$

we get the required derivation

$$\dfrac{\begin{array}{c} {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g \\ \mid M^g \\ A^g \wedge B^g \end{array} \qquad \begin{array}{c} [u^g\colon A^g]\ [v^g\colon B^g] \\ \mid N^g \\ C^g \end{array}}{C^g}\,\wedge^- u^g, v^g$$

$(\vee_0^+)$ If we consider the derivation and the inductive hypothesis

$$\begin{array}{c} u_1\colon C_1 \dots u_n\colon C_n \\ \mid M \\ \dfrac{A}{A \vee B}\,\vee_0^+ \end{array} \quad \text{and} \quad \begin{array}{c} {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g \\ \mid M^g \\ A^g \end{array}$$

we get the required derivation of $(A \vee B)^g \equiv \neg A^g \to \neg B^g \to \bot$

$$\dfrac{\dfrac{\dfrac{u\colon A^g \to \bot \qquad \begin{array}{c} {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g \\ \mid M^g \\ A^g \end{array}}{\bot}\,\to^-}{\neg B^g \to \bot}\,\to^+ v\colon \neg B^g}{\neg A^g \to \neg B^g \to \bot}\,\to^+ u$$

For the rules $\vee_1^+$ we work similarly.
($\vee^-$) Here for simplicity we use the notate $w\colon \Gamma$ for $w_1\colon C_1 \dots w_n\colon C_n$, and $w'\colon \Delta$ for $w_1{}'\colon D_1 \dots w_m{}'\colon D_m$, and $w''\colon E$ for $w_1{}''\colon E_1 \dots w_k{}''\colon E_m$. If we consider the derivation

$$\dfrac{\begin{matrix} w\colon \Gamma \\ \mid M \\ A \vee B \end{matrix} \qquad \begin{matrix} [u\colon A]\ w'\colon \Delta \\ \mid N \\ C \end{matrix} \qquad \begin{matrix} [v\colon B]\ w''\colon E \\ \mid K \\ C \end{matrix}}{C}\ \vee^- u, v$$

with the inductive hypotheses

$$\begin{matrix} w^g\colon \Gamma^g \\ \mid M^g \\ (A \vee B)^g \end{matrix} \qquad \begin{matrix} u^g\colon A^g\ w'^g\colon \Delta^g \\ \mid N^g \\ C^g \end{matrix} \qquad \begin{matrix} v^g\colon B^g\ w''^g\colon E^g \\ \mid K^g \\ C^g \end{matrix}$$

and since by Proposition 2.4.11 there is a derivation Λ of the formula

$$D \equiv (\neg\neg C^g \to C^g) \to (A^g \to C^g) \to (B^g \to C^g) \to A^g \mathbin{\tilde{\vee}} B^g \to C^g,$$

and by Corollary 2.5.6 there is a derivation Ξ of $\neg\neg C^g \to C^g$, then if $D' \equiv (A^g \to C^g) \to (B^g \to C^g) \to A^g \mathbin{\tilde{\vee}} B^g \to C^g$, and $D'' \equiv (B^g \to C^g) \to A^g \mathbin{\tilde{\vee}} B^g \to C^g$, we get from assumptions Γ^g, Δ^g and E^g the required derivation of C^g

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} \mid \Lambda \\ D \end{matrix} \qquad \begin{matrix} \mid \Xi \\ \neg\neg C^g \to C^g \end{matrix}}{D'} \qquad \dfrac{\begin{matrix} [u^g\colon A^g] \\ \mid N^g \\ C^g \end{matrix}}{A^g \to C^g}}{D''} \qquad \dfrac{\begin{matrix} [v^g\colon B^g] \\ \mid K^g \\ C^g \end{matrix}}{B^g \to C^g}}{A^g \mathbin{\tilde{\vee}} B^g \to C^g} \qquad \begin{matrix} \Gamma^g \\ \mid M^g \\ A^g \mathbin{\tilde{\vee}} B^g \end{matrix}}{C^g}$$

where, for convenience, we omitted to write assumptions $w'^g\colon \Delta^g$ above N^g and assumptions $w''^g\colon E^g$ above K^g.
($\exists^+$) If we consider the derivation and the inductive hypothesis

$$\dfrac{r \in \mathtt{Term} \qquad \begin{matrix} u_1\colon C_1 \dots u_n\colon C_n \\ \mid M \\ A(r) \end{matrix}}{\exists_x A}\ \exists^+ \qquad \text{and} \qquad \begin{matrix} u_1{}^g\colon C_1{}^g \dots u_n{}^g\colon C_n{}^g \\ \mid M^g \\ A(r)^g = A^g(r) \end{matrix}$$

we get the required derivation of $(\exists_x A)^g \equiv \forall_x(A^g \to \bot) \to \bot$

$$\dfrac{\dfrac{\dfrac{[u\colon \forall_x(A^g \to \bot)] \qquad r \in \texttt{Term}}{A^g(r) \to \bot}\,\forall^- \qquad \begin{matrix} u_1{}^g\colon C_1{}^g \dots u_n{}^g\colon C_n{}^g \\ \mid M^g \\ A^g(r) \end{matrix}}{\bot}\,\to^-}{\forall_x(A^g \to \bot) \to \bot}\,\to^+ u$$

($\exists^-$) Again for simplicity we use the notate $w\colon \Gamma$ for $w_1\colon C_1 \dots w_n\colon C_n$, and $w'\colon \Delta$ for $w_1{}'\colon D_1 \dots w_m{}'\colon D_m$. Let the derivation

$$\dfrac{\begin{matrix} w\colon \Gamma \\ \mid M \\ \exists_x A \end{matrix} \qquad \begin{matrix} [u\colon A]\ w'\colon \Delta \\ \mid N \\ B \end{matrix}}{B}\,\exists^- x, u$$

such that $x \notin \mathrm{FV}(\Delta)$ and $x \notin \mathrm{FV}(B)$. The inductive hypotheses are

$$\begin{matrix} w^g\colon \Gamma^g \\ \mid M^g \\ \tilde{\exists}_x A^g \end{matrix} \qquad \text{and} \qquad \begin{matrix} u^g\colon A^g\ w'\colon \Delta^g \\ \mid N^g \\ B^g \end{matrix}$$

As $x \notin \mathrm{FV}(B^g) = \mathrm{FV}(B)$, by Proposition 2.4.8(ii) there is a derivation Λ of

$$D \equiv (\neg\neg B^g \to B^g) \to \tilde{\exists}_x A^g \to \forall_x(A^g \to B^g) \to B^g.$$

Since by Corollary 2.5.6 there is a derivation Ξ of $\neg\neg B^g \to B^g$, then if $D' \equiv \tilde{\exists}_x A^g \to \forall_x(A^g \to B^g) \to B^g$ and $D'' \equiv \forall_x(A^g \to B^g) \to B^g$, we get the required derivation of B^g

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mid \Lambda \\ D\end{matrix} \qquad \begin{matrix}\mid \Xi \\ \neg\neg B^g \to B^g\end{matrix}}{D'} \qquad \begin{matrix} w\colon \Gamma^g \\ \mid M^g \\ \tilde{\exists}_x A^g \end{matrix}}{D''}\,\to^- \qquad \dfrac{\dfrac{\begin{matrix} [u^g\colon A^g]\ w'\colon \Delta^g \\ \mid N^g \\ B^g \end{matrix}}{A^g \to B^g}\,\to^+ u^g}{\forall_x(A^g \to B^g)}\,\forall^+ x}{B^g}\,\to^-$$

from assumptions Γ^g and Δ^g. Note that the variable condition is satisfied in the above use of $\forall^+ x$, since $x \notin \mathrm{FV}(\Delta^g) = \mathrm{FV}(\Delta)$. □

Definition 2.5.8. The *height* $|M|$ of M is the maximum length of a branch in M, where if B is a branch of M, then its length is the number of its nodes minus 1.

For the following derivation tree M

$$\dfrac{\dfrac{\dfrac{\forall_x\neg(Ax \to B) \quad x}{\neg(Ax \to B)} \qquad \dfrac{\dfrac{\forall_x Ax \to B \qquad \dfrac{\dfrac{\dfrac{\forall_y(\bot \to Ay) \quad y}{\bot \to Ay} \qquad \dfrac{u_1\colon \neg Ax \quad u_2\colon Ax}{\bot}}{Ay}}{\forall_y Ay}}{B}}{Ax \to B}\, \to^+ u_2}{\bot}}{\neg\neg Ax}\, \to^+ u_1$$

we have that $|M| = 7$, since the length of its longest branch $\{\neg\neg Ax, \bot, Ax \to B, B, \forall_y Ay, Ay, \bot, Ax\}$ is $8 - 1 = 7$. Clearly, $|M_A| = 1$, and $|M| \geq 2$, for all other elements M of $\mathcal{D}$.

Corollary 2.5.9. $\forall_{M \in \mathcal{D}}(|M^g| \geq |M|)$.

Proof. By induction on $\mathcal{D}$ and inspection of the proof of Theorem 2.5.7. $\square$

Definition 2.5.10. We say that minimal logic is *consistent*, if there is no derivation $\vdash \bot$. If there is such derivation, we say that minimal logic is *inconsistent*. Similarly we define the consistency and inconsistency of intuitionisitc and classical logic. A pair of logics, like $(\vdash, \vdash_c)$, or $(\vdash, \vdash_i)$, or $(\vdash_c, \vdash_i)$, is a pair of *equiconsistent* logics, if the consistency of one logic of the pair is equivalent to the consistency of the other.

Corollary 2.5.11. *The following hold.*
(i) *If minimal logic is consistent, then classical logic and intuitionistic logic are consistent.*
(ii) *If classical logic is consistent, then minimal logic and intuitionistic logic are consistent.*
(iii) *If intuitionistic logic is consistent, then minimal logic and classical logic are consistent.*
(iv) *The pairs* $(\vdash, \vdash_c)$, $(\vdash, \vdash_i)$, *and* $(\vdash_c, \vdash_i)$ *are pairs of equiconsistent logics.*

Proof. (i) If in Theorem 2.5.7 we set $\Gamma \equiv \emptyset$ and $A \equiv \bot$, we get

$$(*) \quad \vdash_c \bot \Rightarrow \vdash \bot^g \equiv \bot.$$

Suppose that there is a derivation $\vdash_c \bot$. Then there is a derivation $\vdash \bot$, which contradicts our hypothesis. Hence, there is no $\vdash_c \bot$. We have already shown that Proposition 2.4.2 implies the implications

$$(**) \quad \vdash \bot \Rightarrow \vdash_i \bot \Rightarrow \vdash_c \bot.$$

Using $(*)$ we get $\vdash_i \bot \Rightarrow \vdash_c \bot \Rightarrow \vdash \bot$, hence, if there is a derivation $\vdash_i \bot$, there is a derivation $\vdash \bot$.
(ii) It follows immediately from $(**)$.
(iii) The consistency of minimal logic follows from $(**)$, and the consistency of classical logic follows from $(*)$ and $(**)$.
(iv) It follows immediately from $(i) - (iii)$. $\square$

In general, we cannot show that $\forall_{A \in \mathtt{Form}}\big(\vdash A \leftrightarrow A^g\big)$, since it is not always the case that $\vdash R\vec{t} \to \neg\neg R\vec{t}$; if $R(x, y) \equiv x < y$, then by our discussion after Corollary 1.2.19 we cannot expect to find a constructive i.e., intuitionistic, derivation of $\neg\neg(x < y) \to x < y$, since from $\neg(x \geq y)$ we cannot derive in general that $x < y$. Classically though, we can derive the following equivalence.

LEMMA 2.5.12. $\forall_{A \in \mathtt{Form}^*}\big(\vdash_c A \leftrightarrow A^g\big)$.

PROOF. Exercise. $\square$

The next theorem is a partial converse to Theorem 2.5.7.

THEOREM 2.5.13. *If* $\Gamma \subseteq \mathtt{Form}^*$ *and* $A \in \mathtt{Form}^*$, *then*

$$\Gamma^g \vdash A^g \Rightarrow \Gamma \vdash_c A.$$

PROOF. Without loss of generality we suppose a minimal derivation

$$\begin{array}{c} {u_1}^g\colon {C_1}^g \dots {u_n}^g\colon {C_n}^g \\ \mid M \\ A^g \end{array}$$

and by Proposition 2.2.9(i) and Lemma 2.5.12 there are classical derivations

$$\begin{array}{c} u_1\colon C_1 \\ \mid {M_c}^{(1)} \\ {C_1}^g \end{array} \quad \dots \quad \begin{array}{c} u_n\colon C_n \\ \mid {M_c}^{(n)} \\ {C_n}^g \end{array} \qquad \begin{array}{c} \mid N_c \\ A^g \to A \end{array}$$

The following is a classical derivation of A with assumptions $C_1, \dots, C_n$

$$\dfrac{\begin{array}{c} \mid N_c \\ A^g \to A \end{array} \qquad \dfrac{\dfrac{\begin{array}{c} [{u_1}^g\colon {C_1}^g] \dots [{u_n}^g\colon {C_n}^g] \\ \mid M \\ A^g \end{array}}{{C_1}^g \to \dots \to {C_n}^g \to A^g} \qquad \begin{array}{c} u_1\colon C_1 \\ \mid {M_c}^{(1)} \\ {C_1}^g \end{array} \qquad \begin{array}{c} u_n\colon C_n \\ \mid {M_c}^{(n)} \\ {C_n}^g \end{array}}{A^g}}{A}\to^-$$

where the successive implication-introduction rules $\to^+ {u_n}^g, \dots, \to^+ {u_1}^g$ and the subsequent implication eliminations are written as one rule each. $\square$

PROPOSITION 2.5.14. *The following formulas are derivable.*
(i) $(A \mathbin{\tilde{\vee}} B) \leftrightarrow \neg\neg(A \vee B)$.
(ii) $(\tilde{\exists}_x A \leftrightarrow \neg\neg(\exists_x A)$.

PROOF. Exercise. $\square$

Because of these equivalences, the following translation of Kolmogorov is expected to be equivalent to the Gödel-Gentzen translation.

DEFINITION 2.5.15. The *Kolmogorov negative translation* is the mapping

$$^k : \mathtt{Form} \to \mathtt{Form}$$
$$A \mapsto A^k$$

defined by the following clauses

$$\begin{aligned}
\perp^k &\equiv \perp, \\
R^k &\equiv \neg\neg R, \quad R \in \mathtt{Rel}^{(0)} \setminus \{\perp\}, \\
(R\vec{t}\,)^k &\equiv \neg\neg R\vec{t}, \quad R \in \mathtt{Rel}^{(n)}, n \in \mathbb{N}^+, \vec{t} \in \mathtt{Term}^n, \\
(A \Box B)^k &\equiv \neg\neg(A^k \Box B^k), \quad \Box \in \{\to, \wedge, \vee\}, \\
(\triangle_x A)^k &\equiv \neg\neg(\triangle_x A^k), \quad \triangle \in \{\forall, \exists\}.
\end{aligned}$$

If $\Gamma \subseteq \mathtt{Form}$, we define the set

$$\Gamma^k \equiv \{C^k \mid C \in \Gamma\}.$$

Note that the range $\mathtt{Form}^k$ of the mapping ^{k} is not included in $\mathtt{Form}^*$, as the range $\mathtt{Form}^g$ of the mapping ^{g}.

PROPOSITION 2.5.16. $\forall_{A \in \mathtt{Form}}\big(\vdash (A^g \leftrightarrow A^k)\big)$.

PROOF. Exercise. $\square$

By Proposition 2.4.7 and the above result we get the minimal derivability of the stability of A^k, although $\mathtt{Form}^k$ is not included in $\mathtt{Form}^-$.

COROLLARY 2.5.17. *If* $\Gamma \subseteq \mathtt{Form}$ *and* $A \in \mathtt{Form}$*, then*

$$\Gamma \vdash_c A \Rightarrow \Gamma^k \vdash A^k.$$

PROOF. Exercise. $\square$

DEFINITION 2.5.18. If $A \in \mathtt{Form}$, we define the set

$$\begin{aligned}
O_A &\equiv \{C \in \mathtt{Form} \mid A \vdash C\} \\
&= \{C \in \mathtt{Form} \mid \vdash A \to C\}.
\end{aligned}$$

LEMMA 2.5.19. *Let* $A, C \in \texttt{Form}$.
(i) $A \in O_A$.
(ii) $C \in O_A \Leftrightarrow O_C \subseteq O_A$.
(iii) $\vdash A \leftrightarrow C \Leftrightarrow O_C = O_A$.

PROOF. (i) Since $\vdash A \to A$, we get $A \in O_A$.
(ii) Let $D \in O_C$ i.e., $\vdash C \to D$. Since by hypothesis we also have that $\vdash A \to C$, then by the cut-rule of Proposition 2.2.7 for $\Gamma = \Delta = \{A\}$

$$\frac{\{A\} \vdash C, \quad \{A\} \cup \{C\} \vdash D}{\{A\} \vdash D}\ \texttt{cut}$$

we get $\vdash A \to D$ i.e., $D \in O_A$. Conversely, if $O_C \subseteq O_A$, then by (i) we get $C \in O_C$, hence $C \in O_A$.
(iii) By (ii) the hypothesis $O_C = O_A$ is equivalent to $C \in O_A$ and $A \in O_C$, hence to $\vdash A \leftrightarrow C$. $\square$

PROPOSITION 2.5.20. *The collection of sets*

$$\mathcal{B} \equiv \{O_A \mid A \in \texttt{Form}\} \cup \{\emptyset, \texttt{Form}\}$$

is a basis for a topology $\mathcal{T}(\mathcal{B})$ *on* Form.

PROOF. For this it suffices to show[2] that if $A, B, C \in \texttt{Form}$ such that $C \in O_A \cap O_B$, there is some $D \in \texttt{Form}$ such that

$$C \in O_D \subseteq O_A \cap O_B.$$

The hypothesis $C \in O_A \cap O_B$ implies that $A \vdash C$ and $B \vdash C$ i.e., $C \in O_A$ and $C \in O_B$, hence by Lemma 2.5.19(ii) we get $O_C \subseteq O_A$ and $O_C \subseteq O_B$. Hence $C \in O_C \subseteq O_A \cap O_B$. $\square$

We denote the resulting topological space as $\mathcal{F} \equiv (\texttt{Form}, \mathcal{T}(\mathcal{B}))$. It is easy to see that this space does not behave well with respect to the separation properties. E.g., it is not T_1, since $A \wedge A$ is in the complement $\{A\}^C$ of $\{A\}$, which is not open; if there was some $C \in \texttt{Form}$ such that $A \wedge A \in O_C \subseteq \{A\}^C$, then $O_{A \wedge A} \subseteq O_C \subseteq \{A\}^C$, but $A \in O_{A \wedge A}$ and $A \notin \{A\}^C$.

PROPOSITION 2.5.21. *The Gödel-Gentzen translation* $^g : \texttt{Form} \to \texttt{Form}$ *and the Kolmogorov translation* $^k : \texttt{Form} \to \texttt{Form}$ *are continuous functions from* $\mathcal{F}$ *to* $\mathcal{F}$.

[2] Here we use the basic fact that if a collection $\mathcal{B}$ of subsets of some set X satisfies the property: for every $x \in X$ and $B_i, B_j \in \mathcal{B}$, there is some $B_k \in \mathcal{B}$ such that $x \in B_k \subseteq B_i \cap B_j$, then $\mathcal{B}$ is a basis for some topology $\mathcal{T}(\mathcal{B})$ on X. This topology $\mathcal{T}(\mathcal{B})$ is unique and the smallest topology on X that includes $\mathcal{B}$ (see [**8**], Theorem 3.2).

Proof. We prove the continuity of the Gödel-Gentzen translation and, because of Corollary 2.5.17, the proof of the continuity of the Kolmogorov translation is similar.

By definition, a function $f : X \to Y$ between two topological spaces X, Y is continuous, if the inverse image $f^{-1}(O)$ of every open set O in Y is open in X. If $\mathcal{B}$ is a basis for Y, it is easy to see that f is continuous if and only if inverse image $f^{-1}(B)$ of every basic open set B in $\mathcal{B}$ is open in X. Clearly, ${}^{g-1}(\texttt{Form}) = \texttt{Form} \in \mathcal{T}(\mathcal{B})$ and ${}^{g-1}(\emptyset) = \emptyset \in \mathcal{T}(\mathcal{B})$. If $A \in \texttt{Form}$,

$$\begin{aligned} {}^{g-1}(O_A) &\equiv \{B \in \texttt{Form} \mid B^g \in O_A\} \\ &= \{B \in \texttt{Form} \mid \ A \vdash B^g\}. \end{aligned}$$

Let $B \in {}^{g-1}(O_A)$ i.e., $A \vdash B^g$. We show that

$$B \in O_B \subseteq {}^{g-1}(O_A),$$

hence the set ${}^{g-1}(O_A)$ is open, as the union of the open sets O_B, for every $B \in {}^{g-1}(O_A)$. The membership $B \in O_B$ follows from Lemma 2.5.19(i). Next we fix some $C \in O_B$ i.e., $\vdash B \to C$, and we show that $C \in {}^{g-1}(O_A)$ i.e., $A \vdash C^g$. By Theorem 2.5.7 we get

$$\vdash B \to C \Rightarrow \vdash B^g \to C^g,$$

hence the following derivation tree

$$\dfrac{\begin{matrix} \\ \mid M \\ B^g \to C^g \end{matrix} \qquad \begin{matrix} A \\ \mid N \\ B^g \end{matrix}}{C^g} \to^-$$

is a derivation $A \vdash C^g$. $\square$

2.6. Notes

In this chapter we draw a lot from [**22**], Chapter 1.

The main subject of Mathematical Logic is mathematical proof. In this chapter we deal with the basics of formalizing such proofs. The system we pick for the representation of proofs is Gentzen's natural deduction from [**10**]. As the name says this is a *natural* notion of formal proof, which means that the way proofs are represented corresponds very much to the way a careful mathematician writing out all details of an argument would proceed anyway. Moreover, formal proofs in natural deduction are closely related (via the so-called *Curry-Howard correspondence*) to terms in typed lambda calculus. This provides us not only with a compact notation for logical derivations (which otherwise tend to become somewhat unmanagable tree-like structures), but also opens up a route to applying the computational techniques which underpin lambda calculus.

The Gödel-Gentzen translation was introduced from Gödel in [**11**], and independently from Gentzen in [**9**]. The Kolmogorov translation was introduced even earlier in [**18**], but it was not known neither to Gödel nor to Gentzen. The Curry-Howard correspondence dates back to [**7**] and somewhat later Howard, published only in [**15**], who noted that the types of the combinators used in combinatory logic are exactly the Hilbert style axioms for minimal propositional logic.

CHAPTER 3

Normalization

3.1. The Curry-Howard correspondence

Since natural deduction derivations can be notationally cumbersome, it will be convenient to represent them as typed "derivation terms", where the derived formula is the "type" of the term (and displayed as a superscript). This representation goes under the name of *Curry-Howard correspondence*.

DEFINITION 3.1.1. We associate to each derivation $M \in \mathcal{D}$ a *derivation term* $M^{\mathtt{root}(M)}$ according to Tables 1 and 2. We denote by $\mathtt{Term}(\mathcal{D})$ the set of derivation terms and by $\mathtt{Var}_a$ the set of assumption variables.

Every derivation term carries a formula as its type. However, we shall usually leave these formulas implicit and write derivation terms without them. Notice that every derivation term of Table 1 can be written uniquely in one of the forms

$$u\vec{M} \mid \lambda_v M \mid (\lambda_v M)N\vec{L},$$

where $u \in \mathtt{Var}_a$ is an assumption variable, $v \in \mathtt{Var} \cup \mathtt{Var}_a$,

$$\vec{L} \equiv (L_0, \dots, L_{|\vec{L}|-1}),$$

where each $\vec{L}_i$ and M, N are derivation terms or (object) terms. Moreover we use the following notational conventions:

$$MNK \equiv (MN)K,$$

$$M\vec{L} \equiv M(L_1 \dots L_n) \equiv (\dots((ML_1)L_2)\dots L_{n-1})L_n,$$

$$u\vec{L} \equiv u(L_1 \dots L_n) \equiv (\dots((uL_1)L_2)\dots L_{n-1})L_n.$$

The derivation terms for $\vee$, $\wedge$ and $\exists$ are given in Table 2. To a derivation with assumptions $A_1, \dots, A_n$ we correspond a derivation term according to Table 3.

For simplicity, in the above M stands for both a derivation tree and a derivation term. One could also use the notation t_M for the derivation term

Derivation	Term
$\dfrac{u\colon A}{A}\,\texttt{ax}$	u^A
$\dfrac{\begin{array}{c}[u\colon A]\\ \mid M\\ B\end{array}}{A\to B}\to^+ u$	$(\lambda_{u^A} M^B)^{A\to B}$
$\dfrac{\begin{array}{c}\mid M\\ A\to B\end{array}\qquad\begin{array}{c}\mid N\\ A\end{array}}{B}\to^-$	$(M^{A\to B}N^A)^B$
$\dfrac{\begin{array}{c}\mid M\\ A\end{array}}{\forall_x A}\forall^+ x$ (with var.cond.)	$(\lambda_x M^A)^{\forall_x A}$ (with var.cond.)
$\dfrac{\begin{array}{c}\mid M\\ \forall_x A(x)\end{array}\qquad r\in\texttt{Term}}{A(r)}\forall^-$	$(M^{\forall_x A(x)} r)^{A(r)}$

TABLE 1. Derivation terms for $\to$ and $\forall$

corresponding to the derivation M and define

$$\texttt{Type}(t_M) = \texttt{root}(M).$$

Since a derivation term contains both assumption and object variables, we define recursively its free assumption and object variables. In what follows we restrict our definitions and results to the derivation terms of Table 1.

Derivation	Term
$\dfrac{\begin{array}{c}\mid M\\ A\end{array}}{A\vee B}\,\vee_0^+ \qquad \dfrac{\begin{array}{c}\mid N\\ B\end{array}}{A\vee B}\,\vee_1^+$	$\big(\vee_{0,B}^+ M^A\big)^{A\vee B} \quad \big(\vee_{1,A}^+ N^B\big)^{A\vee B}$
$\dfrac{\begin{array}{c}\mid M\\ A\vee B\end{array} \quad \begin{array}{c}[u\colon A]\\ \mid N\\ C\end{array} \quad \begin{array}{c}[v\colon B]\\ \mid K\\ C\end{array}}{C}\,\vee^- u,v$	$\big(M^{A\vee B}(u^A.N^C, v^B.K^C)\big)^C$
$\dfrac{\begin{array}{c}\mid M\\ A\end{array} \quad \begin{array}{c}\mid N\\ B\end{array}}{A\wedge B}\,\wedge^+$	$\langle M^A, N^B\rangle^{A\wedge B}$
$\dfrac{\begin{array}{c}\mid M\\ A\wedge B\end{array} \quad \begin{array}{c}[u\colon A]\quad [v\colon B]\\ \mid N\\ C\end{array}}{C}\,\wedge^- u,v$	$\big(M^{A\wedge B}(u^A, v^B.N^C)\big)^C$
$\dfrac{r\in \texttt{Term} \quad \begin{array}{c}\mid M\\ A(r)\end{array}}{\exists_x A}\,\exists^+$	$\big(\exists_{x,A}^+ r M^{A(r)}\big)^{\exists_x A}$
$\dfrac{\begin{array}{c}\mid M\\ \exists_x A\end{array} \quad \begin{array}{c}[u\colon A]\\ \mid N\\ B\end{array}}{B}\,\exists^- x,u$ (var.cond.)	$\big(M^{\exists_x A}(x, u^A.N^B)\big)^B$ (var. cond.)

TABLE 2. Derivation terms for $\vee$, $\wedge$ and $\exists$

Derivation	Term
$u_1\colon A_1 \dots u_n\colon A_n$ $\mid M$ B	$(M(u_1^{A_1}, \dots, u_n^{A_n}))^B$

TABLE 3. Derivation term for a derivation with assumptions

DEFINITION 3.1.2. The set of free assumption variables $\mathrm{FV}_a(M)$ of a derivation term in Table 1 is defined by the following clauses:

$$\mathrm{FV}_a(U) \equiv \{u\},$$

$$\mathrm{FV}_a(\lambda_u M) \equiv \mathrm{FV}_a(M) \setminus \{u\},$$

$$\mathrm{FV}_a(MN) \equiv \mathrm{FV}_a(M) \cup \mathrm{FV}_a(N),$$

$$\mathrm{FV}_a(\lambda_x M) \equiv \mathrm{FV}_a(M),$$

$$\mathrm{FV}_a(Mr) \equiv \mathrm{FV}_a(M).$$

The set of free object variables $\mathrm{FV}_o(M)$ of a derivation term in Table 1 is defined by the following clauses:

$$\mathrm{FV}_o(U) \equiv \emptyset,$$

$$\mathrm{FV}_o(\lambda_u M) \equiv \mathrm{FV}_o(M),$$

$$\mathrm{FV}_o(MN) \equiv \mathrm{FV}_o(M) \cup \mathrm{FV}_o(N),$$

$$\mathrm{FV}_o(\lambda_x M) \equiv \mathrm{FV}_o(M) \setminus \{x\},$$

$$\mathrm{FV}_o(Mr) \equiv \mathrm{FV}_o(M) \cup \mathrm{FV}(r).$$

The sets of free assumption variables $\mathrm{FV}_a(M)$ and object variables of a derivation term in Table 3 are defined by

$$\mathrm{FV}_a(M(u_1^{A_1}, \dots, u_n^{A_n})) \equiv \{u_1, \dots, u_n\},$$

$$\mathrm{FV}_o(M(u_1^{A_1}, \dots, u_n^{A_n})) \equiv \bigcup_{i=1}^{n} FV(A_i).$$

Similarly, we can define the free assumption and object variables for the derivation terms in Table 2. Note e.g., that

$$u^A \notin \mathrm{FV}_a(M^{\exists_x A}(x, u^A.N^B)), \quad x \notin \mathrm{FV}_o(M^{\exists_x A}(x, u^A.N^B)).$$

Note that if an assumption variable is cancelled in a derivation M, it is not in $\mathrm{FV}_a(t_M)$. We can formulate now the variable condition for the derivation term $\lambda_x M$ in the language of derivation terms as follows:

$$x \notin \{\mathrm{FV}(\texttt{Type}(u)) \mid u \in \mathrm{FV}_a(M)\}.$$

As in Definition 2.1.15 we define when $w \in \texttt{Var}_a$ is substitutable from some derivation term K in a derivation term M. The idea behind this definition is again the avoidance of "capture", which in this case is turning an open assumption of K into a cancelled one after the substitution in M.

Definition 3.1.3. Let $K \in \texttt{Term}(D)$ and $w \in \texttt{Var}_a$. The function $\mathrm{Free}_{K,w} : \texttt{Term}(\mathcal{D}) \to \mathbf{2}$, for the terms of Table 1, is defined by the clauses

$$\mathrm{Free}_{K,w}(u) \equiv 1,$$

$$\mathrm{Free}_{K,w}(\lambda_u M) \equiv \begin{cases} 0 & , w = u \ \vee \ [w \neq u \ \wedge \ u \in \mathrm{FV}_a(K)] \\ 1, & , w \neq u \ \wedge \ w \notin \mathrm{FV}_a(M) \setminus \{u\} \\ \mathrm{Free}_{K,w}(M) & , w \neq u \wedge \ u \notin \mathrm{FV}_a(K) \wedge w \in \mathrm{FV}_a(M), \end{cases}$$

$$\mathrm{Free}_{K,w}(MN) \equiv \mathrm{Free}_{K,w}(M) \cdot \mathrm{Free}_{K,w}(N),$$

$$\mathrm{Free}_{K,w}(\lambda_x M) \equiv \mathrm{Free}_{K,w}(M) \equiv \mathrm{Free}_{K,w}(Mr).$$

If $s \in \texttt{Term}$, one can define in a similar way the function $\mathrm{Free}_{s,x} : \texttt{Term}(\mathcal{D}) \to \mathbf{2}$, expressing when $x \in \texttt{Var}$ is substitutable from s in some $M \in \texttt{Term}(\mathcal{D})$.

According to Definition 3.1.3, w is substitutable from K in u, since there are no λ-terms in it that can generate a capture.

Definition 3.1.4. If $K \in \texttt{Term}(\mathcal{D})$ and $w \in \texttt{Var}_a$, the function

$$\mathrm{Sub}_{K/w} : \texttt{Term}(\mathcal{D}) \to \texttt{Term}(\mathcal{D})$$

$$M \mapsto M[w := K] \equiv \mathrm{Sub}_{K/w}(M),$$

determines the derivation term generated by substituting w from K in M, and it is defined as follows:

$$\text{if } \mathrm{Free}_{K,w}(M) = 0, \text{ then } M[w := K] \equiv M,$$

while if $\mathrm{Free}_{K,w}(M) = 1$, we use the following clauses:

$$u[w := K] \equiv \begin{cases} K & , w = u \\ u & , w \neq u, \end{cases}$$

$$(\lambda_u M)[w := K] \equiv \lambda_u(M[w := K]),$$

$$(MN)[w := K] \equiv (M[w := K])(N[w := K]),$$
$$(\lambda_x M)[w := K] \equiv \lambda_x (M[w := K]),$$
$$(Mr)[w := K] \equiv (M[w := K])r.$$

As in the case of formulas (Proposition 2.1.18), we have the following.

PROPOSITION 3.1.5. $\forall_{M \in \mathtt{Term}(\mathcal{D})} \big(w \notin \mathrm{FV}_a(M) \Rightarrow M[w := K] \equiv M \big)$.

PROOF. Left to the reader. □

Similarly, one can define the function $\mathrm{Sub}_{s/x} : \mathtt{Term}(\mathcal{D}) \to \mathtt{Term}(\mathcal{D})$,

$$M \mapsto M[x := s] \equiv \mathrm{Sub}_{s/x}(M),$$

which gives the derivation term generated by substituting x from s in M.

3.2. Reductions of derivation terms

DEFINITION 3.2.1. A derivation term of the form $(\lambda_v M)N\vec{L}$ is called a β-*redex* (for "reducible expression"). It can be reduced by a "conversion". A *conversion* removes a detour in a derivation, i.e., an elimination immediately following an introduction. We consider the following conversions, for derivations written in tree notation and also as derivation terms.

$\to$-*conversion.*

$$\dfrac{\dfrac{\begin{array}{c}[u\colon A]\\ \mid M\\ B\end{array}}{A \to B}\,\to^+ u \qquad \begin{array}{c}\mid N\\ A\end{array}}{B}\,\to^- \qquad \mapsto_\beta \qquad \begin{array}{c}\mid N\\ A\\ \mid M\\ B\end{array}$$

or written as derivation terms

$$(\lambda_u M(u^A)^B)^{A \to B} N^A \mapsto_\beta M(N^A)^B,$$

where $M(N) \equiv M[u := N]$.

$\forall$-*conversion.*

$$\dfrac{\dfrac{\begin{array}{c}\mid M\\ A(x)\end{array}}{\forall_x A(x)}\,\forall^+ x \qquad r \in \mathtt{Term}}{A(r)}\,\forall^- \qquad \mapsto_\beta \qquad \begin{array}{c}\mid M'\\ A(r)\end{array}$$

or written as derivation terms

$$(\lambda_x M(x)^{A(x)})^{\forall_x A(x)} r \mapsto_\beta M(r),$$

where $M(r) \equiv M[x := r]$. The *closure* of the conversion relation $\mapsto_\beta$ is defined by

(a) If $M \mapsto_\beta M'$, then $M \to M'$.
(b) If $M \to M'$, then also $MN \to M'N$, $NM \to NM'$, $\lambda_v M \to \lambda_v M'$ (*inner reductions*).

To the definition of $\to$ corresponds the following induction principle:

$$\begin{aligned}
\forall_{M,M' \in \mathtt{Term}(\mathcal{D})}\Big(M \to M' \\
\wedge\ \forall_{M,M' \in \mathtt{Term}(\mathcal{D})}(M \mapsto_\beta M' \Rightarrow P(M,M')) \\
\wedge\ \forall_{M,M',N \in \mathtt{Term}(\mathcal{D})}(M \to M' \wedge P(M,M') \Rightarrow P(MN, M'N)) \\
\wedge\ \forall_{M,M',N \in \mathtt{Term}(\mathcal{D})}(M \to M' \wedge P(M,M') \Rightarrow P(NM, NM')) \\
\wedge\ \forall_{M,M' \in \mathtt{Term}(\mathcal{D})}\forall_{u \in \mathtt{Var}_a}(M \to M' \wedge P(M,M') \Rightarrow P(\lambda_u M, \lambda_u M') \\
\wedge\ \forall_{M,M' \in \mathtt{Term}(\mathcal{D})}\forall_{x \in \mathtt{Var}}(M \to M' \wedge P(M,M') \Rightarrow P(\lambda_x M, \lambda_x M') \\
\Rightarrow P(M,M')\Big),
\end{aligned}$$

where $P \subseteq \mathtt{Term}(\mathcal{D}) \times \mathtt{Term}(\mathcal{D})$. This induction principle expresses that $\to$ is the least relation on $\mathtt{Term}(\mathcal{D})$ that includes $\mapsto_\beta$ and is closed under inner reductions. It is immediate to see with the use of this induction principle that if $M \to M'$, then "*M reduces in one step to M'*", i.e., M' is obtained from M by replacement of (an occurrence of) a redex K of M by a *conversum* K' of M', i.e., by a single conversion. Later we shall give a formal definition of the occurrence of a redex in a term (see Definition 3.2.16). Note also that, both in the definition of $\to$ and the formulation of its induction principle we simplify our writing by suppressing in the condition that M, M' and N have types that make the applications $MN, M'N$ and NM, NM' well-defined. For simplicity in the rest we always assume these conditions without mentioning them.

If we suppose that the types of the assumption variables in the type

$$(\lambda_w \lambda_{w'} \lambda_{w''}(ww''(w'w'')))(\lambda_u \lambda_v\, u)(\lambda_{u'} \lambda_{v'}\, u')$$

are such that its type is well-defined, we have that

$$\begin{aligned}
&(\lambda_w \lambda_{w'} \lambda_{w''}(ww''(w'w'')))(\lambda_u \lambda_v\, u)(\lambda_{u'} \lambda_{v'}\, u') \to \\
&(\lambda_{w'} \lambda_{w''}((\lambda_u \lambda_v\, u)w''(w'w'')))(\lambda_{u'} \lambda_{v'}\, u') \qquad \to \\
&(\lambda_{w'} \lambda_{w''}((\lambda_v\, w'')(w'w'')))(\lambda_{u'} \lambda_{v'}\, u') \qquad \to \\
&(\lambda_{w'} \lambda_{w''}\, w'')(\lambda_{u'} \lambda_{v'}\, u') \qquad \to \\
&\lambda_{w''}\, w''.
\end{aligned}$$

DEFINITION 3.2.2. If $R \subseteq X \times X$ is a relation on X, the *reflexive closure* R^0 of R is defined by the clauses:
(R^0_1) $\forall_{x,y \in X}(R(x,y) \Rightarrow R^0(x,y))$.
(R^0_2) $\forall_{x \in X}(R^0(x,x))$.
The *transitive closure* R^+ of R is defined by the clauses:
(R^+_1) $\forall_{x,y \in X}(R(x,y) \Rightarrow R^+(x,y))$.
(R^+_2) $\forall_{x,y,z \in X}(R(x,y) \wedge R^+(y,z) \Rightarrow R^+(x,z))$.
The *reflexive and transitive closure* R^* of R is defined by the clauses:
(R^*_1) $\forall_{x,y \in X}(R(x,y) \Rightarrow R^*(x,y))$.
(R^*_2) $\forall_{x \in X}(R^*(x,x))$.
(R^*_3) $\forall_{x,y,z \in X}(R(x,y) \wedge R^*(y,z) \Rightarrow R^*(x,z))$.

The induction principle for R^0 is[1]

$$\forall_{x,y \in X}(R^0(x,y) \wedge \forall_{x,y \in X}(R(x,y) \Rightarrow P(x,y)) \wedge \forall_{x \in X}(P(x,x)) \Rightarrow P(x,y)).$$

The induction principle for R^+ is

$$\begin{gathered}\forall_{x,y \in X}\Big(R^+(x,y) \ \wedge\ \forall_{x,y \in X}(R(x,y) \Rightarrow P(x,y)) \\ \wedge\ \forall_{x,y,z \in X}\big(R(x,y) \wedge R^+(y,z) \wedge P(y,z) \Rightarrow P(x,z)\big) \\ \Rightarrow P(x,y)\Big).\end{gathered}$$

PROPOSITION 3.2.3. *If R^+ is the transitive closure of $R \subseteq X \times X$, then*

$$\forall_{x,y,z \in X}(R^+(x,y) \wedge R^+(y,z) \Rightarrow R^+(x,z)).$$

PROOF. Exercise. $\square$

DEFINITION 3.2.4. The relation $\to^+$ ("*properly reduces to*") is the transitive closure of $\to$, and $\to^*$ ("*reduces to*") is the reflexive and transitive closure of $\to$. The relation $\to^*$ is said to be the notion of reduction *generated* by $\mapsto_\beta$.

PROPOSITION 3.2.5. *Let $M, M' \in \mathtt{Term}(\mathcal{D})$.*
(i) $M \to M' \Rightarrow \mathtt{Type}(M) = \mathtt{Type}(M')$.
(ii) $M \to^* M' \Rightarrow \mathtt{Type}(M) = \mathtt{Type}(M')$.

PROOF. Exercise. $\square$

[1]Note that predicate P competes R^+ on the pairs (x,y) that already belong to R^+. In this way we gain maximum generality in the formulation of the induction principle for R^+. The same attitude is followed in the formulation of every induction principle of an inductive definition.

PROPOSITION 3.2.6. *If* $M, M', N \in \mathtt{Term}(\mathcal{D})$*, such that* $M \to^* M'$*, then*
(i) $MN \to^* M'N$*,*
(ii) $NM \to^* NM'$*,*
(iii) $\lambda_u M \to^* \lambda_u M'$*.*

PROOF. Exercise. □

COROLLARY 3.2.7. *If* $M^{A \to B}, M', N^A, N' \in \mathtt{Term}(\mathcal{D})$*, such that* $M \to^* M'$ *and* $N \to^* N'$*, then*

$$MN \to^* M'N'.$$

PROOF. By Proposition 3.2.5(ii) we have that $\mathtt{Type}(M') = A \to B$ and $\mathtt{Type}(N') = A$. By Proposition 3.2.6 we get $MN \to^* M'N$ and $M'N \to^* M'N'$, therefore $MN \to^* M'N'$. □

PROPOSITION 3.2.8. *Let* $M, N, K \in \mathtt{Term}(\mathcal{D})$*,* $w \in \mathrm{FV}_a(M)$*,* $u \neq w$ *and* $\mathrm{Free}_{K,u}(N) = 1$*. Then*

$$\mathrm{Free}_{K,u}(M) = 1 \Rightarrow \mathrm{Free}_{K,u}(M[w := N]) = 1.$$

PROOF. Left to the reader. □

PROPOSITION 3.2.9. *Let* $M, N, K \in \mathtt{Term}(\mathcal{D})$*,* $u, w \in \mathrm{FV}_a(M)$*,* $u \in \mathrm{FV}_a(N)$*,* $u \neq w$ *and* $\mathrm{Free}_{K,u}(M) = 1 = \mathrm{Free}_{K,u}(N)$*. Then*[2]

$$\big(M[w := N]\big)[u := K] \equiv \big(M[u := K]\big)[w := N[u := K]].$$

PROOF. Left to the reader. □

PROPOSITION 3.2.10. *Let* $M, K \in \mathtt{Term}(\mathcal{D})$*,* $u \in \mathrm{FV}_a(M)$*,* $x \in \mathrm{FV}_o(M)$ *and* $r \in \mathtt{Term}$*.Then*

$$\mathrm{Free}_{K,u}(M) = 1 \Rightarrow \mathrm{Free}_{K,u}(M[x := r]) = 1.$$

PROOF. Left to the reader. □

PROPOSITION 3.2.11. *Let* $M, M' \in \mathtt{Term}(\mathcal{D})$*,* $u \in \mathrm{FV}_a(M) \cap \mathrm{FV}_a(M')$*, and* $M \to M'$*.Then*

$$\mathrm{Free}_{K,u}(M) = 1 \Rightarrow \mathrm{Free}_{K,u}(M') = 1.$$

PROOF. Left to the reader. □

THEOREM 3.2.12. *Let* $M, M' \in \mathtt{Term}(\mathcal{D})$*,* $u \in \mathrm{FV}_a(M) \cap \mathrm{FV}_a(M')$*,* $M \to M'$*, and* $\mathrm{Free}_{K,u}(M) = 1$*. Then*[3]

$$M[u := K] \to M'[u := K].$$

[2] A simplified writing of this equality is $(M(N))(K) \equiv (M(K))(N(K))$.

[3] A simplified writing of this one step-reduction is $M(K) \to M'(K)$.

Proof. Left to the reader. □

Theorem 3.2.13. *Let* $M, M' \in \mathtt{Term}(\mathcal{D})$, $x \in \mathrm{FV}_o(M) \cap \mathrm{FV}_o(M')$, $M \to M'$, *and* $\mathrm{Free}_{s,x}(M) = 1$. *Then*[4]

$$M[x := s] \to M'[x := s].$$

Proof. We work as in the proof of Theorem 3.2.12. □

The reason for the appearance of $\to^*$ in the formulation of the next theorem is the fact that u may have many occurrences in M.

Theorem 3.2.14. *Let* $M, N, N' \in \mathtt{Term}(\mathcal{D})$ *and* $\mathrm{Free}_{N,u}(M) = 1 = \mathrm{Free}_{N',u}(M)$. *Then*[5]

$$M[u := N] \to^* M[u := N'].$$

Proof. Left to the reader. □

Proposition 3.2.15. *There is an inductively defined relation* $\lhd$ *on the set* $\mathtt{Term}(\mathcal{D})$, *such that* $M \lhd N$ *expresses that* M *is a "subterm" of* N. *Using the induction principle corresponding to the definition of* $\lhd$ *one shows*

$$M \lhd M' \Rightarrow M \lhd M'N \ \wedge \ M \lhd NM' \ \wedge \ M \lhd \lambda_v M',$$

for every appropriately typed $M, M', N \in \mathtt{Term}(\mathcal{D})$.

Proof. Exercise. □

Definition 3.2.16. If $N \in \mathtt{Term}(\mathcal{D})$, we say that N *contains a redex*, if there are $M \in \mathtt{Term}(\mathcal{D})$, $K \in \mathtt{Term}(\mathcal{D}) \cup \mathtt{Term}$ and $v \in \mathtt{Var}_a \cup \mathtt{Var}$ such that

$$(\lambda_v M)K \lhd N.$$

N is called *normal*, if it doesn't contain a redex. We define

$$\mathtt{Normal} \equiv \{N \in \mathtt{Term}(\mathcal{D}) \mid N \text{ is normal}\}.$$

A derivation M is called *normal*, if its term t_M is in $\mathtt{Normal}$. A term M *has a normal form*, if there is $N \in \mathtt{Normal}$ such that $M \to^* N$.

Proposition 3.2.17. *Let* $N, N' \in \mathtt{Term}(\mathcal{D})$.
(i) $N \to N' \Rightarrow N \neq N'$.
(ii) $N \to N' \Rightarrow N$ *contains a redex.*
(iii) $N \to^* N' \Rightarrow (N \neq N' \Rightarrow \exists_{K \in \mathtt{Term}(\mathcal{D})}(N \to K \to^* N'))$.
(iv) $N \to^* N' \wedge N \in \mathtt{Normal} \Rightarrow N = N'$.

Proof. Exercise. □

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[4] A simplified writing of this one step-reduction is $M(x) \to M'(x)$.
[5] A simplified writing of this one step-reduction is $M(N) \to M(N')$.

3.3. The Church-Rosser property

DEFINITION 3.3.1. If X is an inhabited set i.e., a set with a given element, we define $X^0 \equiv \{\emptyset\}$, $X^n \equiv \{f : \{0, \ldots, n-1\} \to X\}$ and

$$X^{<\mathbb{N}} \equiv \bigcup_{n\in\mathbb{N}} X^n.$$

the length $|u| = 0$, if $u = \emptyset$, and $|u| = n$, if $u \in X^n$, for some (unique) $n \in \mathbb{N}^+$. If $u, w \in X^{<\mathbb{N}}$, we define

$$u \prec w \equiv |u| < |w| \ \wedge \ \forall_{i\in\{0,\ldots,|u|-1\}}(u_i = w_i).$$

If $\alpha \in X^{\mathbb{N}}$ and $n \in \mathbb{N}$, the n-th initial part $\bar{\alpha}(n)$ of α is $\emptyset$, if $n = 0$, and it is $(\alpha_0, \ldots, \alpha_{n-1})$, if $n \in \mathbb{N}^+$. A *tree* T on X is a subset of $X^{<\mathbb{N}}$, which is closed under initial segments i.e.,

$$\forall_{u,w\in X^{<\mathbb{N}}}(w \in T \wedge u \prec w \Rightarrow u \in T).$$

An element of T is called a *node* of T. An *infinite path* of T is an $\alpha \in X^{\mathbb{N}}$ such that $\forall_{n\in\mathbb{N}}(\bar{\alpha}(n) \in T)$. The *body* $[T]$ of T is the set of its infinite paths. If $u \in T$, the set $\mathrm{Succ}(u)$ of immediate successor nodes of u is defined by

$$\mathrm{Succ}(u) \equiv \{w \in T \mid u \prec w \ \wedge \ |w| = |u| + 1\}.$$

A tree T is (in)finite, if it is an (in)finite set, and it is called *well-founded*, if it has no infinite path. T is called finitely branching, or a *fan*, if $\mathrm{Succ}(u)$ is a finite set, for every $u \in T$. Otherwise, it is called infinitely branching.

Clearly, $X^{<\mathbb{N}}$ is a tree on X. If $X \equiv \mathbb{N}$, then $\mathbb{N}^{<\mathbb{N}}$ is called the *Baire tree*, and if $X = \mathbf{2}$, then $\mathbf{2}^{<\mathbb{N}}$ is called the *Cantor tree*. Clearly, $[\mathbb{N}^{<\mathbb{N}}]$ is the *Baire space* $\mathbb{N}^{\mathbb{N}}$ and $[\mathbf{2}^{<\mathbb{N}}]$ is the *Cantor space* $\mathbf{2}^{\mathbb{N}}$. Trivially, $\emptyset \prec u$, for every $u \in X^{<\mathbb{N}} \setminus \{\emptyset\}$, while a tree T on X is inhabited if and only if $\emptyset \in T$. If T is a finite tree on X, then T is well-founded, but the converse is not true. One can show classically, that an infinite fan has an infinite path (Exercise).

DEFINITION 3.3.2. A binary relation $R \subseteq X \times X$ on X has an *infinite descending chain*, if there is $\alpha \in X^{\mathbb{N}}$ such that $\forall_{n\in\mathbb{N}}(\alpha_{n+1} \ R \ \alpha_n)$ i.e.,

$$\ldots \alpha_3 \ R \ \alpha_2 \ R \ \alpha_1 \ R \ \alpha_0,$$

and $\prec$ is called *well-founded*, if it has no infinite descending chain.

Clearly, $<_{\mathbb{N}}$ is a well-founded relation on $\mathbb{N}$, while $<_{\mathbb{Z}}$ is not a well-founded relation on $\mathbb{Z}$. If T is a well-founded tree on X, then the relation $w \ R \ u \equiv u \prec w$ is well-founded relation on T.

Proposition 3.3.3 (Well-founded induction). *If $\prec$ is a well-founded relation on X, then*

$$\Big(\forall_{x\in X}\big(\forall_{y\in X}(y \prec x \Rightarrow P(y)) \Rightarrow P(x)\big)\Big) \Rightarrow \forall_{x\in X}(P(x)).$$

Proof. Suppose that there is some $x \in X$ such that $\neg P(x)$. This implies the (classical) existence of some $x_1 \prec x$ such that $\neg P(x)$. By repeating this step, and using some form of the axiom of choice, we get that $\prec$ has an infinitely descending chain, which contradicts our hypothesis. □

Definition 3.3.4. For $n = 0, 1$ and $n \geq 2$, we define the following sets:

$$\mathrm{Red}^0 \equiv \{\emptyset\} \equiv \mathtt{Term}(\mathcal{D})^0,$$

$$\mathrm{Red}^1 \equiv \{(M) \mid M \in \mathtt{Term}(\mathcal{D})\} \equiv \mathtt{Term}(\mathcal{D})^1,$$

$$\mathrm{Red}^n \equiv \{\vec{M} \in \mathtt{Term}(\mathcal{D})^n \mid \forall_{i\in\{0,\dots,|\vec{M}|-2\}}(M_i \to M_{i+1})\},$$

$$\mathrm{Red}^{<\mathbb{N}} \equiv \bigcup_{n\in\mathbb{N}} \mathrm{Red}^n.$$

If $M \in \mathtt{Term}(\mathcal{D})$, its *reduction tree* $\mathrm{Red}(M)$ is defined by

$$\mathrm{Red}(M) \equiv \{\vec{M} \in \mathrm{Red}^{<\mathbb{N}} \mid \vec{M}_0 = M\} \cup \mathrm{Red}^0.$$

The non-empty elements of $\mathrm{Red}(M)$ are the *reduction sequences*

$$M_0 \to M_1 \to \dots \to M_{|\vec{M}|-1}$$

starting from M i.e., $M_0 = M$. We say that M is *strongly normalizing*, if $\mathrm{Red}(M)$ is finite. We also define their set

$$\mathtt{SNormal} \equiv \{M \in \mathtt{Term}(\mathcal{D}) \mid M \text{ is strongly normalizing}\}.$$

It is immediate to see that $\mathrm{Red}^{<\mathbb{N}}$ and $\mathrm{Red}(M)$ are trees on $\mathtt{Term}(\mathcal{D})$, and that $N \in \mathtt{Normal} \Leftrightarrow \mathrm{Red}(N) = \{\emptyset, (N)\}$. Clearly, if M is strongly normalizing, then $\mathrm{Red}(M)$ is a well-founded fan, and M has a normal form.

Definition 3.3.5. A relation $R \subseteq X \times X$ on X is called *confluent*, or it has the *Church-Rosser property* (CR), if

$$\forall_{x_0,x_1,x_2\in X}\big(R(x_0,x_1) \wedge R(x_0,x_2) \Rightarrow \exists_{x_3\in X}(R(x_1,x_3) \wedge R(x_2,x_3)\big).$$

A relation R on X is called *weakly confluent*, or it has the *weak Church-Rosser property* ($\mathtt{WCR}$), if

$$\forall_{x_0,x_1,x_2\in X}\big(R(x_0,x_1) \wedge R(x_0,x_2) \Rightarrow \exists_{x_3\in X}(R^*(x_1,x_3) \wedge R^*(x_2,x_3)\big).$$

Proposition 3.3.6. *If the relation $\to^*$ is confluent, $M \in \mathtt{Term}(\mathcal{D})$ and $N, N' \in \mathtt{Normal}$ such that $M \to^* N$ and $M \to^* N'$, then $N = N'$.*

$$\begin{tikzcd}
 & & M \arrow[dl] \arrow[dr] & & \\
 & M' \arrow[dl, "*"'] \arrow[dr, "*"] & \texttt{WCR} & M'' \arrow[dl, "*"'] \arrow[dr, "*"] & \\
K \arrow[dr, "*"] & \mathrm{IH}(M') & \exists N' \arrow[dl, "*"'] & \mathrm{IH}(M'') & L \arrow[ddll, "*"] \\
 & \exists N'' \arrow[dr, "*"] & & & \\
 & & \exists N & &
\end{tikzcd}$$

FIGURE 1. Proof of Newman's lemma

PROOF. By the confluence of $\to^*$ there is some $K \in \mathtt{Term}(\mathcal{D})$ such that $N \to^* K$ and $N' \to^* K$. Since $N, N' \in \mathtt{Normal}$, we get $N = K = N'$. $\square$

PROPOSITION 3.3.7 (Newman's lemma). *Assume that $\to$ is weakly confluent. If all derivation terms are strongly normalizing with respect to $\to$, then the relation $\to^*$ is confluent.*

PROOF. We write $N \leftarrow M$ for $M \to N$, and $N \leftarrow^* M$ for $M \to^* N$. Call M *good* if it satisfies the confluence property w.r.t. $\to^*$, i.e., whenever $K \leftarrow^* M \to^* L$, then $K \to^* N \leftarrow^* L$ for some N. We show that every strongly normalizing M is good, by induction on the well-founded relation $\to^+$, restricted to all terms occurring in the reduction tree of M. So let M be given and assume

$$\text{every } M' \text{ with } M \to^+ M' \text{ is good.}$$

We must show that M is good, so assume $K \leftarrow^* M \to^* L$. We may further assume that there are M', M'' such that $K \leftarrow^* M' \leftarrow M \to M'' \to^* L$, for otherwise the claim is trivial (if $M = K$, then use L to complete the diamond, and if $M = L$, use K). But then the claim follows from the assumed weak confluence and the induction hypothesis for M' and M'', as shown in figure 1. $\square$

PROPOSITION 3.3.8. *(i) If $M \to^* M'$ and $M' \neq M$, then*

$$\exists_{n \in \mathbb{N}} \exists_{K_1, \dots, K_n} (M \to K_1 \to \dots \to K_n \to M').$$

$$\begin{tikzcd}[column sep=tiny]
& (\lambda_v M(v))N\vec{L} \arrow[dl] \arrow[dr] & \\
M(N)\vec{L} \arrow[dr] & & (\lambda_v M'(v))N\vec{L} \arrow[dl] \\
& M'(N)\vec{L} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& (\lambda_v M(v))N\vec{L} \arrow[dl] \arrow[dr] & \\
M(N)\vec{L} \arrow[dr, "*"] & & (\lambda_v M(v))N'\vec{L} \arrow[dl] \\
& M(N')\vec{L} &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
& (\lambda_v M(v))N\vec{L} \arrow[dl] \arrow[dr] & \\
M(N)\vec{L} \arrow[dr] & & (\lambda_v M(v))N\vec{L}' \arrow[dl] \\
& M(N)\vec{L}' &
\end{tikzcd}$$

FIGURE 2. Weak concluence of $\to$

(ii) $M \to^* M' \Rightarrow M(N) \to^* M'(N)$.
(iii) $N \to^* N' \Rightarrow M(N) \to^* M(N')$.

PROOF. Left to the reader. $\square$

PROPOSITION 3.3.9. *The one-step reduction $\to$ is weakly confluent.*

PROOF. Assume $N_0 \leftarrow M \to N_1$. We show that $N_0 \to^* N \leftarrow^* N_1$ for some N, by induction on M. If there are two inner reductions both on the same subterm, then the claim follows from the induction hypothesis using Theorems 3.2.12, 3.2.13, 3.2.14 and Proposition 3.3.8(ii) and (iii). If they are on distinct subterms, then the subterms do not overlap and the claim is obvious. It remains to deal with the case of a head reduction together with an inner conversion. This is done in figure 2, where for the lower left arrows we have used substitutivity again. $\square$

3.4. The strong normalization theorem

In this section we define a relation $\mathrm{sn} \subseteq \mathtt{Term}(\mathcal{D}) \times \mathbb{N}$ and a set SN, for which we show that $\mathtt{Term}(\mathcal{D}) \subseteq \mathrm{SN}$ and that for each $M \in \mathrm{SN}$ that is also in $\mathtt{Term}(\mathcal{D})$ there is some $k \in \mathbb{N}$ such that $\mathrm{sn}(M, k)$. By the definition of $\mathrm{sn}(M, k)$ we get as a corollary that M is strongly normalizing. The strong normalizing property of every derivation term is the content of the strong normalization theorem.

DEFINITION 3.4.1. If $M \in \texttt{Term}(\mathcal{D})$, we define the set

$$M^{\rightarrow} \equiv \{M' \in \texttt{Term}(\mathcal{D}) \mid M \rightarrow M'\}.$$

The relation $\mathrm{sn} \subseteq \texttt{Term}(\mathcal{D}) \times \mathbb{N}$ is defined by

$$\begin{aligned} \mathrm{sn}(M, 0) &\equiv M \in \texttt{Normal}, \\ \mathrm{sn}(M, k+1) &\equiv \forall_{M' \in M^{\rightarrow}}(\mathrm{sn}(M', k)). \end{aligned}$$

The relation $\mathrm{sn}(M, k)$ expresses that "M reduces to a normal form after *at most* k one step-reductions".

COROLLARY 3.4.2. $\forall_{k \in \mathbb{N}}\big(\forall_{M \in \texttt{Term}(\mathcal{D})}(\mathrm{sn}(M, k) \Rightarrow M \in \texttt{SNormal})\big)$.

PROOF. By induction on k. If $k = 0$ and $\mathrm{sn}(M, 0)$, then $M \in \texttt{Normal}$, therefore $\mathrm{Red}(M)$ is finite. If we suppose $\forall_{M \in \texttt{Term}(\mathcal{D})}(\mathrm{sn}(M, k) \Rightarrow M \in \texttt{SNormal})$, we show $\forall_{M \in \texttt{Term}(\mathcal{D})}(\mathrm{sn}(M, k+1) \Rightarrow M \in \texttt{SNormal})$. If M such that $\mathrm{sn}(M, k+1)$, then $\mathrm{sn}(M', k)$, for every $M' \in M^{\rightarrow}$. By inductive hypothesis on every $M' \in M^{\rightarrow}$ we get $M' \in \texttt{SNormal}$, and since every $\mathrm{Red}(M')$ is finite, we have that $\mathrm{Red}(M)$ is finite. $\square$

Next we prove some closure properties of the relation sn.

LEMMA 3.4.3. *For the relation* sn *the following hold:*

(a) *If* $\mathrm{sn}(M, k)$*, then* $\mathrm{sn}(M, k+1)$.
(b) *If* $\mathrm{sn}(MN, k)$*, then* $\mathrm{sn}(M, k)$.
(c) *If* $\mathrm{sn}(M_i, k_i)$ *for* $i = 1 \dots n$*, then* $\mathrm{sn}(uM_1 \dots M_n, k_1 + \dots + k_n)$.
(d) *If* $\mathrm{sn}(M, k)$*, then* $\mathrm{sn}(\lambda_v M, k)$.
(e) *If* $\mathrm{sn}(M(N)\vec{L}, k)$ *and* $\mathrm{sn}(N, l)$*, then* $\mathrm{sn}((\lambda_v M(v))N\vec{L}, k + l + 1)$.

PROOF. (a) Induction on k. The case $k = 0$ is trivial, since $M^{\rightarrow} = \emptyset$, if M is normal. Assume $\mathrm{sn}(M, k)$. We show $\mathrm{sn}(M, k+1)$. Let M' with $M \rightarrow M'$ be given; because of $\mathrm{sn}(M, k)$ we must have $k > 0$. We have to show $\mathrm{sn}(M', k)$. Because of $\mathrm{sn}(M, k)$ we have $\mathrm{sn}(M', k-1)$, hence by induction hypothesis $\mathrm{sn}(M', k)$.
(c) Assume $\mathrm{sn}(M_i, k_i)$ for $i = 1 \dots n$. We show $\mathrm{sn}(uM_1 \dots M_n, k)$ with $k \equiv k_1 + \dots + k_n$. Again we employ induction on k. In case $k = 0$ all M_i are normal, hence also $uM_1 \dots M_n$. Let $k > 0$ and $uM_1 \dots M_n \rightarrow M'$. Then $M' = uM_1 \dots M_i' \dots M_n$ with $M_i \rightarrow M_i'$. We have to show $\mathrm{sn}(uM_1 \dots M_i' \dots M_n, k-1)$. Because of $M_i \rightarrow M_i'$ and $\mathrm{sn}(M_i, k_i)$ we have $k_i > 0$ and $\mathrm{sn}(M_i', k_i - 1)$, hence $\mathrm{sn}(uM_1 \dots M_i' \dots M_n, k-1)$ by induction hypothesis.
(b), (d), (e) Exercise. $\square$

The essential idea of the strong normalization proof is to view the last three closure properties of sn from the preceding lemma, without the information on the bounds, as an inductive definition of a new set SN.

Definition 3.4.4. The set SN is defined inductively by the rules:

$$\frac{t \in \mathtt{Term}}{t \in \mathrm{SN}}\ (\mathrm{Term}) \qquad \frac{\vec{M} \in \mathrm{SN}}{u\vec{M} \in \mathrm{SN}}\ (\mathrm{Var})$$

$$\frac{M \in \mathrm{SN}}{\lambda_v M \in \mathrm{SN}}\ (\lambda) \qquad \frac{M(N)\vec{L} \in \mathrm{SN} \qquad N \in \mathrm{SN}}{(\lambda_v M(v))N\vec{L} \in \mathrm{SN}}\ (\beta),$$

where

$$\vec{M} \in \mathrm{SN} \equiv M_0 \in \mathrm{SN} \wedge M_1 \in \mathrm{SN} \wedge \ldots \wedge M_{|\vec{M}|-1} \in \mathrm{SN} \equiv \bigwedge_{i=0}^{|\vec{M}|-1} M_i \in \mathrm{SN}.$$

The induction principle corresponding to the definition of SN is

$$\begin{aligned}\forall_M \Big(M \in \mathrm{SN} \wedge \forall_{t\in\mathtt{Term}}(P(t)) \ \wedge\ \forall_{\vec{M}\in\mathrm{SN}^{<\mathbb{N}}}\forall_{u\in\mathtt{Var}_a}\big(P(\vec{M}) \Rightarrow P(u\vec{M})\big) \\ \wedge\ \forall_{M\in\mathrm{SN}}\forall_{v\in\mathtt{Var}_a\cup\mathtt{Var}}\big(P(M) \Rightarrow P(\lambda_v M)\big) \\ \wedge\ \forall_{M,N\in\mathrm{SN}}\forall_{\vec{L}\in\mathrm{SN}^{<\mathbb{N}}}\forall_{v\in\mathtt{Var}_a\cup\mathtt{Var}}\big(P(M(N)\vec{L}) \wedge P(N) \Rightarrow P((\lambda_v M)N\vec{L})\big) \\ \Rightarrow\ P(M)\Big),\end{aligned}$$

where $P(M)$ is any property, and

$$P(\vec{M}) \equiv P(M_0) \wedge P(M_1) \wedge \ldots \wedge P(M_{|\vec{M}|-1}) \equiv \bigwedge_{i=0}^{|\vec{M}|-1} P(M_i).$$

If $\vec{M} = \emptyset$ and $u \in \mathtt{Var}_a$, the Var-rule implies that $u \in \mathrm{SN}$. Note that there are elements of SN, like $u^{A\to B}r$, that are not in the union $\mathtt{Term}(\mathcal{D}) \cup \mathtt{Term}$.

Corollary 3.4.5. *(i)* $\forall_{M\in\mathrm{SN}}\big(M \in \mathtt{Term}(\mathcal{D}) \Rightarrow \exists_{k\in\mathbb{N}}(\mathrm{sn}(M,k))\big)$.
(ii) $\forall_{M\in\mathrm{SN}}\big(M \in \mathtt{Term}(\mathcal{D}) \Rightarrow M \in \mathtt{SNormal}\big)$.

Proof. (i) By induction on SN. Let $\vec{M}$ with $\bigwedge_{i=0}^{|\vec{M}|-1} \exists_{k_i\in\mathbb{N}}(\mathrm{sn}(M_i,k_i))$. By Lemma 3.4.3(iii) we get $\mathrm{sn}(u\vec{M}, \sum_{i=0}^{|\vec{M}|-1} k_i)$. The rest of the clauses of the induction principle of SN are shown similarly from Lemma 3.4.3.
(ii) Immediately by (i) and Corollary 3.4.2. □

Theorem 3.4.6. *For all formulas A,*

(a) *for all* $M(w) \in \mathrm{SN}$*, if* $N^A \in \mathrm{SN}$*, then* $M(N) \in \mathrm{SN}$*,*
(b) *for all* $M(x) \in \mathrm{SN}$*,* $M(r) \in \mathrm{SN}$*,*
(c) *if* $M \in \mathrm{SN}$ *derives* $A \to B$ *and* $N^A \in \mathrm{SN}$*, then* $MN \in \mathrm{SN}$*,*
(d) *if* $M \in \mathrm{SN}$ *derives* $\forall_x A$*, then* $Mr \in \mathrm{SN}$*.*

Proof. The proof is by *course-of-values induction* on the height $|A|$ of A (Definition 2.1.7), as a primary induction, with a side induction on SN. I.e., we prove the following formula

$$\forall_{n\in\mathbb{N}}\Bigg[\forall_{A\in\texttt{Form}}\Bigg(|A| = n \wedge$$

$$\forall_{M\in\mathrm{SN}}\Bigg(\forall_{w^A\in\texttt{Var}_a}\forall_{N^A}\big(w^A \in \mathrm{FV}_a(M) \wedge N^A \in \mathrm{SN} \Rightarrow M(N) \in \mathrm{SN}\big) \wedge$$

$$\forall_{x,r}\big(x \in \mathrm{FV}_0(M) \Rightarrow M(r) \in \mathrm{SN}\big) \wedge$$

$$\forall_{B\in\texttt{Form}}\forall_{N^A}\big(M \in \texttt{Term}(\mathcal{D}) \wedge \texttt{Type}(M) \equiv A \to B \wedge N^A \in \mathrm{SN} \Rightarrow MN \in \mathrm{SN}\big)$$

$$\wedge\ \forall_r\big(M \in \texttt{Term}(\mathcal{D}) \wedge \texttt{Type}(M) \equiv \forall_x A \Rightarrow Mr \in \mathrm{SN}\big)\Bigg)\Bigg)\Bigg].$$

Note that the course-of-values induction

$$\Big(\forall_{n\in\mathbb{N}}\big(\forall_{m\in\mathbb{N}}(m < n \Rightarrow P(m)) \Rightarrow P(n)\big)\Big) \Rightarrow \forall_{n\in\mathbb{N}}(P(n))$$

is a different name for the well-founded induction on $\mathbb{N}$, since the order $<$ of $\mathbb{N}$ is a well-founded relation on $\mathbb{N}$ (why?). We fix formula A with $|A| = n$ and we suppose the formula for all $m < n$ i.e., for all formulas with height $< n$. Next we use induction on SN.

Case $t \in \mathrm{SN}$ by (Term) from $t \in \texttt{Term}$.
(a) It follows trivially, since $\mathrm{FV}_a(t) = \emptyset$.
(b) Since $t(r) \in \texttt{Term}$, we get from the Term-rule that $t(r) \in \mathrm{SN}$.
(c) and (d) follow trivially, as $t \notin \texttt{Term}(\mathcal{D})$.

Case $u\vec{M} \in \mathrm{SN}$ by (Var) from $\vec{M} \in \mathrm{SN}$.
(a) Let $N^A \in \mathrm{SN}$. The side induction hypothesis (a) for each M_i in $\vec{M}$ yields $M_i(N) \in \mathrm{SN}$. If $u \neq w$, then

$$\frac{\vec{M}(N) \equiv (M_0(N), \ldots, M_{|\vec{M}|-1}(N)) \in \mathrm{SN}}{(u\vec{M})(N) \equiv u\vec{M}(N) \equiv u(M_0(N), \ldots, M_{|\vec{M}|-1}(N)) \in \mathrm{SN}}\ (\mathrm{Var}).$$

If $u = w$, then

$$(u\vec{M})(N) \equiv N(M_0(N), \ldots, M_{|\vec{M}|-1}(N)) \equiv (\ldots(N(M_0(N))\ldots)M_{|\vec{M}|-1}(N).$$

One can show though, that

$$\texttt{Type}(M_i(N)) \equiv D_i \quad \text{and} \quad |D_i| < |A|,$$

since $D_i \lhd_* A$ i.e., D_i is a *strict subformula* of A, for each $i \in \{0, \ldots, |\vec{M}|-1\}$. E.g., if $M_0^{A \to B}$ and $M_1^{A \to C}$, then $(M_0(N))^B$ and $(M_1(N))^C$, therefore the application $N^A(M_0(N))^B$ implies that $A \equiv B \to D$, for some formula D, hence $B \lhd_* A$. Moreover, the application $(N(M_0(N)))^D(M_1(N))^C$ implies that $D \equiv C \to E$, for some formula E, hence $A \equiv B \to (C \to E)$ and $C \lhd_* A$. It is easy to turn this argument into an inductive proof on the length of $\vec{M}$.

With the use of the induction principle corresponding to the inductive definition of the notion of strict subformula $B \lhd_* A$(see also Proposition 2.1.13) one shows that

$$\forall_{B \lhd_* A}(|B| < |A|).$$

Since $|B| < |A|$, applying (c) on $N^{B \to D} \in \mathrm{SN}$ and on $(M_0(N))^B \in \mathrm{SN}$, we get $N(M_0(N)) \in \mathrm{SN}$. Working similarly for $|C| < |A|$, $N(M_0(N))^{C \to E}$ and $(M_1(N))^C \in \mathrm{SN}$, we get again by the corresponding condition (c) that $(NM_0(N))M_1(N) \in \mathrm{SN}$. It is easy to turn this argument into an inductive proof on the length of $\vec{M}$.
(b) Let $r \in \texttt{Term}$. The side induction hypothesis (b) for each M_i in $\vec{M}$ yields $M_i(r) \in \mathrm{SN}$. Hence,

$$\frac{\vec{M}(r) \equiv (M_0(r), \ldots, M_{|\vec{M}|-1}(r)) \in \mathrm{SN}}{(u\vec{M})(r) \equiv u\vec{M}(r) \equiv u(M_0(r), \ldots, M_{|\vec{M}|-1}(r)) \in \mathrm{SN}}\ (\mathrm{Var}).$$

(c) We show that $(u\vec{M})N \in \mathrm{SN}$. If $\vec{M} * N \equiv (M_0, \ldots, M_{|\vec{M}|-1}, N)$, then

$$\frac{\vec{M} * N \in \mathrm{SN}}{(u\vec{M})N \equiv u(\vec{M} * N) \in \mathrm{SN}}\ (\mathrm{Var}).$$

(d) We show that $(u\vec{M})r \in \mathrm{SN}$. If $\vec{M} * r \equiv (M_0, \ldots, M_{|\vec{M}|-1}, r)$, then using the Term-rule, and this is exactly where this rule is crucially used, we get

$$\frac{\vec{M} * r \in \mathrm{SN}}{(u\vec{M})r \equiv u(\vec{M} * r) \in \mathrm{SN}}\ (\mathrm{Var}).$$

Case $\lambda_v M \in \mathrm{SN}$ by (λ) from $M \in \mathrm{SN}$.
(a) Without loss of generality let $M(v, w)$. By inductive hypothesis $M(N) \in \mathrm{SN}$, therefore by the λ-rule we get $(\lambda_v M)(N) \equiv \lambda_v M(N) \in \mathrm{SN}$.
(b) Without loss of generality let $x \in \mathrm{FV}_o(M)$. By inductive hypothesis $M(r) \in \mathrm{SN}$, therefore by the λ-rule we get $(\lambda_v M)(r) \equiv \lambda_v M(r) \in \mathrm{SN}$.

(c) Our goal is $(\lambda_v M(v))N \in \mathrm{SN}$. By the β-rule it suffices to show $M(N) \in \mathrm{SN}$ and $N \in \mathrm{SN}$. The latter holds by assumption, and the former by the side induction hypothesis (a) on M and N.
(d) If $\mathtt{Type}(\lambda_v M) \equiv \forall_x A$, then $v \equiv x$, and by the β-rule in order to show that $(\lambda_x M)r \in \mathrm{SN}$ it suffices to show $M(r) \in \mathrm{SN}$, which follows from the inductive hypothesis (b) on M.

Case $(\lambda_v M(v))K\vec{L} \in \mathrm{SN}$ by (β) from $M(K)\vec{L} \in \mathrm{SN}$ and $K \in \mathrm{SN}$.
(a) Since $\big((\lambda_v M(v))K\vec{L}\big)(N) \equiv (\lambda_v M(N))K(N)\vec{L}(N)$, by the β-rule suffices to show $M(N)(K(N))\vec{L}(N) \in \mathrm{SN}$ and $K(N) \in \mathrm{SN}$. The latter follows from our assumption. The former follows from the inductive hypothesis (a) on $M(K)\vec{L} \in \mathrm{SN}$, since by Proposition 3.2.9

$$(M(K)\vec{L})(N) \equiv (M(K))(N)\vec{L}(N) \equiv M(N)(K(N))\vec{L}(N).$$

(b) Since $((\lambda_v M(v))K\vec{L})(r) \equiv (\lambda_v M(r))K(r)\vec{L}(r)$ by the β-rule it suffices to show $M(r)(K(r))\vec{L}(r) \in \mathrm{SN}$ and $K(r) \in \mathrm{SN}$. The latter follows from the inductive hypothesis (b) on $K \in \mathrm{SN}$. The former follows from the inductive hypothesis (b) on $M(K)\vec{L} \in \mathrm{SN}$ and the term-analogue to Proposition 3.2.9, since

$$\big(M(K)\vec{L}\big)(r) \equiv M(K)(r)\vec{L}(r) \equiv M(r)(K(r))\vec{L}(r).$$

(c) Let $((\lambda_v M(v))K\vec{L})^{A\to B}$. Since $((\lambda_v M(v))K\vec{L})N \equiv (\lambda_v M(v))K(\vec{L} * N)$, by the β-rule it suffices to show that $M(K)(\vec{L} * N) \in \mathrm{SN}$ and $K \in \mathrm{SN}$. The latter follows from our assumption, and the former from the inductive hypothesis (c) on $M(K)\vec{L}$ and N and the fact $M(K)(\vec{L} * N) \equiv (M(K)\vec{L})N$. Notice that by Proposition 3.2.5(i) we have that $\mathtt{Type}(((\lambda_v M(v))K\vec{L})) = \mathtt{Type}(M(K)\vec{L}) \equiv A \to B$.
(d) Let $((\lambda_v M(v))K\vec{L})^{\forall_x A}$. Since $((\lambda_v M(v))K\vec{L})r \equiv (\lambda_v M(v))K(\vec{L} * r)$, by the β-rule it suffices to show that $M(K)(\vec{L}*r) \in \mathrm{SN}$ and $K \in \mathrm{SN}$. The latter follows from our assumption, and the former from the inductive hypothesis (d) on $M(K)\vec{L}$ and r and the fact $M(K)(\vec{L}*r) \equiv (M(K)\vec{L})r$. Notice that by Proposition 3.2.5(i) we have that $\mathtt{Type}(((\lambda_v M(v))K\vec{L})) = \mathtt{Type}(M(K)\vec{L}) \equiv \forall_x A$. $\square$

For the next proof we shall use the induction principle corresponding to the inductive characterization

$$u\vec{M} \mid \lambda_v M \mid (\lambda_v M)N\vec{L}$$

of derivation terms of table 1. If

$$T \equiv \mathtt{Term}(\mathcal{D}) \cup \mathtt{Term},$$

this principle is written as follows:

$$\forall_M \Big(M \in \texttt{Term}(\mathcal{D}) \wedge \forall_{\vec{M} \in T^{<\mathbb{N}}} \forall_{u \in \texttt{Var}_a} \big(P(\vec{M}) \Rightarrow P(u\vec{M})\big)$$

$$\wedge\ \forall_{M \in \texttt{Term}(\mathcal{D})} \forall_{v \in \texttt{Var}_a \cup \texttt{Var}} \big(P(M) \Rightarrow P(\lambda_v M)\big)$$

$$\wedge\ \forall_{M \in \texttt{Term}(\mathcal{D})} \forall_{N \in T} \forall_{\vec{L} \in T^{<\mathbb{N}}} \forall_{v \in \texttt{Var}_a \cup \texttt{Var}} \big(P(M) \wedge P(N) \wedge P(\vec{L})$$

$$\Rightarrow P((\lambda_v M) N \vec{L})\big)$$

$$\Rightarrow P(M) \Big).$$

THEOREM 3.4.7 (Strong normalization theorem). *(i)* $\texttt{Term}(\mathcal{D}) \subseteq \mathrm{SN}$. *(ii)* $\texttt{Term}(\mathcal{D}) \subseteq \texttt{SNormal}$.

PROOF. (i) By induction on $\texttt{Term}(\mathcal{D})$ for $P(M) \equiv M \in \mathrm{SN}$. The case $P(\vec{M}) \Rightarrow P(u\vec{M})$ is the Var-rule of SN, and the case $P(M) \Rightarrow P(\lambda_v M)$ is the λ-rule of SN. We suppose $M, N \in \mathrm{SN}$, $\vec{L} \in \mathrm{SN}$, and we show that $(\lambda_v M) N \vec{L} \in \mathrm{SN}$. First we consider the case $N^A \in \texttt{Term}(\mathcal{D})$. By the β-rule of SN it suffices to show that $M(N)\vec{L} \in \mathrm{SN}$ and $N \in \mathrm{SN}$. The latter is by hypothesis, while the former follows from Theorem 3.4.6; by the formula of Theorem 3.4.6 for $n = |A|$ and $M \in \mathrm{SN}$ we get $M(N) \in \mathrm{SN}$. If $L_1^{A_1}$, then by the formula of Theorem 3.4.6 for $n = |A_1|$ we get $M(N)L_1 \in \mathrm{SN}$. If $L_a \in \texttt{Term}$, we use the induction hypothesis (d) on $M(N) \in \mathrm{SN}$. Continuing similarly (or by the obvious inductive proof on the length of $\vec{L}$) we get $M(N)\vec{L} \in \mathrm{SN}$. Next we consider the case $N \equiv r \in \texttt{Term}$. By the β-rule of SN it suffices to show that $M(r)\vec{L} \in \mathrm{SN}$ and $r \in \mathrm{SN}$. The latter is by the Term-rule, while the former follows as above; first $M(r) \in \mathrm{SN}$ follows from the inductive hypothesis (b) on $M \in \mathrm{SN}$, and then we repeatedly use condition (c) for $M(r)L_1$, or condition (d), if $L_1 \in \texttt{Term}$. We work similarly for $(M(r)L_1)L_2$ (or by the obvious inductive proof on the length of $\vec{L}$).
(ii) Immediately by (i) and by Corollary 3.4.5. $\square$

COROLLARY 3.4.8 (Uniqueness of normal form). *The normal form of a derivation term is unique.*

PROOF. By Proposition 3.3.9 $\to$ is weakly confluent, and by Newman's lemma (Proposition 3.3.7) and Theorem 3.4.7 $\to^*$ is confluent, therefore by Proposition 3.3.6 the normal form of a derivation term is unique. $\square$

Hence we can define the function

$$\mathtt{normal} : \mathtt{Term}(\mathcal{D}) \to \mathtt{Term}(\mathcal{D})$$
$$M \mapsto N_M,$$

where N_M is the unique normal derivation term in which M reduces to.

PROPOSITION 3.4.9. *If $M \in \mathtt{Term}(\mathcal{D})$, let $M^*, {}^*M, B^*, {}^*B$ defined by*

$$M^* \equiv \{K \in \mathtt{Term}(\mathcal{D}) \mid M \to^* K\},$$
$${}^*M \equiv \{K \in \mathtt{Term}(\mathcal{D}) \mid K \to^* M\},$$
$$B^* \equiv \{M^* \mid \in \mathtt{Term}(\mathcal{D})\} \cup \{\emptyset, \mathtt{Term}(\mathcal{D})\},$$
$${}^*B \equiv \{{}^*M \mid \in \mathtt{Term}(\mathcal{D})\} \cup \{\emptyset, \mathtt{Term}(\mathcal{D})\}.$$

(i) B^* and ${}^*B$ are basis for a topology $\mathcal{T}^*$ and ${}^*\mathcal{T}$ on $\mathtt{Term}(\mathcal{D})$, respectively.
(ii) $\mathcal{T}^* \neq {}^*\mathcal{T}$.
(iii) The function

$$\mathtt{normal} : (\mathtt{Term}(\mathcal{D}), \mathcal{T}^*) \to (\mathtt{Term}(\mathcal{D}), \mathcal{T}^*)$$
$$\mathtt{normal} : (\mathtt{Term}(\mathcal{D}), {}^*\mathcal{T}) \to (\mathtt{Term}(\mathcal{D}), {}^*\mathcal{T})$$
$$\mathtt{normal} : (\mathtt{Term}(\mathcal{D}), \mathcal{T}^*) \to (\mathtt{Term}(\mathcal{D}), {}^*\mathcal{T})$$
$$\mathtt{normal} : (\mathtt{Term}(\mathcal{D}), {}^*\mathcal{T}) \to (\mathtt{Term}(\mathcal{D}), \mathcal{T}^*)$$

is in all above cases continuous.

PROOF. Exercise. □

3.5. The subformula property

To analyze normal derivations, it will be useful to introduce the notion of a *track* in a proof tree, which makes sense for non-normal derivations as well. For simplicity, we restrict our study to the $(\to)$-fragment of minimal logic i.e., to derivations where only the rules $\mathtt{ax}$, $\to^-$ and $\to^+$ are used. First we give some intermediate definitions.

DEFINITION 3.5.1. The relation $B \triangleleft A$, "B is a *Gentzen subformula* of A", is defined inductively by the rules:

$$\frac{}{A \triangleleft A}\,(R) \qquad \frac{B \Box C \triangleleft A}{B \triangleleft A, \quad C \triangleleft A}\,(\Box \in \{\to, \wedge, \vee\}),$$

$$\frac{\triangle_x B \triangleleft A, \quad \mathrm{Free}_{s,x}(B) = 1}{B(s) \triangleleft A}\,(\triangle \in \{\exists, \forall\}),$$

The relation $B \lhd_l A$, "B is a *literal subformula* of A", is defined inductively by the rules:

$$\frac{}{A \lhd_l A}\,(R) \qquad \frac{B\Box C \lhd_l A}{B \lhd_l A,\quad C \lhd_l A}\,(\Box \in \{\to, \wedge, \vee\}),$$

$$\frac{\triangle_x B \lhd_l A}{B \lhd_l A}\,(\triangle \in \{\exists, \forall\}).$$

The first part of the next definition is an example of a *simultaneous* inductive definition.

Definition 3.5.2. The relations $B \lhd^+ A$, "B is a *positive Gentzen subformula* of A" and $B \lhd^- A$, "B is a *negative Gentzen subformula* of A" are defined simultaneously and inductively by the rules:

$$\frac{}{A \lhd^+ A}\,(R),$$

$$\frac{B \circ C \lhd^+ A}{B \lhd^+ A,\quad C \lhd^+ A}\,(\circ \in \{\wedge, \vee\}), \qquad \frac{B \circ C \lhd^- A}{B \lhd^- A,\quad C \lhd^- A}\,(\circ \in \{\wedge, \vee\})$$

$$\frac{B \to C \lhd^+ A}{B \lhd^- A,\quad C \lhd^+ A}\,(\to), \qquad \frac{B \to C \lhd^- A}{B \lhd^+ A,\quad C \lhd^- A}\,(\to),$$

$$\frac{\triangle_x B \lhd^+ A,\quad \mathrm{Free}_{s,x}(B) = 1}{B(s) \lhd^+ A}, \qquad \frac{\triangle_x B \lhd^- A,\quad \mathrm{Free}_{s,x}(B) = 1}{B(s) \lhd^- A}\,(\triangle \in \{\exists, \forall\}).$$

The relation $B \lhd^{\oplus} A$, "B is a *strictly positive Gentzen subformula* of A" is defined inductively by the rules:

$$\frac{}{A \lhd^{\oplus} A}\,(R), \qquad \frac{B \circ C \lhd^{\oplus} A}{B \lhd^{\oplus} A,\quad C \lhd^{\oplus} A}\,(\circ \in \{\wedge, \vee\}), \qquad \frac{B \to C \lhd^{\oplus} A}{C \lhd^{\oplus} A}\,(\to),$$

$$\frac{\triangle_x B \lhd^{\oplus} A,\quad \mathrm{Free}_{s,x}(B) = 1}{B(s) \lhd^{\oplus} A}\,(\triangle \in \{\exists, \forall\}).$$

Proposition 3.5.3. *The following hold:*

(i) $\forall_{B,A \in \mathtt{Form}}(B \lhd^{\oplus} A \Rightarrow B \lhd^+ A)$.
(ii) $\forall_{B,A \in \mathtt{Form}}(B \lhd A \Rightarrow B \lhd^+ A$ *or* $B \lhd^- A)$.
(iii) $\forall_{B,A \in \mathtt{Form}}(B \lhd^+ A \Rightarrow B \lhd A$ *and* $B \lhd^- A \Rightarrow B \lhd A)$.
(iv) $\forall_{B,A \in \mathtt{Form}}(B \lhd^+ A \Rightarrow \neg(B \lhd^- A)$ *and* $B \lhd^- A \Rightarrow \neg(B \lhd^+ A))$.

Proof. Exercise. For (iii) and (iv) one needs to use the induction principle that corresponds to the simultaneous inductive definition of $(\lhd^+, \lhd^-)$. Note that in (iv) we actually mean that if the formula occurrence B is a positive subformula of A, then this occurrence cannot be a negative subformula of A. $\Box$

The notion of a (sub)formula occurrence in a formula is intuitively obvious, but for formal proofs of metamathematical properties it is sometimes necessary to use a rigorous formal definition. This may be given via the notion of a formula-context. Roughly speaking, a formula-context is nothing but a formula with an occurrence of a special propositional variable "$*$", a "placeholder". Alternatively, a context is sometimes described as "a formula with a hole in it". We define positive and negative formula-contexts simultaneously by the following inductive definition. The symbol "$*$" in the first rule functions as a special proposition letter (not in the language of first-order logic).

Definition 3.5.4. The sets $\mathcal{P}$ of *positive formula-contexts* and $\mathcal{N}$ of *negative formula-contexts* are defined simultaneously by the following rules

$$\frac{}{* \in \mathcal{P}}\,,$$

$$\frac{B^+ \in \mathcal{P}, \quad B^- \in \mathcal{N}, \quad A \in \texttt{Form}}{A \circ B^+, B^+ \circ A, A \to B^+, B^- \to A, \triangle_x B^+ \in \mathcal{P}}\,,$$

$$\frac{B^+ \in \mathcal{P}, \quad B^- \in \mathcal{N}, \quad A \in \texttt{Form}}{A \circ B^-, B^- \circ A, A \to B^-, B^+ \to A, \triangle_x B^- \in \mathcal{N}}\,.$$

The set of *formula-contexts* is the union of $\mathcal{P}$ of and $\mathcal{N}$. A *positive formula-occurrence in a formula* A is a pair (C, B^+), where $C \triangleleft_l A$ and $B^+ \in \mathcal{P}$ indicating the place where C occurs in A. A *negative formula-occurrence in a formula* A is a pair (D, B^-), where $D \triangleleft_l A$ and $B^- \in \mathcal{N}$ indicating the place where C occurs in A. A formula occurrence in A is either a positive or a negative formula occurrence in A.

Note that a formula-context contains always only a single occurrence of the symbol $*$. We may think of a formula-context as a formula in the language extended by $*$, in which $*$ occurs only once. In a positive (negative) context, $*$ is a positive (negative) subformula. We may also write $B^+[*]$ to denote an arbitrary positive formula-context and $B^-[*]$ to denote an arbitrary negative formula-context. Then $B^+[C]$, or $B^-[D]$, is the formula obtained by replacing $*$ by C (literally, without renaming variables), or $*$ by D, in A. Hence, if (C, B^+) and (D, B^-) are formula-occurrences in A,

$$B^+[C] = A, \quad \text{and} \quad B^-[D] = A.$$

E.g., if

$$A \equiv (P \to Q) \to (R \vee \forall_x S(x)),$$

the pair

$$\big(R,\ (P \to Q) \to (* \vee \forall_x S(x))\big)$$

is positive formula occurrence in A as $R \lhd_l A$ and $(P \to Q) \to (* \vee \forall_x S(x))$ is in $\mathcal{P}$. The notion of context may be generalized to a context with several placeholders $*_1, \ldots, *_n$, which are treated as extra proposition variables, each of which may occur only once in the formula-context.

Generally, a formal occurrence is a formula with a position in some structure. Working similarly to Definition 3.5.4, one can define inductively the notion of a *formal occurrence in a derivation* M, defining first inductively the notion of *derivation-context* i.e., a derivation extended by some placeholder "$*$" that occurs only once in M.

Definition 3.5.5. We define inductively the notion of derivation-context M^* of a derivation M within the $(\to)$-framework of minimal logic.

If M is the `ax`-derivation, we define $M^* \equiv *$.

If M is the derivation

$$\dfrac{\begin{array}{c}[u\colon A], \Gamma \\ \mid N \\ B\end{array}}{A \to B} \to^+ u$$

the following rules are used. If N^* is a derivation-context of N, then

$$\dfrac{N^*}{A \to B} \to^+ u$$

is a derivation-context of M. Moreover,

$$\dfrac{\begin{array}{c}[u\colon A], \Gamma \\ \mid N \\ B\end{array}}{*} \to^+ u$$

is a derivation-context of M.

If M is the derivation

$$\dfrac{\begin{array}{c}\Gamma \\ \mid N \\ A \to B\end{array} \qquad \begin{array}{c}\Delta \\ \mid K \\ A\end{array}}{B} \to^- ,$$

the following rules are used. If N^* is a derivation-context of N, then

$$\dfrac{N^* \qquad \begin{array}{c}\Delta \\ \mid K \\ A\end{array}}{B} \to^-$$

is a derivation-context of M. If K^* is a derivation-context of K, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mid N\\ A \to B\end{array} \qquad K^*}{B}\to^-$$

is a derivation-context of M. Moreover,

$$\dfrac{\begin{array}{c}\Gamma\\ \mid N\\ A \to B\end{array} \qquad \begin{array}{c}\Delta\\ \mid K\\ A\end{array}}{*}\to^- ,$$

is a derivation-context of M.

A formal occurrence in M is going to be a pair (A, M^*), where $A \in \texttt{Form}$ and M^* is a derivation-context indicating the place where A occurs in M. In this case A is called "*a formula in* M".

For example, if we consider the following derivation M

$$\dfrac{u_4\colon A_3 \to B \qquad \dfrac{\dfrac{u_1\colon A_1 \to A_2 \to A_3 \qquad u_2\colon A_1}{A_2 \to A_3}\to^- \qquad u_3\colon A_2}{A_3}\to^-}{B}\to^-$$

and if M_1^* is the derivation-context

$$\dfrac{u_4\colon A_3 \to B \qquad \dfrac{\dfrac{u_1\colon A_1 \to A_2 \to A_3 \qquad *}{A_2 \to A_3}\to^- \qquad u_3\colon A_2}{A_3}\to^-}{B}\to^-$$

the pair (A_1, M_1^*) is a formal occurrence in M. Moreover, one can describe this formal occurrence in M as a *top* formal occurrence. If N^+ is the derivation-context

$$\dfrac{u_4\colon A_3 \to B \qquad \dfrac{\dfrac{u_1\colon A_1 \to A_2 \to A_3 \qquad u_2\colon A_1}{*}\to^- \qquad u_3\colon A_2}{A_3}\to^-}{B}\to^-$$

the formal occurrence $(A_2 \to A_3, M_2^*)$ in M is *directly below* the formal occurrence (A_1, M_1^*).

Of course, we may have different formula occurrences $(A, M^*), (A, N^*)$ of the formula A in some derivation M, and that's why we need to indicate the exact position of A in M when we refer to A in the derivation tree M.

Definition 3.5.6. A *track* of a derivation M is a sequence

$$\vec{t} \equiv ((A_0, M_0^*), \ldots, (A_n, M_n^*))$$

of formula occurrences $(A_0, M_0^*), \ldots, (A_n, M_n^*)$ in M such that

(a) (A_0, M_0^*) is a top formal occurrence in M;
(b) For $i < n$, A_i is not the minor premise of an instance of $\to^-$, and (A_{i+1}, M_{i+1}^*) is directly below (A_i, M_i^*);
(c) A_n is either the minor premise of an instance of $\to^-$, or the $\mathtt{root}(M)$.

For each component (A_i, M_i^*) of $\vec{t}$, we say that (A_i, M_i^*) *belongs* to $\vec{t}$. The *track of order* 0, or the *main track*, in M is the (unique) track ending in $\mathtt{root}(M)$ (the uniqueness of this track is an exercise). A *track of order* $n+1$ is a track ending in the minor premise of an $\to^-$-application, with major premise belonging to a track of order n. We denote by $\mathrm{Track}(M)$ the set of tracks in M and by $\mathrm{ord}(\vec{t})$ the order of $\vec{t} \in \mathrm{Track}(M)$. Moreover, we use the notations

$$(A, M^*) \in \vec{t} \equiv \exists_{i \in \{0,\ldots,|\vec{t}|-1\}}\big((A, M^*) = \vec{t}_i\big),$$

$$A_i \equiv \mathrm{pr}_1(\vec{t}_i), \quad i \in \{0, \ldots, |\vec{t}| - 1\}.$$

If N is the derivation

$$\cfrac{\cfrac{u_4 \colon A_3 \to B \qquad \cfrac{\cfrac{u_1 \colon A_1 \to A_2 \to A_3 \qquad u_2 \colon A_1}{A_2 \to A_3}\to^- \qquad u_3 \colon A_2}{A_3}\to^-}{B}\to^-}{A_1 \to B}\to^+ u_2$$

and if we avoid for simplicity mentioning the derivation-contexts, we have the following tracks of N, where their subscript indicates their order:

$$\vec{t_0} \equiv (A_3 \to B, B, A_1 \to B), \quad \vec{t_1} \equiv (A_1 \to A_2 \to A_3, A_2 \to A_3, A_3),$$

$$\vec{t_2} \equiv (A_1), \quad \vec{s_2} \equiv (A_2).$$

Now the next result is expected.

Lemma 3.5.7. *If M is a derivation within the $(\to)$-fragment of minimal logic, then each formula occurrence in M belongs to some track of M.*

Proof. We use induction on derivations within the $(\to)$-fragment of minimal logic. In the case of M being the $\mathtt{ax}$-derivation we identify it for simplicity with the assumption $v \colon A$, hence the formula occurrence $(A, *)$ belongs to the only track $((A, *))$ of M.

Let M be the derivation

$$\dfrac{\begin{array}{c}[u\colon A]\\ \mid N\\ B\end{array}}{A \to B}\to^{+}u$$

and let every formula occurrence (C, N^*) in N belong to some track of N. Clearly, a derivation-context N^* of N generates a derivation-context $M(N)^*$

$$\dfrac{N^*}{A \to B}\to^{+}u$$

of M, and a derivation-context of M is either one generated by a derivation-context of N, or the derivation-context $\texttt{root}(M)^*$

$$\dfrac{\begin{array}{c}[u\colon A]\\ \mid N\\ B\end{array}}{*}\to^{+}u$$

that corresponds to $\texttt{root}(M)$. A formula-occurrence (C, M^*) in M is either of the form $(C, M(N)^*)$, or the formula occurrence $(A \to B, \texttt{root}(M)^*)$. Suppose that $(C, M^*) \equiv (C, M(N)^*)$, and let $\vec{t} \equiv ((C_0, N_0^*), \ldots, (C_k, N_k^*))$ be a track of N in which (C, N^*) belongs to. If C_k is the minor premise of an $\to^-$-rule, then

$$\vec{t}_M \equiv ((C_0, M(N_0)^*), \ldots, (C_k, M(N_k)^*))$$

is a track of M in which (C, M^*) belongs to. If $C_k \equiv B$, i.e., (C, N^*) belongs to the main track of N, then

$$\vec{t}_M \equiv ((C_0, M(N_0)^*), \ldots, (C_k, M(N_k)^*), (A \to B, \texttt{root}(M)^*))$$

is the main track of M in which $(C, M(N)^*)$ belongs to. If $(C, M^*) \equiv (A \to B, \texttt{root}(M)^*)$, then $(A \to B, \texttt{root}(M)^*)$ belongs to the main track of M.

Let M be the derivation

$$\dfrac{\begin{array}{c}\Gamma\\ \mid N\\ A \to B\end{array} \qquad \begin{array}{c}\Delta\\ \mid K\\ A\end{array}}{B}\to^{-},$$

and let every formula occurrence (C, N^*) in N belong to some track of N and every formula occurrence (D, K^*) in K belong to some track of K. Clearly, a derivation-context N^* of N generates a derivation-context $M(N)^*$

$$\dfrac{N^* \qquad \begin{array}{c}\Delta\\ \mid K\\ A\end{array}}{B}\to^-$$

of M, and a derivation-context K^* of K generates a derivation-context $M(K)^*$

$$\dfrac{\begin{array}{c}\Gamma\\ \mid N\\ A\to B\end{array} \qquad K^*}{B}\to^-$$

of M. A derivation-context of M is either one generated by a derivation-context of N, or by a derivation-context of K, or the derivation-context $\texttt{root}(M)^*$

$$\dfrac{\begin{array}{c}\Gamma\\ \mid N\\ A\to B\end{array} \qquad \begin{array}{c}\Delta\\ \mid K\\ A\end{array}}{*}\to^- \ .$$

A formula-occurrence (C, M^*) in M is either of the form $(C, M(N)^*)$, or of the form $(C, M(K)^*)$, or the formula occurrence $(B, \texttt{root}(M)^*)$. Suppose that $(C, M^*) \equiv (C, M(N)^*)$. Let $\vec{t} \equiv ((C_0, N_0^*), \ldots, (C_k, N_k^*))$ be a track of N in which (C, N^*) belongs to. If C_k is the minor premise of an $\to^-$-rule, we work as above. If $C_k \equiv A \to B$, then $(C, M(N)^*)$ belongs to the main track of M. Suppose that $(C, M^*) \equiv (C, M(K)^*)$. If $\vec{t} \equiv ((C_0, K_0^*), \ldots, (C_m, K_m^*))$ is a track of K in which (C, K^*) belongs to, then

$$\vec{t}_M \equiv ((C_0, M(K_0)^*), \ldots, (C_m, M(K_m)^*))$$

is a track of M in which (C, M^*) belongs to. Finally, the formula occurrence $(A \to B, \texttt{root}(M)^*)$ belongs to the main track of M. $\square$

THEOREM 3.5.8 (subformula property). *Let $N : C_1, \ldots, C_n \vdash A$ be a a normal derivation of A from (uncanceled) assumptions $C_1, \ldots, C_n$ within the $(\to)$-fragment of minimal logic. If B is a formula in N, then B is a Gentzen subformula of some formula in $\{C_1, \ldots, C_n\} \cup \{A\}$.*

PROOF. Let $\vec{t} = ((A_0, N_0^*), \ldots, (A_k, N_k^*)$ be a track of N. If (A, N^*) is a formal occurrence in $\vec{t}$ such that A is the conclusion of an $\to^-$-rule, we say that that *there is an* E-*rule in* $\vec{t}$, while if A is the conclusion of an $\to^+$-rule, we say that that *there is an* I-*rule in* $\vec{t}$. Suppose first that the set of E-rules in $\vec{t}$ is inhabited, and let A_i be the conclusion of this last E-rule in $\vec{t}$ i.e.,

$$\dfrac{B_1 \to A_i \qquad \begin{array}{c}\Delta_1 \\ \mid N_1 \\ B_1\end{array}}{A_i} \to^- \ .$$

By definition of the notion of track the formula-occurrence $(B_1 \to A_i, N^*_{i-1})$ is *immediately above* (A_i, N^*_i) in $\vec{t}$. Either $B_1 \to A_i$ is an assumption, or, because N is normal, $B_1 \to A_i$ is the conclusion of $\to^-$-rule. In the first case, we get that $A_i \triangleleft (B_1 \to A_i)$ and I-rules may follow in $\vec{t}$. In the second case we have

$$\dfrac{\dfrac{B_2 \to B_1 \to A_i \qquad \begin{array}{c}\Delta_2 \\ \mid N_2 \\ B_2\end{array}}{B_1 \to A_i} \to^- \qquad \begin{array}{c}\Delta_1 \\ \mid N_1 \\ B_1\end{array}}{A_i} \to^-$$

i.e., another E-rule precedes the last one in $\vec{t}$. Note that B_1 can be the conclusion of an I-rule, as B_1 is only the minor premise of the last E-rule in $\vec{t}$, but this does not affect our study of $\vec{t}$. Again we get that $A_i \triangleleft (B_1 \to A_i) \triangleleft (B_2 \to B_1 \to A_i)$. Working similarly, we get that

$$A_i \triangleleft A_{i-1} \triangleleft \ldots \triangleleft A_0.$$

Suppose next that the set of I-rules in $\vec{t}$ after A_i is also inhabited. Since there can be no E-rule in between, as A_i is the conclusion of the last E-rule, we have, for some $k \in \mathbb{N}^+$, the following picture of $\vec{t}$ directly below A_i:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{A_i}{D_1 \to A_i} \to^+ v_1}{D_2 \to D_1 \to A_i} \to^+ v_2 \\ \vdots\end{array}}{D_k \to \ldots \to D_2 \to D_1 \to A_i} \to^+ v_k \ .$$

In this case we get

$$A_i \triangleleft A_{i+1} \triangleleft \ldots \triangleleft A_k.$$

To summarize: a track $\vec{t}$ in N is divided into an E-part

$$((A_0, N^*_0), \ldots, (A_{i-1}, N^*_{i-1})),$$

a formula A_i, and an I-part

$$((A_{i+1}, N^*_{i+1}), \ldots, (A_k, N^*_k)),$$

such that A_i is the conclusion of an E-rule and, if $i < n$, a premise of an I-rule. Moreover either A_{j+1} is a subformula of A_j or vice versa. As a result, all formulas in the track are subformulas of A_0 or of A_k. Note that if A_0 is canceled later in the I-part of $\vec{t}$ i.e., if the assumption variable corresponding to A_0 is one of $v_1, \ldots, v_k$, all formulas in the track are Gentzen subformulas of A_k. If the I-part is empty, all formulas in the track are subformulas of A_0, and if the E-part of the track is empty, all formulas of the track are subformulas of A_k. If both the I-part and the E-part of the track are empty, then we just have an assumption A_0 in the track, which is a subformula of itself.

Let B be a formula in N i.e., (B, N^*) is a given formal occurrence in N. We show that is a Gentzen subformula of some formula in $\{C_1, \ldots, C_n\} \cup \{A\}$ by induction on the order of tracks of N. For this we prove inductively the following:

$$\forall_{n \in \mathbb{N}} \Big[\forall_{(B,N^*) \in \mathrm{Context}(N)} \forall_{\vec{t} \in \mathrm{Track}(N)} \big(\mathrm{ord}(\vec{t}) = n \ \wedge \ (B, N^*) \in \vec{t} \ \Rightarrow$$

$$\Rightarrow \exists_{i \in \{1,\ldots,n\}} (B \lhd C_i) \ \vee \ B \lhd A \big) \Big].$$

We suppose the above property for all $m < n$, and we show it for n. Note that if $\mathrm{ord}(\vec{t}) = 0$, we can use the above summary for the main track of M (if A_0 is canceled, it is canceled in the I-part of the main track, hence all its formulas are subformulas of A). We suppose the required property for all tracks of order $< n$ and we show it for every track of order $n > 0$. Let $\vec{s} = ((B_0, K_0^*), \ldots, (B_l, K_l^*) \in \mathrm{Track}(N)$ with $\mathrm{ord}(\vec{s}) = n$, and let (B_j, K_j^*) a formula-occurrence in $\vec{s}$. If $B_j \lhd B_l$, we have the following subtree of N

$$\frac{B_l \to D \qquad B_l}{D} \to^-$$

i.e., there is formula D with $B_l \to D$ in a track $\vec{u} \equiv ((E_0, L_0^*), \ldots, (E_m, L_m^*)$ of order $n - 1$. By the inductive hypothesis on $B_l \to D$ and $\vec{u}$ we get

$$B_j \lhd B_l \lhd (B_l \to D) \lhd E_0,$$

where $E_0 \in \{C_1, \ldots, C_n\}$, or

$$B_j \lhd B_l \lhd (B_l \to D) \lhd A.$$

Suppose next that $B_j \lhd B_0$. If $B_0 \in \{C_1, \ldots, C_n\}$, we are done. If B_0 is canceled, it will be canceled in a track of order less than n (i.e., on the left of the given track $\vec{s}$), therefore $B_j \lhd B_0 \lhd (B_0 \to E)$ and $B_0 \to E$ is in a track of N of order $< n$. Since by the inductive hypothesis $(B_0 \to E) \triangleleft C_j$, for some $j \in \{1, \ldots, n\}$, or $(B_0 \to E) \triangleleft A$, the same holds for B_j. $\square$

COROLLARY 3.5.9. *The* $(\to)$*-fragment of minimal logic is consistent.*

PROOF. Suppose that there is a derivation $M \colon \vdash \bot$. By normalization, there is some normal derivation $N \colon \vdash \bot$. Clearly, the last rule in N can be neither the `ax`-rule, nor the $\to^+$-rule. It cannot be also the $\to^-$-rule, since by the subformula property its major premise, which is an implication, is a subformula of $\bot$ i.e., it is $\bot$ again, which is absurd. $\square$

3.6. Notes

The proof of the strong normalization theorem presented here is based on the SN-method elaborated in [**16**]. The Term-rule in Definition 3.4.4 is introduced for technical reasons related to the proof of Theorem 3.4.6, and it can be avoided if we use a typed version of the set `Term` (see [**16**]). The main concepts and results of section 3.5 extend to the whole of minimal logic. Although normal derivations satisfy the subformula property, they can be of very large height.

CHAPTER 4

Models

4.1. Fan models

It is an obvious question to ask whether the logical rules we have been considering suffice, i.e., whether we have forgotten some necessary rules. To answer this question we first have to fix the *meaning* of a formula, i.e., provide a semantics for the syntax developed in Chapter 2. This will be done by means of fan models. Using this concept of a model we will prove soundness and completeness. First we extend Definition 3.3.1.

DEFINITION 4.1.1. Let T be a tree on some inhabited set X. A *leaf* of T is a node of T without proper successors (equivalently, without immediate successors). We denote by $\mathrm{Leaf}(T)$ the set of leaves of T. T is a *spread*, if $\mathrm{Leaf}(T) = \emptyset$, or equivalently, if every node of T has an immediate successor. A *subtree* T' of T, in symbols $T' \leq T$, is a subset T' of T which is also a tree on X. A *branch* S of T is a linearly ordered subtree of T i.e.,

$$\forall_{u,w \in S}(u \preceq w \vee w \preceq u).$$

A *finite path* of T is a finite branch of T. A *bar* B of a spread S on X is some $B \subseteq S$ such that

$$\forall_{\alpha \in [S]} \exists_{n \in \mathbb{N}} \big(\bar{\alpha}(n) \in B\big).$$

If $\bar{\alpha}(n) \in B$, we say that α *hits* the bar B at the node $\bar{\alpha}(n)$. A bar B of S is called *uniform*, if

$$\exists_{n \in \mathbb{N}} \forall_{\alpha \in [S]} \exists_{m \leq n} \big(\bar{\alpha}(m) \in B\big).$$

Clearly, a path is an infinite branch. The Baire and the Cantor tree are spreads, and the sets

$$L_n \equiv \{u \in 2^{<\mathbb{N}} \mid |u| = n\}$$

are uniform bars of $2^{<\mathbb{N}}$, for every $n \in \mathbb{N}$. Note that $L_0 \equiv \{\emptyset\}$, which is a uniform bar of every spread.

PROPOSITION 4.1.2. *A tree T on X is a subtree of a spread S on X.*

Proof. Since X is inhabited by some x_0, we define

$$S \equiv T \cup \bigcup_{u \in \mathrm{Leaf}(T)} u(x_0),$$

$$u(x_0) \equiv \{u * \vec{x_0} \mid \vec{x_0} \in \{x_0\}^{<\mathbb{N}}\},$$

where $u * \vec{x_0}$ denotes the concatenation of u and the node $\vec{x_0} \equiv (x_0, \ldots, x_0)$ of arbitrary length. $\square$

Proposition 4.1.3. *If F is a fan that is also a spread, then every bar B of F is uniform.*

Proof. Exercise. $\square$

Definition 4.1.4. For every $n \in \mathbb{N}$ we define

$$\mathbf{0} \equiv \emptyset,$$

$$\mathbf{n} \equiv \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n\text{-}1}\}.$$

If D is an inhabited set, the set $D^{\mathbf{n}}$ of all functions $f : \mathbf{n} \to D$ can be identified to the product set D^n. Moreover, we define

$$\mathtt{Rel}^{(n)}(D) \equiv \mathcal{P}(D^{\mathbf{n}}),$$

$$\mathtt{Rel}(D) \equiv \bigcup_{n \in \mathbb{N}} \mathtt{Rel}^{(n)}(D),$$

$$\mathtt{Fun}^{(n)}(D) \equiv \mathbb{F}(D^{\mathbf{n}}, D),$$

$$\mathtt{Fun}(D) \equiv \bigcup_{n \in \mathbb{N}} \mathtt{Fun}^{(n)}(D).$$

If $n > 0$, an element of $\mathtt{Rel}^{(n)}(D)$ is relation on D of arity n, and an element of $\mathtt{Fun}^{(n)}(D)$ is a function $f : D^n \to D$. Since $D^{\mathbf{0}} \equiv \{\emptyset\}$, we get $\mathtt{Rel}^{(0)}(D) \equiv \mathcal{P}(\{\emptyset\}) \equiv \{\emptyset, \{\emptyset\}\} \equiv \mathbf{2}$. The value $\mathbf{0} \equiv \emptyset$ represents falsity, and the value $\mathbf{1} \equiv \{\emptyset\}$ represents truth. Moreover, the set $\mathtt{Fun}^{(0)}(D) \equiv \mathbb{F}(\{\emptyset\}, D)$ can be identified with D.

For the rest of this section we fix a countable formal language $\mathcal{L}$ i.e., the sets $\mathtt{Rel}, \mathtt{Fun}$ in the signature of $\mathcal{L}$ are countable.

Definition 4.1.5. A *fan model* of $\mathcal{L}$ is a structure $\mathcal{M} = (D, F, \mathbf{i}, \mathbf{j})$ satisfying the following clauses:

(a) D is an inhabited set; we may also use the notation $|\mathcal{M}|$ for D.
(b) F is a fan on some inhabited set X.

(c) $\mathbf{i} : \mathtt{Fun} \to \mathtt{Fun}(D)$ such that for every $n \in \mathbb{N}$

$$\mathbf{i}_n \equiv \mathbf{i}_{|\mathtt{Fun}^{(n)}} : \mathtt{Fun}^{(n)} \to \mathtt{Fun}^{(n)}(D).$$

(d) $\mathbf{j} : \mathtt{Rel} \times F \to \mathtt{Rel}(D)$ such that for every $n \in \mathbb{N}$

$$\mathbf{j}_n \equiv \mathbf{j}_{|\mathtt{Rel}^{(n)} \times F} : \mathtt{Rel}^{(n)} \times F \to \mathtt{Rel}^{(n)}(D),$$

and for every $R \in \mathtt{Rel}$ the following monotonicity condition is satisfied:

$$\forall_{u,w \in F}\big(u \preceq w \Rightarrow \mathbf{j}(R, u) \subseteq \mathbf{j}(R, w)\big).$$

We also write $R^{\mathcal{M}}(\vec{d}, u)$ for $\vec{d} \in \mathbf{j}(R, u)$.

If $n > 0$ and $f \in \mathtt{Fun}^{(n)}$, we have that $\mathbf{i}(f) \in \mathtt{Fun}^{(n)}(D)$ i.e.,

$$\mathbf{i}(f) : D^n \to D\ .$$

If $n = 0$ and $f \in \mathtt{Fun}^{(0)} \equiv \mathtt{Const}$, we have that $\mathbf{i}(f) \in \mathtt{Fun}^{(0)}(D)$ i.e.,

$$\mathbf{i}(f) \in D\ .$$

If $n > 0$ and $R \in \mathtt{Rel}^{(n)}$, we have that $\mathbf{j}(R) \in \mathtt{Rel}^{(n)}(D)$ i.e.,

$$\mathbf{j}(R) \text{ is an } n\text{-ary relation on } D.$$

If $n = 0$ and $R \in \mathtt{Rel}^{(0)}$, we have that $\mathbf{j}(R) \in \mathtt{Rel}^{(0)}(D)$ i.e.,

$$\mathbf{j}(R) \in \mathbf{2},$$

hence $\mathbf{j}(R, u)$ is either true or false. Note that we set no special requirement on the value $\mathbf{j}(\perp, u)$, as minimal logic places no particular constraints on falsum $\perp$.

Within $\mathcal{M}$ we may interpret a node $u \in F$ as a "possible world", the relation $u \prec w$ as "the possible world w is a possible future of the possible world u", and the monotonicity condition of $\mathbf{j}_R$, where $u \mapsto \mathbf{j}_R(u) \equiv \mathbf{j}(R, u)$, for some $R \in \mathtt{Rel}^{(0)}$, as "if R is true at u, it is true at w", since, if $\mathbf{j}(R, u) \equiv \emptyset$, then we always have that $\mathbf{j}(R, u) \subseteq \mathbf{j}(R, w)$, while if $\mathbf{j}(R, u) \equiv \{\emptyset\}$, the monotonicity $\mathbf{j}(R, u) \subseteq \mathbf{j}(R, w)$, implies that $\mathbf{j}(R, w) \equiv \{\emptyset\}$ too.

The next simple fact explains why we may assume without loss of generality that the fan in a fan model is also a spread.

PROPOSITION 4.1.6. *If $\mathcal{M} = (D, F, \mathbf{i}, \mathbf{j})$ is a fan model of $\mathcal{L}$, there is a fan model $\mathcal{M}^* = (D, S, \mathbf{i}, \mathbf{j}^*)$ of $\mathcal{L}$ such that S is a spread.*

PROOF. If $x_0 \in D$, we consider S to be the spread of Proposition 4.1.2. We define $\mathbf{j}^*(R, u * \vec{x_0}) \equiv j(R, u)$, for every $u \in \mathrm{Leaf}(F)$, and $\mathbf{j}^*(R, u) \equiv \mathbf{j}(R, u)$, if $u \notin \mathrm{Leaf}(F)$. $\square$

DEFINITION 4.1.7. A *variable assignment* in D is a map $\eta : \mathtt{Var} \to D$. We denote by $[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n]$ the assignment that maps x_i to a_i and every other variable to $d_0 \in D$. If $\eta \in \mathbb{F}(\mathtt{Var}, D)$ and $a \in D$, let η_x^a be the variable assignment in D defined by η and a as follows:

$$\eta_x^a(y) \equiv \begin{cases} \eta(y) & \text{, if } y \neq x, \\ a & \text{, if } y = x. \end{cases}$$

Let a fan model $\mathcal{M} = (D, F, \mathbf{i}, \mathbf{j})$ and a variable assignment η in D be given. We define an *assignment* of $\mathtt{Term}$ in D i.e., a function $\eta_{\mathcal{M}} : \mathtt{Term} \to D$, through $(\mathcal{M}, \eta)$ by the following clauses:

$$\begin{aligned} \eta_{\mathcal{M}}(x) &\equiv \eta(x), \\ \eta_{\mathcal{M}}(c) &\equiv \mathbf{i}(c), \\ \eta_{\mathcal{M}}(f(t_1, \ldots, t_n)) &\equiv \mathbf{i}(f)(\eta_{\mathcal{M}}(t_1), \ldots, \eta_{\mathcal{M}}(t_n)). \end{aligned}$$

We often write $t^{\mathcal{M}}[\eta]$ for $\eta_{\mathcal{M}}(t)$, and when $\mathcal{M}$ is fixed, we may even use the same symbol $\eta(t)$ for $\eta_{\mathcal{M}}(t)$. If $\vec{t} \in \mathtt{Term}^{<\mathbb{N}}$, we define

$$\eta_{\mathcal{M}}(\vec{t}) \equiv \begin{cases} \emptyset & \text{, if } \vec{t} = \emptyset, \\ (\eta_{\mathcal{M}}(t_0), \ldots, \eta_{\mathcal{M}}(t_{|\vec{t}|-1})) & \text{, if } \vec{t} = (t_0, \ldots, t_{|\vec{t}|-1}). \end{cases}$$

In the rest of this chapter we use the following notation for some formula ϕ of our metalanguage:

$$\forall_{u' \succeq_n u} \phi \equiv \forall_{u' \succeq u} (|u'| = |u| + n \Rightarrow \phi).$$

DEFINITION 4.1.8 (Tarski/Beth). Let $\mathcal{M} = (D, F, \mathbf{i}, \mathbf{j})$ be a fan model of $\mathcal{L}$, such that F is a spread. We define inductively the relation " the formula A is *true* in $\mathcal{M}$ at the node u under the variable assignment η", in symbols

$$\mathcal{M}, u \Vdash A[\eta], \quad (\text{or simpler } u \Vdash A[\eta]),$$

by the following rules:

$$\begin{aligned} u \Vdash (R\vec{t})[\eta] &\equiv \exists_n \forall_{u' \succeq_n u} R^{\mathcal{M}}(\vec{t}^{\mathcal{M}}[\eta], u'), \\ u \Vdash (A \vee B)[\eta] &\equiv \exists_n \forall_{u' \succeq_n u} (u' \Vdash A[\eta] \vee u' \Vdash B[\eta]), \\ u \Vdash (\exists_x A)[\eta] &\equiv \exists_n \forall_{u' \succeq_n u} \exists_{a \in D} (u' \Vdash A[\eta_x^a]), \\ u \Vdash (A \to B)[\eta] &\equiv \forall_{u' \succeq u} (u' \Vdash A[\eta] \Rightarrow u' \Vdash B[\eta]), \\ u \Vdash (A \wedge B)[\eta] &\equiv u \Vdash A[\eta] \wedge u \Vdash B[\eta], \\ u \Vdash (\forall_x A)[\eta] &\equiv \forall_{a \in D} (u \Vdash A[\eta_x^a]). \end{aligned}$$

In this definition, the logical connectives $\rightarrow, \wedge, \vee, \forall, \exists$ on the left hand side are part of the object language, whereas the same connectives on the right hand side are to be understood in the usual sense: they belong to the "metalanguage". It should always be clear from the context whether a formula is part of the object or the metalanguage.

Note that if $R \in \mathtt{Rel}^{(0)}$, then

$$\begin{aligned} u \Vdash R[\eta] &\equiv \exists_n \forall_{u' \succeq_n u} R^{\mathcal{M}}(\emptyset^{\mathcal{M}}[\eta], u') \\ &\equiv \exists_n \forall_{u' \succeq_n u} R^{\mathcal{M}}(\emptyset, u') \\ &\equiv \exists_n \forall_{u' \succeq_n u}(\emptyset \in \mathbf{j}(R, u') \\ &\equiv \exists_n \forall_{u' \succeq_n u}(\mathbf{j}(R, u') = \mathbf{1}). \end{aligned}$$

If $R \in \mathtt{Rel}^{(n)}$, for some $n > 0$, then

$$\begin{aligned} u \Vdash (R\vec{t})[\eta] &\equiv \exists_n \forall_{u' \succeq_n u} R^{\mathcal{M}}(\vec{t}^{\mathcal{M}}[\eta], u') \\ &\equiv \exists_n \forall_{u' \succeq_n u}(\vec{t}^{\mathcal{M}}[\eta] \in \mathbf{j}(R, u') \\ &\equiv \exists_n \forall_{u' \succeq_n u}((\eta_{\mathcal{M}}(t_0), \dots, \eta_{\mathcal{M}}(t_{|\vec{t}|-1})) \in \mathbf{j}(R, u')) \end{aligned}$$

i.e., if we define

$$S_F(u) \equiv \{w \in F \mid u \preceq w\} \cup \{w \in F \mid w \preceq u\},$$

the set

$$\{w \in S_F(u) \mid R^{\mathcal{M}}(\vec{t}^{\mathcal{M}}[\eta], w)\}$$

is a uniform bar of the spread subfan $S_F(u)$ of F, with uniform bound $|u|+n$.

Note that the use of η_x^a in the definition of $u \Vdash (\exists_x A)[\eta]$ and $u \Vdash (\forall_x A)[\eta]$ makes sure that no capture occurs when we infer $u \Vdash (\exists_x A)[\eta]$ and $u \Vdash (\forall_x A)[\eta]$ from $\exists_n \forall_{u' \succeq_n u} \exists_{a \in D}(u' \Vdash A[\eta_x^a])$ and $\forall_{a \in D}(u \Vdash A[\eta_x^a])$, respectively.

PROPOSITION 4.1.9. *Let F be a a fan on X and G a fan on Y such that F, G are spreads.*

(i) If $u \in F$, we define

$$B(u) \equiv \{\alpha \in [F] \mid u \prec \alpha\},$$

$$u \prec \alpha \equiv \exists_{n \in \mathbb{N}}(\bar{\alpha}(n) = u).$$

The family $\{B(u) \mid u \in F\} \cup \{\emptyset\}$ is a basis for a topology T_F on $[F]$.

(ii) Let $\phi : F \to G$ satisfying the following properties:

$$\forall_{u,w \in F}(u \preceq w \Rightarrow \phi(u) \preceq \phi(w)),$$

$$\forall_{\alpha \in [F]}\big(\lim_{n \to \infty} |\phi(\bar{\alpha}(n))| = \infty\big).$$

Then, the function $[\phi] : [F] \to [G]$, *defined by*

$$\phi(\alpha) \equiv \bigvee_{n\in\mathbb{N}} \phi(\bar{\alpha}(n)),$$

where $u \vee w \equiv \sup_{\preceq}\{u, w\}$, *is continuous with respect to topologies* T_F, T_G.

PROOF. Exercise. □

PROPOSITION 4.1.10 (Extension). *Let* $A \in \texttt{Form}$, $\mathcal{M} \equiv (D, F, \mathbf{i}, \mathbf{j})$ *be a fan model of* $\mathcal{L}$, η *a variable assignment in* D, *and* $u, w \in F$. *Then*

$$u \preceq w \wedge u \Vdash A[\eta] \Rightarrow w \Vdash A[\eta].$$

PROOF. Exercise. □

PROPOSITION 4.1.11. *Let* $\mathcal{M} \equiv (D, F, \mathbf{i}, \mathbf{j})$ *be a fan model of* $\mathcal{L}$, η *a variable assignment in* D *and* $A, B \in \texttt{Form}$.
(i) *The set*

$$[\![A]\!]_{\mathcal{M},\eta} \equiv \{\alpha \in [F] \mid \exists_{n\in\mathbb{N}}\big(\bar{\alpha}(n) \Vdash A[\eta]\big)\}$$

is open in T_F.
(ii) *The following hold:*

$$[\![A \wedge B]\!]_{\mathcal{M},\eta} \equiv [\![A]\!]_{\mathcal{M},\eta} \cap [\![B]\!]_{\mathcal{M},\eta},$$

$$[\![A \vee B]\!]_{\mathcal{M},\eta} \equiv [\![A]\!]_{\mathcal{M},\eta} \cup [\![B]\!]_{\mathcal{M},\eta}.$$

PROOF. Exercise. □

Next proposition is a kind of converse to Proposition 4.1.10. According to it, in order to infer the truth of A at some node u from the truth of A in the possible future of u, we need to know that A is true at all future-nodes of u of some common level above the level of u.

PROPOSITION 4.1.12 (Covering). *Let* $A \in \texttt{Form}$, $\mathcal{M} \equiv (D, F, \mathbf{i}, \mathbf{j})$ *be a fan model of* $\mathcal{L}$ *and* η *a variable assignment in* D

$$\forall_{u' \succeq_n u}\big(u' \Vdash A[\eta]\big) \Rightarrow u \Vdash A[\eta].$$

PROOF. By induction on `Form`. *Case* $R\vec{s}$. Assume

$$\forall_{u' \succeq_n u}(u' \Vdash R\vec{s}).$$

Since F is a fan,there are finitely many nodes u' such that $u' \succeq_n u$. Let their set be $N = \{u_1, \ldots, u_l\}$. By definition we have that for each $u_k \in N$

$$\exists_{n_k}\forall_{w_k \succeq_{n_k} u_k} R^{\mathcal{M}}(\vec{s}^{\mathcal{M}}[\eta], w_k).$$

Let $m \equiv \max\{n_1, \ldots, n_l\}$. Then we have that

$$\forall_{w \succeq_{n+m} u} R^{\mathcal{T}}(\vec{s}^{\mathcal{T}}[\eta], w),$$

hence $u \Vdash R\vec{s}$. For this we argue as follows. If $w \succeq_{n+m} u$, then $w \succeq w_k \succeq_{n_k} u_k$, for some $k \in \{1, \ldots, l\}$. Since by hypothesis, $\eta_{\mathcal{M}}(\vec{s}) \in \mathbf{j}(R, w_k)$, by monotonicity of $\mathbf{j}_R$ we get $\eta_{\mathcal{M}}(\vec{s}) \in \mathbf{j}(R, w) \equiv R^{\mathcal{T}}(\vec{s}^{\mathcal{T}}[\eta], w)$.

The cases $A \vee B$ and $\exists_x A$ are handled similarly.

Case $A \to B$. Let $N = \{u_1, \ldots, u_l\}$ be the set of all $u' \succeq u$ with $|u'| = |u| + n$ such that $u' \Vdash (A \to B)[\eta]$. We show that

$$\forall_{w \succeq u}(w \Vdash A[\eta] \Rightarrow w \Vdash B[\eta]).$$

Let $w \succeq u$ and $w \Vdash A[\eta]$. We must show $w \Vdash B[\eta]$. If $|w| \geq |u| + n$, then $w \succeq u_k$, for some $k \in \{1, \ldots, l\}$. Hence, by the hypothesis on u_k and the definition of $u_k \Vdash (A \to B)[\eta]$, we get $w \Vdash B[\eta]$. If $|u| \leq |w| < |u| + n$, then by Proposition 4.1.10 for the set N' of all elements u_j of N that extend w we have that each $u_j \Vdash A[\eta]$. Hence, we also have that $u_j \Vdash B[\eta]$. But N' is the set of all successors of w with length $|w| + m$, where $m \equiv |u| + n - |w|$. By the induction hypothesis on the formula B, we get the required $w \Vdash B[\eta]$.

The cases $A \wedge B$ and $\forall_x A$ are straightforward. □

4.2. Soundness of minimal logic

LEMMA 4.2.1 (Coincidence). *Let $\mathcal{M}$ be a fan model, t a term, A a formula and η, ξ assignments in $|\mathcal{M}|$.*

(a) *If $\eta(x) = \xi(x)$ for all $x \in \mathrm{FV}(t)$, then $\eta_{\mathcal{M}}(t) = \xi_{\mathcal{M}}(t)$.*
(b) *If $\eta(x) = \xi(x)$ for all $x \in \mathrm{FV}(A)$, then $\mathcal{M}, u \Vdash A[\eta]$ if and only if $\mathcal{M}, u \Vdash A[\xi]$.*

PROOF. By Induction on `Term` and `Form`, respectively. The details are left to the reader. □

LEMMA 4.2.2 (Substitution). *Let $\mathcal{M}$ be a tree model, $t, r(x)$ terms, $A(x)$ a formula and η an assignment in $|\mathcal{M}|$. Then*

(a) $\eta_{\mathcal{M}}(r(t)) = \eta_x^{\eta_{\mathcal{M}}(t)}(r(x))$.
(b) $\mathcal{M}, u \Vdash A(t)[\eta]$ *if and only if* $\mathcal{M}, u \Vdash A(x)[\eta_x^{\eta_{\mathcal{M}}(t)}]$.

PROOF. By Induction on `Term` and `Form`, respectively. The details are left to the reader. □

Next theorem expresses the fact that minimal derivations are sound i.e., they respect truth in a fan model. As usual, if $\Gamma \subseteq$ `Form`, we define

$$u \Vdash \Gamma[\eta] \equiv \forall_{C \in \Gamma}(u \Vdash C[\eta])$$

$$\equiv \forall_{C\in\texttt{Form}}(C \in \Gamma \Rightarrow u \Vdash C[\eta]).$$

Consequently,

$$u \Vdash \emptyset[\eta]$$

is always true.

THEOREM 4.2.3 (Soundness of minimal logic). *Let* $\Gamma \cup \{A\} \subseteq \texttt{Form}$ *such that* $\Gamma \vdash A$. *If* $\mathcal{M} = (D, F, \mathbf{i}, \mathbf{j})$ *is a fan model,* $u \in F$ *and* η *is a variable assignment in* D*, then*

$$\mathcal{M}, u \Vdash \Gamma[\eta] \Rightarrow \mathcal{M}, u \Vdash A[\eta].$$

PROOF. We work as in the proof of Theorem 2.5.7. We prove the formula

$$\forall_{M\in\mathcal{D}}\Big(\forall_{\eta\in\mathbb{F}(\mathrm{Var},D)}\forall_{u\in F}\big(u \Vdash \texttt{Assumptions}(M)[\eta] \Rightarrow u \Vdash \texttt{root}(M)[\eta]\big)\Big)$$

by induction on $\mathcal{D}$, and having fixed $\mathcal{M}$. The required implication $\mathcal{M}, u \Vdash \Gamma[\eta] \Rightarrow \mathcal{M}, u \Vdash A[\eta]$ follows then immediately.

Case $\texttt{ax}$. The validity of $u \Vdash A[\eta] \Rightarrow u \Vdash A[\eta]$ is immediate.

Case $\to^+$. Let the derivation

$$\cfrac{\begin{array}{c}[A],\ C_1, \dots, C_n \\ \mid N \\ B\end{array}}{A \to B}\to^+$$

and suppose $u \Vdash \{C, \dots, C_n\}[\eta]$. We show $u \Vdash (A \to B)[\eta] \equiv \forall_{u'\succeq u}(u' \Vdash A[\eta] \Rightarrow u' \Vdash B[\eta])$ under the inductive hypothesis on N:

$$\mathrm{IH}(N): \quad \forall_\eta\forall_w(w \Vdash \{A, C_1, \dots, C_n\}[\eta] \Rightarrow w \Vdash B[\eta]).$$

We fix u' such that $u' \succeq u$ and we suppose $u' \Vdash A[\eta]$. By Extension (Proposition 4.1.10) we get $u' \Vdash \{C_1, \dots, C_n\}[\eta]$, hence $u' \Vdash \{A, C_1, \dots, C_n\}[\eta]$. Hence, by IH($N$) we get $u' \Vdash B[\eta]$.

Case ($\to^-$). Let the derivation

$$\cfrac{\begin{array}{c}C_1, \dots, C_n \\ \mid N \\ A \to B\end{array} \qquad \begin{array}{c}D_1, \dots, D_m \\ \mid K \\ A\end{array}}{B}\to^-$$

and suppose $u \Vdash \{C_1, \dots, C_n, D_1, \dots, D_m\}[\eta]$. We show $u \Vdash B[\eta]$ under the inductive hypotheses on N and K:

$$\mathrm{IH}(N): \quad \forall_\eta\forall_w(w \Vdash \{C_1, \dots, C_n\}[\eta] \Rightarrow w \Vdash (A \to B)[\eta]),$$

$$\mathrm{IH}(K): \quad \forall_\eta\forall_w(w \Vdash \{D_1, \dots, D_m\}[\eta] \Rightarrow w \Vdash A[\eta]).$$

By IH(N) we have that $u \Vdash (A \to B)[\eta]$, and by IH(K) we get $u \Vdash A[\eta]$, hence $u \Vdash B[\eta]$.

Case ($\forall^+$) Let the derivation

$$\dfrac{\begin{array}{c} C_1, \dots, C_n \\ \mid N \\ A \end{array}}{\forall_x A}\, \forall^+ x$$

with the variable condition $x \notin \mathrm{FV}(C_1) \wedge \ldots \wedge x \notin \mathrm{FV}(C_n)$, and suppose $u \Vdash \{C_1, \dots, C_n\}[\eta]$. We show $u \Vdash (\forall_x A)[\eta] \equiv \forall_{a \in D}(u \Vdash A[\eta_x^a])$ under the inductive hypothesis on N:

$$\mathrm{IH}(N): \quad \forall_\eta \forall_w (w \Vdash \{C_1, \dots, C_n\}[\eta] \Rightarrow w \Vdash A[\eta]).$$

Let $a \in D$. By the variable condition we get $\eta_{|\mathrm{FV}(C_i)} = (\eta_x^a)_{|\mathrm{FV}(C_i)}$, for every $i \in \{1, \dots, n\}$, hence by Coincidence (Lemma 4.2.1) we conclude that $u \Vdash \{C, \dots, C_n\}[\eta_x^a]$. By IH($N$) on η_x^a and u we get $u \Vdash A[\eta_x^a]$.

Case ($\forall^-$). Let the derivation

$$\dfrac{\begin{array}{c} C_1, \dots, C_n \\ \mid N \\ \forall_x A \end{array} \qquad r \in \mathtt{Term}}{A(r)}\, \forall^-$$

and suppose $u \Vdash \{C_1, \dots, C_n\}[\eta]$. We show $u \Vdash A(r)[\eta]$ under the inductive hypotheses on N:

$$\mathrm{IH}(N): \quad \forall_\eta \forall_w (w \Vdash \{C_1, \dots, C_n\}[\eta] \Rightarrow w \Vdash (\forall_x A)[\eta]).$$

Applying IH(N) on u we get $\forall_{a \in D}(u \Vdash A[\eta_x^a])$, hence $u \Vdash A[\eta_x^{\eta_{\mathcal{M}}(r)}]$, and by Substitution (Lemma 4.2.2) we conclude that $u \Vdash A(r)[\eta]$.

Case ($\wedge^+$) and *Case* ($\wedge^-$) are straightforward.

Case ($\vee_0^+$) and *Case* ($\vee_1^+$) are straightforward.

Case ($\vee^-$). Let the derivation

$$\dfrac{\begin{array}{c} C_1, \dots, C_n \\ \mid N \\ A \vee B \end{array} \qquad \begin{array}{c} [A], D_1, \dots, D_m \\ \mid K \\ C \end{array} \qquad \begin{array}{c} [B], E_1, \dots, E_l \\ \mid L \\ C \end{array}}{C}\, \vee^-$$

and suppose $u \Vdash \{C, \dots, C_n, D_1, \dots, D_m, E_1, \dots, E_l\}[\eta]$. We show $u \Vdash C[\eta]$ under the inductive hypotheses on N, K and L:

$$\mathrm{IH}(N): \quad \forall_\eta \forall_w (w \Vdash \{C_1, \dots, C_n\}[\eta] \Rightarrow w \Vdash (A \vee B)[\eta]),$$

$$\mathrm{IH}(K): \quad \forall_\eta \forall_w (w \Vdash \{A, D_1, \dots, D_m\}[\eta] \Rightarrow w \Vdash C[\eta]),$$

$$\mathrm{IH}(L): \quad \forall_\eta \forall_w (w \Vdash \{B, E_1, \dots, E_l\}[\eta] \Rightarrow w \Vdash C[\eta]).$$

By $\mathrm{IH}(N)$ we get $u \Vdash (A \vee B)[\eta] \equiv \exists_n \forall_{u' \succeq_n u}(u' \Vdash A[\eta]$ or $u' \Vdash B[\eta])$. By Covering (Proposition 4.1.12) it suffices to show for this $n \in \mathbb{N}$:

$$\forall_{u' \succeq_n u}(u' \Vdash C[\eta]).$$

We fix u' such that $u' \succeq_n u$. If $u' \Vdash A[\eta]$, then by Extension and $\mathrm{IH}(K)$ we get $u' \Vdash C[\eta]$. If $u' \Vdash B[\eta]$, then by Extension and $\mathrm{IH}(L)$ we get $u' \Vdash C[\eta]$.

Case $(\exists^+)$ is straightforward.

Case $(\exists^-)$. Let the derivation

$$\dfrac{\begin{array}{c} C_1, \dots, C_n \\ \mid N \\ \exists_x A \end{array} \qquad \begin{array}{c} [A], D_1, \dots, D_m \\ \mid K \\ B \end{array}}{B}\ \exists^- x$$

with the variable condition $x \notin \mathrm{FV}(D_1) \wedge \ldots \wedge x \notin \mathrm{FV}(D_m)$ and $x \notin \mathrm{FV}(B)$, and suppose $u \Vdash \{C, \dots, C_n, D_1, \dots, D_m\}$. We show $u \Vdash B[\eta]$ under the inductive hypotheses on N and K:

$$\mathrm{IH}(N): \quad \forall_\eta \forall_w (w \Vdash \{C_1, \dots, C_n\}[\eta] \Rightarrow w \Vdash (\exists_x A)[\eta]),$$

$$\mathrm{IH}(K): \quad \forall_\eta \forall_w (w \Vdash \{A, D_1, \dots, D_m\}[\eta] \Rightarrow w \Vdash B[\eta]).$$

By $\mathrm{IH}(N)$ we get that $u \Vdash (\exists_x A)[\eta] \equiv \exists_n \forall_{u' \succeq_n u} \exists_{a \in D}(u' \Vdash A[\eta_x^a])$. By Covering it suffices to show for this $n \in \mathbb{N}$:

$$\forall_{u' \succeq_n u}(u' \Vdash B[\eta]).$$

We fix u' such that $u' \succeq_n u$, and let $a \in D$ such that $u' \Vdash A[\eta_x^a]$. Since by the variable condition we get $\eta_{|FV(D_i)} = (\eta_x^a)_{|FV(D_i)}$, and since by Extension $u' \Vdash \{D_1, \dots, D_m\}[\eta]$, by Coincidence we get $u' \Vdash \{A, D_1, \dots, D_m\}[\eta_x^a]$. By $\mathrm{IH}(K)$ on η_x^a and u' we get $u' \Vdash B[\eta_x^a]$. Since by the variable condition we get $\eta_{|FV(B)} = (\eta_x^a)_{|FV(B)}$, we conclude that $u' \Vdash B[\eta]$. $\square$

Consequently, if $\vdash A$, then for every $u \in F$ and assignment η we get

$$u \Vdash A[\eta],$$

hence

$$[\![A]\!]_{\mathcal{M},\eta} = [F].$$

PROPOSITION 4.2.4. *If $\mathcal{M} = (D, F, \mathbf{i}, \mathbf{j})$ is a fan model, $u \in F$, and η is a variable assignment in D, then*

$$[\![u]\!]_{\mathcal{M},\eta} \equiv \{A \in \texttt{Form} \mid u \Vdash A[\eta]\}$$

is an open set in the topology $\mathcal{T}(\mathcal{B})$ on Form *(defined in Proposition 2.5.20).*

PROOF. Let $A \in [\![u]\!]_{\mathcal{M},\eta}$. We show that $A \in O_A \subseteq [\![u]\!]_{\mathcal{M},\eta}$. Let $B \in O_A$ i.e., $A \vdash B$. We show that $B \in [\![u]\!]_{\mathcal{M},\eta}$. By soundness theorem we have that $u \Vdash A[\eta] \Rightarrow u \Vdash B[\eta]$, and since by hypothesis $u \Vdash A[\eta]$, we get $u \Vdash B[\eta]$. $\square$

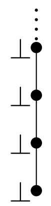
The main application of the soundness theorem is its use in the proof of underivability results.

DEFINITION 4.2.5. A *countermodel* to some derivation $\Gamma \vdash A$ is a triple $(\mathcal{M}, \eta, u)$, where $\mathcal{M} = (D, F, \mathbf{i}, \mathbf{j})$ is a fan model, η is a variable assignment in D, and $u \in F$ such that

$$u \Vdash \Gamma[\eta] \text{ and } u \nVdash A[\eta].$$

By soundness theorem, if $(\mathcal{M}, \eta, u)$ is a countermodel to the derivation $\Gamma \vdash A$, we can conclude that $\Gamma \nvdash A$, since if there was such a derivation we should have $u \Vdash \Gamma[\eta] \Rightarrow u \Vdash A[\eta]$, which contradicts the existence of a countermodel.

A countermodel to the derivation $\vdash \bot \to R$, where $R \in \texttt{Rel}^{(0)} \setminus \{\bot\}$, is constructed as follows: Take $F = \{x_0\}^{<\mathbb{N}}$, D any inhabited set, and define $\mathbf{j}(\bot, \emptyset) \equiv \mathbf{1}$, and $\mathbf{j}(R, \emptyset) \equiv \mathbf{0}$.



By extension we get $\mathbf{j}(\bot, u) \equiv \mathbf{1}$, for every $u \in F$. Moreover, we get $\mathbf{j}(R, u) \equiv \mathbf{0}$, for every $u \in F$; if there was some $u \in F \setminus \{\emptyset\}$ such that $\mathbf{j}(R, u) \equiv \mathbf{1}$, then, since this is the only node $u' \in F$ such that $u' \succeq_{|u'|} \emptyset$, by Covering we would get $\mathbf{j}(R, \emptyset) \equiv \mathbf{1}$ too. We show that $\emptyset \nVdash (\bot \to R)[\eta]$, where η is arbitrary. Suppose that $\emptyset \Vdash (\bot \to R)[\eta] \equiv \forall_u (u \Vdash \bot[\eta] \Rightarrow u \Vdash R[\eta])$. For every $u \in F$ though, we have that $u \Vdash \bot[\eta]$ and $u \nVdash R[\eta]$.

DEFINITION 4.2.6. An *intuitionistic fan model* of a countable first-order language $\mathcal{L}$ is a fan model $\mathcal{M}_i \equiv (D, F, \mathbf{i}, \mathbf{j})$ of $\mathcal{L}$ such that

$$\forall_{u \in F} (\mathbf{j}(\bot, u) \equiv \mathbf{0}).$$

It is immediate to see that if $\mathcal{M}_i$ is an intuitionistic fan model, then $\mathcal{M}_i, u \Vdash (\bot \to A)[\eta]$, for every $u \in F$ and assignment η in D.

LEMMA 4.2.7. *A fan model* $\mathcal{M} \equiv (D, F, \mathbf{i}, \mathbf{j})$ *is intuitionistic if and only if* $\forall_\eta \forall_{u\in F}(u \nVdash \bot[\eta])$.

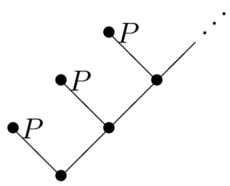
PROOF. Suppose first that $\mathcal{M}$ is intuitionistic, and let $u \Vdash \bot[\eta]$ i.e., $\exists_n \forall_{u' \succeq_n u}(\mathbf{j}(\bot, u') = 1$, which contradicts the definition of an intuitionistic fan model. For the converse, we suppose that there is some $u \in F$ such that $j(\bot, u) \equiv \mathbf{1}$. By Extension we get $\forall_{u'\succeq_n u}(\mathbf{j}(\bot, u') = 1 \Leftrightarrow \forall_{u'\succeq_n u}(u' \Vdash \bot[\eta])$, hence by Covering we get $u \Vdash \bot[\eta]$, which contradicts our hypothesis on $\mathcal{M}$. $\square$

PROPOSITION 4.2.8. *Let* $\mathcal{M}_i \equiv (D, F, \mathbf{i}, \mathbf{j})$ *be an intuitionistic fan model of* $\mathcal{L}$*,* η *a variable assignment,* $u \in F$ *and* $A \in \texttt{Form}$*.*
(i) $u \Vdash (\neg A)[\eta] \Leftrightarrow \forall_{u'\succeq u}(u' \nVdash A[\eta])$.
(ii) $u \Vdash (\neg\neg A)[\eta] \Leftrightarrow \forall_{u'\succeq u}\neg\forall_{u''\succeq u'}(u'' \nVdash A[\eta])$.

DEFINITION 4.2.9. An *intuitionistic countermodel* to some derivation $\Gamma \vdash_i A$ is a triple $(\mathcal{M}_i, \eta, u)$, where $\mathcal{M}_i = (D, F, \mathbf{i}, \mathbf{j})$ is an intuitionisitc fan model, η is a variable assignment in D, and $u \in F$ such that $u \Vdash \Gamma[\eta]$ and $u \nVdash A[\eta]$.

Since the soundness theorem of intuitionistic logic follows immediately from the soundness theorem of minimal logic, we can use it to conclude intuitionistic underivability $\Gamma \nvdash_i A$ from an intuitionistic countermodel to $\Gamma \vdash_i A$.

As an example we give an intuitionistic countermodel to the derivation $\vdash_i \neg\neg P \to P$. We describe the desired fan model by means of a diagram below. Next to every node we write all propositions forced at that node.



This is a fan model because monotonicity clearly holds. Observe also that $\mathbf{j}(\bot, u) \equiv \mathbf{0}$, for every node u i.e., it is an intuitionistic fan model, and moreover $\emptyset \nVdash P$. Using Proposition 4.2.8(ii), it is easily seen that $\emptyset \Vdash \neg\neg P$. Thus $\emptyset \nVdash (\neg\neg P \to P)$, and hence $\nvdash_i (\neg\neg P \to P)$.

4.3. Completeness of minimal logic

THEOREM 4.3.1 (Completeness of minimal logic). *Let* $\Gamma \cup \{A\} \subseteq$ `Form`. *The following are equivalent.*

(a) $\Gamma \vdash A$.

(b) $\Gamma \Vdash A$, *i.e., for all fan models* $\mathcal{M}$, *assignments* η *in* $|\mathcal{M}|$ *and nodes* u *in the tree of* $\mathcal{M}$

$$\mathcal{M}, u \Vdash \Gamma[\eta] \Rightarrow \mathcal{M}, u \Vdash A[\eta].$$

PROOF. Soundness of minimal logic already gives "(a) implies (b)".

The main idea in the proof of the other direction is the construction of a fan model $\mathcal{M}$ over the Cantor tree $2^{<\mathbb{N}}$ with domain D the set `Term` of all terms of the underlying language such that the following property holds:

$$\Gamma \vdash B \Leftrightarrow \mathcal{M}, \emptyset \Vdash B[\mathrm{id}].$$

We assume here that $\Gamma \cup \{A\}$ is a set of closed formulas.

In order to define $\mathcal{M}$, we will need an enumeration $A_0, A_1, A_2, \dots$ of the underlying language $\mathcal{L}$ (assumed countable), in which every formula occurs infinitely often. We also fix an enumeration $x_0, x_1, \dots$ of distinct variables. Since Γ is countable it can we written $\Gamma = \bigcup_n \Gamma_n$ with finite sets Γ_n such that $\Gamma_n \subseteq \Gamma_{n+1}$. With every node $u \in 2^{<\mathbb{N}}$, we associate a finite set Δ_u of formulas and a set V_u of variables, by induction on the length of u.

We write $\Delta \vdash_n B$ to mean that there is a derivation of height $\leq n$ of B from Δ.

Let $\Delta_\emptyset \equiv \emptyset$ and $V_\emptyset \equiv \emptyset$. Take a node u such that $|u| = n$ and suppose that Δ_u, V_u are already defined. We define Δ_{u*0}, V_{u*0} and Δ_{u*1}, V_{u*1} as follows:

Case 0. $\mathrm{FV}(A_n) \not\subseteq V_u$. Then let

$$\Delta_{u*0} \equiv \Delta_{u*1} \equiv \Delta_u \quad \text{and} \quad V_{u*0} := V_{u*1} := V_u.$$

Case 1. $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \nvdash_n A_n$. Let

$$\begin{aligned} &\Delta_{u*0} \equiv \Delta_u \quad \text{and} \quad \Delta_{u*1} \equiv \Delta_u \cup \{A_n\}, \\ &V_{u*0} \equiv V_{u*1} := V_u. \end{aligned}$$

Case 2. $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \vdash_n A_n = A'_n \vee A''_n$. Let

$$\begin{aligned} &\Delta_{u*0} \equiv \Delta_u \cup \{A_n, A'_n\} \quad \text{and} \quad \Delta_{u*1} \equiv \Delta_u \cup \{A_n, A''_n\}, \\ &V_{u*0} := V_{u*1} := V_u. \end{aligned}$$

Case 3. $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \vdash_n A_n = \exists_x A'_n(x)$. Let

$$\Delta_{u*0} \equiv \Delta_{u*1} \equiv \Delta_u \cup \{A_n, A'_n(x_i)\} \quad \text{and} \quad V_{u*0} \equiv V_{u*1} \equiv V_u \cup \{x_i\},$$

where x_i is the first variable $\notin V_u$.

Case 4. $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \vdash_n A_n$, with A_n neither a disjunction nor an existentially quantified formula. Let

$$\Delta_{u*0} \equiv \Delta_{u*1} \equiv \Delta_u \cup \{A_n\} \quad \text{and} \quad V_{u*0} \equiv V_{u*1} \equiv V_u.$$

The following remarks (R1)-(R3) are clear.

(R1) Δ_u, V_u are finite sets.

(R2) $\mathrm{FV}(\Delta_u) \subseteq V_u$.

(R3) $u \preceq w \Rightarrow \Delta_u \subseteq \Delta_w$ and $V_u \subseteq V_w$.

(R4) $\forall_{x_i \in \mathtt{Var}} \exists_m \forall_{u \in 2^{<\mathbb{N}}} (|u| = m \Rightarrow x_i \in V_u)$.

Remark (R4) is shown as follows: Let the derivation $\vdash \exists_x(\bot \to \bot)$ with height m_0. Suppose that for every x_j with $j < i$, there is some m_j such that $\forall_{u \in 2^{<\mathbb{N}}}(|u| = m_j \Rightarrow x_j \in V_u)$. Let $n \geq \max\{m_0, m_1, \ldots, m_{i-1}\}$ such that $A_n \equiv \exists_x(\bot \to \bot)$ (this n can be found, as the formula $\exists_x(\bot \to \bot)$ occurs infinitely often in the fixed enumeration of formulas). Sinve $n \geq m_0$, if $|u| = n$, then $\Gamma_n, \Delta_u \vdash_n \exists_x(\bot \to \bot)$. By definition of n and (R3) we get that $x_1, \ldots, x_{i-1} \in V_u$. If $x_i \in V_u$, then $x_i \in V_{u*j}$, with $j \in \mathbf{2}$. If $x_i \notin V_u$, and since $\mathrm{FV}(\exists_x(\bot \to \bot)) = \emptyset \subseteq V_u$, by Case 3 we have that $x_i \in V_{u*j}$, since x_i is the first variable in the fixed enumeration of $\mathtt{Var}$ that does not occur in V_u. Hence $m_i \equiv n + 1$ satisfies the required property.

We also have the following:

$$\forall_{u' \succeq_n u} (\Gamma, \Delta_{u'} \vdash B) \Rightarrow \Gamma, \Delta_u \vdash B, \quad \text{provided } \mathrm{FV}(B) \subseteq V_u. \tag{4.1}$$

It is sufficient to show that, for $\mathrm{FV}(B) \subseteq V_u$,

$$(\Gamma, \Delta_{u*0} \vdash B) \wedge (\Gamma, \Delta_{u*1} \vdash B) \Rightarrow (\Gamma, \Delta_u \vdash B).$$

In cases 0, 1 and 4, this is obvious. For case 2, the claim follows immediately from the axiom schema $\vee^-$. In case 3, we have $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \vdash_n A_n \equiv \exists_x A'_n(x)$. Assume $\Gamma, \Delta_u \cup \{A_n, A'_n(x_i)\} \vdash B$ with $x_i \notin V_u$, and $\mathrm{FV}(B) \subseteq V_u$. Then $x_i \notin \mathrm{FV}(\Delta_u \cup \{A_n, B\})$, hence $\Gamma, \Delta_u \cup \{A_n\} \vdash B$ by $\exists^-$ and therefore $\Gamma, \Delta_u \vdash B$.

Next, we show

$$\Gamma, \Delta_u \vdash B \Rightarrow \exists_n \forall_{u' \succeq_n u} (B \in \Delta_{u'}), \quad \text{provided } \mathrm{FV}(B) \subseteq V_u. \tag{4.2}$$

Choose $n \geq |u|$ such that $B \equiv A_n$ and $\Gamma_n, \Delta_u \vdash_n A_n$. For all $u' \succeq u$, if $|u'| = n + 1$ then $A_n \in \Delta_{u'}$ (we work as above for Cases 2-4).

Using the sets Δ_u we define the fan model $\mathcal{M} \equiv (\mathtt{Term}, \mathbf{i}, \mathbf{j})$ as follows. If $f \in \mathtt{Fun}^{(n)}$, then $\mathbf{i}(f) : \mathtt{Term}^n \to \mathtt{Term}$ is defined by

$$\mathbf{i}(f)(t_1, \ldots, t_n) \equiv f(t_1, \ldots, t_n).$$

Obviously, $t^{\mathcal{M}}[\mathrm{id}] = t$ for all $t \in \mathtt{Term}$. If $R \in \mathtt{Rel}^{(n)}$, then $\mathbf{j}(R, u) \subseteq \mathtt{Term}^n$ is defined by

$$\mathbf{j}(R, u) \equiv \{(t_1, \ldots, t_n) \in \mathtt{Term}^n \mid R(t_1, \ldots, t_n) \in \Delta_u\}.$$

Hence, if $R \in \mathtt{Rel}^{(0)}$, $\mathbf{j}(R, u) \equiv \mathbf{0}$, for every $u \in 2^{<\mathbb{N}}$. We write $u \Vdash B$ for $\mathcal{M}, u \Vdash B[\mathrm{id}]$, and we show:

Claim. $\Gamma, \Delta_u \vdash B \Leftrightarrow u \Vdash B$, *provided* $\mathrm{FV}(B) \subseteq V_u$.

The proof is by induction on the well-founded relation $C \lhd_* B$, "C is a proper Gentzen subformula of B" (see Proposition 3.3.3). I.e., if

$$P(B) \equiv \forall_u \big(\mathrm{FV}(B) \subseteq V_u \Rightarrow (\Gamma, \Delta_u \vdash B \Leftrightarrow u \vdash B)\big),$$

we show by induction on `Form` that

$$\forall_{B \in \mathtt{Form}} \big(\forall_{C \lhd_* B}(P(C)) \Rightarrow P(B)\big),$$

and we conclude that $\forall_{B \in \mathtt{Form}}(P(B))$.

Case $R\vec{s}$. Assume $\mathrm{FV}(R\vec{s}) \subseteq V_u$. The following are equivalent:

$$\begin{array}{ll}
\Gamma, \Delta_u \vdash R\vec{s}, & \\
\exists_n \forall_{u' \succeq_n u} (R\vec{s} \in \Delta_{u'}) & \text{by (4.2) and (4.1),} \\
\exists_n \forall_{u' \succeq_n u} R^{\mathcal{M}}(\vec{s}, u') & \text{by definition of } \mathcal{M}, \\
k \Vdash R\vec{s} & \text{by definition of } \Vdash, \text{ since } t^{\mathcal{M}}[\mathrm{id}] = t.
\end{array}$$

Case $B \vee C$. Assume $\mathrm{FV}(B \vee C) \subseteq V_u$. For the implication $(\Rightarrow)$ let $\Gamma, \Delta_u \vdash B \vee C$. Choose an $n \geq |u|$ such that $\Gamma_n, \Delta_u \vdash_n A_n = B \vee C$. Then, for all $u' \succeq u$ such that $|u'| = n$,

$$\Delta_{u*0} = \Delta_{u'} \cup \{B \vee C, B\} \quad \text{and} \quad \Delta_{u'*1} = \Delta_{u'} \cup \{B \vee C, C\},$$

and therefore by hypothesis on B and C

$$u' * 0 \Vdash B \quad \text{and} \quad u' * 1 \Vdash C.$$

Then by definition we have $u \Vdash B \vee C$. For the reverse implication $(\Leftarrow)$ we argue as follows:

$$\begin{array}{ll}
u \Vdash B \vee C, & \\
\exists_n \forall_{u' \succeq_n u}(u' \Vdash B \vee u' \Vdash C), & \\
\exists_n \forall_{u' \succeq_n u}((\Gamma, \Delta_{u'} \vdash B) \vee (\Gamma, \Delta_{u'} \vdash C)) & \text{by hypothesis on } B, C, \\
\exists_n \forall_{u' \succeq_n u} (\Gamma, \Delta_{u'} \vdash B \vee C), & \\
\Gamma, \Delta_u \vdash B \vee C & \text{by (4.1).}
\end{array}$$

Case $B \wedge C$. This is easy.

Case $B \to C$. Assume $\mathrm{FV}(B \to C) \subseteq V_k$. For ($\Rightarrow$) let $\Gamma, \Delta_u \vdash B \to C$. We must show $u \Vdash B \to C$, i.e.,

$$\forall_{u' \succeq u}(u' \Vdash B \to u' \Vdash C).$$

Let $u' \succeq u$ be such that $u' \Vdash B$. By hypothesis on B, it follows that $\Gamma, \Delta_{u'} \vdash B$. Hence $\Gamma, \Delta_{u'} \vdash C$ follows by assumption. Then again by hypothesis on C we get $u' \Vdash C$.

For ($\Leftarrow$) let $u \Vdash B \to C$, i.e., $\forall_{u' \succeq u}(u' \Vdash B \to u' \Vdash C)$. We show that $\Gamma, \Delta_u \vdash B \to C$, using (4.1). Choose $n \geq \mathrm{lh}(k)$ such that $B \equiv A_n$. For all $u' \succeq_m u$ with $m \equiv n - |u|$ we show that $\Gamma, \Delta_{u'} \vdash B \to C$.

If $\Gamma_n, \Delta_{u'} \vdash_n A_n$, then $u' \Vdash B$ by induction hypothesis, and $u' \Vdash C$ by assumption. Hence $\Gamma, \Delta_{u'} \vdash C$ again by hypothesis on C and thus $\Gamma, \Delta_{u'} \vdash B \to C$.

If $\Gamma_n, \Delta_{u'} \not\vdash_n A_n$, then by definition $\Delta_{u'*1} = \Delta_{u'} \cup \{B\}$. Hence $\Gamma, \Delta_{u'*1} \vdash B$, and thus $u' * 1 \Vdash B$ by hypothesis on B. Now $u' * 1 \Vdash C$ by assumption, and finally $\Gamma, \Delta_{u'*1} \vdash C$ by hypothesis on C. From $\Delta_{u'*1} = \Delta_{u'} \cup \{B\}$ it follows that $\Gamma, \Delta_{u'} \vdash B \to C$.

Case $\forall_x B(x)$. Assume $\mathrm{FV}(\forall_x B(x)) \subseteq V_u$. For ($\Rightarrow$) let $\Gamma, \Delta_u \vdash \forall_x B(x)$. Fix a term t. Then $\Gamma, \Delta_u \vdash B(t)$. Choose $n \geq |k|$ such that $\mathrm{FV}(B(t)) \subseteq V_{u'}$ for all u' with $|u'| = n$. Then $\forall_{u' \succeq_m u}(\Gamma, \Delta_{u'} \vdash B(t))$ with $m := n - |k|$, hence $\forall_{u' \succeq_m u}(u' \Vdash B(t))$ by hypothesis on $B(t)$, hence $u \Vdash B(t)$ by the covering lemma. This holds for every term t, hence $k \Vdash \forall_x B(x)$.

For ($\Leftarrow$) assume $u \Vdash \forall_x B(x)$. Pick $u' \succeq_n u$ such that $A_m \equiv \exists_x(\bot \to \bot)$, for $m \equiv |u| + n$. Then at height m we put some x_i into the variable sets: for $u' \succeq_n u$ we have $x_i \notin V_{u'}$ but $x_i \in V_{u'*j}$. Clearly $u' * j \Vdash B(x_i)$, hence $\Gamma, \Delta_{u'*j} \vdash B(x_i)$ by hypothesis on $B(x_i)$), hence (since at this height we consider the trivial formula $\exists_x(\bot \to \bot)$) also $\Gamma, \Delta_{u'} \vdash B(x_i)$. Since $x_i \notin V_{u'}$ we obtain $\Gamma, \Delta_{u'} \vdash \forall_x B(x)$. This holds for all $u' \succeq_n u$, hence $\Gamma, \Delta_u \vdash \forall_x B(x)$ by (4.1).

Case $\exists_x B(x)$. Assume $\mathrm{FV}(\exists_x B(x)) \subseteq V_u$. For ($\Rightarrow$) let $\Gamma, \Delta_u \vdash \exists_x B(x)$. Choose an $n \geq |u|$ such that $\Gamma_n, \Delta_u \vdash_n A_n = \exists_x B(x)$. Then, for all $u' \succeq u$ with $|u'| = n$

$$\Delta_{u'*0} = \Delta_{u'*1} = \Delta_{u'} \cup \{\exists_x B(x), B(x_i)\}$$

where $x_i \notin V_{u'}$. Hence by hypothesis on $B(x_i)$ (applicable since $\mathrm{FV}(B(x_i)) \subseteq V_{u'*j}$)

$$u' * 0 \Vdash B(x_i) \quad \text{and} \quad u' * 1 \Vdash B(x_i).$$

It follows by definition that $u \Vdash \exists_x B(x)$.

For ($\Leftarrow$) assume $u \Vdash \exists_x B(x)$. Then $\forall_{u' \succeq_n u} \exists_{t \in \mathtt{Term}}(u' \Vdash B(x)[\mathrm{id}_x^t])$ for some n, hence $\forall_{u' \succeq_n u} \exists_{t \in \mathtt{Term}}(u' \Vdash B(t))$. For each of the finitely many

$u' \succeq_n u$ pick an m such that $\forall_{u'' \succeq_m u'} (\mathrm{FV}(B(t)) \subseteq V_{u''})$. Let m_0 be the maximum of all these m. Then

$$\forall_{u'' \succeq_{m_0+n} u} \exists_{t \in \mathtt{Term}} ((u'' \Vdash B(t)) \wedge \mathrm{FV}(B(t)) \subseteq V_{u''}).$$

The hypothesis on $B(t)$ yields

$$\begin{aligned} &\forall_{u'' \succeq_{m_0+n} k} \exists_{t \in \mathtt{Term}} (\Gamma, \Delta_{u''} \vdash B(t)), \\ &\forall_{u'' \succeq_{m_0+n} k} (\Gamma, \Delta_{u''} \vdash \exists_x B(x)), \\ &\Gamma, \Delta_u \vdash \exists_x B(x) \qquad \text{by (4.1)}, \end{aligned}$$

and this completes the proof of the claim.

Now we finish the proof of the completeness theorem by showing that (b) implies (a). We apply (b) to the tree model $\mathcal{M}$ constructed above from Γ, the empty node $\emptyset$ and the assignment $\eta = \mathrm{id}$. Then $\mathcal{M}, \emptyset \Vdash \Gamma[\mathrm{id}]$ by the claim (since each formula in Γ is derivable from Γ). Hence $\mathcal{M}, \emptyset \Vdash A[\mathrm{id}]$ by (b) and therefore $\Gamma \vdash A$ by the claim again. □

Completeness of intuitionistic logic follows as a corollary.

COROLLARY 4.3.2 (Completeness of intuitionistic logic). *Let* $\Gamma \cup \{A\} \subseteq$ `Form`. *The following are equivalent.*

(a) $\Gamma \vdash_i A$.

(b) $\Gamma, \mathrm{Efq} \Vdash A$, *i.e., for all intuitionistic fan models* $\mathcal{M}_i$, *assignments* η *in* $|\mathcal{M}_i|$ *and nodes* u *in the fan of* $\mathcal{M}_i$

$$\mathcal{M}_i, u \Vdash \Gamma[\eta] \to \mathcal{M}_i, u \Vdash A[\eta].$$

PROOF. It follows immediately from Theorem 4.3.1. □

4.4. Soundness and completeness of classical logic

We give a proof of completeness of classical logic which relies on the above completeness proof for minimal logic.

We define the notion of a (classical) model (or more accurately, $\mathcal{L}$-model), and what the value of a term and the meaning of a formula in a model should be. The latter definition is by induction on formulas, where in the quantifier case we need a quantifier in the definition.

For the rest of this section, fix a countable formal language $\mathcal{L}$; we do not mention the dependence on $\mathcal{L}$ in the notation. Since we deal with classical logic, we only consider formulas built without $\vee, \exists$.

DEFINITION 4.4.1. A *model* is a triple $\mathcal{M} = (D, \mathbf{i}, \mathbf{j})$ such that

(a) D is an inhabited set;
(b) for every n-ary function symbol f, $\mathbf{i}$ assigns to f a map $\mathbf{i}(f)\colon D^n \to D$;
(c) for every n-ary relation symbol R, $\mathbf{j}$ assigns to R an n-ary relation on D^n. In case $n = 0$, $\mathbf{j}(R)$ is either true or false. We require that $\mathbf{j}(\bot)$ is false.

We write $|\mathcal{M}|$ for the carrier set D of $\mathcal{M}$ and $f^{\mathcal{M}}$, $R^{\mathcal{M}}$ for the interpretations $\mathbf{i}(f)$, $\mathbf{j}(R)$ of the function and relation symbols. *Assignments* η and their extensions on `Term` are defined as in section 4.1. We write $t^{\mathcal{M}}[\eta]$ for $\eta_{\mathcal{M}}(t)$.

Definition 4.4.2 (Validity). For every model $\mathcal{M}$, assignment η in $|\mathcal{M}|$ and formula A such that $\mathrm{FV}(A) \subseteq \mathrm{dom}(\eta)$ we define $\mathcal{M} \models A[\eta]$ (read: A is *valid* in $\mathcal{M}$ under the assignment η) by induction on A.

$$\begin{aligned}
\mathcal{M} &\models (R\vec{s})[\eta] &&\equiv R^{\mathcal{M}}(\vec{s}^{\mathcal{M}}[\eta]),\\
\mathcal{M} &\models (A \to B)[\eta] &&\equiv ((\mathcal{M} \models A[\eta]) \to (\mathcal{M} \models B[\eta])),\\
\mathcal{M} &\models (A \wedge B)[\eta] &&\equiv ((\mathcal{M} \models A[\eta]) \wedge (\mathcal{M} \models B[\eta])),\\
\mathcal{M} &\models (\forall_x A)[\eta] &&\equiv \forall_{a \in |\mathcal{M}|}(\mathcal{M} \models A[\eta_x^a]).
\end{aligned}$$

Since $\mathbf{j}(\bot)$ is false, we have $\mathcal{M} \not\models \bot[\eta]$.

Lemma 4.4.3 (Coincidence). *Let $\mathcal{M}$ be a model, t a term, A a formula and η, ξ assignments in $|\mathcal{M}|$.*
(a) *If $\eta(x) = \xi(x)$ for all $x \in \mathrm{FV}(t)$, then $\eta(t) = \xi(t)$.*
(b) *If $\eta(x) = \xi(x)$ for all $x \in \mathrm{FV}(A)$, then $\mathcal{M} \models A[\eta]$ if and only if $\mathcal{M} \models A[\xi]$.*

Proof. By induction on terms and formulas. □

Lemma 4.4.4 (Substitution). *Let $\mathcal{M}$ be a model, $t, r(x)$ terms, $A(x)$ a formula and η an assignment in $|\mathcal{M}|$. Then*
(a) $\eta(r(t)) = \eta_x^{\eta(t)}(r(x))$.
(b) $\mathcal{M} \models A(t)[\eta]$ *if and only if* $\mathcal{M} \models A(x)[\eta_x^{\eta(t)}]$.

Proof. By induction on terms and formulas. □

Definition 4.4.5. A model $\mathcal{M}$ is called *classical*, if

$$\neg\neg R^{\mathcal{M}}(\vec{a}\,) \Rightarrow R^{\mathcal{M}}(\vec{a}),$$

for all relation symbols R and all $\vec{a} \in |\mathcal{M}|$.

The above definition makes sense when our metatheory is *not* classical. First we prove that every formula derivable in classical logic is valid in an arbitrary classical model.

THEOREM 4.4.6 (Soundness of classical logic). *Let* $\Gamma \cup \{A\} \subseteq \texttt{Form}$ *such that* $\Gamma \vdash_c A$. *If* $\mathcal{M}$ *is a classical model and* η *an assignment in* $|\mathcal{M}|$, *then* $\mathcal{M} \models \Gamma[\eta]$ *implies* $\mathcal{M} \models A[\eta]$.

PROOF. By induction on derivations. $\mathcal{M} \models C[\eta]$ is abbreviated $\mathcal{M} \models C$ when η is known from the context.

Case Stab. For the stability axiom $\forall_{\vec{x}}(\neg\neg R\vec{x} \to R\vec{x}\,)$ the claim follows from our assumption that $\mathcal{M}$ is classical, i.e., $\neg\neg R^{\mathcal{M}}(\vec{a}\,) \to R^{\mathcal{M}}(\vec{a}\,)$ for all $\vec{a} \in |\mathcal{M}|$.

Case ax. It is immediate.

Case $\to^+$. Assume $\mathcal{M} \models \Gamma$. We show $\mathcal{M} \models (A \to B)$. So assume in addition $\mathcal{M} \models A$. We must show $\mathcal{M} \models B$. By induction hypothesis (with $\Gamma \cup \{A\}$ instead of Γ) this clearly holds.

Case $\to^-$. Assume $\mathcal{M} \models \Gamma$. We must show $\mathcal{M} \models B$. By induction hypothesis, $\mathcal{M} \models (A \to B)$ and $\mathcal{M} \models A$. The claim follows from the definition of $\models$.

Case $\wedge^+$ and *Case* $\wedge^-$. This are easy.

Case $\forall^+$. Assume $\mathcal{M} \models \Gamma[\eta]$ and $x \notin \mathrm{FV}(\Gamma)$. We show $\mathcal{M} \models (\forall_x A)[\eta]$, i.e., $\mathcal{M} \models A[\eta_x^a]$ for an arbitrary $a \in |\mathcal{M}|$. We have

$$\begin{array}{ll} \mathcal{M} \models \Gamma[\eta_x^a] & \text{by the coincidence lemma, since } x \notin \mathrm{FV}(\Gamma), \\ \mathcal{M} \models A[\eta_x^a] & \text{by induction hypothesis.} \end{array}$$

Case $\forall^-$. Let $\mathcal{M} \models \Gamma[\eta]$. We show that $\mathcal{M} \models A(t)[\eta]$. This follows from

$$\begin{array}{ll} \mathcal{M} \models (\forall_x A(x))[\eta] & \text{by induction hypothesis,} \\ \mathcal{M} \models A(x)[\eta_x^{\eta(t)}] & \text{by definition,} \\ \mathcal{M} \models A(t)[\eta] & \text{by the substitution lemma.} \end{array}$$

□

Next we give a constructive analysis of the completeness of classical logic by using, in the metatheory below, constructively valid arguments only, mentioning explicitly any assumptions which go beyond. When dealing with the classical fragment we of course need to restrict to classical models.

The only non-constructive principle used in the following proof is the *axiom of dependent choice* for the weak existential quantifier:

$$\tilde{\exists}_x A(0,x) \wedge \forall_{n,x}\bigl(A(n,x) \Rightarrow \tilde{\exists}_y A(n+1,y)\bigr) \Rightarrow \tilde{\exists}_f \forall_n A(n,fn).$$

Recall that we only consider formulas without $\vee, \exists$.

THEOREM 4.4.7 (Completeness of classical logic). *Let* $\Gamma \cup \{A\} \subseteq \texttt{Form}$. *Assume that for all classical models* $\mathcal{M}$ *and assignments* η,

$$\mathcal{M} \models \Gamma[\eta] \Rightarrow \mathcal{M} \models A[\eta].$$

Then there must exist a derivation of A *from* $\Gamma \cup \mathrm{Stab}$.

PROOF. Since "there must exist a derivation" expresses the weak existential quantifier in the metalanguage, we need to prove a contradiction from the assumption $\Gamma, \mathrm{Stab} \nvdash A$.

By the completeness theorem for minimal logic, there must be a tree model $\mathcal{M} = (\texttt{Term}, 2^{<\mathbb{N}}, \mathbf{i}, \mathbf{j}$ and a node u_0 such that $u_0 \Vdash \Gamma, \mathrm{Stab}$ and $u_0 \nVdash A$.

Call a node u *consistent* if $\nVdash \bot$, and *stable* if $u \Vdash \mathrm{Stab}$. We prove

$$u \nVdash B \to \tilde{\exists}_{u' \succeq u}(u' \Vdash \neg B \wedge u' \nVdash \bot) \qquad (u \text{ stable}). \tag{4.3}$$

Let u be a stable node, and B a formula (without $\vee$, $\exists$). Then $\mathrm{Stab} \vdash \neg\neg B \to B$ by the stability theorem, and therefore $u \Vdash \neg\neg B \to B$. Hence from $u \nVdash B$ we obtain $u \nVdash \neg\neg B$. By definition this implies $\neg\forall_{u' \succeq u}(u' \Vdash \neg B \Rightarrow u' \Vdash \bot)$, which proves (4.3) (since $u' \nVdash \bot$ implies $u' * 0 \nVdash \bot$ or $u' * 1 \nVdash \bot$).

Let α be a branch in the underlying tree $2^{<\mathbb{N}}$. We define

$$\begin{aligned} \alpha \Vdash A \quad &\equiv \tilde{\exists}_{u \in \alpha}(u \Vdash A), \\ \alpha \text{ is consistent} &\equiv \alpha \nVdash \bot, \\ \alpha \text{ is stable} \quad &\equiv \tilde{\exists}_{u \in \alpha}(u \Vdash \mathrm{Stab}). \end{aligned}$$

Note that from $\alpha \Vdash \vec{A}$ and $\vdash \vec{A} \to B$ it follows that $\alpha \Vdash B$. To see this, consider $\alpha \Vdash \vec{A}$. Then $u \Vdash \vec{A}$ for a $u \in \alpha$, since α is linearly ordered. From $\vdash \vec{A} \to B$ it follows that $u \Vdash B$, i.e., $\alpha \Vdash B$.

A branch α is *generic* (in the sense that it generates a classical model) if it is consistent and stable, if in addition for all formulas B

$$(\alpha \Vdash B) \,\tilde{\vee}\, (\alpha \Vdash \neg B), \tag{4.4}$$

and if for all formulas $\forall_{\vec{y}} B(\vec{y})$ with $B(\vec{y})$ not a universal formula

$$\forall_{\vec{s} \in \texttt{Term}^{|\vec{s}|}}(\alpha \Vdash B(\vec{s})) \to \alpha \Vdash \forall_{\vec{y}} B(\vec{y}). \tag{4.5}$$

For a branch α, we define a classical model $\mathcal{M}^{\alpha} = (\texttt{Term}, \mathbf{i}, \mathbf{j}^{\alpha})$ as

$$\mathbf{j}^{\alpha}(R)(\vec{s}) \equiv \tilde{\exists}_{u \in \alpha} \mathbf{j}(R, u)(\vec{s}) \qquad (R \neq \bot).$$

Since $\tilde{\exists}$ is used in this definition, $\mathcal{M}^{\alpha}$ is stable.

We show that for every generic branch α and formula B (without $\vee$, $\exists$)

$$\alpha \Vdash B \Leftrightarrow \mathcal{M}^{\alpha} \models B. \tag{4.6}$$

The proof is by induction on the logical complexity of B.

Case $R\vec{s}$ with $R \neq \bot$. Then (4.6) holds for all α.

Case $\bot$. We have $\alpha \not\Vdash \bot$ since α is consistent.

Case $B \to C$. Let $\alpha \Vdash B \to C$ and $\mathcal{M}^\alpha \models B$. We must show that $\mathcal{M}^\alpha \models C$. Note that $\alpha \Vdash B$ by induction hypothesis, hence $\alpha \Vdash C$, hence $\mathcal{M}^\alpha \models C$ again by induction hypothesis. Conversely let $\mathcal{M}^\alpha \models B \to C$. Clearly $(\mathcal{M}^\alpha \models B) \tilde{\vee} (\mathcal{M}^\alpha \not\models B)$. If $\mathcal{M}^\alpha \models B$, then $\mathcal{M}^\alpha \models C$. Hence $\alpha \Vdash C$ by induction hypothesis and therefore $\alpha \Vdash B \to C$. If $\mathcal{M}^\alpha \not\models B$ then $\alpha \not\Vdash B$ by induction hypothesis. Hence $\alpha \Vdash \neg B$ by (4.4) and therefore $\alpha \Vdash B \to C$, since α is stable (and $\vdash (\neg\neg C \to C) \to \bot \to C$). [Note that for this argument to be constructively valid one needs to observe that the formula $\alpha \Vdash B \to C$ is a negation, and therefore one can argue by the case distinction based on $\tilde{\vee}$. This is because, with $P_1 := \mathcal{M}^\alpha \models B$, $P_2 := \mathcal{M}^\alpha \not\models B$ and $Q := \alpha \Vdash B \to C$, the formula $(P_1 \tilde{\vee} P_2) \to (P_1 \to Q) \to (P_2 \to Q) \to Q$ is derivable in minimal logic.]

Case $B \wedge C$. Easy.

Case $\forall_{\vec{y}} B(\vec{y}\,)$ ($\vec{y}$ not empty) where $B(\vec{y}\,)$ is not a universal formula. The following are equivalent.

$$
\begin{array}{ll}
\alpha \Vdash \forall_{\vec{y}} B(\vec{y}\,), & \\
\forall_{\vec{s} \in \mathrm{Ter}}(\alpha \Vdash B(\vec{s}\,)) & \text{by (4.5)}, \\
\forall_{\vec{s} \in \mathtt{Term}^{|\vec{s}|}}(\mathcal{M}^\alpha \models B(\vec{s}\,)) & \text{by induction hypothesis}, \\
\mathcal{M}^\alpha \models \forall_{\vec{y}} B(\vec{y}\,). &
\end{array}
$$

This concludes the proof of (4.6).

Next we show that for every consistent and stable node u there must be a generic branch containing u:

$$
u \not\Vdash \bot \to u \Vdash \mathrm{Stab} \to \tilde{\exists}_\alpha(\alpha \text{ generic} \wedge u \in \alpha). \tag{4.7}
$$

For the proof, let $A_0, A_1, \dots$ enumerate all formulas. We define a sequence $u \equiv u_0 \preceq u_1 \preceq u_2 \dots$ of consistent stable nodes by dependent choice. Assume that u_n is defined. We write A_n in the form $\forall_{\vec{y}} B(\vec{y}\,)$ (with $\vec{y}$ possibly empty) where B is not a universal formula. In case $u_n \Vdash \forall_{\vec{y}} B(\vec{y}\,)$ let $u_{n+1} \equiv u_n$. Otherwise we have $u_n \not\Vdash B(\vec{s}\,)$ for some $\vec{s}$, and by (4.3) there must be a consistent node $u' \succ u_n$ such that $u' \Vdash \neg B(\vec{s}\,)$. Let $u_{n+1} \equiv u'$. Since $u_n \preceq u_{n+1}$, the node u_{n+1} is stable.

Let $\alpha \equiv \{\, l \mid \exists_n (l \preceq u_n)\,\}$, hence $u \in \alpha$. We show that α is generic. Clearly α is consistent and stable. We now prove both (4.4) and (4.5). Let $C = \forall_{\vec{y}} B(\vec{y}\,)$ (with $\vec{y}$ possibly empty) where $B(\vec{y}\,)$ is not a universal formula, and choose n such that $C = A_n$. In case $u_n \Vdash \forall_{\vec{y}} B(\vec{y}\,)$ we are done. Otherwise by construction $u_{n+1} \Vdash \neg B(\vec{s})$ for some $\vec{s}$. For (4.4) we get $u_{n+1} \Vdash \neg\forall_{\vec{y}} B(\vec{y})$ since $\vdash \forall_{\vec{y}} B(\vec{y}) \to B(\vec{s})$, and (4.5) follows from the consistency of α. This concludes the proof of (4.7).

Now we can finalize the completeness proof. Recall that $u_0 \Vdash \Gamma, \mathrm{Stab}$ and $u_0 \nVdash A$. Since $u_0 \nVdash A$ and u_0 is stable, (4.3) yields a consistent node $u \succeq u_0$ such that $u \Vdash \neg A$. Evidently, u is stable as well. By (4.7) there must be a generic branch α such that $u \in \alpha$. Since $u \Vdash \neg A$ it follows that $\alpha \Vdash \neg A$, hence $\mathcal{M}^\alpha \models \neg A$ by (4.6). Moreover, $\alpha \Vdash \Gamma$, thus $\mathcal{M}^\alpha \models \Gamma$ by (4.6). This contradicts our assumption. □

Since in the above proof the carrier set of the classical model M^α is the countable set `Term`, we conclude the following.

Remark 4.4.8. The hypothesis of completeness theorem can be replaced by the the following:
"Assume that for all classical models $\mathcal{M}$ with a countable carrier set, for all assignments η, $\mathcal{M} \models \Gamma[\eta] \Rightarrow \mathcal{M} \models A[\eta]$".

Definition 4.4.9. A set Γ of formulas is *consistent*, if $\Gamma \nvdash_c \bot$, and *satisfiable*, if there is (in the weak sense) a classical model $\mathcal{M}$ and an assignment η in $|\mathcal{M}|$ such that $\mathcal{M} \models \Gamma[\eta]$.

Corollary 4.4.10. *Let Γ be a set of formulas.*
(a) *If Γ is consistent, then Γ is satisfiable.*
(b) *(Compactness) If each finite subset of Γ is satisfiable, Γ is satisfiable.*

Proof. (a) Assume $\Gamma \nvdash_c \bot$ and that for all classical models $\mathcal{M}$ we have $\mathcal{M} \not\models \Gamma$, hence $\mathcal{M} \models \Gamma$ implies $\mathcal{M} \models \bot$. Then the completeness theorem yields a contradiction.

(b) Otherwise by the completeness theorem there must be a derivation of $\bot$ from $\Gamma \cup \mathrm{Stab}$, hence also from $\Gamma_0 \cup \mathrm{Stab}$ for some finite subset $\Gamma_0 \subseteq \Gamma$. This contradicts the assumption that Γ_0 is satisfiable. □

4.5. Notes

Fan models were introduced by Beth in [**3**]. The proof of completeness of minimal logic is due to Harvey Friedman. The proof of completeness of classical logic from the completeness of minimal logic is due to Ulrich Berger.

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